Advances in Differential Equations Volume 9, Numbers 1-2, January-February 2004, Pages 53–84

BACKWARD UNIQUENESS OF SEMIGROUPS ARISING IN COUPLED PARTIAL DIFFERENTIAL EQUATIONS SYSTEMS OF STRUCTURAL ACOUSTICS

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Abstract. In this paper we consider two established structural acoustic models: Mathematically, the first model couples a *hyperbolic* (wave) equation, defined within a two- or three-dimensional acoustic chamber, with an elastic plate (or beam) equation, possibly with structural damping (*parabolic* type), defined on its elastic (flat) wall. Instead, the second model couples the same *hyperbolic* equation this time with a thermoelastic plate (or beam), either of *parabolic* type or else of *hyperbolic-dominated* type, defined on its flexible (flat) wall. The thermoelastic component may be supplemented by any canonical boundary conditions (B.C.), including the coupled free B.C. Moreover, its differential operators may have variable coefficients (in space). In either of the two models, coupling takes place on the elastic wall. This coupled PDE system (possibly with hyperbolic/parabolic interaction) generates a strongly continuous contraction semigroup e^{At} on a natural energy space Y. The main result of the present paper is a *backward uniqueness* theorem for such structural acoustic semigroups: $e^{AT}y_0 = 0$ for some $T > 0$ and $y_0 \in Y$ implies $y_0 = 0$.

1. Introduction. Backward uniqueness. Literature

Structural acoustic models as coupled PDE systems. The present paper considers two established, rather general, structural acoustic models [10], [24], [1], [2], [3], [15], [16], [5]. Mathematically, the first model couples a *hyperbolic* (wave) equation, defined within a two- or three-dimensional acoustic chamber, with an elastic plate (or beam) equation, possibly with structural damping (*parabolic* type, case $\frac{1}{2} \le \alpha \le 1$ below in (2.1.4)), defined on its elastic (flat) wall. The second model couples the same *hyperbolic* equation this time with a thermoelastic plate (or beam). The latter may be either of *parabolic* type (rotational inertia is not accounted for), or else

Accepted for publication: May 2003.
AMS Subject Classifications: 46, 47, 35.
Research partially supported by the National Science Foundation under Grant DMS-9804056 and the Army Research Office, under Grant DAAD 19-02-1-0179.

of *hyperbolic-dominated* type [21] (rotational inertia is accounted for). The thermoelastic component has variable (in space) coefficients and may be supplemented by any of the canonical boundary conditions (B.C.), including the coupled free B.C. In either of the two models, coupling takes place on the elastic wall, by virtue of the velocity z_t of the wave and of the velocity v_t of the elastic displacement, see equations (2.1.1c–d) or equations (2.2.1c–d) below.

In all of the above cases, the resulting coupled PDE system generates a strongly continuous contraction semigroup e^{At} on a natural state (energy) space Y (equations (3.2) below for the elastic-wall case, or (3.11) below for the thermoelastic-wall case, respectively).

Goal. The main result of the present paper is a *backward uniqueness* theorem for such a structural acoustic semigroup for both models, that is, the property that $e^{AT}y_0 = 0$ for some $T > 0$ and $y_0 \in Y$ implies $y_0 = 0$.

Backward uniqueness literature. Backward uniqueness is trivial for strongly continuous groups; simple to prove for strongly continuous analytic semigroups; and possibly patently false, as in the case of nilpotent semigroups; see, e.g., [17, Remark 1.1, p. 220]. In our present two cases, dealing with structural acoustic models, our coupled PDE systems display a hyperbolic/parabolic interaction (case $\frac{1}{2} \leq \alpha \leq 1$, below in (2.1.4)) or a hyperbolic/Gevrey class interaction (case $0 < \alpha < \frac{1}{2}$, below in (2.1.4)) for the first model; or else a hyperbolic/thermoelastic interaction for the second model, where the thermoelastic model may be either of analytic *(parabolic)* type, or of *hyperbolic-dominated* type [21]. Thus, the present structural acoustic semigroup does not fit into the above cases, for any of the aforementioned combinations: it is neither analytic, nor a group.

Our proof of backward uniqueness for the structural acoustic semigroups is based on the application of a recent new, abstract backward uniqueness theorem for strongly continuous semigroups in a Banach space [17, Theorem 3.1, p. 225], reported as Theorem 3.3 below. This theorem was motivated by the desire to show backward uniqueness results for strongly continuous semigroups arising from (uncoupled) *thermoelastic* plate systems which account for *rotational forces*, under all canonical boundary conditions. This result was shown, in fact, to hold true in [17] by relying on the aforementioned abstract backward uniqueness theorem [17, Theorem 3.1]. Its assumptions are tailored to PDE systems, such as thermoelastic plates with rotational forces, which display a hyperbolic/parabolic interaction. In all such cases, one generally expects two parts of the spectrum of the generator: one which aligns vertically along a parallel to the imaginary axis (hyperbolic component), and one which aligns along the negative real axis (parabolic component). (In our case, for the case $0 < \alpha < \frac{1}{2}$, there is no parabolic component, but rather a Gevrey-class component (Lemma 3.2(iii) below), whose spectrum is then contained in a curved sector [26].) Thus, on a line which goes at an angle in between, one expects to have a bounded resolvent operator.

This is the content of [17, Theorem 3.1], reported here as Theorem 3.3. In the case of thermoelastic plates with rotational forces, under all canonical boundary conditions the rate of decay of the resolvent operator is actually $\mathcal{O}(\frac{1}{|\lambda|})$ along said rays $\lambda = |\lambda|e^{\pm ia}$, $\frac{\pi}{2} < a < \pi$ [17]. In our present structural acoustic case with thermoelastic wall, we use *critically* such a decay rate, along said rays, of the thermoelastic component, to achieve the same rate $\mathcal{O}(\frac{1}{|\lambda|})$, along said rays, of the overall structure. A similar decay rate holds true for a structural acoustic chamber with elastic wall. Thus, in both cases, the assumptions of the abstract backward uniqueness result (Theorem 3.3) are *a fortiori* fulfilled.

It was the encouragement and expectation of the referee of [17] that the abstract backward uniqueness result there provided should have applications to other coupled PDE systems, beyond thermoelastic systems. This paper provides a positive response to such an expectation, by adding the class of structural acoustic semigroups with elastic or else thermoelastic flexible wall.

2. Mathematical PDE models and statement of main results

2.1. **Structural acoustic model with elastic wall. The mathematical model.** In qualitative terms, the mathematical model under consideration consists of a wave equation, active within an acoustic chamber, which is then strongly coupled with a dynamic abstract plate equation acting only on the elastic, flat wall of the chamber. More precisely, let $\Omega \subset \mathbb{R}^3$ be an open, bounded domain ("the acoustic chamber") with boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, where Γ_0 and Γ_1 are open, connected, disjoint parts, $\Gamma_0 \cap \Gamma_1 = \emptyset$ in $\mathbb{R}^2$, of positive measure. The sub-boundary Γ_0 is flat and is referred to as the elastic or flexible wall. Instead, Γ_1 is referred to as the rigid or hard wall. The interaction between wave and plate takes place on Γ_0. We also assume that either Ω is sufficiently smooth (say, Γ is of class C^2), or else Ω is convex. This assumption guarantees that solutions to classical elliptic equations with $L_2(\Omega)$ forcing term are in $H^2(\Omega)$ [12], or that the domain of the Laplacian in Ω, with (either Dirichlet or) Neumann B.C., is contained in $H^2(\Omega)$: see (4.1.1) in our case. The acoustic medium in the chamber is described by the wave equation in the variable z with acoustic pressure $\rho_1 z_t$, where ρ_1 is the density of the fluid. Moreover, let c^2 be the speed of sound. Finally, denote by v the "abstract deflection" of the abstract plate equation on Γ_0.

In this subsection, we consider the following coupled PDE system (where with no loss of generality, we have normalized to 1 both the speed of sound c and the density ρ_1 of the fluid):

$$z_{tt} = \Delta z - dz_t \qquad \text{in } Q \equiv (0,T] \times \Omega; \qquad (2.1.1a)$$

$$\frac{\partial z}{\partial \nu} + bz = 0 \qquad \text{in } \Sigma_1 = (0,T] \times \Gamma_1; \qquad (2.1.1b)$$

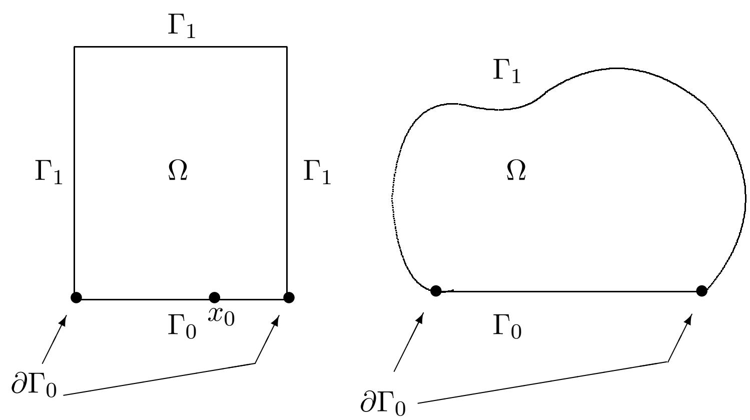
$$\frac{\partial z}{\partial \nu} + \beta D_0 z = v_t \qquad \text{in } \Sigma_0 = (0,T] \times \Gamma_0; \qquad (2.1.1c)$$

$$v_{tt} + \mathcal{A}v + \mathcal{B}v_t + z_t|_{\Gamma_0} = 0; \qquad (2.1.1d)$$

$$z(0,\,\cdot\,) = z_0,\ z_t(0,\,\cdot\,) = z_1 \qquad \text{in } \Omega; \qquad (2.1.1e)$$

$$v(0,\,\cdot\,) = v_0,\ v_t(0,\,\cdot\,) = v_1 \qquad \text{in } \Gamma_0; \qquad (2.1.1f)$$

see [1], [2], [3], [15], [16], [5], and [22, Volume II, Sections 9.10 and 9.11], under the following assumptions to be held throughout the paper.



Assumptions (H.1). $\mathcal{A}$ (the elastic operator): $L_2(\Gamma_0) \supset \mathcal{D}(\mathcal{A}) \to L_2(\Gamma_0)$ is a positive, self-adjoint operator. Moreover,

$$d = d(\,\cdot\,) \in L_\infty(\Omega); \quad b = \text{constant} > 0; \quad \beta = \text{constant} \geq 0. \qquad (2.1.2)$$

(H.2): $D_0 : L_2(\Gamma_0) \supset \mathcal{D}(D_0) \to L_2(\Gamma_0)$ is a positive, self-adjoint operator such that $H^1(\Gamma_0) \subset \mathcal{D}(D_0^{\frac{1}{2}})$; that is,

$$D_0^{\frac{1}{2}} : \ \text{continuous } H^1(\Gamma_0) \to L_2(\Gamma_0). \qquad (2.1.3)$$

(H.3): $\mathcal{B}$ (the dissipation operator): $L_2(\Gamma_0) \supset \mathcal{D}(\mathcal{B}) \to L_2(\Gamma_0)$ is a positive, self-adjoint operator satisfying the following property: there exist a constant $0 < \alpha \leq 1$ and two constants $0 < c_1 < c_2 < \infty$, such that

$$c_1(\mathcal{A}^\alpha x, x) \leq (\mathcal{B}x, x) \leq c_2(\mathcal{A}^\alpha x, x), \qquad x \in \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}) \subset \mathcal{D}(\mathcal{B}^{\frac{1}{2}}), \qquad (2.1.4)$$

where the inner products are in $L_2(\Gamma_0)$, equivalently, that the self-adjoint operator $\mathcal{A}^{-\frac{\alpha}{2}}\mathcal{B}\mathcal{A}^{-\frac{\alpha}{2}}$ is bounded and boundedly invertible on $L_2(\Gamma_0)$.

Remark 2.1.1. What is relevant here is that (H.1) and (H.3) imply that the operator A_2 in (3.8) below generates a strongly continuous semigroup which is analytic for $\frac{1}{2} \le \alpha \le 1$, and of Gevrey class for $0 < \alpha < \frac{1}{2}$; at any rate, which satisfies estimate (3.9) on suitable rays. Weaker assumptions than (H.3) suffice to this end [9], [22, Volume I, Appendix 3B], but we shall not insist on this point. □

The following well-posedness result is contained as a special case of [5, Theorem 1.3.1] (which adds also the damping term $D_0 z_t$ on the left side of (2.1.1c); see equation (6.1) below). See also [22, Chapter 7, Proposition 7.6.2.1, p. 664].

Theorem 2.1.1 (Well-posedness). Assume (H.1), (H.2), and (H.3). With reference to the above problem (2.1.1), we have that the map

$$y_0 \equiv \{z_0, z_1, v_0, v_1\} \to [z(t), z_t(t), v(t), v_t(t)] \equiv e^{At} y_0$$

defines a strongly continuous contraction semigroup on the state space Y defined in (3.2) below, where the operator A is identified in (3.3) below. □

The main result of the present subsection is the following.

Theorem 2.1.2 (Backward uniqueness). *With reference to the structural acoustic semigroup e^{At} on the state space Y guaranteed by Theorem* 2.1.1, *the following backward uniqueness property holds true:*

$$e^{AT} y_0 = 0, \text{ for some } T > 0, \; y_0 \in Y \qquad \Rightarrow y_0 = 0. \tag{2.1.5}$$

The proof of Theorem 2.1.2 will be given in Sections 3 through 5.

2.2. **Structural acoustic model with thermoelastic wall. The model.** In the present section, Ω is again a two- or three-dimensional acoustic chamber with rigid wall Γ_1 and flexible wall Γ_0, assumed flat. We introduce one main change over the illustrative model of Section 2.1. We assume that the flexible wall Γ_0 accounts now also for thermal effects, and is therefore modeled by a thermoelastic beam or plate equation, with or without rotational-inertia term. The first case is of hyperbolic-dominated type [21]. The second case is of parabolic type [23], [18], [19], [20], [22, Chapter 3, Appendices D–I]. More specifically, let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be an open, bounded domain with boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, where Γ_0 and Γ_1 are open, connected, and disjoint parts, $\Gamma_0 \cap \Gamma_1 = \emptyset$ in $\mathbb{R}^{n-1}$. The sub-boundary Γ_0 is assumed *flat*. As in Subsection 2.1, we allow either Γ to be sufficiently smooth, say of class C^2, or else Ω to be convex: this assumption will then guarantee that solutions to classical elliptic equations with $L_2(\Omega)$-nonhomogeneous terms are in $H^2(\Omega)$ [12], or that the domain of the Laplacian in Ω, with the appropriate homogeneous B.C., is contained in $H^2(\Omega)$. In the present subsection, the mathematical model is given by the following coupled system of partial differential equations:

$$
\begin{cases}
\text{acoustic chamber} \begin{cases}
z_{tt} = \Delta z - d z_t & \text{in } Q \equiv (0,T] \times \Omega, \quad (2.2.1\text{a}) \\
\dfrac{\partial z}{\partial \nu} + bz = 0,\ b > 0 & \text{in } \Sigma_1 \equiv (0,T] \times \Gamma_1, \quad (2.2.1\text{b}) \\
\dfrac{\partial z}{\partial \nu} + \beta D_0 z = v_t,\ \beta \geq 0 & \text{in } \Sigma_0 \equiv (0,T] \times \Gamma_0, \quad (2.2.1\text{c})
\end{cases} \\
\text{thermo-elastic wall} \begin{cases}
v_{tt} - \gamma \mathcal{A}(x,\partial) v_{tt} + \mathcal{A}^2(x,\partial) v \\
\qquad + \operatorname{div}(\alpha(x) \nabla \theta) + z_t|_{\Gamma_0} \equiv 0 \quad \text{in } \Sigma_0, & (2.2.1\text{d}) \\
\theta_t - \mathcal{A}(x,\partial)\theta - \operatorname{div}(\alpha(x) \nabla v_t) \equiv 0 \quad \text{in } \Sigma_0, & (2.2.1\text{e})
\end{cases} \\
z(0,\cdot\,) = z_0,\ z_t(0,\cdot\,) = z_1 \quad \text{in } \Omega, \quad (2.2.1\text{f}) \\
v(0,\cdot\,) = v_0,\ v_t(0,\cdot\,) = v_1,\ \theta(0,\cdot\,) = \theta_0 \quad \text{in } \Gamma_0. \quad (2.2.1\text{g})
\end{cases}
$$

In (2.2.1c), D_0 is the same as in (H.2) = (2.1.3). Model (2.2.1d–e) couples a plate equation in v with a heat equation in θ. It is supplemented by boundary conditions to be given below. In (2.2.1d) γ is a nonnegative constant, $\gamma \geq 0$, so that the plate model accounts ($\gamma > 0$), or does not account ($\gamma = 0$), for rotational forces. Moreover, $-\mathcal{A}(x,\partial)$ denotes a second-order, strongly elliptic operator (canonically, in the constant-coefficient case, $\mathcal{A}(x,\partial) = \Delta$), with sufficiently smooth, space-dependent coefficients, say of class C^2. It is assumed that $\mathcal{A}(x,\partial)$ is formally negative self-adjoint; more precisely, that when equipped with the required boundary conditions, the realization of $-\mathcal{A}(x,\partial)$ is a positive, self-adjoint operator on $L_2(\Gamma_0)$. Finally, $\alpha(x)$ (thermal expansion) is a space-dependent function, say of class C^1, which varies with the properties of the plate material.

With model (2.2.1d–e), we associate several canonical boundary conditions, which are listed in the order of progressive mathematical difficulty.

Hinged mechanical boundary conditions/Dirichlet thermal boundary conditions. They are the following *uncoupled* boundary conditions:

$$v|_{\partial\Sigma_0} \equiv 0,\ \mathcal{A}(x,\partial)v|_{\partial\Sigma_0} \equiv 0,\ \theta|_{\partial\Sigma_0} \equiv 0 \text{ on } \partial\Sigma_0 = (0,T] \times \partial\Gamma_0. \qquad (2.2.2)$$

Clamped mechanical boundary conditions/either Dirichlet or else Neumann boundary conditions. They are the following *uncoupled* boundary conditions:

$$v|_{\partial\Sigma_0} \equiv 0, \ \left.\frac{\partial v}{\partial \tilde{\nu}_{\mathcal{A}}}\right|_{\partial\Sigma_0} \equiv 0, \text{ and either } \theta\Big|_{\partial\Sigma_0} \equiv 0, \text{ or else } \left.\frac{\partial \theta}{\partial \tilde{\nu}_{\mathcal{A}}}\right|_{\partial\Sigma_0} \equiv 0. \tag{2.2.3}$$

In (2.2.3), $(\partial/\partial\tilde{\nu}_{\mathcal{A}})$ denotes the co-normal derivative to the curve $\partial\Gamma_0$, when $\dim \Omega = 3$.

Hinged mechanical boundary conditions/Neumann thermal boundary conditions. They are the following boundary conditions, which are *coupled* on the boundary:

$$\begin{cases} v|_{\partial\Sigma_0} \equiv 0, \ [\mathcal{A}(x,\partial)v + \theta]_{\partial\Sigma_0} \equiv 0, \\ \left[\frac{\partial\theta}{\partial\tilde{\nu}_{\mathcal{A}}} + \tilde{b}\theta\right]_{\partial\Sigma_0} \equiv 0, \ \tilde{b}(x) \geq b_0 > 0, \ \partial\Sigma_0 = (0,T] \times \partial\Gamma_0, \end{cases} \tag{2.2.4}$$

where $\tilde{b}(x)$ is a positive function of class $L_\infty(\partial\Gamma_0)$.

Free boundary conditions. Here, for simplicity, we consider the boundary conditions that arise when $\mathcal{A}(x,\partial) = \Delta$; they are the following coupled boundary conditions [13]:

$$\begin{cases} \Delta v + B_1 v + \alpha(x)\theta \equiv 0 & \text{on } \partial\Sigma_0 = (0,T] \times \partial\Gamma_0; & \text{(2.2.5a)} \\ \dfrac{\partial \Delta v}{\partial\tilde{\nu}} + B_2 v - \gamma \dfrac{\partial v_{tt}}{\partial \tilde{\nu}} + \alpha(x)\dfrac{\partial\theta}{\partial\tilde{\nu}} \equiv 0 & \text{on } \partial\Sigma_0; & \text{(2.2.5b)} \\ \dfrac{\partial\theta}{\partial\nu} + \tilde{b}\theta \equiv 0, \ \tilde{b}(x) \geq b_0 > 0 & \text{on } \partial\Sigma_0, & \text{(2.2.5c)} \end{cases}$$

where the boundary operators B_1 and B_2 are given by [22, Chapter 3], [13], [14]

$$\begin{cases} B_1 = (1-\mu)\left[2\tilde{\nu}_1\tilde{\nu}_2 \dfrac{\partial^2}{\partial x_1 \partial x_2} - \tilde{\nu}_1^2 \dfrac{\partial^2}{\partial x_2^2} - \tilde{\nu}_2^2 \dfrac{\partial^2}{\partial x_1^2}\right] & \text{(2.2.6a)} \\ B_2 = (1-\mu)\dfrac{\partial}{\partial\tau}\left[(\tilde{\nu}_1^2 - \tilde{\nu}_2^2)\dfrac{\partial^2}{\partial x_1 \partial x_2} - \tilde{\nu}_1\tilde{\nu}_2\left(\dfrac{\partial^2}{\partial x_2^2} - \dfrac{\partial^2}{\partial x_1^2}\right)\right] + I. & \text{(2.2.6b)} \end{cases}$$

In (2.2.6), $0 < \mu < 1$ is the Poisson's modulus, $\tilde{\nu} = [\tilde{\nu}_1, \tilde{\nu}_2]$ is the unit outward normal on Γ_0, and $\tau = [-\tilde{\nu}_2, \tilde{\nu}_1]$ is the tangential unit vector along Γ_0, oriented counterclockwise, where $\dim \Omega = 3$.

Theorem 2.2.1. (Well-posedness) Assume the above setting of Section 2.2. With reference to the above problem (2.2.1), with $\gamma \geq 0$, under all B.C. (2.2.2) through (2.2.5), we have that the map

$$y_0 = [z_0, z_1, v_0, v_1, \theta_0] \rightarrow [z(t), z_t(t), v(t), v_t(t), \theta(t)] \equiv e^{At} y_0$$

defines a strongly continuous contraction semigroup on the state space $Y \equiv Y_\gamma$ defined in (3.11), where A (which depends on γ) is given by (3.12), and more specifically in Section 4.2. □

The proof of Theorem 2.2.1 is given in [22, Volume III, Chapter 18].

Theorem 2.2.2. (Backward uniqueness) With reference to the structural acoustic semigroup e^{At} on the space Y_γ guaranteed by Theorem 2.2.1, we have that the backward-uniqueness property (2.1.5) holds true. □

3. Proof of Theorem 2.1.2 and Theorem 2.2.2: Strategy

Orientation. Essentially without loss of generality, we shall assume that $d = 0$ and $\beta = 0$ in (2.1.1a) and (2.1.1c) (elastic wall), as well as in (2.2.1a) and (2.2.1c) (thermoelastic wall).

The strategy of the proof of Theorems 2.1.2 and 2.2.2 may be divided in two main parts which are condensed in the present section.

Part I is centered on Theorem 3.1 (elastic wall) and Theorem 3.2 (thermoelastic wall) given below, which provide an abstract first-order model for the structural acoustic problems (2.1.1a-f) (elastic wall) and (2.2.1a-f) (thermoelastic wall). The thermoelastic wall is supplemented by any of the canonical boundary conditions (B.C.), such as (2.2.2) through (2.2.5). These abstract models exhibit special properties, which will be critical for achieving the desired backward-uniqueness conclusion. The proof of Theorems 3.1 and 3.2 will be given in the forthcoming Section 4.

Starting from Theorems 3.1 and 3.2, Part II gives the main core of the proof of the backward-uniqueness property of the strongly continuous structural acoustic semigroup for both cases, with either an elastic or else a thermoelastic wall. The key strategy is to use a recent abstract theorem on the backward uniqueness [17, Theorem 3.1, p. 225], which is recalled here as Theorem 3.3 below. Thus, the core of our proof will be to verify, in Section 5 below, the assumptions of the abstract backward-uniqueness Theorem 3.3. To this end, we shall use Theorems 3.1 and 3.2 on the abstract models, and in particular we shall make use of the critical properties (3.7) and (3.9) (elastic wall), and (3.7) and (3.13) (thermoelastic wall) (in the nontrivial case $\gamma > 0$, the latter was obtained in [17]), giving resolvent estimates on suitable rays.

Part I. Theorem 3.1. Under assumptions (H.1), (H.2), and (H.3) of Section 2.1, the structural acoustic problem (2.1.1a–f) with $d = 0$ and $\beta = 0$ (elastic wall) can be rewritten abstractly as the first-order equation

$$\dot{y} = Ay, \quad y(0) = y_0 \in Y, \tag{3.1}$$

where (i) $y(t) = [y_1(t), y_2(t)]$, $y_1(t) = [z(t), z_t(t)]$, and $y_2(t) = [v(t), v_t(t)]$;
(ii)

$$Y \equiv Y_1 \times Y_2;\ Y_1 = \mathcal{D}(A_N^{\frac{1}{2}}) \times L_2(\Omega);\ Y_2 = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Gamma_0); \tag{3.2}$$

(iii)

$$A = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{A_1}} & 0 & 0 \\ & & 0 & A_N N_0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{A_2}} \\ 0 & -N_0^* A_N & & \end{array}\right] :\ Y \supset \mathcal{D}(A) \to Y, \tag{3.3a}$$

where the operators A_1 and A_2 are defined in (3.6) and (3.8) below, so that

$$\mathcal{D}(A) = \{[y_1, y_2, y_3, y_4] \in Y : y_2 \in \mathcal{D}(A_N^{\frac{1}{2}}), [y_1 - N_0 y_4] \in \mathcal{D}(A_N), [y_3, y_4] \in \mathcal{D}(A_2)\}. \tag{3.3b}$$

(iii1) In (3.2), the operator $\mathcal{A}$ is the one in equation (2.1.1d) subject to assumption (H.1).

(iii2) Moreover, in (3.3), $A_N : L_2(\Omega) \supset \mathcal{D}(A_N) \to L_2(\Omega)$ is the strictly positive, self-adjoint operator, with constant $b > 0$ as assumed, defined by

$$A_N h = -\Delta h,\ \mathcal{D}(A_N) = \left\{ h \in H^2(\Omega) : \frac{\partial h}{\partial \nu}\Big|_{\Gamma_0} = 0, \left[\frac{\partial h}{\partial \nu} + bh\right]_{\Gamma_1} = 0 \right\}. \tag{3.4}$$

Thus, $A_N^{-1} \in \mathcal{L}(L_2(\Omega))$. This is the reason why we are taking $b > 0$ throughout.

(iii3) Finally, in (3.3), N_0 is the Neumann map [22] from $L_2(\Gamma_0)$ to $L_2(\Omega)$, defined by

$$\psi = N_0 g \Longleftrightarrow \left\{ \Delta\psi = 0 \text{ in } \Omega; \frac{\partial \psi}{\partial \nu}\Big|_{\Gamma_0} = g, \left[\frac{\partial \psi}{\partial \nu} + b\psi\right]_{\Gamma_1} = 0 \right\}. \tag{3.5}$$

Further properties will be given in Section 4.1 below.

(iv) The operator

$$A_1 = \begin{bmatrix} 0 & I \\ -A_N & 0 \end{bmatrix} : \; Y_1 \supset \mathcal{D}(A_1) = \mathcal{D}(A_N) \times \mathcal{D}(A_N^{\frac{1}{2}}) \to Y_1 \tag{3.6}$$

generates a strongly continuous, unitary group $e^{A_1 t}$ on Y_1 (defined in (3.2)), $t \geq 0$. Thus, its resolvent $R(\lambda, A_1)$ satisfies *a fortiori* the estimate

$$\|R(\lambda, A_1)\|_{\mathcal{L}(Y_1)} \leq \frac{C_a}{|\lambda|}, \quad \text{on fixed rays } 0 \neq \lambda = |\lambda| e^{\pm ia}, \tag{3.7}$$

$|\lambda|$ large enough, where a is any fixed constant satisfying $\frac{\pi}{2} < a < \pi$.

(v) The operator

$$A_2 = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{B} \end{bmatrix} : \; Y_2 \supset \mathcal{D}(A_2) \to Y_2; \tag{3.8a}$$

$$\mathcal{D}(A_2) = \{[x_1, x_2] \in Y_2 : \; x_1 \in \mathcal{D}(\mathcal{A}^{1-\frac{\alpha}{2}}); \; x_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}); \; [\mathcal{A}^{1-\frac{\alpha}{2}} x_1 + (\mathcal{A}^{-\frac{\alpha}{2}} \mathcal{B} \mathcal{A}^{-\frac{\alpha}{2}}) \mathcal{A}^{\frac{\alpha}{2}} x_2] \in \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}})\} \tag{3.8b}$$

generates a strongly continuous contraction semigroup $e^{A_2 t}$ on Y_2, $t \geq 0$. Moreover, (v1) $e^{A_2 t}$ is analytic and uniformly stable on Y_2, $t > 0$, for $\frac{1}{2} \leq \alpha \leq 1$ in (2.1.4); (v2) of Gevrey class $\delta > \frac{1}{(2\alpha)}$, and hence *a fortiori* differentiable on Y_2 for all $t > 0$ if $0 < \alpha < \frac{1}{2}$; (v3) finally, a group if $\alpha = 0$.

The resolvent operator $R(\lambda, A_2)$ is compact on Y_2, for $0 < \alpha < 1$. Thus, its resolvent $R(\lambda, A_2)$ satisfies the following estimate: there exists a constant a, satisfying $\frac{\pi}{2} < a < \pi$, and a constant C_a, such that

$$\|R(\lambda, A_2)\|_{\mathcal{L}(Y_2)} \leq \frac{C_a}{|\lambda|}, \text{ on fixed rays } 0 \neq \lambda = |\lambda| e^{\pm ia}. \quad \Box \tag{3.9}$$

The proof of Theorem 3.1 will be given in Section 4.1.

Theorem 3.2. *The structural acoustic problem* (2.2.1a–g) *with thermoelastic wall and* $\gamma \geq 0$ *can be rewritten abstractly as the first-order equation*

$$\dot{y} = Ay, \qquad y(0) = y_0 \in Y \tag{3.10}$$

under any of the canonical B.C.: (2.2.2) (*hinged mechanical/Dirichlet thermal B.C.*); (2.2.3) (*clamped mechanical/either Dirichlet or Neumann B.C.*); (2.2.4) (*coupled hinged mechanical/Neumann thermal B.C.*); *finally,* (2.2.5) (*coupled free B.C.*). *Here we have (i)* $y(t) = [y_1(t), y_2(t)]$, $y_1(t) = [z(t), z_t(t)]$, *and* $y_2(t) = [v(t), v_t(t), \theta(t)]$;

(ii)

$$Y \equiv Y_\gamma \equiv Y_1 \times Y_{2,\gamma}; \quad Y_1 = \mathcal{D}(A_N^{\frac{1}{2}}) \times L_2(\Omega) \;\; \text{(as in (3.2))}, \tag{3.11}$$

while the Hilbert space $Y_{2,\gamma}$ *depends on* $\gamma > 0$ *or* $\gamma = 0$ *as well as on the B.C. considered, and is identified explicitly in the various cases in Section* 4.2 *below; see* (4.2.23).

(*iii*) A *is the generator of a strongly continuous contraction semigroup* e^{At} *on* Y, *and has the following structure:*

$$A = \left[\begin{array}{cc|ccc} & & 0 & 0 & 0 \\ A_1 & & & & \\ & & 0 & A_N N_0 & 0 \\ \hline 0 & 0 & & & \\ & & & A_2 & \\ 0 & -N_0^* A_N & & & \\ 0 & 0 & & & \end{array}\right] : \; Y \supset \mathcal{D}(A) \to Y, \tag{3.12}$$

to be further identified in Section 4.2. *The operator* A *depends on* γ, *since* A_2 *depends on* γ: *see* (4.2.26)–(4.2.28) *or different B.C. Moreover, the operators* A_1, A_N, *and* N_0 *are the same as those in* (3.6), (3.4), *and* (3.5). *The operator* A_2 *is the generator of a strongly continuous contraction semigroup* $e^{A_2 t}$ *on* $Y_{2,\gamma}$, $t \geq 0$. *Moreover, for* $\gamma = 0$, $e^{A_2 t}$ *is analytic on* $Y_{2,\gamma=0}$ (*identified below in Section* 4.2 *for each of the canonical B.C.*) [23], [18], [19], [20], [22, Chapter 3, Appendix]. *For* $\gamma > 0$, $e^{A_2 t}$ *is hyperbolic-dominated, in the sense of* [21]. *In all cases, its resolvent* $R(\lambda, A_2)$ *satisfies the following estimate: for* $\gamma \geq 0$, *and for any fixed arbitrary a with* $\frac{\pi}{2} < a < \pi$, *there exist constants* $r_0 > 0$ *and* $C > 0$, *possibly depending on* $\gamma \geq 0$, *such that*

$$\|R(\lambda, A_2)\|_{\mathcal{L}(Y_{2,\gamma})} \leq \frac{C}{|\lambda|}, \; \text{on fixed rays } 0 \neq \lambda = |\lambda| e^{\pm ia}, \; |\lambda| \geq r_0 > 0, \tag{3.13}$$

(*counterpart of* (3.9)).

The proof of Theorem 3.2 will be given in Section 4.2. We note here, however, that the critical resolvent estimate (3.13) on rays was established in [17] in the challenging case $\gamma > 0$, while it is obvious in the analytic case $\gamma = 0$.

Part II. Step 1. Our proof of Theorems 2.1.2 and 2.2.2 relies on the following abstract backward-uniqueness result.

Theorem 3.3. [17, Theorem 3.1] *Let* A *be the infinitesimal generator of a strongly continuous semigroup in a Banach space* X. *Assume that there exist constants* $a \in (\frac{\pi}{2}, \pi)$, $r_0 > 0$, *and* $C > 0$, *such that*

$$\left\|R(re^{\pm ia}, A)\right\| = \left\|(re^{\pm ia} I - A)^{-1}\right\| \leq C \tag{3.14}$$

in the norm of $\mathcal{L}(X)$, *for all* $r \geq r_0$. *Then the backward-uniqueness property holds true; that is,* $e^{AT}x_0 = 0$ *for some* $T > 0$ *and* $x_0 \in X$ *implies* $x_0 = 0$.

Step 2. We next verify that the assumptions of Theorem 3.3 are fulfilled in the two cases of the structural acoustic semigroups e^{At} of Theorems 2.1.1 and 2.2.1 on the state space Y, defined in (3.2) in the case of elastic wall, and in (3.11) and (4.2.23) in the case of thermoelastic wall, respectively. In fact, even more is true in these cases: on such rays $re^{\pm ia}$, the resolvent of A grows as $\mathcal{O}(\frac{1}{|\lambda|})$ in the norm of $\mathcal{L}(Y)$.

Theorem 3.4. Let $R(\lambda, A)$ be the resolvent operator of either the operator A in (3.3) (the structural acoustic generator with elastic wall) on the state space Y defined by (3.2), in the case of problem (2.1.1a–f); or else of the operator A in (3.12) (the structural acoustic generator with thermoelastic wall, $\gamma \geq 0$), on the state space $Y \equiv Y_\gamma$ defined by (3.11) and (4.2.23) in the case of problem (2.2.1a–g). (More generally, let $R(\lambda, A)$ be the resolvent of an operator A given by either (3.3) or (3.12), satisfying the estimates (3.7) and (3.9), and respectively (3.7) and (3.13), for its components A_1 and A_2, on rays $\lambda = re^{\pm ia}$, $\frac{\pi}{2} < a < \pi$.) Then there exist constants $r_0 > 0$ and $C > 0$ (depending possibly on a, $\gamma \geq 0$), such that the following estimate holds true:

$$\|R(\lambda, A)\|_{\mathcal{L}(Y)} \leq \frac{C}{|\lambda|}, \text{ on rays } \lambda = |\lambda| e^{\pm ia}, \ |\lambda| \geq r_0 > 0. \tag{3.15}$$

Thus, *a fortiori*, assumption (3.14) holds true, and Theorem 3.3 applies. □

4. Abstract setting. Proof of Theorems 3.1 and 3.2

4.1. **Structural acoustic model with elastic wall. Theorem 3.1.** In this subsection, we introduce the abstract set-up for the coupled system (2.1.1), (2.1.2). We follow closely [5, Section 2], of which the present setting is a simplified version.

Operators acting on Ω. (i) Let $A_N : L_2(\Omega) \supset \mathcal{D}(A_N) \to L_2(\Omega)$ be the strictly positive, self-adjoint operator, with constant $b > 0$ as assumed, defined as in (3.4) by

$$A_N h = -\Delta h, \ \mathcal{D}(A_N) = \left\{ h \in H^2(\Omega) : \frac{\partial h}{\partial \nu}\Big|_{\Gamma_0} = 0, \ \left[\frac{\partial h}{\partial \nu} + bh\right]_{\Gamma_1} = 0 \right\}. \tag{4.1.1}$$

Thus, $A_N^{-1} \in \mathcal{L}(L_2(\Omega))$. This is the reason why we are taking $b > 0$ throughout.

(ii) Let N_0 be the Neumann map [22] from $L_2(\Gamma)$ to $L_2(\Omega)$, defined as in (3.6) by

$$\psi = N_0 g \Longleftrightarrow= \left\{ \Delta\psi = 0 \text{ in } \Omega;\ \frac{\partial\psi}{\partial\nu}\Big|_{\Gamma_0} = g,\ \left[\frac{\partial\psi}{\partial\nu} + b\psi\right]_{\Gamma_1} = 0 \right\}; \tag{4.1.2}$$

$$N_0 \text{ continuous: } L_2(\Gamma_0) \to H^{\frac{3}{2}}(\Omega) \subset \mathcal{D}(A_N^{\frac{3}{4}-\epsilon}), \quad \epsilon > 0, \tag{4.1.3a}$$

$$\text{so that } A_N^{\frac{3}{4}-\epsilon} N_0 \text{ continuous: } L_2(\Gamma_0) \to L_2(\Omega); \tag{4.1.3b}$$

$$\text{more generally, } N_0 \text{ continuous } H^s(\Gamma_0) \to H^{s+\frac{3}{2}}(\Omega),\ s \in \mathbb{R}. \tag{4.1.3c}$$

Moreover, by Green's second theorem, the following trace results hold true [22]:

$$N_0^* A_N h = \begin{cases} h|_{\Gamma_0} & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1, \end{cases} \qquad h \in \mathcal{D}(A_N), \tag{4.1.4}$$

and the validity of (4.1.4) may be extended to all $h \in H^1(\Omega) \equiv \mathcal{D}(A_N^{\frac{1}{2}})$, as $\mathcal{D}(A_N^{\frac{1}{2}})$ is dense in $\mathcal{D}(A_N)$.

Second-order abstract model. By using the Green operator introduced above, the coupled PDE problem (2.1.1) can be rewritten as the following abstract second-order system:

$$\begin{cases} z_{tt} + A_N z + \beta A_N N_0 D_0 N_0^* A_N z + d z_t - A_N N_0 v_t = 0, & (4.1.5a) \\ v_{tt} + \mathcal{A} v + \mathcal{B} v_t + N_0^* A_N z_t = 0, & (4.1.5b) \end{cases}$$

the first equation to be read in $[\mathcal{D}(A_N)]'$, the latter one on $[\mathcal{D}(\mathcal{A})]'$, see [22], [5].

Function spaces and operators. The state space for problem (4.1.5) is then

$$Y \equiv Y_{1,\beta} \times Y_2 \equiv Z_\beta \times L_2(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Gamma_0); \tag{4.1.6a}$$

$$Y_1 \equiv Y_{1,\beta} \equiv Z_\beta \times L_2(\Omega);\ Y_2 \equiv \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Gamma_0); \tag{4.1.6b}$$

$$Z_\beta \equiv \left\{ h \in \mathcal{D}(A_N^{\frac{1}{2}}) = H^1(\Omega):\ N_0^* A_N h = h|_{\Gamma_0} \in \mathcal{D}(D_0^{\frac{1}{2}}) \right\}; \tag{4.1.6c}$$

$$\|h\|^2_{Z_\beta} = \|A_N^{\frac{1}{2}} h\|^2_{L_2(\Omega)} + \beta \|D_0^{\frac{1}{2}} N_0^* A_N h\|^2_{L_2(\Gamma_0)}. \tag{4.1.6d}$$

Notice that, for $\beta = 0$, we then have $Z_\beta \equiv \mathcal{D}(A_N^{\frac{1}{2}})$.

Accordingly, we define the operators $A_1 = A_{1,\beta} : Y_1 \supset \mathcal{D}(A_1) \to Y_1$, and $A_2 : Y_2 \supset \mathcal{D}(A_2) \to Y_2$, as follows:

$$A_1 = A_{1,\beta} = \begin{bmatrix} 0 & I \\ -A_N - \beta A_N N_0 D_0 N_0^* A_N & -dI \end{bmatrix}; \tag{4.1.7a}$$

$$\mathcal{D}(A_1) = \{[h_1, h_2] : h_1, h_2 \in \mathcal{D}(A_N^{\frac{1}{2}}) : h_1 + \beta N_0 D_0 N_0^* A_N h_1 \in \mathcal{D}(A_N)\}; \tag{4.1.7b}$$

$$A_2 = \begin{bmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{B} \end{bmatrix}; \tag{4.1.8a}$$

$$\mathcal{D}(A_2) = \{[h_1, h_2] : h_1 \in \mathcal{D}(\mathcal{A}^{1-\frac{\alpha}{2}}), h_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}}); \\ [\mathcal{A}^{1-\frac{\alpha}{2}} h_1 + (\mathcal{A}^{-\frac{\alpha}{2}} \mathcal{B} \mathcal{A}^{-\frac{\alpha}{2}}) \mathcal{A}^{\frac{\alpha}{2}} h_2] \in \mathcal{D}(\mathcal{A}^{\frac{\alpha}{2}})\}, \tag{4.1.8b}$$

as in (3.8). The Y_1- and Y_2-adjoint operators are given, respectively, by

$$A_1^* = \begin{bmatrix} 0 & -I \\ A_N + \beta A_N N_0 D_0 N_0^* A_N & -dI \end{bmatrix}, \quad A_2^* = \begin{bmatrix} 0 & -I \\ \mathcal{A} & -\mathcal{B} \end{bmatrix}, \tag{4.1.9}$$

with domains analogously defined.

Generation results for A_1 and A_2. The following generation result is contained in [5, Lemma 2.1] (which includes also the damping term $D_0 z_t$ on the left side of (2.1.1c) as in (6.1) below). See also [22, Chapter 7, Proposition 7.6.2.1, p. 664].

Lemma 4.1.1. Assume (H.1) and (H.2) as they pertain to A_1. Then, the operators A_1 and A_1^* defined by (4.1.7)–(4.1.9) are maximal dissipative on the space Y_1 in (4.1.6b), and hence the generators of strongly continuous semigroups $e^{A_1 t}$ and $e^{A_1^* t}$ of contractions on Y_1, $t \geq 0$. □

The following result is given in [6], [7], and [8] for $\frac{1}{2} \leq \alpha \leq 1$, and in [9] for $0 < \alpha < \frac{1}{2}$. See also [22, Chapter 3, Appendix B].

Lemma 4.1.2. Assume (2.1.4). (i) The operators A_2 and A_2^* defined by (4.1.8) and (4.1.9) are maximal dissipative on the space Y_2 defined by (4.1.6b) and hence the generators of strongly continuous semigroups $e^{A_2 t}$ and $e^{A_2^* t}$ of contractions on Y_2, $t \geq 0$, which are uniformly stable.

(ii) For $\frac{1}{2} \leq \alpha \leq 1$, $e^{A_2 t}$ and $e^{A_2^* t}$ are, moreover, analytic on Y_2, $t > 0$.

(iii) For $0 < \alpha < \frac{1}{2}$, $e^{A_2 t}$ and $e^{A_2^* t}$ are, moreover, of Gevrey class $\delta > 1/(2\alpha)$, and hence *a fortiori* differentiable for all $t > 0$ on Y.

(iv) For $\alpha = 0$, $e^{A_2 t}$ and $e^{A_2^* t}$ are strongly continuous groups.

In all cases, $A_2^{-1} \in \mathcal{L}(Y_2)$. In fact, for $0 < \alpha < 1$, the resolvent $R(\lambda, A_2)$ is compact on Y_2. Thus, estimate (3.9) holds true. □

Coupling. Finally, we introduce the densely defined (unbounded, unclosable) trace operator C: $Y_1 \supset \mathcal{D}(C) \to Y_2$, defined by

$$C \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ N_0^* A_N z_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & N_0^* A_N \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{4.1.10a}$$

with domain (see (4.1.3b)

$$\begin{aligned} \mathcal{D}(C) = \{[z_1, z_2] \in Y_1 :\ & N_0^* A_z z_2 = z_2|_{\Gamma_0} \in L_2(\Gamma_0)\} \\ & \supset \mathcal{D}(A_N^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{4}+\epsilon}), \quad \epsilon > 0, \end{aligned} \tag{4.1.10b}$$

so that $\mathcal{D}(A_N^{\frac{1}{2}}) \times \mathcal{D}(A_N^{\frac{1}{2}}) \subset \mathcal{D}(C)$. Its adjoint $C^* :\ Y_2 \to \mathcal{D}(A_N^{\frac{1}{2}}) \times [\mathcal{D}(A_N^{\frac{1}{4}+\epsilon})]'$, in the sense that $(Cy_1, y_2)_{Y_2} = (y_1, C^* y_2)_{Y_1}$, is given by

$$C^* \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ A_N N_0 v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A_N N_0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{4.1.11}$$

where $A_N N_0 : L_2(\Gamma_0) \to [\mathcal{D}(A_N^{\frac{1}{4}+\epsilon})]'$, recalling property (4.1.3b).

First-order abstract model. Dynamics operator. Finally, from (4.1.7), (4.1.8), (4.1.10), and (4.1.11), we define the operator

$$\begin{aligned} A = \begin{bmatrix} A_1 & C^* \\ -C & A_2 \end{bmatrix} &: Y \supset \mathcal{D}(A) \to Y = Y_1 \times Y_2 \\ &= \mathcal{D}(A_N^{\frac{1}{2}}) \times L_2(\Omega) \times \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times L_2(\Gamma_0), \end{aligned} \tag{4.1.12}$$

which is explicitly given by

$$A = \left[\begin{array}{cc|cc} 0 & I & 0 & 0 \\ -A_N - \beta A_N N_0 D_0 N_0^* A_N & -dI & 0 & A_N N_0 \\ \hline 0 & 0 & 0 & I \\ 0 & -N_0^* A_N & -\mathcal{A} & -\mathcal{B} \end{array}\right], \tag{4.1.13a}$$

with domain

$$\begin{aligned} \mathcal{D}(A) = \{[z_1, z_2, v_1, v_2] \in Y :\ & z_2 \in \mathcal{D}(A_N^{\frac{1}{2}}),\ \{v_1, v_2\} \\ & \text{as in (4.1.8b)},\ z_1 + \beta N_0 D_0 N_0^* A_N z_1 - N_0 v_2 \in \mathcal{D}(A_N)\}. \end{aligned} \tag{4.1.13b}$$

Finally, returning to the *second*-order abstract model (4.1.5), we see that these equations can be rewritten as the following *first*-order abstract equation in the variable $y(t) = [z(t), z_t(t), v(t), v_t(t)]$:

$$\dot{y}(t) = Ay(t), \quad y(0) = y_0 \in Y, \tag{4.1.14}$$

where A is defined by (4.1.13).

Theorem 2.1.1 claims well-posedness of (4.1.14), in the sense that A in (4.1.13) is maximal dissipative and hence generates a strongly continuous semigroup of contractions e^{At} on the space Y. A stronger generation result for a more general PDE problem—which includes the damping term $D_0 z_t$ on the left side of (2.1.1c) as in (6.1) below—is given in [5, Theorem 1.3.1]. All the statements of Theorem 3.1 hold true.

4.2. **Structural acoustic model with thermoelastic wall. Theorem 3.2.** In this subsection we introduce the abstract equations of the structural acoustic model with thermoelastic wall (2.2.1), under each of the canonical sets of boundary conditions (2.2.2)–(2.2.5). We refer to [1], [3], [18], [19], [20], [21], and [22, Chapter 3, Sections 11–13] for details and derivations.

Operators acting on Ω. These are the same A_N and N_0 as in (4.1.1)–(4.1.4) of Section 4.1.

Operators acting on Γ_0. Depending on the type of B.C. associated with each beam/plate model on Γ_0, we introduce corresponding elliptic, self-adjoint operators.

Hinged beams or plates/Dirichlet thermal B.C. (2.2.2): dim $\Gamma_0 = 1, 2$. We define the following positive, self-adjoint operators on $L_2(\Gamma_0)$:

$$\mathcal{A} = \mathcal{A}_D^2 : L_2(\Gamma_0) \supset \mathcal{D}(\mathcal{A}) \to L_2(\Gamma_0),\ \mathcal{A}h = \mathcal{A}^2(x,\partial)h; \tag{4.2.1a}$$

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_D^2) = \left\{ h \in H^4(\Gamma_0) :\ h|_{\partial\Gamma_0} = \mathcal{A}(x,\partial)h|_{\partial\Gamma_0} = 0 \right\}; \tag{4.2.1b}$$

$$\mathcal{A}_D h = -\mathcal{A}(x,\partial)h,\ \mathcal{D}(\mathcal{A}_D) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0); \tag{4.2.2}$$

$$D_\alpha h = \operatorname{div}(\alpha(x)\nabla h), \quad \mathcal{D}(D_\alpha) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0); \tag{4.2.3}$$

$$C_\gamma = I + \gamma\mathcal{A}_D,\ (f,g)_{\mathcal{D}(C_\gamma^{\frac{1}{2}})} \equiv (C_\gamma f, g)_{L_2(\Gamma_0)} = ((I+\gamma C)f, g)_{L_2(\Gamma_0)}. \tag{4.2.4}$$

Clamped beams or plates either Dirichlet or else Robin thermal B.C. (2.2.3): dim $\Gamma_0 = 1, 2$. We introduce the following positive, self-adjoint operator on $L_2(\Gamma_0)$:

$$\begin{aligned} \mathcal{A}h &= \mathcal{A}^2(x,\partial)h,\ \mathcal{D}(\mathcal{A}) = H^4(\Gamma_0) \cap H_0^2(\Gamma_0) \\ &= \left\{ h \in H^4(\Gamma_0) :\ h|_{\partial\Gamma_0} = \left.\frac{\partial h}{\partial\tilde{\nu}}\right|_{\partial\Gamma_0} = 0 \right\}; \end{aligned} \tag{4.2.5}$$

$$\begin{aligned} \mathcal{A}_R h &= -\mathcal{A}(x,\partial)h,\ \mathcal{D}(\mathcal{A}_R) \\ &= \left\{ h \in H^2(\Gamma_0) :\ \frac{\partial\theta}{\partial\tilde{\nu}} + \tilde{b}\theta = 0 \text{ on } \partial\Gamma_0 \right\}; \end{aligned} \tag{4.2.6}$$

$$D_\alpha \text{ as in (4.2.3)};\ \ F_\alpha h = \operatorname{div}(\alpha(x)\nabla h),\ \mathcal{D}(F_\alpha) = \mathcal{D}(\mathcal{A}_R). \tag{4.2.7}$$

Coupled hinged mechanical/thermal Neumann B.C. (2.2.4). Here we need the operators $\mathcal{A}_D$, D_α, and F_α, in (4.2.2), (4.2.3), and (4.2.7), as well as the operator G_1.

$$h = G_1 g \iff \{\mathcal{A}(x,\partial)h = 0 \text{ on } \Gamma_0,\ h = g \text{ on } \partial\Gamma_0\}; \tag{4.2.8a}$$

$$G_1 : \text{ continuous } H^s(\partial\Gamma_0) \to H^{s+\frac{1}{2}}(\Gamma_0), \quad s \in \mathbb{R}. \tag{4.2.8b}$$

Free coupled B.C. (2.2.5): $\dim \Gamma_0 = 2$. Here we introduce the positive, self-adjoint operator of [22, Chapter 3, Section 3.13, equation (3.13.5)], [21, Section 1.3], and [20], on $L_2(\Gamma_0)$; that is,

$$\mathcal{A}h = \Delta^2 h; \tag{4.2.9a}$$

$$\mathcal{D}(\mathcal{A}) = \left\{ h \in H^4(\Gamma_0) : [\Delta h + B_1 h]_{\partial\Gamma_0} = 0; \left[\frac{\partial \Delta h}{\partial \tilde{\nu}} + B_2 h - h\right]_{\partial\Gamma_0} = 0 \right\}, \tag{4.2.9b}$$

where the operators B_1 and B_2 are defined in (2.2.6). For (strict) positive self-adjointness of $\mathcal{A}$, we refer to [22, Chapter 3, Appendix 3C, Proposition 3C.5], [13], and [14]. We likewise introduce the Green operators $\mathcal{G}_i$, $i = 1, 2$, as in [22, Chapter 3, Section 3.3, equations (3.13.11) and (3.13.13)], respectively [21, Section 1.3]; that is,

$$h = \mathcal{G}_1 g \iff \begin{cases} \Delta^2 h = 0 \text{ in } \Gamma_0, & (4.2.10\text{a}) \\ [\Delta h + B_1 h]_{\partial\Gamma_0} = g \text{ in } \partial\Gamma_0, & (4.2.10\text{b}) \\ \left[\dfrac{\partial \Delta h}{\partial \tilde{\nu}} + B_2 h - h\right]_{\partial\Gamma_0} = 0; & (4.2.10\text{c}) \end{cases}$$

$$h = \mathcal{G}_2 h \iff \begin{cases} \Delta^2 h = 0 \text{ in } \Gamma_0, & (4.2.11\text{a}) \\ [\Delta h + B_1 h]_{\partial\Gamma_0} = 0, & (4.2.11\text{b}) \\ \left[\dfrac{\partial \Delta h}{\partial \tilde{\nu}} + B_2 h - h\right]_{\partial\Gamma_0} = g \text{ in } \partial\Gamma_0; & (4.2.11\text{c}) \end{cases}$$

which, as noted in [14], [22, Chapter 3, Section 3.13], and [20], are regular elliptic problems for $0 < \mu < 1$. Thus, elliptic regularity yields, as in [22, Chapter 3, Section 3.13, equations (3.13.12) and (3.13.14)],

$$\begin{aligned} &\mathcal{G}_1 : \text{ continuous } L_2(\partial\Gamma_0) \to H^{\frac{5}{2}}(\Gamma_0) \subset H^{\frac{5}{2}-4\epsilon}(\Gamma_0) \equiv \mathcal{D}(\mathcal{A}^{\frac{5}{8}-\epsilon}),\ \epsilon > 0, \\ &\mathcal{G}_2 : \text{ continuous } L_2(\partial\Gamma_0) \to H^{\frac{7}{2}}(\Gamma_0) \subset H^{\frac{7}{2}-4\epsilon}(\Gamma_0) \equiv \mathcal{D}(\mathcal{A}^{\frac{7}{8}-\epsilon}),\ \epsilon > 0. \end{aligned} \tag{4.2.12}$$

Moreover, [22, Lemma 3.13.1 and Lemma 3.13.2 of Chapter 3, Section 3.13] yield

$$\mathcal{G}_1^*\mathcal{A}h = \frac{\partial h}{\partial \tilde{\nu}},\ h \in \mathcal{D}(\mathcal{A});\ \mathcal{G}_2^*\mathcal{A}h = -h|_{\partial\Gamma_0},\ h \in \mathcal{D}(\mathcal{A}), \tag{4.2.13}$$

where these relations may be extended to $h \in \mathcal{D}(\mathcal{A}^{\frac{3}{8}+\epsilon}) = H^{\frac{3}{2}+4\epsilon}(\Gamma_0)$, and to all $h \in \mathcal{D}(\mathcal{A}^{\frac{1}{8}+\epsilon}) = H^{\frac{1}{2}+4\epsilon}(\Gamma_0)$, respectively.

Second-order abstract models. Hinged or clamped B.C. (2.2.2), (2.2.3). In this case, with reference to (2.2.1a–c) with $g = -\beta D_0(z|_{\Gamma_0}) + v_t$ on Γ_0, recalling (4.1.3), we obtain first

$$\begin{cases} z_{tt} = \Delta(z - N_0 g) - dz_t \text{ on } (0,T] \times \Omega; \\ g = -\beta D_0(z|_{\Gamma_0}) + v_t \text{ on } \Gamma_0; & (4.2.14a) \\ \dfrac{\partial}{\partial\nu}[z - N_0 g]_{\Sigma_1} = 0,\ \dfrac{\partial}{\partial\nu}[z - N_0 g]_{\Sigma_0} = g - g = 0; & (4.2.14b) \end{cases}$$

hence, by (3.4), the abstract version $z_{tt} = -A_N(z - N_0 g) - dz_t$; and, finally, recalling also (4.1.4), (4.2.1), (4.2.3), (4.2.7), and (2.2.1d–e), we obtain the definitive second-order abstract version of problem (2.2.1):

$$\begin{cases} z_{tt} + A_N z + \beta A_N N_0 D_0 N_0^* A_N z + dz_t - A_N N_0 v_t \\ \qquad = 0,\ \text{on } [\mathcal{D}(A_N)]'; & (4.2.15a) \\ v_{tt} + \gamma\mathcal{A}_D v_{tt} + \tilde{\mathcal{A}} v + \tilde{D}_\alpha\theta + N_0^* A_N z_t = 0,\ \text{on } [\mathcal{D}(\mathcal{A})]'; & (4.2.15b) \\ \theta_t + \mathcal{A}_D\theta - D_\alpha v_t = 0,\ \text{on } [\mathcal{D}(\mathcal{A}_D)]'. & (4.2.15c) \end{cases}$$

$$\begin{aligned} &\tilde{\mathcal{A}} = \mathcal{A}_D^2,\ \text{see (4.2.1) for hinged B.C.,} \\ &\tilde{\mathcal{A}} = \mathcal{A} \text{ as in (4.2.5) for clamped B.C.} \end{aligned} \tag{4.2.16}$$

$$\tilde{D}_\alpha = D_\alpha,\ \text{see (4.2.3) for thermal Dirichlet B.C.} \tag{4.2.17}$$

$$\tilde{D}_\alpha = F_\alpha,\ \text{see (4.2.7) for thermal Neumann (Robin) B.C.} \tag{4.2.18}$$

Second-order abstract model. Coupled hinged/thermal Neumann B.C. (2.2.4). Under the hinged mechanical/thermal Neumann boundary conditions (2.2.4), the structural acoustic problem (2.2.1) with thermoelastic wall admits the following abstract model [22, Chapter 3, Section 12], [21, equation (1.1.5), (1.1.6)]

$$\begin{cases} z_{tt} + A_N z + \beta A_N N_0 D_0 N_0^* A_N z + dz_t - A_N N_0 v_t = 0; & (4.2.19a) \\ v_{tt} + \gamma\mathcal{A}_D v_{tt} + \mathcal{A}_D^2 v + F_\alpha\theta + N_0^* A_N z_t = \mathcal{A}_D G_1(\theta|_\Gamma); & (4.2.19b) \\ \theta_t + \mathcal{A}_R\theta - D_\alpha v_t = 0, & (4.2.19c) \end{cases}$$

where $\mathcal{A}_D$ and D_α are the operators defined in (4.2.2) and (4.2.3), while $\mathcal{A}_R$ and F_α are the operators defined in (4.2.6) and (4.2.7). Finally, G_1 is the Green (Dirichlet) map defined by (4.2.8).

Free B.C. (2.2.5). In this case, with reference to (2.2.1d–e) with $\mathcal{A}(x,\partial) = \Delta$ and (2.2.5) with $\alpha(x) \equiv 1$ for simplicity of notation, where we set $g_1 = -\theta$, $g_2 = -\dfrac{\partial\theta}{\partial\tilde{\nu}} = \tilde{b}\theta$ on $\partial\Gamma_0$, recalling (4.2.10) and (4.2.11), we first obtain

$$\begin{cases} v_{tt} - \gamma\Delta v_{tt} + \Delta^2(v - \mathcal{G}_1 g_1 - \mathcal{G}_2 g_2) + \Delta\theta + z_t \\ \qquad = 0, \text{ in } (0,T] \times \Gamma_0; & (4.2.20a) \\ [v - \mathcal{G}_1 g_1 - \mathcal{G}_2 g_2]_{\partial\Sigma_1} = 0, \ [v - \mathcal{G}_1 g_1 - \mathcal{G}_2 g_2]_{\partial\Sigma_0} = 0; & (4.2.20b) \end{cases}$$

hence, by (4.2.6) and (4.2.7), the abstract version $v_{tt}+\gamma A_N v_{tt}+\mathcal{A}(v-\mathcal{G}_1 g_1-\mathcal{G}_2 g_2)-\mathcal{A}_R\theta+z_t|_{\Gamma_0} = 0$ on Σ_0; and finally, recalling also (2.2.1a–c), (4.2.16a), (4.1.4), and (4.2.6),

$$\begin{cases} z_{tt} + A_N z + \beta A_N N_0 D_0 N_0^* A_N z - A_N N_0 v_t \\ \qquad = -d z_t, \text{ on } [\mathcal{D}(A_N)]'; & (4.2.21a) \\ v_{tt} + \gamma\mathcal{A}_N v_{tt} + \mathcal{A}[v + \mathcal{G}_1(\theta|_{\partial\Gamma_0}) - \tilde{b}\mathcal{G}_2(\theta|_{\partial\Gamma_0})] + F_\alpha\theta + N^* A_N z_t \\ \qquad = 0, \text{ on } [\mathcal{D}(\mathcal{A})]'; & (4.2.21b) \\ \theta_t + \mathcal{A}_R\theta - D_\alpha v_t = 0, \text{ on } [\mathcal{D}(\mathcal{A}_R)]'; & (4.2.21c) \end{cases}$$

$$\mathcal{A}_N h = -\Delta h, \ \mathcal{D}(\mathcal{A}_N) = \left\{ h \in H^2(\Gamma_0) : \frac{\partial h}{\partial\tilde{\nu}}\Big|_{\partial\Gamma_0} = 0 \right\}. \qquad (4.2.21d)$$

We notice that in the case of free B.C., the coupling between mechanical and thermal variables occurs also on the boundary $\partial\Gamma_0$.

Function spaces and operators. Next, we introduce the following function spaces for $\{z, z_t\}$ and $\{v, v_t, \theta\}$ respectively:

$$Y_1 \equiv \mathcal{D}(A_N^{\frac{1}{2}}) \times L_2(\Omega); \qquad Y_{2,\gamma} = \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \times \mathcal{D}(C_\gamma^{\frac{1}{2}}) \times L_2(\Gamma_0); \qquad (4.2.22a)$$

$$\begin{cases} C_\gamma = I + \gamma\mathcal{A}_D \text{ for B.C. (2.2.2), (2.2.3) and (2.2.4);} & (4.2.22b) \\ C_\gamma = I + \gamma\mathcal{A}_N \text{ for (free) B.C. (2.2.5)} & (4.2.22c) \end{cases}$$

$$Y_\gamma = Y_1 \times Y_{2,\gamma}, \quad \mathcal{D}(A_N^{\frac{1}{2}}) = H^1(\Omega). \qquad (4.2.23)$$

Accordingly, we define the following operators: $A_1 : Y_1 \supset \mathcal{D}(A_1) \to Y_1$, and $A_2 : Y_{2,\gamma} \supset \mathcal{D}(A_2) \to Y_{2,\gamma}$ as follows:

(i) for all three models, (4.2.16) (hinged/clamped B.C.), (4.2.20) (hinged/Neumann B.C.), and (4.2.22) (free B.C.):

$$A_1 = \begin{bmatrix} 0 & I \\ -A_N - \beta A_N N_0 D_0 N_0^* A_N & -dI \end{bmatrix};$$

$$A_1^* = \begin{bmatrix} 0 & -I \\ A_N - \beta A_N N_0 D_0 N_0^* A_N & -dI \end{bmatrix}; \tag{4.2.24a}$$

$$Y_1 \supset \mathcal{D}(A_1) = \mathcal{D}(A_1^*) \to Y_1;$$

$$\mathcal{D}(A_1) = \mathcal{D}(A_1^*) = \{\{h_1, h_2\} :\ h_1, h_2 \in \mathcal{D}(A_N^{\frac{1}{2}}) = H^1(\Omega); \\ h_1 + N_0 D_0 N_0^* A_N h_2 \in \mathcal{D}(A_N)\}; \tag{4.2.24b}$$

(ii$_1$) for *model* (4.2.16) (*hinged/clamped* B.C.): for $C_\gamma = I + \gamma\mathcal{A}_D$ as in (4.2.23b) and

$$A_2 = \begin{bmatrix} 0 & I & 0 \\ -C_\gamma^{-1}\tilde{\mathcal{A}} & 0 & -C_\gamma^{-1}\tilde{D}_\alpha \\ 0 & D_\alpha & -\mathcal{A}_D \end{bmatrix} :\ Y_{2,\gamma} \supset \mathcal{D}(A_2) \to Y_{2,\gamma}; \tag{4.2.25a}$$

$$\mathcal{D}(A_2) = \mathcal{D}(A_2^*) = \mathcal{D}(\mathcal{A}^{\frac{3}{4}}) \times [\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \cap \mathcal{D}(\mathcal{A}_D)] \times \mathcal{D}(\mathcal{A}_D); \tag{4.2.25b}$$

(ii$_2$) while for *model* (4.2.20) (*hinged/thermal Neumann B.C.*): here $C_\gamma = I + \gamma\mathcal{A}_D$ again and

$$A_2 = \begin{bmatrix} 0 & I & 0 \\ -C_\gamma^{-1}\mathcal{A}_D^2 & 0 & C_\gamma^{-1}[-F_\alpha + \mathcal{A}_D G_1(\,\cdot\,|_{\partial\Gamma_0})] \\ 0 & D_\alpha & -\mathcal{A}_R \end{bmatrix}; \tag{4.2.26a}$$

$$\mathcal{D}(A_2) = \{v_1, v_2 \in \mathcal{D}(\mathcal{A}_D), \theta \in \mathcal{D}(\mathcal{A}_R) : [\mathcal{A}_D v_1 - \mathcal{G}_1(\theta|_{\partial\Gamma_0})] \in \mathcal{D}(\mathcal{A}_D^{\frac{1}{2}})\}; \tag{4.2.26b}$$

(ii$_3$) while for *model* (4.2.21) (*free* B.C.): here $C_\gamma = I + \gamma\mathcal{A}_N$ as in (4.2.23c) and

$$A_2 = \begin{bmatrix} 0 & I & 0 \\ -C_\gamma^{-1}\mathcal{A} & 0 & C_\gamma^{-1}[-\mathcal{A}\mathcal{G}_1(\,\cdot\,|_{\partial\Gamma_0}) + \tilde{b}\mathcal{A}\mathcal{G}_2(\,\cdot\,|_{\partial\Gamma_0}) - F_\alpha]; \\ 0 & D_\alpha & -\mathcal{A}_R \end{bmatrix}; \tag{4.2.27a}$$

$$\mathcal{D}(A_2) = \{v_1 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}});\ v_2 \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}});\ \theta \in \mathcal{D}(\mathcal{A}_R) : \\ [v_1 + \mathcal{G}_1(\theta|_{\Gamma_0}) - b\mathcal{G}_2(\theta|_{\Gamma_0})] \in \mathcal{D}(C_\gamma^{-1}\mathcal{A})\} = \mathcal{D}(\mathcal{A}^{\frac{3}{4}}), \tag{4.2.27b}$$

to be interpreted as in [20] and [21]. To orient our presentation, we preliminarily note—and this will be formally stated in Theorem 4.2.1(i), (ii) below—that

(i) For all $d \geq 0$, the operator A_1 in (4.2.25) is maximal dissipative on Y_1, and hence the infinitesimal generator of a strongly continuous semigroup of contractions $e^{A_1 t}$ on Y_1 (wave equation with absorbing B.C.);

(ii) the operator A_2, in either form (4.2.26) (hinged/clamped B.C.) or in form (4.2.27) (hinged/Neumann B.C.), or else in form (4.2.28) (free B.C.) is maximal dissipative on $Y_{2,\gamma}$, and hence the generator of a strongly continuous semigroup $e^{A_2 t}$ of contractions on $Y_{2,\gamma}$ [22, Chapter 3, Section 3.13], which, moreover, is analytic on Y_2 for $\gamma = 0$ (see Chapter 3, Appendices 3D, or 3E, or 3F (for hinged/clamped B.C.), and Appendix I for free B.C.)), as well as the original references [18], [19], [20], and [23].

Coupling. Finally, we introduce the densely defined (unbounded, unclosable) trace operator $\mathcal{C} : Y_1 \supset \mathcal{D}(\mathcal{C}) \to Y_{2,\gamma}$, defined by

$$\mathcal{C}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ N_0^* A_N z_2 \\ 0 \end{bmatrix};\ \mathcal{D}(A_N^{\frac{1}{2}}) \times \mathcal{D}(A_N^{\frac{1}{4}+\epsilon}) \subset$$

$$\mathcal{D}(\mathcal{C}) = \{[z_1, z_2] \in Y_1 : N_0^* A_N z_2 = z_2|_{\Gamma_0} \in L_2(\Gamma_0)\}, \tag{4.2.28}$$

so that $\mathcal{D}(A_N^{\frac{1}{2}}) \times \mathcal{D}(A_N^{\frac{1}{2}}) \subset \mathcal{D}(\mathcal{C})$, whose adjoint $\mathcal{C}^* : Y_2 \supset \mathcal{D}(C^*) \to \mathcal{D}(A_N^{\frac{1}{2}}) \times [\mathcal{D}(A_N^{\frac{1}{4}+\epsilon})]'$, in the sense that $(\mathcal{C}y_1, y_2)_{Y_2} = (y_1, \mathcal{C}^* y_2)_{Y_1}$, is given, recalling (4.1.3b), by

$$\mathcal{C}^* \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ A_N N v_2 \end{bmatrix},\ A_N N :\ L_2(\Gamma_0) \to [\mathcal{D}(A_N^{\frac{1}{4}+\epsilon})]'. \tag{4.2.29}$$

First-order abstract model. Finally, from (4.2.25)–(4.2.30), we define in all cases (hinged/clamped/free B.C.)

$$A = \begin{bmatrix} A_1 & \mathcal{C}^* \\ -\mathcal{C} & A_2 \end{bmatrix} :\ Y \supset \mathcal{D}(A) \to Y. \tag{4.2.30}$$

More specifically, in the case of *model* (4.2.16) (*hinged/clamped B.C.*)

$$A = \left[\begin{array}{cc|ccc} 0 & I & 0 & 0 & 0 \\ -A_N - \beta A_N N_0 D_0 N_0^* A_N & -dI & 0 & A_N N & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ 0 & -N_0^* A_N & -C_\gamma^{-1}\tilde{\mathcal{A}} & 0 & -C_\gamma^{-1}\tilde{D}_\alpha \\ 0 & 0 & 0 & D_\alpha & -\mathcal{A}_D \end{array}\right], \tag{4.2.31}$$

while in the case of model (4.2.20) (hinged/thermal Neumann B.C.)

$$A = \left[\begin{array}{c|c} (1) & (2) \\ \hline (3) & (4) \end{array}\right];$$

$$(1) = \begin{bmatrix} 0 & I \\ -A_N - \beta A_N N_0 D_0 N_0^* A_N & -dI \end{bmatrix};\ (2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_N N & 0 \end{bmatrix};$$

$$(3) = \begin{bmatrix} 0 & 0 \\ 0 & -N_0^* A_N \\ 0 & 0 \end{bmatrix};\ (4) = \begin{bmatrix} 0 & I & 0 \\ -C_\gamma^{-1}\mathcal{A}_D^2 & 0 & C_\gamma^{-1}[-F_\alpha + \mathcal{A}_D G_1(\,\cdot\,|_{\partial\Gamma_0})] \\ 0 & D_\alpha & -\mathcal{A}_R \end{bmatrix};$$

(4.2.32)

with natural domains [21], and similarly for the model (4.2.21) (free B.C.) [21], [20]. The above details—which are not all strictly necessary for the main proof in Section 5—prove Theorem 3.2(i), (ii), (iii), where generation by A of a strongly continuous contraction semigroup follows via the Lumer-Phillips theorem. Most importantly, the estimate (3.13) for $\gamma = 0$ is obvious since in this case $e^{A_2 t}$ is analytic on Y_2 under all B.C. [18], [19], [20], [23], [22, Chapter 3, Appendices D–I]. Instead, for $\gamma > 0$, estimate (3.13) in all cases is the challenging result proved in [17]. □

5. Proof of Theorem 3.4

We shall provide an explicit proof in the more-challenging case of Theorem 3.4: the one which refers to problem (2.2.1) with thermoelastic wall.

Step 1. Preliminaries. We return to Theorem 3.2, in particular to the operator A, and space $Y \equiv Y_\gamma$ given by (3.12) and (3.11); or (4.2.31), via A_1 in (4.2.25) and A_2 in (4.2.26)–(4.2.28). Accordingly, the solution of problem (2.2.1a–g), or its abstract version (3.10) with $y_0 = [z_0, z_1, v_0, v_1, \theta_0] \in Y_\gamma$, may be written as

$$y_1(t) \equiv \begin{bmatrix} z(t) \\ z_t(t) \end{bmatrix} = e^{A_1 t}\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \int_0^t e^{A_1(t-\tau)}\begin{bmatrix} 0 \\ A_N N_0 v_t(\tau) \end{bmatrix} d\tau; \qquad (5.1)$$

$$y_2(t) \equiv \begin{bmatrix} v(t) \\ v_t(t) \\ \theta(t) \end{bmatrix} = e^{A_2 t}\begin{bmatrix} v_0 \\ v_1 \\ \theta_0 \end{bmatrix} + \int_0^t e^{A_2(t-\tau)}\begin{bmatrix} 0 \\ -N_0^* A_N z_t(\tau) \\ 0 \end{bmatrix} d\tau \qquad (5.2)$$

in the t-domain; or, in the Laplace transform versions of (5.1) and (5.2), as

$$\hat{y}_1(\lambda) = \begin{bmatrix} \hat{z}(\lambda) \\ \hat{z}_t(\lambda) \end{bmatrix} = R(\lambda, A_1)\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + R(\lambda, A_1)\begin{bmatrix} 0 \\ A_N N_0 \hat{v}_t(\lambda) \end{bmatrix}; \qquad (5.3)$$

$$\hat{y}_2(\lambda) = \begin{bmatrix} \hat{v}(\lambda) \\ \hat{v}_t(\lambda) \\ \hat{\theta}(\lambda) \end{bmatrix} = R(\lambda, A_2) \begin{bmatrix} v_0 \\ v_1 \\ \theta_0 \end{bmatrix} + R(\lambda, A_2) \begin{bmatrix} 0 \\ -N_0^* A_N \hat{z}_t(\lambda) \\ 0 \end{bmatrix}, \quad (5.4)$$

for all $\lambda \in \mathbb{C}$ for which they make sense, in particular, for Re $\lambda > 0$. The estimates below will establish that the relevant expressions (5.3) and (5.4) are well defined.

The explicit expressions of $R(\lambda, A_1)$ and $R(\lambda, A_2)$ are available. However, in the present proof, where without loss of generality we take $d = \beta = 0$, we shall make use of the expression for $R(\lambda, A_1)$ of A_1 in (3.6), or (4.2.25), with $d = \beta = 0$, which is given by

$$\begin{aligned} R(\lambda, A_1) &= \begin{bmatrix} \lambda R(\lambda^2, -A_N) & R(\lambda^2, -A_N) \\ -A_N R(\lambda^2, -A_N) & \lambda R(\lambda^2, -A_N) \end{bmatrix}; \\ R(\lambda^2, -A_N) &= (\lambda^2 I + A_N)^{-1}, \end{aligned} \quad (5.5)$$

while it will suffice to employ only the element $R_{22}(\lambda)$ (second row, second column) of $R(\lambda, A_2)$:

$$R(\lambda, A_2) = \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & R_{22}(\lambda) & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}, \quad (5.6a)$$

without identifying it explicitly, but only subject to estimate (5.6b) below. We recall that $R(\lambda, A_1)$ and $R(\lambda, A_2)$ satisfy the key estimates (3.7) and (3.13), respectively, on rays $\lambda = |\lambda| e^{\pm ia}$, in the case of the thermoelastic wall. In particular, the latter estimate implies that

$$\|\lambda R_{22}(\lambda)\|_{\mathcal{L}(\mathcal{D}(C_\gamma^{\frac{1}{2}}))} \leq C, \quad \text{on rays } \lambda = |\lambda| e^{\pm ia}, \quad |\lambda| \geq r_0 > 0, \quad (5.6b)$$

where a, $\frac{\pi}{2} < a < \pi$, is the constant in (3.13), recalling the space $Y_{2,\gamma}$ in (4.2.23).

Step 2. Lemma 5.1. With reference to problem (5.3), (5.4), we have that there exists some $r_0 > 0$ such that, for all $\lambda = re^{\pm ia}$, with $\frac{\pi}{2} < a < \pi$ as in (3.13) and with $|\lambda| = r \geq r_0 > 0$, we have recalling Y_γ in (4.2.24),

$$\|N_0^* A_N \hat{z}_t(\lambda)\|_{L_2(\Gamma_0)} \leq \frac{C_{r_0,a,\epsilon}}{|\lambda|^{\frac{1}{2} - 2\epsilon}} \|\{z_0, z_1, v_0, v_1, \theta_0\}\|_{Y_\gamma},$$
$$\forall\ \lambda = re^{\pm ia},\ r \geq r_0 > 0. \quad (5.7)$$

Proof of Lemma 5.1. (i) We extract $\hat{v}_t(\lambda)$ explicitly from (5.4), using (5.6a) and $y_{0,2} = [v_0, v_1, \theta_0]$, thus obtaining

$$\hat{v}_t(\lambda) = \{R(\lambda, A_2) y_{0,2}\}_2 - R_{22}(\lambda) N_0^* A_N \hat{z}_t(\lambda), \quad (5.8)$$

where, henceforth, $\{\ \}_2$ denotes the second row entry of the argument. We then substitute $\hat{v}_t(\lambda)$ given by (5.8) into the second row of (5.3), rewritten explicitly for $y_{0,1} = [z_0, z_1]$, as

$$\hat{z}_t(\lambda) = \{R(\lambda, A_1)y_{0,1}\}_2 + \lambda R(\lambda^2, -A_N)A_N N_0 \hat{v}_t(\lambda), \tag{5.9}$$

by use of (5.5); apply $N_0^* A_N$ to the result, and obtain

$$\begin{aligned} N_0^* A_N \hat{z}_t(\lambda) &= N_0^* A_N \{R(\lambda, A_1)y_{0,1}\}_2 \\ &+ \lambda N_0^* A_N R(\lambda^2, -A_N) A_N N_0 \{R(\lambda, A_2)y_{0,2}\}_2 \\ &- \lambda N_0^* A_N R(\lambda^2, -A_N) A_N N_0 R_{22}(\lambda) N_0^* A_N \hat{z}_t(\lambda), \end{aligned} \tag{5.10}$$

or

$$\begin{aligned} [I + N_0^* A_N R(\lambda^2, -A_N) A_N N_0 \lambda R_{22}(\lambda)] N_0^* A_N \hat{z}_t(\lambda) &= N_0^* A_N \{R(\lambda, A_1)y_{0,1}\}_2 \\ &+ N_0^* A_N R(\lambda^2, -A_N) A_N N_0 \{\lambda R(\lambda, A_2)y_{0,2}\}_2. \end{aligned} \tag{5.11}$$

(ii) We next seek to invert, in $\mathcal{L}(L_2(\Gamma_0))$, the square-bracket term [ ] on the left side of (5.11), for all $\lambda = re^{\pm ia}$ with $|\lambda| = r$ sufficiently large, and hence extract and estimate $N_0^* A_N \hat{z}_t(\lambda)$ for all such λ's. Indeed, in this step we shall show that there exists some $r_0 > 0$, and a corresponding constant c_{r_0}, such that, for all $\lambda = re^{\pm ia}$, with $|\lambda| = r \geq r_0 > 0$,

$$\begin{aligned} \|[I + N_0^* A_N R(\lambda^2, -A_N) A_N N_0 \lambda R_{22}(\lambda)]^{-1}\|_{\mathcal{L}(L_2(\Gamma_0))} &\leq C_{r_0}, \\ \forall\ \lambda = re^{\pm ia},\ r \geq r_0 > 0. \end{aligned} \tag{5.12}$$

To prove (5.12) we shall invoke two bounds: the uniform bound on $\|\lambda R_{22}(\lambda)\|$ given by (5.6b) for $\lambda = re^{\pm ia}$, as well as the bound

$$\begin{cases} \|A_N^s R(\mu, -A_N)\|_{\mathcal{L}(L_2(\Omega))} \leq \frac{C_s}{|\mu|^{1-s}},\ 0 < s < 1,\ \text{for all } \mu \in \Sigma_N \\ \equiv \{\mu \in \mathbb{C} : |\arg(\mu)| \leq \pi - \delta,\ \text{for some } \delta > 0 \text{ arbitrary but fixed}\}. \end{cases} \tag{5.13}$$

Estimate (5.13) is well-known [25], since the (negative, self-adjoint) operator $-A_N$ in (4.1.1) generates a strongly continuous analytic semigroup $e^{-A_N t}$ on $L_2(\Omega)$, $t > 0$. Applying (5.13) for $s = \frac{1}{2} + 2\epsilon$, $\forall\ \epsilon > 0$ sufficiently small, and for $\lambda = re^{\pm ia}$, hence $\mu = \lambda^2 = r^2 e^{\pm i2a} \in \Sigma_N$, yields

$$\|A_N^{\frac{1}{2}+2\epsilon} R(\lambda^2, -A_N)\|_{\mathcal{L}(L_2(\Omega))} \leq \frac{C_{a,\epsilon}}{(|\lambda^2|)^{1-(\frac{1}{2}+2\epsilon)}} = \frac{C_{a,\epsilon}}{|\lambda|^{1-4\epsilon}}, \quad \forall\ \lambda = re^{\pm ia}. \tag{5.14}$$

Hence, (5.14), as well as (4.1.3b), lead to

$$\|N_0^* A_N R(\lambda^2, -A_N) A_N N_0\|_{\mathcal{L}(L_2(\Gamma_0))} \tag{5.15}$$

$$= \|[N_0^* A_N^{\frac{3}{4}-\epsilon}][A_N^{\frac{1}{2}+2\epsilon} R(\lambda^2, -A_N)][A_N^{\frac{3}{4}-\epsilon} N_0]\|_{\mathcal{L}(L_2(\Gamma_0))} \leq \frac{C_{a,\epsilon}}{|\lambda|^{1-4\epsilon}}, \ \forall \lambda = re^{\pm ia}.$$

Thus, invoking the bound on $\lambda R_{22}(\lambda)$ given by (5.6b) for all $\lambda = re^{\pm ia}$, we obtain from (5.15),

$$\|N_0^* A_N R(\lambda^2, -A_N) A_N N_0 \lambda R_{22}(\lambda)\|_{\mathcal{L}(\mathcal{D}(C_\gamma^{\frac{1}{2}}); L_2(\Gamma_0))} \leq \frac{C_{a,\epsilon}}{|\lambda|^{1-4\epsilon}}, \quad \forall \ \lambda = re^{\pm ia}. \tag{5.16}$$

Thus, for all $|\lambda|$ sufficiently large, say $|\lambda| \geq$ some $r_0 > 0$, the bound on (5.16) becomes strictly less than one, and then estimate (5.12) is achieved.

(iii) Using (5.12), we obtain from (5.11), for all $\lambda = re^{\pm ia}$, $|\lambda| = r \geq r_0 > 0$,

$$\begin{aligned} & N_0^* A_N \hat{z}_t(\lambda) \\ & \quad = \quad [I + N_0^* A_N R(\lambda^2, -A_N) A_N N_0 \lambda R_{22}(\lambda)]^{-1} [N_0^* A_N \{R(\lambda, A_1) y_{0,1}\}_2 \\ & \qquad + N_0^* A_N R(\lambda^2, -A_N) A_N N_0 \{\lambda R(\lambda, A_2) y_{0,2}\}_2], \end{aligned} \tag{5.17}$$

from which we get via (5.12)

$$\begin{aligned} \|N_0^* A_N \hat{z}_t(\lambda)\|_{L_2(\Gamma_0)} \leq C_{r_0} [\| & [N_0^* A_N \{R(\lambda, A_1) y_{0,1}\}_2\|_{L_2(\Gamma_0)} \\ & + \|N_0^* A_N R(\lambda^2, -A_N) A_N N_0 \{\lambda R(\lambda, A_2) y_{0,2}\}_2\|_{L_2(\Gamma_0)}], \\ & \forall \ \lambda = re^{\pm ia}, \ |\lambda| = r > r_0 > 0. \end{aligned} \tag{5.18}$$

(iv) As to the first term on the right side of (5.18), we shall show in this step that with $y_{0,1} = \{z_0, z_1\| \in Y_1$, see (4.2.23), we have $\forall \ \lambda = re^{\pm ia}$

$$\left\| N_0^* A_N \left\{ R(\lambda, A_1) \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \right\}_2 \right\|_{L_2(\Gamma_0)} \leq \frac{C_{a,\epsilon}}{|\lambda|^{\frac{1}{2}-2\epsilon}} \|\{z_0, z_1\}\|_{Y_1}, . \tag{5.19}$$

Indeed, to establish (5.19), we shall first invoke (5.5) and write explicitly for $y_{0,1} = \{z_0, z_1\}$ and $\lambda = re^{\pm ia}$,

$$\begin{aligned} & N_0^* A_N \left\{ R(\lambda, A_1) \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \right\}_2 \\ & \quad = \quad -N_0^* A_N A_N^{\frac{1}{2}} R(\lambda^2, -A_N) A_N^{\frac{1}{2}} z_0 + \lambda N_0^* A_N R(\lambda^2, -A_N) z_1, \end{aligned} \tag{5.20}$$

so that in the norm of $L_2(\Gamma_0)$,

$$\begin{aligned} \left\| N_0^* A_N \left\{ R(\lambda, A_1) \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \right\}_2 \right\|_{L_2(\Gamma_0)} & \leq \|N_0^* A_N A_N^{\frac{1}{2}} R(\lambda^2, -A_N) A_N^{\frac{1}{2}} z_0\|_{L_2(\Gamma_0)} \\ + \|\lambda N_0^* A_N R(\lambda^2, -A_N) z_1\|_{L_2(\Gamma_0)}. & \end{aligned} \tag{5.21}$$

As to the first term on the right side of (5.21), we invoke (4.1.3b) and (5.13) for $s = \frac{3}{4} + \epsilon$, and obtain

$$\begin{aligned} &\|N_0^* A_N A_N^{\frac{1}{2}} R(\lambda^2, -A_N) A_N^{\frac{1}{2}} z_0\|_{L_2(\Gamma_0)} \\ &= \|[N_0^* A_N^{\frac{3}{4}-\epsilon}][A_N^{\frac{3}{4}+\epsilon} R(\lambda^2, -A_N)][A_N^{\frac{1}{2}} z_0]\|_{L_2(\Gamma_0)} \\ &\le \frac{C_{a,\epsilon}}{|\lambda^2|^{1-(\frac{3}{4}+\epsilon)}} \|A_N^{\frac{1}{2}} z_0\|_{L_2(\Omega)} = \frac{C_{a,\epsilon}}{|\lambda|^{\frac{1}{2}-2\epsilon}} \|A_N^{\frac{1}{2}} z_0\|_{L_2(\Omega)}, \quad \forall\ \lambda = re^{\pm ia}. \end{aligned} \tag{5.22}$$

As to the second term on the right side of (5.21), we invoke again (4.1.3b) and (5.13) this time for $s = \frac{1}{4} + \epsilon$, and obtain

$$\begin{aligned} &\|\lambda N_0^* A_N R(\lambda^2, -A_N) z_1\|_{L_2(\Gamma_0)} = \|\lambda [N_0^* A_N^{\frac{3}{4}-\epsilon}][A_N^{\frac{1}{4}+\epsilon} R(\lambda^2, -A_N)] z_1]\|_{L_2(\Gamma_0)} \\ &\le |\lambda| \frac{C_{a,\epsilon}}{(|\lambda^2|)^{1-(\frac{1}{4}+\epsilon)}} \|z_1\|_{L_2(\Omega)} = \frac{C_{a,\epsilon}}{|\lambda|^{\frac{1}{2}-2\epsilon}} \|z_1\|_{L_2(\Omega)}, \quad \forall\ \lambda = re^{\pm ia}. \end{aligned} \tag{5.23}$$

Using (5.22) and (5.23) on the right side of (5.21) yields estimate (5.19), as desired, after recalling Y_1 from (3.2), or (3.11), or (4.2.23).

(v) Returning to the second term on the right side of (5.18), we invoke estimate (5.15) as well as estimate (5.6b), to obtain with $y_{0,2} = \{v_0, v_1, \theta_0\} \in Y_{2,\gamma}$ given by (4.2.23),

$$\begin{aligned} &\|N_0^* A_N R(\lambda^2, -A_N) A_N N_0 \{\lambda R(\lambda, A_2) y_{0,2}\}_2\|_{L_2(\Gamma_0)} \\ &\le \frac{C_{a,\epsilon}}{|\lambda|^{1-4\epsilon}} \|y_{0,2}\|_{Y_{2,\gamma}}, \ \forall\ \lambda = re^{\pm ia}. \end{aligned} \tag{5.24}$$

(vi) Finally, using estimates (5.19) and (5.24) on the right side of (5.18) with $y_0 = \{y_{0,1}, y_{0,2}\} = \{z_0, z_1, v_0, v_1, \theta_0\} \in Y_\gamma$, we obtain

$$\|N_0^* A_N \hat{z}_t(\lambda)\|_{L_2(\Gamma_0)} \le \frac{C_{r_0}}{|\lambda|^{\frac{1}{2}-2\epsilon}} \|\{z_0, z_1, v_0, v_1, \theta_0\}\|_{Y_\gamma}, \quad \forall\ \lambda = re^{\pm ia}, \tag{5.25}$$

recalling the definition of Y_γ from (3.11), or (4.2.24). Thus, (5.7) of Lemma 5.1 is established.

Step 3. In this step we show that there exists some $r_0 > 0$ and a constant C (depending on r_0 and a) such that, for all $\lambda = re^{\pm ia}$, $|\lambda| = r \ge r_0 > 0$,

$$\|\{\hat{v}(\lambda), \hat{v}_t(\lambda), \hat{\theta}(\lambda)\}\|_{Y_{2,\gamma}} \le \frac{C_{r_0,a,\epsilon}}{|\lambda|} \|\{z_0, z_1, v_0, v_1, \theta_0\}\|_{Y_\gamma}. \tag{5.26}$$

Indeed, we return to (5.4), and invoke here estimates (3.13) and (5.7), to obtain

$$\begin{aligned}\left\|\begin{bmatrix}\hat{v}(\lambda)\\ \hat{v}_t(\lambda)\\ \hat{\theta}(\lambda)\end{bmatrix}\right\|_{Y_{2,\gamma}} &\le \|R(\lambda,A_2)\|_{\mathcal{L}(Y_{2,\gamma})}\left\|\begin{bmatrix}v_0\\ v_1\\ \theta_0\end{bmatrix}\right\|_{Y_{2,\gamma}}\\ &\quad+\|R(\lambda,A_2)\|_{\mathcal{L}(Y_{2,\gamma})}\|N_0^*A_N\hat{z}_t(\lambda)\|_{L_2(\Gamma_0)} \qquad (5.27)\\ &\le \frac{C_a}{|\lambda|}\|\{v_0,v_1,\theta_0\}\|_{Y_{2,\gamma}}+\frac{C_a}{|\lambda|}\frac{C_{r_0,a,\epsilon}}{|\lambda|^{\frac{1}{2}-2\epsilon}}\|\{z_0,z_1,v_0,v_1,\theta_0\}\|_{Y_\gamma},\\ &\qquad \forall\ \lambda=re^{\pm ia},\ |\lambda|=r\ge r_0>0, \qquad (5.28)\end{aligned}$$

and (5.28) yields (5.26), as desired.

Step 4. In this step we show that there exists some $r_0>0$ and a corresponding constant $C>0$ such that, for all $\lambda=re^{\pm ia}$, $|\lambda|=r\ge r_0>0$,

$$\|\{\hat{z}(\lambda),\hat{z}_t(\lambda)\}\|_{Y_1}\le\frac{C}{|\lambda|}\ \|\{z_0,z_1,v_0,v_1,\theta_0\}\|_{Y_\gamma}. \qquad (5.29)$$

Indeed, we return to (5.3) and preliminarily estimate

$$\left\|\begin{bmatrix}\hat{z}(\lambda)\\ \hat{z}_t(\lambda)\end{bmatrix}\right\|_{Y_1}\le\left\|R(\lambda,A_1)\begin{bmatrix}z_0\\ z_1\end{bmatrix}\right\|_{Y_1}+\left\|R(\lambda,A_1)\begin{bmatrix}0\\ A_NN_0\hat{v}_t(\lambda)\end{bmatrix}\right\|_{Y_1}. \qquad (5.30)$$

As to the first term on the right side of (5.30), we readily estimate from (3.7)

$$\left\|R(\lambda,A_1)\begin{bmatrix}z_0\\ z_1\end{bmatrix}\right\|_{Y_1}\le\frac{C}{|\lambda|}\ \|\{z_0,z_1\}\|_{Y_1},\quad \forall\ \lambda=re^{\pm ia},\quad |\lambda|\ge r_0>0. \qquad (5.31)$$

As to the second term on the right side of (5.30), we invoke (5.5) and rewrite it explicitly as

$$R(\lambda,A_1)\begin{bmatrix}0\\ A_NN_0\hat{v}_t(\lambda)\end{bmatrix}=\begin{bmatrix}R(\lambda^2,-A_N)A_NN_0\hat{v}_t(\lambda)\\ \lambda R(\lambda^2,-A_N)A_NN_0\hat{v}_t(\lambda)\end{bmatrix}. \qquad (5.32)$$

Invoking (4.1.3b) and (5.13) with $s=\frac{3}{4}+\epsilon$ and $s=\frac{1}{4}+\epsilon$, we estimate since $Y_1=\mathcal{D}(A_N^{\frac{1}{2}})\times L_2(\Gamma_0)$, see (4.2.23),

$$\begin{aligned}&\left\|R(\lambda,A_1)\begin{bmatrix}0\\ A_NN_0\hat{v}_t(\lambda)\end{bmatrix}\right\|_{Y_1}\\ &\le \left\|\left[A_N^{\frac{3}{4}+\epsilon}R(\lambda^2,-A_N)\right]\left[A_N^{\frac{3}{4}-\epsilon}N_0\right]\hat{v}_t(\lambda)\right\|_{L_2(\Omega)}\\ &+ \left\|\lambda\left[A_N^{\frac{1}{4}+\epsilon}R(\lambda^2,-A_N)\right]\left[A_N^{\frac{3}{4}-\epsilon}N_0\right]\hat{v}_t(\lambda)\right\|_{L_2(\Omega)} \qquad (5.33)\end{aligned}$$

$$
\begin{aligned}
\text{(by (4.1.3), (5.13))} \quad &\le \frac{C_a}{(|\lambda^2|)^{1-(\frac{3}{4}+\epsilon)}}\|\hat{v}_t(\lambda)\|_{L_2(\Omega)} \\
&+ \frac{|\lambda|C_a}{(|\lambda^2|)^{1-(\frac{1}{4}+\epsilon)}}\|\hat{v}_t(\lambda)\|_{L_2(\Omega)} \qquad (5.34) \\
\text{(by (5.26))} \quad &\le \left[\frac{C_a}{|\lambda|^{\frac{1}{2}-2\epsilon}} + \frac{C_a}{|\lambda|^{\frac{1}{2}-2\epsilon}}\right]\frac{C_{r_0}, a}{|\lambda|}\ \|\{z_0, z_1, v_0, v_1, \theta_0\}\|_{Y_\gamma}, \qquad (5.35)
\end{aligned}
$$

where in the last step, we have invoked (5.26).

Thus, in conclusion, by (5.35), we have

$$
\left\|R(\lambda, A_1)\begin{bmatrix} 0 \\ A_N N_0 \hat{v}_t(\lambda)\end{bmatrix}\right\|_{Y_1} \le \frac{C_{a,r_0}}{|\lambda|^{\frac{3}{2}-2\epsilon}}\|\{z_0, z_1, v_0, v_1, \theta_0\}\|_{Y_\gamma},
$$

$$
\forall\ \lambda = re^{\pm ia},\ |\lambda| = r \ge r_0 > 0. \qquad (5.36)
$$

Finally, (5.31) and (5.36), used in (5.30), establish (5.29), as desired.

Step 5. Finally, by Theorem 2.2.1, we can write

$$
[z(t), z_t(t), v(t), v_t(t), \theta(t)] = e^{At}[z_0, z_1, v_0, v_1, \theta_0], \qquad (5.37)
$$

or in the Laplace transform form,

$$
[\hat{z}(\lambda), \hat{z}_t(\lambda), \hat{v}(\lambda), \hat{v}_t(\lambda), \hat{\theta}(\lambda)] = R(\lambda, A)[z_0, z_1, v_0, v_1, \theta_0]. \qquad (5.38)
$$

Finally, recalling the definition of Y_γ in (3.11), or (4.1.6) with $\beta = 0$, we use (5.29) and (5.26) in (5.38), and obtain

$$
\|R(\lambda, A)\|_{\mathcal{L}(Y_\gamma)} \le \frac{C}{|\lambda|} \text{ on rays } \lambda = |\lambda|e^{\pm ia},\ |\lambda| \ge r_0 > 0, \qquad (5.39)
$$

which is conclusion (3.15). Theorem 3.4—at least in the case of the thermoelastic wall, system (2.2.1) with any of the B.C. (2.2.2) through (2.2.5)—is thus proved. □

Remark 5.1. The case of the structural acoustic generator corresponding (via Theorem 2.1.1) to system (2.1.2) with elastic wall can be proved in exactly the same way. Here one uses estimate (3.9) in place of estimate (3.13). □

6. The case of damping on the boundary Γ_0

Suppose now that the B.C. (2.1.1c) on Γ_0 is replaced by the following one:

$$
\frac{\partial z}{\partial \nu} + D_0 z_t + \beta D_0 z = v_t \text{ on } \Sigma_0, \quad D_0 \neq 0. \qquad (6.1)
$$

A well-posedness result in the semigroup sense for the corresponding structural acoustic problem (2.1.1a–b), (6.1), (2.1.1d–f) is given in [5, Theorem 1.3.1] for D_0 a boundary operator satisfying natural hypotheses including (H.2) = (2.1.3).

Now, one may not generally expect to extend the above backward-uniqueness result of Theorem 2.1.2 in the presence of damping z_t on Γ_0. The same negative expectation applies to just the uncoupled wave-equation component. For instance, consider the one-dimensional wave problem on the unit interval $\Omega = (0, 1)$:

$$\begin{cases} z_{tt} = z_{xx}, \ t > 0, \ 0 < x < 1; & (6.2a) \\ \left[\dfrac{\partial z}{\partial \nu} - hz_t\right]_{x=0} = -[z_x + hz_t]_{x=0} \equiv 0, \ t > 0, & (6.2b) \\ z|_{x=1} \equiv 0, \ t > 0, & (6.2c) \\ z|_{t=0} = f(x), \ z_t|_{t=0} = g(x), & (6.2d) \end{cases}$$

where h is a nonzero constant.

(a) If $h = 1$, then problem (6.2) is not well-posed. In fact, let, say, $f = g = 0$. Then there are infinitely many solutions, say on $0 < t < 1$: see [22, Volume 2, equations (9.8.41a-b-c), p. 857, combined with equation (9.9.4.3), p. 883] for their explicit form depending on the arbitrary nonhomogeneous term u in equation (9.9.4.1), p. 882.

(b) If $h = -1$, then it is well known that, for any I.C. $\{f, g\} \in H^1(\Omega) \times L_2(\Omega)$, the corresponding solution $\{z, z_t\}$ vanishes at all $t \geq 2$; in particular,

$$z(2, x) \equiv 0, \ z_t(2, x) \equiv 0, \quad 0 \leq x \leq 1; \tag{6.3}$$

that is, the corresponding semigroup is *nilpotent* for $t \geq 2$. Thus, the backward-uniqueness property is patently false in this case. Indeed, the method of characteristics shows in this case that the solution of problem (6.2) corresponding to $h = -1$ is given by (Figure 1)

$$z(t, x) = \frac{1}{2} f(x + t) + \frac{1}{2} \int_0^{x+t} g(\overline{x}) d\overline{x} + \frac{1}{2} f(0), \quad x \leq t \leq 1 - x; \tag{6.4}$$

$$z(t, x) = \frac{1}{2}[f(x + t) + f(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} g(\overline{x}) d\overline{x}, \\ 0 \leq x + t \leq 1 \text{ and } 0 \leq x - t \leq 1; \tag{6.5}$$

$$z(t, x) = \frac{1}{2} f(x - t) - \frac{1}{2} f(2 - (x + t)) + \frac{1}{2} \int_0^{2-(x+t)} g(\overline{x}) d\overline{x} \\ - \frac{1}{2} \int_0^{x-t} g(\overline{x}) d\overline{x}, \ 1 - x \leq t \leq x; \tag{6.6}$$

$$z(t,x) = \frac{1}{2}f(0) - \frac{1}{2}f(2-(x+t)) + \frac{1}{2}\int_0^{2-(x+t)} g(\overline{x})d\overline{x},$$
$$x - t < 0 \text{ and } 1 \le x+t \le 2; \tag{6.7}$$
$$z(t,x) \equiv 0, \quad 2 \le x+t, \tag{6.8}$$

so that, in particular, (6.3) holds true.

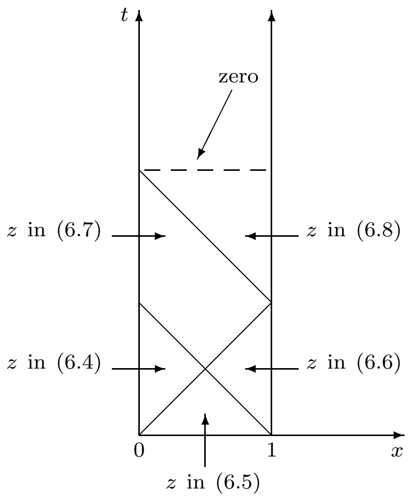


Figure 1

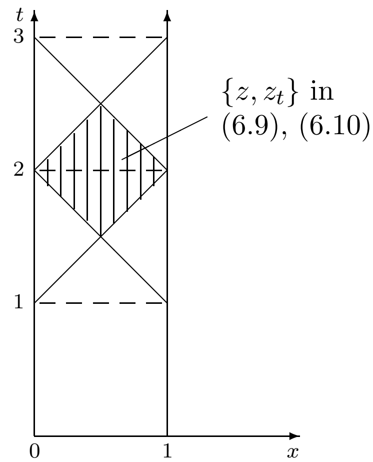


Figure 2

(c) Let $h \neq -1, 1$. Then, the method of characteristics shows that for $2 \le x+t \le 3$ and $-2 \le x-t \le -1$ (see Figure 2),

$$\begin{cases} z(t,x) = -\frac{1}{2}\frac{1+h}{1-h}\left\{f(x+t-2) + f(2+x-t) + \int_{2+x-t}^{x+t-2} g(\bar{x})d\bar{x}\right\}; & (6.9) \\ z_t(t,x) = -\frac{1}{2}\frac{1+h}{1-h}\left\{f'(x+t-2) - f'(2+x-t) \right. \\ \qquad \left. + g(x+t-2) + g(2+x-t)\right\}. & (6.10) \end{cases}$$

Thus, for $t = 2$ and $0 < x < 1$, we have by (6.9) and (6.10),

$$z(2,x) = \frac{1+h}{h-1}\, f(x), \; z_t(2,x) = \frac{1+h}{h-1}\, g(x), \quad 0 \le x \le 1. \tag{6.11}$$

Hence, by (6.11),

$$z(2,x) \equiv 0, \; z_t(2,x) \equiv 0, \; 0 \le x \le 1 \Rightarrow f(x) \equiv 0, \; g(x) \equiv 0, \tag{6.12}$$

and the backward-uniqueness property *does hold true* in this case, with $T = 2$.

Acknowledgment. This work was completed while the author was visiting the Scuola Normale Superiore, Pisa, Italy, whose generous hospitality is acknowledged and greatly appreciated. The author wishes to warmly thank, in particular, Professor G. Da Prato for stimulating discussions.

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