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SCHRÖDINGER GROUP ON ZHIDKOV SPACES

Clément Gallo
UMR de Mathématiques, Bat. 425
Université Paris-Sud, 91405 Orsay, France

(Submitted by: Jerry L. Bona)

Abstract. We consider the Cauchy problem for nonlinear Schrödinger equations on $\mathbb{R}^n$ with nonzero boundary condition at infinity, a situation which occurs in stability studies of dark solitons. We prove that the Schrödinger operator generates a group on Zhidkov spaces $X^k(\mathbb{R}^n)$ for $k > n/2$, and that the Cauchy problem for NLS is locally well-posed on the same Zhidkov spaces. We justify the conservation of classical invariants which implies in some cases the global well-posedness of the Cauchy problem.

1. Introduction

This paper is devoted to the Cauchy problem for the nonlinear Schrödinger equation (NLS)

$$\begin{cases} iu_t + \Delta u + f(|u|^2)u = 0, \ (t,x) \in \mathbb{R} \times \mathbb{R}^n \\ u(0) = u_0 \end{cases} \tag{1.1}$$

with nonzero boundary condition at infinity. Such boundary conditions occur in the "defocusing" case (e.g. $f(|u|^2) = 1 - |u|^2$), and are pertinent to many physical contexts. In nonlinear optics, the so-called dark soliton (see [11]) is a solution of (1.1) of the form $u(x,t) = u_v(x - vt)$. For instance, for $n = 1$ and $f(r) = 1 - r$, we can compute that, for $v \in (-\sqrt{2}, \sqrt{2})$, $u_v(x - vt)$ solves (1.1), where u_v is given by

$$u_v(x) = \sqrt{1 - \frac{v^2}{2}} \tanh\Big(\sqrt{1 - \frac{v^2}{2}} \frac{x}{\sqrt{2}}\Big) + i\frac{v}{\sqrt{2}}. \tag{1.2}$$

The Gross-Pitaevskii equation (see [3], [4], and references therein)

$$iu_t + \Delta u + (1 - |u|^2)u = 0$$

with the boundary condition $u \to 1$ as $|x| \to \infty$ is a model for Superfluid Helium II at a temperature near zero and for Bose-Einstein condensation.

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More generally, NLS with the boundary condition $|u| \to \rho_0^{1/2}$ as $|x| \to \infty$ where ρ_0 is a positive constant such that $f(\rho_0) = 0$ occurs in several physical contexts (see [1]), an especially interesting particular case being the cubic-quintic "$\psi^3 - \psi^5$" NLS.

These nonlinear Schrödinger equations possess solitons or solitary waves (see [3], [2], [12], and [11]), and it is natural to study the Cauchy problem in spaces the solitary waves belong to. Of course they can not be the usual Sobolev spaces $H^s(\mathbb{R}^n)$ because of the boundary condition at infinity. A possibility would be to work in the affine space $1 + H^s(\mathbb{R}^n)$, when $u \to 1$ as $|x| \to \infty$, and this actually was done in [3]. This approach obviously fails when only $|u|$ (but not u) tends to a constant at infinity, as is the case for the dark soliton (1.2). Also, the solitary wave ϕ could be only slowly decaying at infinity, implying that $\phi - 1 \notin L^2(\mathbb{R}^n)$ (see [9] for the travelling wave of Gross-Pitaevskii equation in the two-dimensional case).

In [14] Zhidkov introduces in the one-dimensional case the spaces X^k (with k a natural number), which consist of functions defined on $\mathbb{R}$, bounded and uniformly continuous, with derivatives up to order k in L^2, and proved that the Cauchy problem for NLS is locally well-posed in X^k.

Our aim here is to complete and generalize Zhidkov's results. We introduce the Zhidkov spaces $X^k(\mathbb{R}^n)$ in higher dimensions and prove that the linear Schrödinger equation defines a strongly continuous group on $X^k(\mathbb{R}^n)$ if and only if $k > n/2$, and consequently that the Cauchy problem for NLS is locally well-posed in $X^k(\mathbb{R}^n)$ if $k > n/2$. We also justify rigorously the conservation of natural invariants of the NLS yielding some global well-posedness in $X^k(\mathbb{R}^n)$ for some defocusing NLS. A byproduct is the complete justification of the result of Zhiwu Lin in [18] that gives a criterion of stability for dark solitons of a class of NLS equations.

This paper is organized as follows. In Section 2, we define the Zhidkov spaces $X^k(\mathbb{R}^n)$ and state some useful properties of these spaces. In Section 3, we prove that the linear Schrödinger equation is well posed in $X^k(\mathbb{R}^n)$ if and only if $k > n/2$. In Section 4, we show that the Cauchy problem for NLS is locally well-posed in $X^k(\mathbb{R}^n)$, if $k > n/2$. In Section 5, we introduce the renormalized energy for (1.1) and prove its conservation, for $n = 1$ or 2, under some hypothesis on f and u_0, and we show that in dimension 1, it implies the globalness of the solution of (1.1) in a defocusing case.

Notation. Throughout this paper, C denotes a constant that can change from line to line. If j is a positive integer and u is a map of class C^j from $\mathbb{R}^n$ into $\mathbb{C}$, and $x, v_1, \ldots, v_j \in \mathbb{R}^n$, we denote by $D^j u(x)(v_1, \ldots, v_j)$ the j^{th} differential of u at x applied to $(v_1, \ldots, v_j)$. If E is a Banach space, we denote by $C_b(\mathbb{R}, E)$ the space of continuous, bounded functions from $\mathbb{R}$ into E.

2. Some properties of Zhidkov spaces

Definition 2.1. *Let $n, k \in \mathbb{N}^*$. We define the space $X^k(\mathbb{R}^n)$ as the closure for the norm*

$$\|u\|_{X^k(\mathbb{R}^n)} := \|u\|_{L^\infty} + \sum_{1 \leqslant |\alpha| \leqslant k} \|\partial^\alpha u\|_{L^2}$$

of the space $\{u \in C^k(\mathbb{R}^n)$, bounded, uniformly continuous, with $\nabla u \in H^{k-1}\}$.

Note that a function in X^k is uniformly continuous.

Proposition 2.1. *Let $n, k \in \mathbb{N}^*$. Then $X^{k+1}(\mathbb{R}^n)$ is dense in $X^k(\mathbb{R}^n)$.*

Proof. Let $u \in X^k(\mathbb{R}^n)$, $(\rho_l)_{l \geqslant 1}$ a mollifier sequence (i.e., $\rho_l \in C_c^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \rho_l = 1$, $\rho_n \geqslant 0$, *Supp* $\rho_l \subset B(0, 1/l)$). Then $\rho_l * u$ is a sequence in X^{k+1} that converges to u in X^k. □

We show now a regularity result for functions in $X^k(\mathbb{R}^n)$, for $k > n/2$.

Proposition 2.2. *Let $n \in \mathbb{N}^*$, let $k = \lfloor n/2 \rfloor + 1$, and $p \in \mathbb{R}$ such that*

$$\begin{cases} p > n & \text{if } n \text{ is even} \\ p = 2n & \text{if } n \text{ is odd .} \end{cases}$$

Then $\nabla u \in L^p(\mathbb{R}^n)$ for $u \in X^k(\mathbb{R}^n)$ and there exists $C > 0$ such that

$$|u(x) - u(y)| \leqslant C|x-y|^{1-n/p} ||\nabla u||_{L^p}, \; x, y \in \mathbb{R}^n \; . \tag{2.1}$$

Proof. By our choice of p and k, Sobolev's embedding implies that

$$||\nabla u||_{L^p(\mathbb{R}^n)} \leqslant C||\nabla u||_{H^{k-1}(\mathbb{R}^n)} \leqslant C||u||_{X^k(\mathbb{R}^n)}, \; u \in X^k(\mathbb{R}^n) \; . \tag{2.2}$$

Following the proof of Morrey's theorem given in [7], we can show that there exists a constant $C > 0$ that depends only on n and p such that for any compact set $K \subset \mathbb{R}^n$, and any cube $Q = [-r, r]^n$ containing K,

$$|u(x) - u(y)| \leqslant C|x-y|^{1-n/p} ||\nabla u||_{L^p(Q)}, \; x, y \in K, \; u \in H^k_{loc}(\mathbb{R}^n) \; . \tag{2.3}$$

Since $X^k(\mathbb{R}^n) \subset H^k_{loc}(\mathbb{R}^n)$, (2.2) and (2.3) prove the announced result. □

Remark. In our proof, we did not use the fact that the elements of $X^k(\mathbb{R}^n)$ are uniformly continuous. Therefore, for $k > n/2$, an equivalent definition for X^k could be $X^k(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n), \; \nabla u \in H^{k-1}(\mathbb{R}^n)\}$. In particular, for $k > n/2$, $H^k(\mathbb{R}^n) \subset X^k(\mathbb{R}^n)$.

3. The Schrödinger group on $X^k(\mathbb{R}^n)$

In this section, we prove that if $k > n/2$, the Schrödinger operator defines a group on $X^k(\mathbb{R}^n)$. More precisely,

Theorem 3.1. *Let $n \in \mathbb{N}^*$, $k > n/2$, and $u_0 \in X^k(\mathbb{R}^n)$. For $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, the quantity*

$$S(t)u_0(x) = \begin{cases} e^{-in\pi/4}\pi^{-n/2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}u_0(x+2\sqrt{t}z)dz & \text{if } t \geqslant 0 \\ e^{in\pi/4}\pi^{-n/2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(-i-\varepsilon)|z|^2}u_0(x+2\sqrt{-t}z)dz & \text{if } t \leqslant 0 \end{cases} \quad (3.1)$$

makes sense, and the family of operators $(S(t))_{t\in\mathbb{R}}$ defines a strongly continuous group on $X^k(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$ that depends only on n and k, such that for every $u_0 \in X^k(\mathbb{R}^n)$ and $t \in \mathbb{R}$,

$$||S(t)u_0||_{X^k} \leqslant C(1+|t|^\rho)||u_0||_{X^k}, \quad (3.2)$$

where

$$\rho = \begin{cases} 1/2 & \text{if } n \text{ is even} \\ 1/4 & \text{if } n \text{ is odd} \end{cases}.$$

For convenience, we also denote $S(t)u_0(x)$ by $u(t,x)$.

All the computations below will be performed with $t \geqslant 0$. The case $t \leqslant 0$ is similar. Before starting the proof of the theorem itself, we need to prove some technical lemmas.

Lemma 3.1. *Let $n, k \geqslant 1$, $u_0 \in X^k(\mathbb{R}^n)$, $\beta > 0$, $x \in \mathbb{R}^n$ and $t > 0$. We define $g : (\beta,\infty) \to \mathbb{C}$ by*

$$g(r) = \int_{\mathbb{S}^{n-1}} u_0(x+2\sqrt{t}rv)dv; \quad (3.3)$$

then $g \in X^k(\beta,\infty)$ (the definition of which is clear), and for all $j \in \{1,\dots,k\}$, $g^{(j)} \in L^2(\beta,\infty,r^{n-1}dr)$ with

$$||g^{(j)}||_{L^2(\beta,\infty,r^{n-1}dr)} \leqslant (2\sqrt{t})^{j-n/2}|\mathbb{S}^{n-1}|^{1/2}||u_0||_{X^k(\mathbb{R}^n)}. \quad (3.4)$$

Proof. For any $r \in (\beta,\infty)$, $|g(r)| \leqslant |\mathbb{S}^{n-1}|||u_0||_{L^\infty}$; hence, $g \in L^\infty(\beta,\infty)$.

Let $\varepsilon > 0$. Since u_0 is uniformly continuous, there exists some $\delta > 0$ such that $|y-z| \leqslant \delta$ implies $|u_0(y)-u_0(z)| \leqslant \varepsilon$. Let r_1 and r_2 be such that $|r_1 - r_2| \leqslant \delta/(2\sqrt{t})$. Then we get $|g(r_1)-g(r_2)| \leqslant |\mathbb{S}^{n-1}|\varepsilon$, and g is uniformly continuous.

For $j \in \{1, \dots, k\}$, the Cauchy-Schwarz inequality and a change of variables yield

$$\begin{aligned}
&\int_\beta^\infty |g^{(j)}(r)|^2 r^{n-1} dr \\
&\quad \leqslant \quad (2\sqrt{t})^{2j} |\mathbb{S}^{n-1}| \int_\beta^\infty \int_{\mathbb{S}^{n-1}} |D^j u_0(x + 2\sqrt{t} r v)(v \dots v)|^2 r^{n-1} dv\, dr \\
&\quad = \quad (2\sqrt{t})^{2j} |\mathbb{S}^{n-1}| \int_{2\sqrt{t}\beta}^\infty \int_{\mathbb{S}^{n-1}} |D^j u_0(x + r v)(v \dots v)|^2 \frac{r^{n-1} dv\, dr}{(2\sqrt{t})^n} \\
&\quad \leqslant \quad (2\sqrt{t})^{2j-n} |\mathbb{S}^{n-1}| ||u_0||^2_{X^k(\mathbb{R}^n)},
\end{aligned}$$

which is the announced inequality. □

We state next two elementary lemmas, which are straightforward consequences of the Leibniz formula.

Lemma 3.2. *Let $k \in \mathbb{N}$. There exists constants $(b_{l,k})_{0 \leqslant l \leqslant k}$ such that*

$$\left(\frac{d}{dr}\left(\frac{1}{r}\cdot\right)\right)^k \quad = \quad \sum_{l=0}^k b_{l,k} \frac{1}{r^{2k-l}} \frac{d^l}{dr^l} \ . \tag{3.5}$$

Lemma 3.3. *There exists constants $(a_{k,j})_{0 \leqslant j \leqslant k}$ such that*

$$\left(\frac{d}{dr}\left(\frac{1}{r}\cdot\right)\right)^k (r^{n-1}\cdot) \quad = \quad \sum_{j=0}^k a_{k,j} \frac{1}{r^{2k-n+1-j}} \frac{d^j}{dr^j}. \tag{3.6}$$

We are now ready to prove Theorem 3.1. We fix $n \in \mathbb{N}^*$, and we introduce a function $\chi \in C^\infty(\mathbb{R}^n)$ such that

- χ is radial
- χ increases along any half-line issued at 0
- $\chi \equiv 0$ on $\{x, |x| \leqslant 1\}$
- $\chi \equiv 1$ on $\{x, |x| \geqslant 2\}$.

For any $\beta > 0$, we define $\chi_\beta = \chi(\cdot/\beta)$. For convenience, we will also use the notation $\chi_\beta(|\cdot|) = \chi_\beta(\cdot)$.

Proof of Theorem 3.1. We assume first that $k = \lfloor n/2 \rfloor + 1$. The first step of the proof is to show that the limit in formula (3.1) is well defined. Let us fix $t > 0$, $\beta > 0$ and $\varepsilon > 0$. We split the integral in (3.1) into two parts:

$$\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} u_0(x + 2\sqrt{t} z) dz \quad = \quad \underbrace{\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} (1 - \chi_\beta(z)) u_0(x + 2\sqrt{t} z) dz}_{(I)}$$

$$+\underbrace{\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\chi_\beta(z)u_0(x+2\sqrt{t}z)dz}_{(II)}.$$

We first consider the term (I):

$$\left|e^{(i-\varepsilon)|z|^2}(1-\chi_\beta(z))u_0(x+2\sqrt{t}z)\right| \leqslant ||u_0||_{L^\infty}\mathbf{1}_{B(0,2\beta)}(z).$$

Then by Lebesgue's theorem, the limit as $\varepsilon \to 0$ of (I) exists and

$$\left|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(1-\chi_\beta(z))u_0(x+2\sqrt{t}z)dz\right| \leqslant ||u_0||_{L^\infty}|B(0,1)|(2\beta)^n. \tag{3.7}$$

We now consider the term (II). We compute it by using polar coordinates, and with the notation of Lemma 3.1, we integrate by parts (which is justified by the regularity results on g proved in Lemma 3.1) using Lemma 3.3:

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\chi_\beta(z)u_0(x+2\sqrt{t}z)dz \qquad (3.8)\\
&= \int_0^\infty e^{(i-\varepsilon)r^2}\chi_\beta(r)r^{n-1}\left(\int_{\mathbb{S}^{n-1}} u_0(x+2\sqrt{t}rv)dv\right)dr\\
&= \int_\beta^\infty \left(\frac{1}{2(i-\varepsilon)r}\frac{d}{dr}.\right)^k\left(e^{(i-\varepsilon)r^2}\right)\chi_\beta(r)r^{n-1}\left(\int_{\mathbb{S}^{n-1}} u_0(x+2\sqrt{t}rv)dv\right)dr\\
&= \left(\frac{-1}{2(i-\varepsilon)}\right)^k\int_\beta^\infty e^{(i-\varepsilon)r^2}\sum_{j=0}^k a_{k,j}\frac{1}{r^{2k-n+1-j}}\frac{d^j}{dr^j}\left[\chi_\beta(r)g(r)\right]dr\\
&= \left(\frac{-1}{2(i-\varepsilon)}\right)^k\sum_{j=0}^k a_{k,j}\sum_{l=0}^j\binom{j}{l}\int_\beta^\infty e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g^{(l)}(r)\chi_\beta^{(j-l)}(r)r^{n-1}dr,
\end{aligned}$$

where we have used the Leibniz formula in the last equality. We will now apply Lebesgue's theorem to each term of this sum.

For $l=0$ and $j\in\{0,\dots,k\}$, we have

$$\begin{aligned}
&\left|e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g(r)\chi_\beta^{(j)}(r)r^{n-1}\right|\\
&\qquad\leqslant \frac{|\mathbb{S}^{n-1}|||u_0||_{L^\infty}}{r^{2k-n+1}}\begin{cases} ||\chi||_{L^\infty} & \text{if } j=0\\ \frac{r^j}{\beta^j}||\chi^{(j)}||_{L^\infty} & \text{if } j\in\{1,\dots,k\} \text{ and } r\leqslant 2\beta\\ 0 & \text{if } j\in\{1,\dots,k\} \text{ and } r>2\beta\end{cases}\\
&\qquad\leqslant \frac{|\mathbb{S}^{n-1}|||u_0||_{L^\infty}}{r^{2k-n+1}}2^j||\chi^{(j)}||_{L^\infty}.
\end{aligned}$$

Since $k > n/2$, $\int_\beta^\infty dr/r^{2k-n+1} = \beta^{n-2k}/(2k-n) < \infty$ and we can pass to the limit as $\varepsilon \to 0$ by Lebesgue's theorem. We obtain

$$\left|\lim_{\varepsilon\to 0}\int_\beta^\infty e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g(r)\chi_\beta^{(j)}(r)r^{n-1}dr\right| \leqslant \frac{|\mathbb{S}^{n-1}|||u_0||_{L^\infty}2^j||\chi^{(j)}||_{L^\infty}}{(2k-n)\beta^{2k-n}}. \tag{3.9}$$

For $l \geqslant 1$ and $j \in \{l,\dots,k\}$, we have

$$\left|e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g^{(l)}(r)\chi_\beta^{(j-l)}(r)r^{n-1}\right| \leqslant ||\chi^{(j-l)}||_{L^\infty}\frac{1}{\beta^{j-l}}\frac{1}{r^{2k-j}}|g^{(l)}(r)|r^{n-1}.$$

Notice that $2(2k-j)-(n-1) > 1$, so that $r \to 1/r^{2k-j} \in L^2(\beta,\infty,r^{n-1}dr)$. The Cauchy-Schwarz inequality together with (3.4) lead to

$$\begin{aligned}\int_\beta^\infty &\frac{1}{r^{2k-j}}|g^{(l)}(r)|r^{n-1}dr\\ &\leqslant \frac{1}{(4k-2j-n)^{1/2}\beta^{2k-j-n/2}}(2\sqrt{t})^{l-n/2}|\mathbb{S}^{n-1}|^{1/2}||u_0||_{X^k}\ .\end{aligned}$$

So Lebesgue's theorem can be applied, and

$$\begin{aligned}&\left|\lim_{\varepsilon\to 0}\int_\beta^\infty e^{(i-\varepsilon)r^2}\frac{1}{r^{2k-j}}g^{(l)}(r)\chi_\beta^{(j-l)}(r)r^{n-1}dr\right|\\ &\qquad\leqslant \frac{|\mathbb{S}^{n-1}|^{1/2}||\chi^{(j-l)}||_{L^\infty}}{(4k-2j-n)^{1/2}}||u_0||_{X^k}\frac{(2\sqrt{t})^{l-n/2}}{\beta^{2k-l-n/2}}.\end{aligned} \tag{3.10}$$

We fix now $\beta = 1$. For $j \in \{1,\dots,k\}$, the quantities $(2\sqrt{t})^{j-n/2}$ are majorized by $C(t^{(1-n/2)/2} + t^{(k-n/2)/2})$, where C is a positive constant. Therefore, using (3.7), (3.9), and (3.10), there exists a positive constant C such that for all $u_0 \in X^k$, for all $t > 0$,

$$||u(t)||_{L^\infty(\mathbb{R}^n)} \leqslant C(1 + t^{(1-n/2)/2} + t^{(k-n/2)/2})||u_0||_{X^k(\mathbb{R}^n)}. \tag{3.11}$$

The second step of the proof consists in proving that $u(t) \to u_0$ in L^∞ as $t \to 0$. We first introduce some definitions.

Definition 3.1. *For $l \in \{0,\dots,k\}$ and $h \in L^2(\beta,\infty,r^{n-1}dr)$, we define*

$$T_h^l := \int_\beta^\infty e^{(i-\varepsilon)r^2}h(r)g^{(l)}(r)r^{n-1}dr,$$

and we will say that such a quantity is "of type $T^{(l)}$."

If h can be written $h(r) = \chi_\beta^{(p)}(r)/r^q$ with $2q-n > 0$, we will say that the "order" of T_h^l in β is $p+q-n/2$.

If h is given as a linear combination of such terms, we define the "order" of T_h^l as the lowest order of nonzero monomials in the expression of h (remark that our "definition" of the order of T_h^l depends on the decomposition of h).

Let $\alpha \in (0, 1/(2n+1))$, and $m > 0$ such that $m \geqslant (n/2-1)/\alpha$. The following technical lemma gives a new expression of the term (II). It consists in doing as much integration by parts as necessary, in order to express (II) as a sum of terms of orders $\geqslant m$, which we are able to estimate in an appropriate way (see estimate (3.14) below).

Lemma 3.4. (II) *can be written as a linear combination of terms of type $T^{(l)}$ with $l \in \{0, \dots, k\}$, such that for $l \in \{1, \dots, k-1\}$, the order of the terms of type $T^{(l)}$ in this linear combination is $\geqslant m$.*

Proof. Let $l \in \{1, \dots, k-1\}$, $p, q \in \mathbb{N}$ such that $2q - n > 0$, and $h(r) = \chi_\beta^{(p)}(r) r^q$. We transform T_h^l by integrations by parts, as in (3.8). After some computations, we get (this is actually (3.8), where we have replaced k by $k-l$, g by $g^{(l)}$, and χ_β by $\chi_\beta^{(p)}(r)/r^q$)

$$\begin{aligned} T_h^l = \Big(\frac{-1}{2(i-\varepsilon)}\Big)^{k-l} \sum_{j=0}^{k-l} a_{k-l,j} \sum_{c=0}^{j} \binom{j}{c} \sum_{b=0}^{j-c} \binom{j-c}{b} \\ \times (-q) \cdots (-q-(j-c-b-1)) \int_\beta^\infty \frac{e^{(i-\varepsilon)r^2} \chi_\beta^{(p+b)}(r) g^{(l+c)}(r)}{r^{2(k-l)+q-c-b}} r^{n-1} dr. \end{aligned} \tag{3.12}$$

Since $2(2(k-l)+q-c-b)-n \geqslant 2(k-l)+2q-n$, $k \geqslant l$, and $2q-n > 0$, we have written T_h^l as a linear combination of terms of types $T^{(l)}, T^{(l+1)}, \dots, T^{(k)}$. Moreover, the order of the terms of type $T^{(l)}$ (that correspond to $c = 0$) in this sum is

$$p + b + 2(k-l) + q - b - n/2 = \underbrace{p + q - n/2}_{\text{order of } T_h^l} + 2\underbrace{(k-l)}_{\geqslant 1}.$$

The conclusion of this computation is that passing from the canonical expression of T_h^l to its new expression, the order of type $T^{(l)}$ terms increases at least by 2.

We now use the above calculation to show the result by induction on $l \in \{0, \dots, k-1\}$.

Let us consider, for $l \in \{0, \dots, k-1\}$, the induction hypothesis H_l: "(II) can be written as a sum of terms of type $T^{(\gamma)}$, $0 \leqslant \gamma \leqslant k$, so that if $1 \leqslant \gamma \leqslant l$, the term of type $T^{(\gamma)}$ in this sum has order $\geqslant m$." Formula (3.8) implies that

$$(\mathrm{II}) = \sum_{l=0}^{k} T^l_{h^{(l)}}, \text{ with } h^{(l)}(r) = \sum_{j=l}^{k} a_{kj} \binom{j}{l} \frac{\chi_\beta^{(j-l)}(r)}{r^{2k-j}} \ .$$

We have $2(2k-j)-n \geqslant 2k-n > 0$, so that H_0 is true thanks to (3.8). Let us now take $l \in \{1,\dots,k-1\}$ and suppose H_{l-1}. The induction hypothesis implies that (II) can be written in the form $(\mathrm{II}) = \sum_\gamma^k \lambda_\gamma T^\gamma_{h_\gamma}$, where $\lambda_\gamma \in \mathbb{C}$, h_γ is a linear combination of $\chi_\beta^{(p)}(r)/r^q$, and for $\gamma \in \{1,\dots,l-1\}$, $T^\gamma_{h_\gamma}$'s order is at least m. Applying the former calculation to the term $T^l_{h_l}$, we get a new expression of (II) where the terms of type $T^{(\gamma)}$ with $\gamma \leqslant l-1$ are unchanged (in particular, the new expression still satisfies H_{l-1}) and the order of the term of type $T^{(l)}$ has increased by $2(k-l) \geqslant 2$. We can start this process again, as long as it is necessary (but with a finite number of steps) to ensure that the term of type $T^{(l)}$ is of order $\geqslant m$, and hence H_l is true. So we have proved that H_{k-1} is true, which is the result of the lemma. □

We give now an upper bound on $|T^l_h|$, for $l \in \{1,\dots,k\}$, with $h(r) = \chi_\beta^{(p)}(r)/r^q$ and $2q-n>0$:

$$\begin{aligned} |T^l_h| &\leqslant \frac{\|\chi^{(p)}\|_\infty}{\beta^p} \int_\beta^\infty \frac{1}{r^q} |g^{(l)}(r)| r^{n-1} dr \\ &\leqslant \underbrace{\frac{\|\chi^{(p)}\|_\infty}{\sqrt{2q-n}} \|u_0\|_{X^k} |\mathbb{S}^{n-1}|^{1/2}}_{=:C} \frac{(2\sqrt{t})^{l-n/2}}{\beta^{p+q-n/2}}, \end{aligned} \tag{3.13}$$

where we have used the Cauchy-Schwarz inequality and Lemma 3.1. If we assume that $\beta \geqslant 1$ and T^l_h's order is at least m, then

$$|T^l_h| \leqslant C \frac{(2\sqrt{t})^{l-n/2}}{\beta^m} \ . \tag{3.14}$$

We write β in the form $\beta = \tilde{\beta}/(2\sqrt{t})^\alpha$, for $t \leqslant 1$ and $\tilde{\beta} > 2^\alpha$. Let us fix $\delta > 0$. The choice of m and the fact that $l \geqslant 1$ ensure that $l - n/2 + \alpha m \geqslant 0$. We choose $\tilde{\beta} > 2^\alpha$ large enough such that for all $t \leqslant 1$,

$$\frac{(2\sqrt{t})^{l-n/2+\alpha m}}{\tilde{\beta}^m} \leqslant \delta \ , \ l \in \{1,\dots,k\}$$

and

$$\frac{|\mathbb{S}^{n-1}| \|u_0\|_\infty 2^j \|\chi^{(j)}\|_{L^\infty}}{(2k-n)\tilde{\beta}^{2k-n}} (2\sqrt{t})^{\alpha(2k-n)} \leqslant \delta \ , \ j \in \{0,\dots,k\} \ .$$

The property H_{k-1} proven in Lemma 3.4, (3.14), and (3.9) imply that there exists a constant $C > 0$ (which does not depend on δ) such that

$$\forall \varepsilon > 0, \left| \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \chi_{\tilde{\beta}/(2\sqrt{t})^\alpha}(z) u_0(x + 2\sqrt{t}z) dz \right| \leqslant C\delta \,. \quad (3.15)$$

Proposition 2.2 with $p = 2n$ yields

$$\begin{aligned} &\left| \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} (1 - \chi_\beta(z)) u_0(x + 2\sqrt{t}z) dz \right. \\ &\qquad \left. - \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} (1 - \chi_\beta(z)) u_0(x) dz \right| \\ &\leqslant \int_{\mathbb{R}^n} (1 - \chi_\beta(z)) |u_0(x + 2\sqrt{t}z) - u_0(x)| dz \\ &\leqslant \int_{|z|\leqslant 2\beta} C(2\sqrt{t}2\beta)^{1/2} ||\nabla u_0||_{L^p} dz \\ &\leqslant C(2\sqrt{t})^{1/2} \beta^{n+1/2} = C(2\sqrt{t})^{1/2-\alpha(n+1/2)} \tilde{\beta}^{n+1/2} \,. \end{aligned} \quad (3.16)$$

By the choice of α, $1/2 - \alpha(n + 1/2) > 0$. So we can take $t_0 < 1$ such that

$$\forall t \leqslant t_0, \quad C(2\sqrt{t})^{1/2-\alpha(n+1/2)} \tilde{\beta}^{n+1/2} \leqslant \delta. \quad (3.17)$$

Therefore, for $t \leqslant t_0$,

$$\begin{aligned} &\left| \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} u_0(x + 2\sqrt{t}z) dz - \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} u_0(x) dz \right| \\ &\leqslant \left| \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \chi_\beta(z) u_0(x + 2\sqrt{t}z) dz \right| + \left| \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \chi_\beta(z) u_0(x) dz \right| \\ &+ \left| \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} (1 - \chi_\beta(z)) (u_0(x + 2\sqrt{t}z) - u_0(x)) dz \right| \\ &\leqslant C\delta + C\delta + \delta \end{aligned} \quad (3.18)$$

by (3.15), (3.15) applied to $z \to u_0(x)$ instead of u_0 and (3.17).

Inequality (3.18) and the well-known identity

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} dz = \pi^{n/2} e^{in\pi/4}$$

imply that $u(t) \to u_0 = u(0)$ in L^∞ as $t \to 0$, $t > 0$. We could prove in the same way the left continuity at 0, and this, combined with the group property $S(t + s) = S(t)S(s)$, $t, s \in \mathbb{R}$ (which is easy to verify) shows that for all $u_0 \in X^k$, $S(\cdot)u_0 \in C(\mathbb{R}, L^\infty(\mathbb{R}^n))$.

We consider now the general case $k > n/2$. For any multi-index α such that $1 \leqslant |\alpha| \leqslant k$, for any $t > 0$ (we have a similar formula for $t < 0$),

$$\partial^\alpha u(t) = \frac{e^{-in\pi/4}}{\pi^{n/2}} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \partial^\alpha u_0(x + 2\sqrt{t}z) dz \ .$$

Here, $\partial^\alpha u_0 \in L^2$. Since (3.1) defines classically a unitary group on L^2, the map $t \to \partial^\alpha u(t)$ belongs to $C(\mathbb{R}, L^2(\mathbb{R}^n))$, with $||\partial^\alpha u(t)||_{L^2} = ||\partial^\alpha u_0||_{L^2}$, $t \in \mathbb{R}$. Therefore, we can conclude that for all $u_0 \in X^k$, $t \to S(t)u_0 \in C(\mathbb{R}, X^k(\mathbb{R}^n))$. Moreover, since for all $u_0 \in X^k$, $S(\cdot)u_0$ is bounded on $[-1, 1]$, the Banach-Steinhaus theorem implies that $||S(t)||_{\mathcal{L}(X^k)}$ is bounded on $[-1, 1]$. Combining this with (3.11), we get (3.2). □

We give now the infinitesimal generator of $S(t)$.

Theorem 3.2. *Let $k > n/2$. The generator of the group $(S(t))_{t\in\mathbb{R}}$ on $X^k(\mathbb{R}^n)$ defined in Theorem* 3.1 *is $i\Delta$; its domain is $X^{k+2}(\mathbb{R}^n)$.*

Proof. We denote by $A = S'(0)$ the generator of $(S(t))_t$. We split the proof into three steps. In the first step, we show that if $u_0 \in X^{k+4}(\mathbb{R}^n)$, $(S(t)u_0 - u_0)/t \underset{t\to 0}{\longrightarrow} i\Delta u_0$ in $X^k(\mathbb{R}^n)$, and therefore $(X^{k+4}(\mathbb{R}^n), i\Delta) \subset (D(A), A)$. In the second step, we prove that $(X^{k+2}(\mathbb{R}^n), i\Delta) \subset (D(A), A)$. We conclude in the third step.

1st step. Let $u_0 \in X^{k+4}(\mathbb{R}^n) \subset C^4(\mathbb{R}^n)$. We want to show that

$$\frac{S(t)u_0 - u_0}{t} \underset{t\to 0}{\longrightarrow} i\Delta u_0 \quad \text{in } X^k \ . \tag{3.19}$$

We will prove (3.19) for $t \to 0,\ t > 0$. The proof is similar in the case $t \to 0,\ t < 0$. By Taylor's formula

$$u_0(x+2\sqrt{t}z) - u_0(x) = \nabla u_0(x)\cdot 2\sqrt{t}z + \int_0^1 (1-s) D^2 u_0(x+2\sqrt{t}z)(2\sqrt{t}z, 2\sqrt{t}z) ds,$$

the fact that

$$\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} \nabla u_0(x) \cdot 2\sqrt{t}z \, dz = 0 \ ,$$

and

$$i\Delta u_0(x) = \frac{1}{2} \lim_{\varepsilon\to 0} \frac{e^{-in\pi/4}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} D^2 u_0(x)(2\sqrt{t}z, 2\sqrt{t}z) dz \ ,$$

we obtain

$$\Big(\frac{S(t)u_0 - u_0}{t} - i\Delta u_0\Big)(x) = 4 \frac{e^{-in\pi/4}}{\pi^{n/2}} \lim_{\varepsilon\to 0} \int_0^1 (1-s)$$

$$\times \int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(D^2u_0(x+2s\sqrt{t}z)-D^2u_0(x))(z,z)dz\,ds.$$

Next,

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(D^2u_0(x+2s\sqrt{t}z)-D^2u_0(x))(z,z)dz\\ &= \left(\frac{-1}{2(i-\varepsilon)}\right)^2\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}\Delta^2u_0(x+2s\sqrt{t}z)(2s\sqrt{t})^2dz\\ &\quad -\frac{1}{2(i-\varepsilon)}\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2}(\Delta u_0(x+2s\sqrt{t}z)-\Delta u_0(x))dz\ ;\end{aligned}$$

hence,

$$\begin{aligned}\Big(\frac{S(t)u_0-u_0}{t}-i\Delta u_0\Big)(x) &= -4\Big[t\int_0^1(1-s)s^2S(ts^2)\Delta^2u_0(x)ds \\ &+\frac{1}{2i}\int_0^1(1-s)(S(ts^2)\Delta u_0-\Delta u_0)(x)ds\Big].\end{aligned} \tag{3.20}$$

Since $k>n/2$, $\Delta^2u_0\in H^k(\mathbb{R}^n)\subset X^k(\mathbb{R}^n)$. Therefore, $\big(S(ts^2)\Delta^2u_0\big)_{t,s\in[0,1]}$ is bounded in X^k and the first term on the right-hand side of (3.20) tends to 0 in X^k as $t\to 0$. Since $\Delta u_0\in H^{k+2}(\mathbb{R}^n)\subset X^k(\mathbb{R}^n)$, $S(ts^2)\Delta u_0-\Delta u_0 \underset{t\to 0}{\longrightarrow} 0$ in X^k, uniformly in $s\in[0,1]$. Thus (3.19) has been proven. Therefore, if $u_0\in X^{k+4}(\mathbb{R}^n)$, $u_0\in D(A)$ and $Au_0=i\Delta u_0$.

2nd step. Let $u_0\in X^{k+2}(\mathbb{R}^n)$. Thanks to Proposition 2.1, there exists a sequence $(v_0^l)_{l\in\mathbb{N}}\subset X^{k+4}(\mathbb{R}^n)$ such that $v_0^l\to u_0$ in $X^{k+2}\subset X^k$. Hence $i\Delta v_0^l\to i\Delta u_0$ in $H^k\subset X^k$. Therefore $(u_0,i\Delta u_0)$ belongs to the closure of $\{(v_0,i\Delta v_0),\ v_0\in X^{k+4}(\mathbb{R}^n)\}$ in $X^k\times X^k$, and since the infinitesimal generator of a strongly continuous semigroup is a closed operator (see Corollary 2.5 in [13], p. 5), this implies that $u_0\in D(A)$ and $Au_0=i\Delta u_0$; i.e., $(X^{k+2}(\mathbb{R}^n),i\Delta)\subset(D(A),A)$.

3rd step. Let $u_0\in D(A)$. We want to show that $Au_0=i\Delta u_0$ in $\mathcal{D}'$. Since $X^{k+2}(\mathbb{R}^n)$ is dense in $X^k(\mathbb{R}^n)$, there exists a sequence $(v_0^l)_{l\in\mathbb{N}}\in X^{k+2}(\mathbb{R}^n)$ that tends to u_0 in $X^k(\mathbb{R}^n)$, and then

$$Av_0^l=i\Delta v_0^l \underset{l\to\infty}{\longrightarrow} i\Delta u_0 \text{ in } \mathcal{D}'\ .$$

Let us show that $Av_0^l\to Au_0$ in $\mathcal{D}'$. Let $\phi\in C_c^\infty(\mathbb{R}^n)$. We have

$$\begin{aligned}&\langle Av_0^l,\phi\rangle_{\mathcal{D}',\mathcal{D}}=\langle\phi,Av_0^l\rangle_{(X^k)',X^k}=\langle A^*\phi,v_0^l\rangle_{(X^k)',X^k}\\ &\to\langle A^*\phi,u_0\rangle_{(X^k)',X^k}=\langle\phi,Au_0\rangle_{(X^k)',X^k}=\langle Au_0,\phi\rangle_{\mathcal{D}',\mathcal{D}}.\end{aligned} \tag{3.21}$$

To justify this calculation, we need to prove that $\phi \in D(A^*)$. Since the Schrödinger group is continuous in the Schwartz space $\mathcal{S}$, one has for any $u \in D(A)$,

$$\begin{aligned}\langle Au, \phi\rangle_{\mathcal{D}',\mathcal{D}} &= \lim_{t\to 0}\Big\langle \frac{S(t)-id}{t}u, \phi\Big\rangle_{\mathcal{D}',\mathcal{D}} \\ &= \lim_{t\to 0}\int_{\mathbb{R}^n} u(y)\Big(\frac{S(-t)-id}{t}\overline{\phi}\Big)(y)dy = -\langle i\Delta\phi, u\rangle_{\mathcal{D}',\mathcal{D}},\end{aligned}$$

and then

$$|\langle S'(0)u, \phi\rangle_{\mathcal{D}',\mathcal{D}}| \leqslant ||\Delta\phi||_{L^1}||u||_{X^k},\ u \in D(A)\ .$$

It follows that $\phi \in D(A^*)$ and (3.21) is justified.

Therefore, $Au_0 = i\Delta u_0$ in $\mathcal{D}'$. Now, $u_0 \in X^k$ and $Au_0 = i\Delta u_0 \in X^k$; hence, $\Delta\nabla u_0 \in \big(H^{k-1}\big)^n$, and since $\nabla u_0 \in \big(H^{k-1}\big)^n$, this yields $\nabla u_0 \in \big(H^{k+1}\big)^n$, and we conclude that $u_0 \in X^{k+2}$. □

Finally, we show that the assumption $k > n/2$ we made in Theorem 3.1 is sharp. More precisely, we have

Proposition 3.1. *Let us take $n \in \mathbb{N}^*$. For $x \in \mathbb{R}^n$, we define*

$$u_0(x) = \frac{e^{-i|x|^2}}{(1+|x|^2)^{n/2}\log\sqrt{2+|x|^2}}\ .$$

Then $u_0 \in X^{\lfloor n/2\rfloor}(\mathbb{R}^n)$, but if we define $u(t,x)$ by formula (3.1), $u(\frac{1}{4}, 0) = \infty$.

Proof. It is clear that $u_0 \in L^\infty(\mathbb{R}^n)$ and is uniformly continuous (because ∇u_0 is bounded). Let α be a multi-index with $1 \leqslant |\alpha| \leqslant \lfloor n/2\rfloor$. The "worst term" in $\partial^\alpha u_0$ is the one obtained when deriving $e^{-i|x|^2}$ $|\alpha|$ times. This term is $(-2ix)^\alpha e^{-i|x|^2}(1+|x|^2)^{-n/2}/\log\sqrt{2+|x|^2}$. Plainly,

$$\begin{aligned}\int_{\mathbb{R}^n}\Big|\frac{(-2ix)^\alpha e^{-i|x|^2}}{(1+|x|^2)^{n/2}\log\sqrt{2+|x|^2}}\Big|^2 dx &\leqslant 2^{|\alpha|}\int_{\mathbb{R}^n}\frac{(|x|^2)^{|\alpha|}}{(1+|x|^2)^n\log^2\sqrt{2+|x|^2}}dx \\ &= 2^{|\alpha|}|\mathbb{S}^{n-1}|\int_0^\infty \frac{r^{2|\alpha|+n-1}dr}{(1+r^2)^n\log^2\sqrt{2+r^2}}.\end{aligned}$$

The right-hand side term in the above inequality is finite, because $2n-(2|\alpha|+n-1) \geqslant 1$ and $\int^\infty \frac{dr}{rlog^2 r} < \infty$. Therefore $u_0 \in X^{\lfloor n/2\rfloor}(\mathbb{R}^n)$. Moreover, since $u_0 \in L^2$, $u_0 \in H^{\lfloor n/2\rfloor}(\mathbb{R}^n)$, and $u \in C(\mathbb{R}, H^{\lfloor n/2\rfloor})$ is a solution of the linear Schrödinger equation.

For $t = 1/4,\ x = 0$,

$$\int_{\mathbb{R}^n} e^{(i-\varepsilon)|z|^2} u_0(x + 2\sqrt{t}z)dz = \int_{\mathbb{R}^n} \frac{e^{-\varepsilon|y|^2}dy}{(1+|y|^2)^{n/2}\log\sqrt{2+|y|^2}}$$
$$= |\mathbb{S}^{n-1}| \int_0^\infty \frac{e^{-\varepsilon r^2} r^{n-1} dr}{(1+r^2)^{n/2}\log\sqrt{2+r^2}},$$

which tends to $+\infty$ as $\varepsilon \to 0$, by the monotone convergence theorem and since $\int^\infty dr/(r\log r) = \infty$.

An amusing fact is that for $(t,x) \neq (1/4, 0)$, $u(t,x)$ is well defined by formula (3.1) and then u is a continuous function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n \backslash (\frac{1}{4}, 0)$. □

Remark. The "ill-posedness" result in Proposition 3.1 pertains to the dispersive blow-up phenomena described in [5] for KdV-type equations and in [6] for general dispersive equations.

4. Local existence for NLS in $X^k(\mathbb{R}^n)$

We consider now the Cauchy problem (1.1) with a more general nonlinearity:

$$iu_t + \Delta u + F(u) = 0, \quad u(0) = u_0 \in X^k(\mathbb{R}^n), \tag{4.1}$$

where $k > n/2$ and $F : X^k(\mathbb{R}^n) \to X^k(\mathbb{R}^n)$ is of class C^1. We have shown in Section 3 that the Schrödinger group $S(t)$ defines a strongly continuous group on $X^k(\mathbb{R}^n)$. By a classical fixed-point argument one obtains the local well-posedness of the Cauchy problem (4.1). Namely, we have

Theorem 4.1. *Let $M > 0$. Then there exists $T_+(M) > 0$ and $T_-(M) < 0$ such that for all $u_0 \in X^k(\mathbb{R}^n)$ with $||u_0||_{X^k} \leqslant M$, there exists a unique mild solution $u \in C([T_-(M), T_+(M)], X^k)$ of (4.1). We recall that a mild solution of (4.1) is a solution of the integral equation*

$$u(t) = S(t)u_0 + i\int_0^t S(t-s)F(u(s))ds\ ,\ t \in [T_-, T_+]\ . \tag{4.2}$$

We also recall (see [13] or [8]) that if u solves (4.2), it is a solution of (4.1) in the space $C(\mathbb{R}, H^{k-2}_{loc})$, and that if $u_0 \in D(S'(0)) = X^{k+2}$, $u \in C^1([T_-, T_+], X^k) \cap C([T_-, T_+], X^{k+2})$ is the classical solution of (4.1).

Proof. It is a direct application of Proposition 4.3.3 in [8]. □

We have furthermore

Theorem 4.2. *For every $u_0 \in X^k(\mathbb{R}^n)$, there exists $T_*(u_0) \in [-\infty, 0)$ and $T^*(u_0) \in (0, +\infty]$ such that*

- *there exists a maximal solution* $u \in C(T_*(u_0), T^*(u_0), X^k)$ *which is the unique solution of* (4.2), *for all* $T_\pm$, $T_*(u_0) < T_- < 0 < T_+ < T^*(u_0)$,
- *either* $T^*(u_0) = +\infty$ *or* $||u(t)||_{X^k} \underset{t\uparrow T^*(u_0)}{\longrightarrow} +\infty$,
- *either* $T_*(u_0) = -\infty$ *or* $||u(t)||_{X^k} \underset{t\downarrow T_*(u_0)}{\longrightarrow} +\infty$.

Proof. It is a direct application of Theorem 4.3.4 in [8]. □

Theorem 4.3. *If* $u_0 \in X^{k+2}(\mathbb{R}^n)$, *the solution* $u \in C(T_*(u_0), T^*(u_0), X^k)$ *of* (4.2) *given by Theorems* 4.1 *and* 4.2 *is a classical solution of* (4.1) *in* X^k, *which means that* $u \in C(T_*(u_0), T^*(u_0), X^{k+2}) \cap C^1(T_*(u_0), T^*(u_0), X^k)$.

Proof. It is a direct application of Theorem 1.5 in [13], because $F : X^k \to X^k$ is of class C^1, and because the domain of $i\Delta$ (which is the generator of the group $S(t)$) on X^k is X^{k+2}. □

Here is a typical example to which Theorem 4.1 applies.

Proposition 4.1. *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be of class* C^{k+1}. *Then* $F : X^k(\mathbb{R}^n) \to X^k(\mathbb{R}^n)$ *defined by* $F(u) = f(|u|^2)u$ *is of class* C^1.

Proof. Let $u \in X^k(\mathbb{R}^n)$. It is clear that $F(u) \in L^\infty$.

For any multi-index α such that $1 \leqslant |\alpha| \leqslant k$, $\partial^\alpha F(u)$ can be written as a linear combination of terms of type $u^a \overline{u}^b f^{(l)}(|u|^2)\partial^{\alpha_1} u \cdots \partial^{\alpha_r} u \partial^{\beta_1}\overline{u} \cdots \partial^{\beta_s}\overline{u}$, where a, b, r, s, and l are integers and α_i and β_j are multi-indices such that $l \leqslant |\alpha| \leqslant k$ and $\sum |\alpha_i| + \sum |\beta_j| \leqslant |\alpha| \leqslant k$.

By Sobolev's embeddings and generalized Hölder's inequality, it can be shown easily that $\partial^\alpha F(u) \in L^2$, and hence $F(u) \in X^k$.

The proof that for $u \in X^k$, $F'(u)$ maps X^k into X^k and that F' is continuous is similar. □

We end up this paragraph by proving the persistency of higher regularity.

Proposition 4.2. *Let* $n \in \mathbb{N}^*$, $k > n/2 + 1$, *and* $u_0 \in X^k(\mathbb{R}^n)$. *We suppose* $f \in C^{k+1}(\mathbb{R}_+)$. *For* $l \in \{\lfloor n/2 \rfloor + 1, \ldots, k\}$, *we shall denote by* $u \in C(T_*(l), T^*(l), X^l)$ *the solution of* (4.2) *with* $F(u) = f(|u|^2)u$, *where* $(T_*(l), T^*(l))$ *is the maximal existence interval of* u *in* $X^l(\mathbb{R}^n)$. *Then*

$$T^* := T^*(\lfloor n/2 \rfloor + 1) = \cdots = T^*(k)$$

and similarly, $T_* := T_*(\lfloor n/2 \rfloor + 1) = \cdots = T_*(k)$.

Proof. We make the proof for T^*; it is similar for T_*. A priori, for $l > \lfloor \frac{n}{2} \rfloor + 1$, $T^* \geqslant T^*(l)$. We suppose for the sake of contradiction that $T^* > T^*(l)$.

For $t \in (T_*, T^*)$,

$$u(t) = S(t)u_0 + i \int_0^t S(t-s) \left(f(|u(s)|^2)u(s) \right) ds;$$

hence, for $t \in (T_*(l), T^*(l))$,

$$\|u(t)\|_{X^l} \leqslant \|S(t)\|_{\mathcal{L}(X^l,X^l)} \|u_0\|_{X^l} + \int_0^t \|S(t-s)\|_{\mathcal{L}(X^l,X^l)} \|f(|u(s)|^2)u(s)\|_{X^l} ds.$$

We know by (3.2) that $||S(t)||_{\mathcal{L}(X^l,X^l)}$ is bounded on $[0, T^*(l)]$, and that u is continuous from $[0, T^*(l)]$ into $X^{\lfloor n/2 \rfloor +1}(\mathbb{R}^n)$, so there exists $M > 0$ such that for all $t \in [0, T^*(l)]$, $||u(t)||_{X^{\lfloor n/2 \rfloor +1}} \leqslant M$. In particular, $||u(t)||_{L^\infty} \leqslant M$, which implies that there exists $\tilde{M} > 0$ such that $||f(|u(s)|^2)u(s)||_{X^l} \leqslant \tilde{M}||u(s)||_{X^l}$. Finally,

$$||u(t)||_{X^l} \leqslant C||u_0||_{X^l} + \int_0^t C\tilde{M}||u(s)||_{X^l} ds$$

and Gronwall's lemma imply as usual that $||u(t)||_{X^l}$ can not blow up at $T^*(l)$, which is a contradiction with the definition of $T^*(l)$. □

5. CONSERVED QUANTITIES AND GLOBAL WELL-POSEDNESS OF NLS IN $X^k(\mathbb{R}^n)$

We first show the conservation of the renormalized energy for (1.1), for a "regular" initial data in $X^k(\mathbb{R}^n)$, in the case $n = 1$ or 2.

Proposition 5.1. *Let $n = k = 1, 2$, $u_0 \in X^{k+2}(\mathbb{R}^n)$, $f \in C^{k+1}(\mathbb{R}_+)$. We denote by $u \in C(T_*, T^*, X^{k+2}(\mathbb{R}^n)) \cap C^1(T_*, T^*, X^k(\mathbb{R}^n))$ the solution to the Cauchy problem* (1.1) *obtained in Theorem* 4.3, *where (T_*, T^*) is its maximal existence interval. Let $V(r) := -\int^r f(s)ds$. We assume that $\int_{\mathbb{R}^n} V(|u_0(x)|^2)dx$ converges (in a sense we will make precise in the proof). Then for all $t \in (T_*, T^*)$, $\int_{\mathbb{R}^n} V(|u(t,x)|^2)dx$ converges (in the same sense), and the energy is conserved:*

$$\int_{\mathbb{R}^n} \left[|\nabla u(t,x)|^2 + V(|u(t,x)|^2) \right] dx = \int_{\mathbb{R}^n} \left[|\nabla u_0(x)|^2 + V(|u_0(x)|^2) \right] dx. \tag{5.1}$$

Proof. Let us multiply (1.1) by $\bar{u_t}$ and take the real part:

$$2Re(\Delta u \bar{u_t}) = \partial_t V(|u|^2) \ .$$

We fix $t \in (T_*, T^*)$ and $x \in \mathbb{R}^n$ and integrate over $[0,t]$:

$$2Re \int_0^t \Delta u(s,x)\bar{u_t}(s,x)ds = V(|u(t,x)|^2) - V(|u_0(x)|^2). \tag{5.2}$$

We choose a nonincreasing function $\theta \in C_c^\infty(\mathbb{R})$ such that

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x \leqslant 1 \\ 0 & \text{if} \quad x \geqslant 2. \end{cases}$$

For $R > 0$, we define $\theta_R(x) := \theta(|x|/R),\ x \in \mathbb{R}^n$. We have then

$$||\nabla \theta_R||_{L^2(\mathbb{R}^n)} = R^{n/2-1} \left(\int_{\mathbb{R}^n} |\theta'(|y|)|^2 dy \right)^{1/2} .$$

The last integral is finite because $\theta'(|\cdot|) \in C_c^\infty(\mathbb{R}^n)$. In particular, $\{\nabla\theta_R\}_{R\geqslant 1}$ is bounded in $L^2(\mathbb{R}^n)$.

We now multiply (5.2) by $\theta_R(x)$ and integrate over $\mathbb{R}^n$:

$$\begin{aligned} &2Re \int_{\mathbb{R}^n} \int_0^t \Delta u(s,x)\bar{u_t}(s,x)ds\, \theta_R(x)dx \\ &= \int_{\mathbb{R}^n} V(|u(t,x)|^2)\theta_R(x)dx - \int_{\mathbb{R}^n} V(|u_0(x)|^2)\theta_R(x)dx . \end{aligned} \tag{5.3}$$

Using Fubini's theorem and an integration by parts, we calculate the left-hand side of (5.3):

$$\begin{aligned} 2Re \int_{\mathbb{R}^n} \int_0^t \Delta u(s,x)\bar{u_t}(s,x)ds\, \theta_R(x)dx \\ = -\int_{\mathbb{R}^n} \left(|\nabla u(t,x)|^2 - |\nabla u_0(x)|^2\right) \theta_R(x)dx \\ -2Re \int_0^t \int_{\mathbb{R}^n} \nabla u(s,x)\nabla\theta_R(x)\bar{u_t}(s,x)dx\, ds. \end{aligned}$$

For convenience, we assume $t > 0$. Since $\nabla\theta_R$ is supported in $\{x, |x| \geqslant R\}$, the Cauchy-Schwarz inequality implies

$$\begin{aligned} &\Big| \int_0^t \int_{\mathbb{R}^n} \nabla u(s,x)\nabla\theta_R(x)\bar{u_t}(s,x)dx\, ds\Big| \\ &\leqslant \sup_{s\in[0,t]} ||u_t(s)||_{L^\infty} \Big(\int_0^t ||\nabla u(s)||_{L^2(|x|\geqslant R)} ds\Big) ||\nabla\theta_R||_{L^2(\mathbb{R}^n)} , \end{aligned}$$

and this last quantity converges to 0 as $R \to \infty$, because $||\nabla\theta_R||_{L^2}$ is bounded. Moreover, $\int_{\mathbb{R}^n} \left(|\nabla u(t,x)|^2 - |\nabla u_0(x)|^2\right) \theta_R(x)dx$ clearly converges to $||\nabla u(t)||_{L^2}^2 - ||\nabla u_0||_{L^2}^2$ as $R \to \infty$. Hence, if we assume that

$$\int_{\mathbb{R}^n} V(|u_0(x)|^2)dx := \lim_{R\to\infty} \int_{\mathbb{R}^n} V(|u_0(x)|^2)\theta_R(x)dx$$

exists (it a priori depends on the choice of θ), taking the limit in (5.3), for all $t \in (T_*, T^*)$, we obtain that

$$\int_{\mathbb{R}^n} V(|u(t,x)|^2)dx := \lim_{R\to\infty} \int_{\mathbb{R}^n} V(|u(t,x)|^2)\theta_R(x)dx$$

exists, and the energy is conserved. □

Remark. This proof does not seem to work for $n \geqslant 3$, because we have used the existence of $\{\theta_R\}_{R\geqslant 1} \subset C_c^\infty(\mathbb{R}^n)$, with $\theta_R(x) \equiv 1$ for $|x| \leqslant R$ and $\nabla\theta_R$ bounded in $L^2(\mathbb{R}^n)$, and it can be shown that such a sequence does not exist for $n \geqslant 3$.

Next, we give a variant of Proposition 5.1 in dimension 1, which will be useful later.

Proposition 5.2. *Let $\phi, \phi_0 : \mathbb{R} \to \mathbb{R}$ be nonincreasing functions of class C^∞, such that*

$$\phi(x) = \phi_0(x) \equiv \begin{cases} 1 & if \quad x \leqslant 0 \\ 0 & if \quad x \geqslant 1. \end{cases}$$

For $R > 0$, we define $\phi_R^+(x) = \phi(x-R)\phi_0(-x)$ and $\phi_R^-(x) = \phi_R^+(-x)$. Then if $\lim\limits_{R\to\infty} \int_{\mathbb{R}} V(|u_0(x)|^2)\phi_R^\pm(x)dx$ exists, so does $\lim\limits_{R\to\infty} \int_{\mathbb{R}} V(|u(t,x)|^2)\phi_R^\pm(x)dx$.

Proof. It suffices to replace θ_R by $\phi_R^\pm$ in the proof of Proposition 5.1. We obtain in the same way that

$$\int_{\mathbb{R}} \left[V(|u(t,x)|^2) - V(|u_0(x)|^2) + \left(|\nabla u(t,x)|^2 - |\nabla u_0(x)|^2\right)\right] \phi_R^\pm(x)dx =$$

$$\mp 2Re\int_0^t\int_{\mathbb{R}}\nabla u(s,x)(\nabla\phi(\pm x-R)\phi_0(\mp x)-\phi(\pm x-R)\nabla\phi_0(\mp x))\overline{u_t}(s,x)dx\,ds.$$

Passing to the limit as $R \to \infty$, the fact that $(\nabla\phi(\pm x - R))_{R\geqslant 0}$ is bounded in L^2 and the assumption that $\int_{\mathbb{R}} V(|u_0(x)|^2)\phi_R^\pm(x)dx$ converges as $R \to \infty$ imply that $\lim\limits_{R\to\infty} \int_{\mathbb{R}} V(|u(t,x)|^2)\phi_R^\pm(x)dx$ exists, with

$$\lim_{R\to\infty} \int_{\mathbb{R}} V(|u(t,x)|^2)\phi_R^\pm(x)dx - \lim_{R\to\infty} \int_{\mathbb{R}} V(|u_0(x)|^2)\phi_R^\pm(x)dx$$
$$= -\int_{\mathbb{R}} \left(|\nabla u(t,x)|^2 - |\nabla u_0(x)|^2\right) \phi_0(\mp x)dx$$

$$\pm 2Re \int_0^t \int_{\mathbb{R}} \nabla u(s,x) \nabla \phi_0(\mp x) \overline{u_t}(s,x) dx\, ds \ .$$

Remark that the limit $\lim_{R\to\infty} \int_{\mathbb{R}} V(|u(t,x)|^2)\phi_R^{\pm}(x)dx$ depends on ϕ_0 but not on ϕ if $\lim_{R\to\infty} \int_{\mathbb{R}} V(|u_0(x)|^2)\phi_R^{\pm}(x)dx$ does not. □

We want now to improve Proposition 5.1, to make it work for $u_0 \in X^k(\mathbb{R}^n)$, with $k > n/2$, and not only $k > n/2 + 2$. The price to pay is an extra assumption on the nonlinearity f:

Theorem 5.1. *Let $n = k = 1$ or 2. We suppose that $f \in C^{k+1}(\mathbb{R}_+)$ and that there exists some $\rho_0 > 0$ such that $f(\rho_0) = 0$ and $f'(\rho_0) < 0$. We define*

$$V(r) := -\int_{\rho_0}^r f(s)ds \ .$$

If $n = 1$, we assume that $\{r,\ V(r) = 0\}$ is discrete, and if $n = 2$, we assume that V is nonnegative on $\mathbb{R}_+$.

Let $0 < C_1 < 1 < C_2$, and $\delta_0 > 0$ ($\delta_0 < \rho_0$) such that

$$|r-\rho_0| \leqslant \delta_0 \Rightarrow C_1 \frac{V''(\rho_0)}{2}(r-\rho_0)^2 \leqslant V(r) \leqslant C_2 \frac{V''(\rho_0)}{2}(r-\rho_0)^2$$

(the assumptions on f ensure that such a δ_0 does exist).

Finally, we assume that $u_0 \in X^k(\mathbb{R}^n)$, $x \to V(|u_0(x)|^2) \in L^1(\mathbb{R}^n)$, and there exists $0 < \delta_1 < \delta_0$ and $A > 0$ such that $|x| \geqslant A$ implies $||u_0(x)|^2 - \rho_0| \leqslant \delta_1$. Then, denoting by (T_*, T^*) the maximal existence interval of the solution of (1.1) associated to u_0, the energy

$$E(u(t)) \ := \ \int_{\mathbb{R}^n} \left[|\nabla u(t,x)|^2 + V(|u(t,x)|^2) \right] dx \qquad (5.4)$$

is finite and conserved for all $t \in (T_*, T^*)$.

Examples. We give some examples for which Theorem 5.1 is valid:

- the cubic defocusing NLS equation:

  $$iu_t + \Delta u + (\rho_0 - |u|^2)u = 0, \quad x \in \mathbb{R}^n,\ n = 1 \text{ or } 2. \qquad (5.5)$$

  In this case, $f(r) = \rho_0 - r$, $V(r) = (\rho_0 - r)^2/2 \geqslant 0$, and the assumptions of Theorem 5.1 are satisfied. In fact, here, the assumption of the existence of δ_1 and A can be relaxed, because it is a consequence of $V(|u_0|^2) \in L^1$.
- more generally, the "pure power case":

  $$iu_t + \Delta u + \alpha(\rho_0^p - |u|^{2p})u = 0, \quad x \in \mathbb{R}^n,\ n = 1 \text{ or } 2, \qquad (5.6)$$

where $\alpha > 0$ and $p \geqslant 1/2$ if $n = 1$, $p \geqslant 1$ if $n = 2$. Here, $f(r) = \alpha(\rho_0^p - r^p)$, $V(r) \geqslant 0$, and $V(r) = 0$ if and only if $r = \rho_0$. The assumption $f \in C^{k+1}(\mathbb{R}^+)$ in Theorem 5.1 is satisfied only if $p = 1$ or $p \geqslant 2$ in the one-dimensional case, and if $p = 1, 2$ or $p \geqslant 3$ in the case $n = 2$. However, in other cases, $X^k \ni u \to f(|u|^2)u \in X^k$ is of class C^1, and it suffices for the conclusion of Theorem 5.1.

- the cubic-quintic NLS equation:

$$iu_t + \Delta u - \alpha_1 u + \alpha_3 u|u|^2 - \alpha_5 u|u|^4 = 0, \ x \in \mathbb{R}, \tag{5.7}$$

where α_1, α_3, and α_5 are positive constants, such that $3/16 < \alpha_1\alpha_5/\alpha_3^2 < 1/4$. In this case, using some scale transformations (see [1]), (5.7) can be rewritten as

$$iu_t + \Delta u + (|u|^2 - \rho_0)(2a + \rho_0 - 3|u|^2)u = 0 \tag{5.8}$$

with $0 < a < \rho_0$. Here, $f(r) = (r - \rho_0)(2a + \rho_0 - 3r)$ and $V(r) = (r - \rho_0)^2(r - a)$, and the assumptions of Theorem 5.1 are satisfied. They are also satisfied for other values of the parameters α_1, α_3, and α_5 (for instance $a \leqslant 0$), but this seems to be less interesting from a physical point of view (see [1]).

Proof of Theorem 5.1. Let us take a mollifier sequence $(\rho_l)_{l\geqslant 1}$ (with $\int \rho_l = 1$, *Supp* $\rho_l \subset B(0, 1/l)$ and $\rho_l \geqslant 0$).

In a first step, we will control $|\rho_0 - |\rho_l * u_0(x)|^2|$ for l large and $|x| \geqslant A$.

$$\begin{aligned}&\big|\rho_0 - |\rho_l * u_0(x)|^2\big| \\ &\leqslant (\sqrt{\rho_0} + ||u_0||_\infty)\left\{\begin{array}{lll} |\rho_l * u_0(x)| - \rho_0 & \text{if } |\rho_l * u_0(x)| \geqslant \sqrt{\rho_0} & \text{(case 1)} \\ \rho_0 - |\rho_l * u_0(x)| & \text{if } |\rho_l * u_0(x)| < \sqrt{\rho_0} & \text{(case 2).}\end{array}\right.\end{aligned}$$

In case 1, we have

$$\begin{aligned}0 \leqslant |\rho_l * u_0(x)| - \sqrt{\rho_0} &\leqslant \int_{\mathbb{R}^n} \rho_l(x-y)(|u_0(y)| - \sqrt{\rho_0})dy \\ &\leqslant \int_{\mathbb{R}^n} \rho_l(x-y)||u_0(y)| - \sqrt{\rho_0}|dy \ .\end{aligned} \tag{5.9}$$

In case 2, note that for $|x| \geqslant A$,

$$|u_0(x)|^2 \geqslant \rho_0 - |\rho_0 - |u_0(x)|^2| \geqslant \rho_0 - \delta_1 > 0 \ . \tag{5.10}$$

Let $\alpha := \sqrt{\rho_0 - \delta_1}$ and $v \in \mathbb{C}$ such that $|v| = 1$ and $v\overline{u_0(x)} \in i\mathbb{R}$. For $y \in \mathbb{R}^n$, the decomposition of $u_0(y)$ on the $\mathbb{R}$ basis $(u_0(x), v)$ of $\mathbb{C}$ can be written as

$$u_0(y) = Re[u_0(y)\overline{u_0(x)}]\frac{u_0(x)}{|u_0(x)|^2} + P(u_0(y))v \ .$$

Then, because of Pythagoras' theorem,

$$\begin{aligned} 0 &\leqslant \sqrt{\rho_0} - |\rho_l * u_0(x)| \\ &\leqslant \sqrt{\rho_0} - \left| \left(\int_{\mathbb{R}^n} \rho_l(x-y) Re[u_0(y)\overline{u_0(x)}]dy \right) \frac{u_0(x)}{|u_0(x)|^2} \right| . \end{aligned}$$

We choose p as in Proposition 2.2, and $l_0 \in \mathbb{N}^*$ such that

$$|x-y| \leqslant 1/l_0 \text{ implies } |u_0(x) - u_0(y)| \leqslant C|x-y|^{1-n/p} ||\nabla u_0||_{L^p} \leqslant \alpha/2 \ .$$

So for $l \geqslant l_0$ and $y \in B(x, 1/l)$, we have

$$\begin{aligned} Re[u_0(y)\overline{u_0(x)}] &= |u_0(x)|^2 + Re[(u_0(y) - u_0(x))\overline{u_0(x)}] \\ &\geqslant |u_0(x)|^2 - |u_0(y) - u_0(x)||u_0(x)| \geqslant \alpha^2/2 > 0 \ , \end{aligned}$$

and therefore, for $l \geqslant l_0$,

$$\begin{aligned} 0 &\leqslant \sqrt{\rho_0} - |\rho_l * u_0(x)| \leqslant \sqrt{\rho_0} - \int_{\mathbb{R}^n} \rho_l(x-y) Re[u_0(y)\overline{u_0(x)}]dy \frac{1}{|u_0(x)|} \\ &= \sqrt{\rho_0} - |u_0(x)| + \frac{1}{|u_0(x)|} \int_{\mathbb{R}^n} \rho_l(x-y) Re[(u_0(x) - u_0(y))\overline{u_0(x)}]dy \\ &\leqslant |\sqrt{\rho_0} - |u_0(x)|| + \int_{\mathbb{R}^n} \rho_l(x-y)|u_0(x) - u_0(y)|dy. \end{aligned} \tag{5.11}$$

Let $\varepsilon > 0$. Since $\rho_l * u_0 \to u_0$ in L^∞, there exists $l_1 \geqslant l_0$ such that $l \geqslant l_1$ implies $|||\rho_l * u_0|^2 - |u_0|^2||_{L^\infty} \leqslant \delta_0 - \delta_1$, and then for $|x| \geqslant A$, $||\rho_l * u_0(x)|^2 - \rho_0| \leqslant \delta_0$. Therefore, for $|x| \geqslant A$ and $l \geqslant l_1$, we have

$$0 \ \leqslant \ V(|\rho_l * u_0(x)|^2) \ \leqslant \ C_2 \frac{V''(\rho_0)}{2} ||\rho_l * u_0(x)|^2 - \rho_0|^2 \ . \tag{5.12}$$

Let $B \geqslant A+1$. (5.9), (5.11), and (5.12) imply, for $l \geqslant l_1$,

$$\begin{aligned} &\int_{|x|\geqslant B} V(|\rho_l * u_0(x)|^2)dx \leqslant C_2 \frac{V''(\rho_0)}{2} (\sqrt{\rho_0} + ||u_0||_\infty)^2 \\ &\times \int_{|x|\geqslant B} \Big[\Big(\int_{\mathbb{R}^n} \rho_l(x-y)||u_0(y)| - \sqrt{\rho_0}|dy \Big)^2 \mathbf{1}_{|\rho_l * u_0(x)| - \sqrt{\rho_0} \geqslant 0} \\ &+ \Big(|\sqrt{\rho_0} - |u_0(x)|| + \int_{\mathbb{R}^n} \rho_l(x-y)|u_0(y) - u_0(x)|dy \Big)^2 \mathbf{1}_{|\rho_l * u_0(x)| - \sqrt{\rho_0} \leqslant 0} \Big] dx \\ &\leqslant C_2 \frac{V''(\rho_0)}{2} (\sqrt{\rho_0} + ||u_0||_\infty)^2 \int_{|x|\geqslant B} \Big[\int_{\mathbb{R}^n} \rho_l(x-y)||u_0(y)| - \sqrt{\rho_0}|^2 dy \\ &+ 2||u_0(x)| - \sqrt{\rho_0}|^2 + 2\Big(\int_{\mathbb{R}^n} \rho_l(x-y)|u_0(y) - u_0(x)|dy \Big)^2 \Big] dx. \end{aligned} \tag{5.13}$$

We will now control each integral on the right-hand side of (5.13). We begin with the first one:

$$\begin{aligned}
&\int_{|x|\geqslant B}\int_{\mathbb{R}^n}\rho_l(x-y)||u_0(y)|-\sqrt{\rho_0}|^2dy\,dx \qquad (5.14)\\
&\leqslant \int_{|x|\geqslant B}\int_{|y|\geqslant B-1/l}\rho_l(x-y)\frac{||u_0(y)|^2-\rho_0|^2}{\rho_0}dy\,dx\\
&\leqslant \frac{1}{\rho_0}\int_{|y|\geqslant B-1/l}||u_0(y)|^2-\rho_0|^2dy \leqslant \frac{2}{C_1\rho_0V''(\rho_0)}\int_{|y|\geqslant B-1/l}V(|u_0|^2)dy.
\end{aligned}$$

Next,

$$\begin{aligned}
\int_{|x|\geqslant B}2|\sqrt{\rho_0}-|u_0(x)||^2dx &\leqslant \frac{2}{\rho_0}\int_{|x|\geqslant B}|\rho_0-|u_0(x)|^2|^2dx\\
&\leqslant \frac{4}{C_1\rho_0V''(\rho_0)}\int_{|x|\geqslant B}V(|u_0|^2)dx. \qquad (5.15)
\end{aligned}$$

It is a bit more difficult to find an upper bound to the third one. We provisionally admit the following lemma:

Lemma 5.1. *Let $n\in\mathbb{N}^*$, $k>n/2$, and $u_0\in X^k(\mathbb{R}^n)$. Then for all (x,y), the function $f:[0,1]\to\mathbb{C}$, $t\to u_0(x+t(y-x))$ is absolutely continuous.*

Thanks to this lemma, we can write

$$|u_0(y)-u_0(x)| = \Big|\int_0^1(y-x).\nabla u_0(x+t(y-x))dt\Big|\,,$$

and

$$\begin{aligned}
&\int_{|x|\geqslant B}\left(\int_{\mathbb{R}^n}\rho_l(x-y)|u_0(y)-u_0(x)|dy\right)^2dx\\
&\leqslant \int_{|x|\geqslant B}\int_{\mathbb{R}^n}\rho_l(x-y)|y-x|^2\int_0^1|\nabla u_0(x+t(y-x))|^2dt\,dy\,dx\\
&\leqslant \frac{1}{l^2}\int_{|x|\geqslant B}\int_0^1\int_{\mathbb{R}^n}\rho_l(\frac{x-\tilde{y}}{t})|\nabla u_0(\tilde{y})|^2\frac{d\tilde{y}}{t^n}dx\,dt\\
&\leqslant \frac{1}{l^2}\int_0^1\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\rho_l(\tilde{x})|\nabla u_0(\tilde{y})|^2d\tilde{x}\,d\tilde{y}\,dt = \frac{||\nabla u_0||_{L^2}}{l^2} \qquad (5.16)
\end{aligned}$$

(here we have made successively the changes of variables $\tilde{y} = ty + (1-t)x$ and $\tilde{x} = \frac{x-\tilde{y}}{t}$). By possibly enlarging B one may assume that

$$(\sqrt{\rho_0} + ||u_0||_\infty)^2 \frac{C_2}{C_1 \rho_0} \Big[\int_{|y| \geqslant B-1} V(|u_0|^2) dy + 2 \int_{|x| \geqslant B} V(|u_0|^2) dx \Big] \leqslant \varepsilon/2. \tag{5.17}$$

We also choose $l_2 \geqslant l_1$ such that

$$2C_2 \frac{V''(\rho_0)}{2} (\sqrt{\rho_0} + ||u_0||_\infty)^2 \frac{||\nabla u_0||_{L^2}}{l_2^2} \quad \leqslant \quad \varepsilon/2 \ . \tag{5.18}$$

Combining (5.13), (5.14), (5.15), (5.16), (5.17), and (5.18), we obtain finally, for $l \geqslant l_2$,

$$\int_{|x| \geqslant B} V(|(\rho_l * u_0)(x)|^2) dx \leqslant \varepsilon \ .$$

In particular, $V(|(\rho_l * u_0)|^2) \in L^1(\mathbb{R}^n)$. By possibly enlarging B one may assume that

$$\int_{|x| \geqslant B} V(|u_0(x)|^2) dx \leqslant \varepsilon \ .$$

Moreover, there exists $l_3 \geqslant l_2$ such that for $l \geqslant l_3$,

$$\int_{|x| \leqslant B} |V(|(\rho_l * u_0)(x)|^2) - V(|u_0(x)|^2)| dx \leqslant \varepsilon$$

because $\rho_l * u_0 \to u_0$ in L^∞. The last three inequalities and the triangle inequality show that for $l \geqslant l_3$,

$$\int_{\mathbb{R}^n} |V(|(\rho_l * u_0)(x)|^2) - V(|u_0(x)|^2)| dx \leqslant 3\varepsilon,$$

and therefore $V(|\rho_l * u_0|^2) \to V(|u_0|^2)$ in L^1.

For $l \in \mathbb{N}^*$, we denote by $u_l(t)$ the solution of (1.1) with initial data $u_l(0) = \rho_l * u_0$, and we denote by $(T_*(l), T^*(l))$ its maximal existence interval. Proposition 5.1 ensures that for $t \in (T_*(l), T^*(l))$

$$\int_{\mathbb{R}^n} \left[|\nabla u_l(t)|^2 + V(|u_l(t)|^2) \right] dx = \int_{\mathbb{R}^n} \left[|\nabla \rho_l * u_0|^2 + V(|\rho_l * u_0|^2) \right] dx. \tag{5.19}$$

Let $T_* < \tilde{T}_1 < \tilde{T}_2 < T^*$. By continuity with respect to the initial data (see for example [8]), there exist $K > 0$ and $\delta > 0$ such that $||\rho_l * u_0 - u_0||_{X^k} \leqslant \delta$ implies $T^*(l) > \tilde{T}_2,\ T_*(l) < \tilde{T}_1$, and

$$||u_l(t) - u(t)||_{X^k} \leqslant K ||\rho_l * u_0 - u_0||_{X^k}, \qquad t \in [\tilde{T}_1, \tilde{T}_2] \ .$$

In particular, since $\rho_l * u_0 \to u_0$ in X^k, we have that $\nabla u_l(t) \to \nabla u(t)$ in L^2. Moreover, $\nabla \rho_l * u_0 = \rho_l * \nabla u_0 \to \nabla u_0$ in L^2 and we have already shown that $V(|\rho_l * u_0|^2) \to V(|u_0|^2)$ in L^1. In order to take the limit as $l \to \infty$ in (5.19), it remains to take care of the term $\int_{\mathbb{R}^n} V(|u_l(t)|^2)dx$. We distinguish the cases $n = 2$ and $n = 1$.

In the case $n = 2$ one has $V \geqslant 0$, and for $l \geqslant l_3$, $x \to V(|u_l(t)|^2)$ is an L^1 function and $\int_{\mathbb{R}^n} V(|u_l(t)|^2)dx$ has a limit as $l \to \infty$. We apply Fatou's lemma to the sequence $(V(|u_l(t,\cdot)|^2))_{l\in\mathbb{N}}$. We already know that it is bounded in L^1, and the continuity with respect to the initial data ensures that $u_l(t) \to u(t)$ in X^k. Therefore $V(|u_l(t)|^2) \to V(|u(t)|^2)$ in L^∞. Thus $V(|u(t,\cdot)|^2) \in L^1$ and

$$\int V(|u(t,x)|^2)dx \leqslant \liminf_{l\to\infty} \int V(|u_l(t,x)|^2) \ .$$

We can now pass to the limit in (5.19):

$$\int_{\mathbb{R}^n} \left[|\nabla u(t)|^2 + V(|u(t)|^2)\right] dx \leqslant \int_{\mathbb{R}^n} \left[|\nabla u_0|^2 + V(|u_0|^2)\right] dx. \tag{5.20}$$

By reversing time, we get the inverse inequality, and the conservation of the energy (5.1) has been proved.

In the case $n = 1$, we also show that $V(|u_l(t,\cdot)|^2) \in L^1$. For $l \geqslant l_3$, we know that $V(|\rho_l * u_0|^2) \in L^1(\mathbb{R})$. Hence for all choices of ϕ as in Proposition 5.2,

$$\lim_{R\to\infty} \int_{\mathbb{R}} V(|\rho_l * u_0(x)|^2)\phi_R^{\pm}(x)dx = \int_{\mathbb{R}} V(|\rho_l * u_0(x)|^2)\phi_0(\mp x)dx \ ,$$

and Proposition 5.2 ensures that $\lim\limits_{R\to\infty} \int_{\mathbb{R}} V(|u_l(t,x)|^2)\phi_R^{\pm}(x)dx$ exists and does not depend on the choice of ϕ. This implies that $V(|u_l(t,x)|^2) \to 0$ as $x \to \pm\infty$. Indeed, we have the following lemma (we will prove it later):

Lemma 5.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function such that for all ϕ as in Proposition 5.2, $\lim\limits_{R\to\infty} \int_{\mathbb{R}} f(x)\phi_R^{\pm}(x)dx$ exists. Then $f(x) \underset{x\to\pm\infty}{\to} 0$.*

Since $\{r, V(r) = 0\}$ is discrete and $u_l(t) \in C_b(\mathbb{R})$, there exists $r_\pm^l(t)$ such that

$$V(r_\pm^l(t)) = 0 \quad \text{and} \quad |u_l(t,x)|^2 \underset{x\to\pm\infty}{\to} r_\pm^l(t) \ .$$

Let us show that $r_\pm^l$ is continuous on $(T_*(l), T^*(l))$. Let $t \in (T_*(l), T^*(l))$ and h be such that $t + h \in (T_*(l), T^*(l))$, $\varepsilon > 0$, and $x \in \mathbb{R}$. One has

$$\begin{aligned}&|r_\pm^l(t+h) - r_\pm^l(t)|\\ &\leqslant |r_\pm^l(t+h) - u_l(t+h,x)| + |u_l(t+h,x) - u_l(t,x)| + |u_l(t,x) - r_\pm^l(t)|.\end{aligned}$$

$u_l \in C((T_*(l),T^*(l)),X^k) \subset C((T_*(l),T^*(l)),L^\infty)$; thus, we can choose h small enough in order that

$$||u_l(t+h)-u_l(t)||_{L^\infty} \leqslant \varepsilon/3\ .$$

We also choose $|x|$ large enough such that

$$|r^l_\pm(t+h)-u_l(t+h,x)| \leqslant \varepsilon/3 \quad \text{and} \quad |u_l(t,x)-r^l_\pm(t)| \leqslant \varepsilon/3\ .$$

Hence $|r^l_\pm(t+h)-r^l_\pm(t)| \leqslant \varepsilon$. Therefore $r^l_\pm$ is continuous with value in a discrete set, which means that it is constant. For $l \geqslant l_3$, the fact that $|||\rho_l * u_0|^2-|u_0|^2||_{L^\infty} \leqslant \delta_0-\delta_1$ and the assumption on u_0 imply that $r^l_\pm(0)=\rho_0$; hence,

$$\forall l \geqslant l_3,\ \forall t \in (T_*(l),T^*(l)),\ |u_l(t,x)|^2 \underset{x\to\pm\infty}{\to} \rho_0\ .$$

We choose $l_4 \geqslant l_3$ such that for $l \geqslant l_4$,

$$\begin{aligned}
&|||u_l(t)|^2-|u(t)|^2||_{L^\infty} \\
&\leqslant ||u_l(t)-u(t)||_{L^\infty}(||u_l(t)-u(t)||_{L^\infty}+2||u(t)||_{L^\infty}) \\
&\leqslant K||\rho_l * u_0-u_0||_{X^k}(K||\rho_l * u_0-u_0||_{X^k}+2\sup_{t\in[\tilde T_1,\tilde T_2]}||u(t)||_{X^k}) \leqslant \frac{\delta_0-\delta_1}{2}.
\end{aligned}$$

Let $D>0$ be such that $|x| \geqslant D$ implies $||u_{l_4}(t,x)|^2-\rho_0| \leqslant \delta_1$. Then for $l \geqslant l_4,\ |x| \geqslant D$,

$$\begin{aligned}
&||u_l(t,x)|^2-\rho_0| \\
&\leqslant ||u_l(t,x)|^2-|u(t,x)|^2|+||u(t,x)|^2-|u_{l_4}(t,x)|^2|+||u_{l_4}(t,x)|^2-\rho_0| \\
&\leqslant 2\frac{\delta_0-\delta_1}{2}+\delta_1=\delta_0.
\end{aligned}$$

This implies that $V(|u_l(t,x)|)$ is nonnegative on $\{x,|x| \geqslant D\}$, and now (5.19) yields

$$\begin{aligned}
\int_{|x|\geqslant D} V(|u_l(t,x)|^2)dx &= \int_{\mathbb{R}} \left[|\nabla \rho_l * u_0|^2+V(|\rho_l * u_0|^2)\right] dx \\
&\quad -\int_{\mathbb{R}} |\nabla u_l(t,x)|^2 dx-\int_{|x|\leqslant D} V(|u_l(t,x)|^2)dx \\
\underset{l\to\infty}{\to} \int_{\mathbb{R}} &\left[|\nabla u_0|^2+V(|u_0|^2)\right] dx-\int_{\mathbb{R}} |\nabla u(t,x)|^2 dx-\int_{|x|\leqslant D} V(|u(t,x)|^2)dx\ ,
\end{aligned}$$

and $\left(V(|u_l(t,\cdot)|^2)\mathbf{1}_{\{|x|\geqslant D\}}\right)_{l\geqslant l_4}$ is bounded in L^1. We apply Fatou's lemma to this sequence, and we can conclude similarly to the case $n=2,\ V \geqslant 0$. $\square$

To complete the proof of Theorem 5.1, it just remains to prove Lemmas 5.1 and 5.2.

Proof of Lemma 5.2. We argue by contradiction. Assume that there exists $\varepsilon > 0$ such that for all $A > 0$, there exists $x > A$ such that $|f(x)| > \varepsilon$. Since f is uniformly continuous, there exists $\delta \in (0, 2)$, such that $|x - y| < \delta$ implies $|f(x) - f(y)| \leqslant \varepsilon/2$. We may thus construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to \infty$ and $|f(y)| \geqslant \varepsilon/2$ as soon as $|y - x_n| \leqslant \delta$. We may assume moreover that the intervals $(x_n - \delta, x_n + \delta)$ are disjoint, and that for instance $f(x_n) > 0$. We choose ϕ as in Proposition 5.2, namely,

$$\phi(x) = \begin{cases} 1 & \text{if} \quad x \leqslant 0 \\ 0 & \text{if} \quad x \geqslant \delta/2 \end{cases} .$$

Then

$$\begin{aligned} &\Big| \int_{\mathbb{R}} f(y)\phi_{x_n-\delta}(y)dy - \int_{\mathbb{R}} f(y)\phi_{x_n+\delta/2}(y)dy \Big| \\ &= \int_{\mathbb{R}} f(y)(\phi_{x_n+\delta/2}(y) - \phi_{x_n-\delta}(y))dy \geqslant \int_{x_n-\delta/2}^{x_n+\delta/2} f(y)dy \geqslant \delta\varepsilon/2, \end{aligned}$$

and this is a contradiction. □

Proof of Lemma 5.1. Let us fix $x, y \in \mathbb{R}^n$ with $x \neq y$. It suffices to show that $f : t \in [0,1] \to u_0(x + t(y - x))$ is absolutely continuous. By definition of X^k, u_0 is the limit in X^k of a sequence $(v_l)_{l\in\mathbb{N}}$ of functions of class C^k. Therefore the functions $f_l : t \in [0,1] \to v_l(x + t(y - x))$ are absolutely continuous.

We are reduced to proving that the moduli of absolute continuity of f_l's are uniformly bounded. Indeed, if we assume this fact, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all positive integers m, and for all choices of $(\alpha_j, \beta_j)_{1\leqslant j\leqslant m}$ with $0 \leqslant \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m \leqslant 1$ and $\sum_{j=1}^m (\beta_j - \alpha_j) \leqslant \delta$, we have for all l,

$$\sum_{j=1}^{m} |f_l(\beta_j) - f_l(\alpha_j)| \leqslant \varepsilon/2.$$

Then

$$\begin{aligned} &\sum_{j=1}^{m} |f(\beta_j) - f(\alpha_j)| \\ &\leqslant \sum_{j=1}^{m} |f_l(\beta_j) - f_l(\alpha_j)| + \sum_{j=1}^{m} \left[|f(\beta_j) - f_l(\beta_j)| + |f(\alpha_j) - f_l(\alpha_j)|\right]; \end{aligned}$$

choosing l large enough, $||f - f_l||_\infty \leqslant \frac{\varepsilon}{4m}$. We infer that

$$\sum_{j=1}^{m} |f(\beta_j) - f(\alpha_j)| \leqslant \varepsilon \ ,$$

and therefore f is absolutely continuous.

We now prove that the moduli of absolute continuity of the f_l's are uniformly bounded. Since $v_l \to u_0$ in X^k, v_l is bounded in X^k. We choose m, α_j, and β_j as above. For all l, we have

$$\begin{aligned}\sum_{j=1}^{m} |f_l(\beta_j) - f_l(\alpha_j)| &= \sum_{j=1}^{m} \Big| \int_{\alpha_j}^{\beta_j} f_l'(s)ds \Big| \leqslant \int_{\bigcup_{j=1}^{m}(\alpha_j,\beta_j)} |f_l'(s)|ds \\ &\leqslant \Big(\int_{\bigcup_{j=1}^{m}(\alpha_j,\beta_j)} ds \Big)^{1/2} \Big(\int_{\bigcup_{j=1}^{m}(\alpha_j,\beta_j)} |f_l'(s)|^2 ds \Big)^{1/2} \\ &\leqslant \Big(\sum_{j=1}^{m} (\beta_j - \alpha_j) \Big)^{1/2} \Big(\int_0^1 |f_l'(s)|^2 ds \Big)^{1/2} . \end{aligned} \tag{5.21}$$

Observe that

$$\begin{aligned}\int_0^1 |f_l'(s)|^2 ds &= \int_0^1 |(y-x)\cdot \nabla v_l(x + s(y-x))|^2 ds \\ &\leqslant |y-x| \int_0^{|y-x|} \Big| \nabla v_l \Big(x + s\frac{y-x}{|y-x|} \Big) \Big|^2 ds. \end{aligned}$$

Since $\nabla u_0 \in H^{k-1}(\mathbb{R}^n)$, the trace theorem ensures that the mapping that sends a function of $H^{k-1}(\mathbb{R}^n)$ on its trace on a line D of $\mathbb{R}^n$ is continuous from $H^{k-1}(\mathbb{R}^n)$ to $H^{(k-1)-(n-1)/2}(D) \subset L^2(D)$ because $k - 1 - (n-1)/2 \geqslant \lfloor n/2 \rfloor - (n-1)/2 \geqslant 0$. Therefore, there exists a constant $C > 0$ such that

$$\begin{aligned}&\Big(\int_0^1 |f_l'(s)|^2 ds \Big)^{1/2} \leqslant |y-x|^{1/2} ||\nabla v_l \,{}_{\big|\{x+t\frac{y-x}{|y-x|}, t\in\mathbb{R}\}}||_{L^2(\{x+t\frac{y-x}{|y-x|}, t\in\mathbb{R}\})} \\ &\leqslant C|y-x|^{1/2} ||\nabla v_l||_{H^{k-1}(\mathbb{R}^n)} \leqslant C|y-x|^{1/2} \sup_l ||v_l||_{X^k(\mathbb{R}^n)}. \end{aligned} \tag{5.22}$$

Finally, (5.21) and (5.22) ensure that the f_ls' absolute continuity moduli are uniformly bounded. □

Remark. The only reason for which Theorem 5.1 in the case $V \geqslant 0$ is not valid for $n \geqslant 3$ is that we did not prove Proposition 5.1 for $n \geqslant 3$. The whole proof of Theorem 5.1 with $V \geqslant 0$ would be valid for $n \geqslant 3$ if Proposition 5.1 were.

We justify next the conservation of the momentum, in the one-dimensional case.

Theorem 5.2. *Let $n = k = 1$. The assumptions on the nonlinearity f are as in Theorem 5.1. Let $u_0 \in X^1(\mathbb{R})$ be such that $V(|u_0|^2) \in L^1$ and $|u_0| \geqslant a > 0$. Let $0 \in (\tilde{T}_*, \tilde{T}^*) \subset (T_*, T^*)$ be the maximal interval on which $|u(t,x)| > 0$, $t \in (\tilde{T}_*, \tilde{T}^*)$, $x \in \mathbb{R}$. Then $\forall t \in (\tilde{T}_*, \tilde{T}^*)$, $P(u(t)) = P(u_0)$, where the renormalized momentum is given by*

$$P(u) \quad = \quad Im \int_{-\infty}^{\infty} \frac{\overline{u}_x u}{|u|^2}(|u|^2 - \rho_0). \tag{5.23}$$

Proof. It follows from the proof of Theorem 5.1 that for $t \in (\tilde{T}_*, \tilde{T}^*)$, $|u(t)|^2 - \rho_0 \in L^2$ and that $|u(t)| \geqslant a(t)$, where $a(t) \in \mathbb{R}_+^*$ (because $|u(t,x)|^2 \underset{x\to\infty}{\longrightarrow} \rho_0$ and $|u(t,x)| > 0$). Moreover, $u_x(t,\cdot) \in L^2$. Hence $P(u(t))$ is well defined.

If $u_0 \in X^3(\mathbb{R})$, the formal proof of the conservation of P on $(\tilde{T}_*, \tilde{T}^*)$ (see [18] and [10]) is valid, since $u \in C(\tilde{T}_*, \tilde{T}^*, X^3(\mathbb{R})) \cap C^1(\tilde{T}_*, \tilde{T}^*, X^1(\mathbb{R}))$.

Proceeding as in the proof of Theorem 5.1, we approach $u_0 \in X^1$ by $\rho_l * u_0 \in X^3$. Let us fix $t \in (\tilde{T}_*, \tilde{T}^*)$. Since $u_l(t) \to u(t)$ in X^1, for l large enough we have $|u_l(t)| \geqslant a(t)/2 > 0$, and then $(\tilde{T}_*, \tilde{T}^*) \subset (\tilde{T}_*(l), \tilde{T}^*(l))$. It follows from the proof of Theorem 5.1 that $(|u_l(t)|^2 - \rho_0) \to (|u(t)|^2 - \rho_0)$ in L^2 and $\partial_x u_l(t) \to \partial_x u(t)$ in L^2. Hence $P(u_l(t)) \to P(u(t))$, and the conservation of the momentum, which is true for u_l, is also true for u. □

Remark. The above analysis fills a gap in the proof of Theorem 1.1 in [18]. Namely, Zhiwu Lin proves a criterion of stability for the traveling-bubbles solution of NLS in the one-dimensional case, for a nonlinearity f satisfying

(1) $f(\rho_0) = 0,\ \eta_0 = \sup\{\eta,\ 0 < \eta < \rho_0,\ V(\eta) = 0\}$ exists, $0 < \eta_0 < \rho_0,\ f(\eta_0) < 0$;
(2) $f'(\rho_0) < 0$.

This proof consists in applying Theorem 3 in [10] to the hydrodynamical problem corresponding to (1.1) (i.e. with the complex unknown u replaced by $(r,v) = (\rho_0 - |u|^2, \partial_x \arg u)$). Theorems 5.1 and 5.2 ensure that Assumption 1 of Theorem 3 in [10] is satisfied. Namely, in a neighborhood of the soliton, our results imply the following facts which were not discussed in [18]:

- the local existence for the hydrodynamical Cauchy problem with $(r,v) \in H^1 \times L^2$ (and not only $X^1 \times L^2$) if the energy is finite at initial time;
- the conservation of energy and momentum.

In dimension 1, we prove next that the conservation of energy implies a global existence result for a solution of (1.1) in X^1.

Theorem 5.3. *Let $n = k = 1$. The assumptions on f and u_0 are as in Theorem 5.1, and we assume moreover that there exists some $C > 0$ such that $V(r) \geqslant C(\rho_0 - r)^2$. Then $u \in C_b(\mathbb{R}, X^k)$.*

Proof. We define the energy at initial time

$$E_0 = \int_{\mathbb{R}^n} \left[|\nabla u_0(x)|^2 + V(|u_0(x)|^2) \right] dx \ .$$

We know by Theorem 5.1 that the energy is conserved for $t \in (T_*, T^*)$ and that if T^* is finite, $||u(t)||_{X^k} \to \infty$ as $t \to T^*$ (and we have a similar result by replacing T^* by T_*) (see [8]). The conservation of the energy and the fact that $V \geqslant 0$ imply that for all t, $\int_{\mathbb{R}} |\partial_x u(t)|^2 dx \leqslant E_0$. To prove that $||u(t)||_{X^k}$ can not blow up in finite time, it suffices then to show that $||u(t)||_{L^\infty}$ can not blow up in finite time. By the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$,

$$\begin{aligned} ||u(t)||_{L^\infty}^2 &\leqslant \rho_0 + |||u(t)|^2 - \rho_0||_{L^\infty} \leqslant \rho_0 + C|||u(t)|^2 - \rho_0||_{H^1} \\ &= \rho_0 + C\sqrt{|||u(t)|^2 - \rho_0||_{L^2}^2 + ||\partial_x(|u(t)|^2 - \rho_0)||_{L^2}^2}. \end{aligned}$$

We now use the additional assumption that $V(r) \geqslant C(1-r)^2$:

$$\begin{aligned} ||u(t)||_{L^\infty}^2 &\leqslant \rho_0 + C\sqrt{E_0 + 4||u(t)||_{L^\infty}^2 \int |\partial_x u(t)|^2 dx} \\ &\leqslant \rho_0 + C\sqrt{E_0} + 2C||u(t)||_{L^\infty}\sqrt{E_0} \ . \end{aligned}$$

Therefore, $||u(t)||_{L^\infty}$ can not blow up in finite time; thus, $(u(t))_{t\in\mathbb{R}}$ is global. Moreover, $||u(t)||_{L^\infty}$ is bounded on $\mathbb{R}$, and therefore $(u(t))_{t\in\mathbb{R}}$ is bounded in $X^1(\mathbb{R})$. □

Example. In NLS with a pure defocusing power (5.6), with $n = 1$ and $p \geqslant 1$, we have $V(r) \geqslant \alpha\rho_0^{p-1}(\rho_0 - r)^2/2$; hence, the assumptions of Theorem 5.3 are satisfied.

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