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SCATTERING STATES FOR THE NONLINEAR WAVE EQUATION WITH SMALL DATA

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Dedicated to the memory of Professor Tsutomu Arai

Abstract. We investigate the energy nondecay and existence of scattering states for solutions to the initial-boundary-value problem for the nonlinear wave equation in exterior domains. When the space dimension is odd, the domain meets no geometrical condition. Otherwise, we assume that the obstacle is convex. For odd-dimensional general domains, taking into account the effective dissipation in trapping regions, we can derive the existence of scattering states. In particular, we can obtain also an L^2 bound of solutions. The method in deriving the energy nondecay is to utilize Huyghens' principle. For even-dimensional domains outside the convex obstacle, the asymptotics stated in the odd-dimensional case are also valid.

1. Introduction

Let us start with a study of the initial-boundary-value problem

$$\text{(P)} \begin{cases} u_{tt} - \Delta u + a(x)u_t = f(u), & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases}$$

where Ω is an exterior domain outside a compact obstacle $\mathcal{O}$ in $\mathbb{R}^n$, $n \geq 3$, with a smooth boundary $\partial\Omega$. We assume that $0 \notin \overline{\Omega}$ and $\mathcal{O} \subset B_{\rho_0}$ for some $\rho_0 > 0$, where we set $B_{\rho_0} = \{x \in \mathbb{R}^n; |x| \leq \rho_0\}$, and $\overline{\Omega}$ denotes the closure of Ω. The function $a(x)$ is localized near $\mathcal{O}$, and the nonlinear term $f(u)$ satisfies $f(u) = \mathrm{O}(|u|^{\alpha+1})$ near $u = 0$ for some large $\alpha > 0$ specified later. *Our concern in this paper is whether the solution u is asymptotically free or not as the time goes to infinity.*

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The global-in-time existence theorem of the Cauchy problem was proved by John [6] in $\mathbb{R}^{3+1}$ and Glassy [2] in $\mathbb{R}^{2+1}$, etc. But, as concerns the exterior problem, there are not so many works as on the Cauchy problem. For example, when Ω is a nontrapping domain in the sense of Vainberg [22], Shibata and Tsutsumi [19, 20] proved the global existence theorem of nonlinear wave equations with sufficiently smooth data. Recently, Nakao [13] proved the global-in-time existence theorem in $\mathbb{R}^{n+1}$, $n \geq 2$, under the effective dissipation near the boundary. But then, as for the exterior problem, it seems that the asymptotics were not sufficiently studied. Furthermore, L^2 estimates of u itself seem to be unknown for the problem (P). In this paper we want to treat the asymptotic behaviour for the problem (P) in exterior domains.

In general, we need the geometrical condition on the shape of $\mathcal{O}$ to discuss asymptotics. For example, if $\mathcal{O}$ is star-shaped, Mochizuki and Nakazawa [11] treated the homogeneous problem ($f(u) \equiv 0$) to prove the energy nondecay and existence of scattering states. With regard to the general exterior domain, it is technically difficult to study their behaviour. *In this paper we are concerned with L^2 estimates and asymptotics for the problem (P) without any assumption on the shape of $\mathcal{O}$.* Our first aim is to obtain L^2 estimates. Combining L^p estimates, $2 < p \leq \frac{2(n+1)}{n-1}$, obtained by Nakao [13] with the method devised in Ikehata [4] (cf. Ikehata and Matsuyama [5]), we can obtain L^2 estimates (Theorem 1). We must restrict ourselves to the space dimensions to $n \geq 3$ because L^p decay rates are too weak in two space dimensions.

Motivated by the existence theorem of L^p decaying solutions, we have as a natural question how solutions of the problem (P) behave as the time goes to infinity, which is the second and main problem in this paper. For the Cauchy problem in $\mathbb{R}^n$ ($n \geq 2$), if $a(x) \equiv 0$, there are many works concerned with the scattering theory. For example, Ginibre and Velo [1], Klainerman [7], Pecher [15], and Strauss [21] constructed the scattering theory for small data, and Mochizuki and Motai [9, 10] weakened the lower bound of the nonlinearity condition in [7, 15, 21]. For another nonlinearity we should refer to Hidano [3] and Shatah [18]. In [18] he developed the theory in the case $f(u, \partial_t u, \nabla u, \partial_{x_i}\partial_{x_j} u)$ with $f(u, 0, 0, 0) = 0$, and Hidano treated a cubic convolution. As for the exterior problem, since the appropriate Strichartz estimates are not known, the method of [9, 10, 15, 18, 21] cannot be available to the present problem, and there seem to be very few results on the asymptotics. By virtue of L^p estimates for smooth solutions, the dissipative term au_t and nonlinear term $f(u)$ decay to 0 as $t \to \infty$, and we can expect the existence of scattering states. *We will prove in Theorems* 2 *and* 4 *that the energy does not in general decay and scattering states exist, respectively, without any assumption on the shape of* $\mathcal{O}$ *in an odd-dimensional space.* Of course, the same conclusion remains valid if the obstacle $\mathcal{O}$ is an even-dimensional and convex domain. For our purpose, we need a geometrical observation on the shape of $\mathcal{O}$, and we will state it in Theorem 3, the proof of which is given by the second author.

For odd-dimensional domains, the crucial idea will lie in utilizing Huyghens' principle in order to obtain the energy nondecay. According to Huyghens' principle, the boundary condition of the free-wave equation becomes compatible with the initial condition, which enables us to analyze the dynamics of a perturbed state starting from an unperturbed state. Moreover, as will be seen in Section 4, the hypersurface $\widetilde{\Gamma}$ introduced in [8], a part of the boundary of a compact and star-shaped set containing $\mathcal{O}$, will play an important role in deriving the space-time integrability of solutions. The discussion of the existence of scattering states is deeply based on this estimate. Needless to say, we need not take into account the hypersurface $\widetilde{\Gamma}$ for even-dimensional domains outside the convex obstacle $\mathcal{O}$.

For convenience of the readers, we list several function spaces often used in this paper. Let $W^{k,p}(\Omega)$ ($k = 0, 1, 2, \ldots$; $p \geq 1$) be the usual Sobolev space with the norm

$$\|f\|_{k,p} = \|f\|_{W^{k,p}(\Omega)} = \Big\{ \sum_{|\alpha| \leq k} \int_\Omega |\nabla^\mu f(x)|^p \, dx \Big\}^{1/p},$$

where $\nabla^\mu = (\partial/\partial x_1)^{\mu_1} \cdots (\partial/\partial x_n)^{\mu_n}$ and μ denotes the multi-index $\mu = (\mu_1, \ldots, \mu_n)$. We write $H^k(\Omega) = W^{k,2}(\Omega)$, $L^p(\Omega) = W^{0,p}(\Omega)$, $H^0(\Omega) = L^2(\Omega)$, and $\|f\|_{L^p(\Omega)} = \|f\|_p$. In particular, we write $\|f\|_{W^{k,p}(\mathbb{R}^n)} = \|f\|'_{k,p}$ and $\|f\|_{L^p(\mathbb{R}^n)} = \|f\|'_p$. Function spaces $H^1_0(\Omega)$ and $H_\nabla(\Omega)$ are the completion of $C_0^\infty(\Omega)$ (the space of all test functions on Ω) in the norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\nabla \cdot\|_2$, respectively. Let E be the energy space of all pairs $\boldsymbol{f} \equiv \{f_1, f_2\}$ of functions belonging to $H_\nabla(\Omega) \times L^2(\Omega)$ such that

$$\|\boldsymbol{f}\|_E^2 = \|\{f_1, f_2\}\|_E^2 = \frac{1}{2} \int_\Omega (|\nabla f_1|^2 + |f_2|^2) \, dx < \infty.$$

For the solution u of (P) we simply write $\|u(t)\|_E = \|\{u(t), u_t(t)\}\|_E$. Let $M = [n/2] + 1$. Then we set

$$X_{2M+m}(\Omega)=\bigcap_{j=0}^{2M+m-1} C^j([0,\infty);H^{2M+m-j}(\Omega)\cap H_0^1(\Omega))\cap C^{2M+m}([0,\infty);L^2(\Omega))$$

for a nonnegative integer m and $V=\{u\in X_{2M}(\Omega)\,;\ \|u\|_V<\infty\}$, where

$$\|u\|_V\equiv \sup_{t\geq 0}\big((1+t)^b\|u(t)\|_p+(1+t)^d\|u(t)\|_\infty\big),$$

$$b=\begin{cases}(n-1)\Big(\dfrac{1}{2}-\dfrac{1}{p}\Big) & \text{if} \quad n\geq 4,\\ 2\Big(\dfrac{1}{2}-\dfrac{1}{p}\Big)-\delta & \text{if} \quad n=3,\end{cases}\qquad d=\begin{cases}\dfrac{n-1}{2} & \text{if} \quad n\geq 4,\\ 1-\delta & \text{if} \quad n=3,\end{cases}$$

and $\delta>0$ is arbitrarily small.

We conclude this section by stating our plan. In Section 2 we will give the results concerned with L^2 estimates, energy nondecay, and existence of scattering states. In Section 3 we will prove L^2 estimates. In Section 4 some inequalities will be given in order to deduce energy nondecay and existence of scattering states, which will be discussed in Section 6 and Section 7, respectively. In Section 5 we will prove the geometric result concerning the regularity of $\mathcal{O}$, and extend this to the scattering theory.

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2. Statement of results

In order to state the results, we make the following hypotheses.

Hypothesis A. *The function $a(x)$ belongs to $\mathfrak{B}^\infty(\overline{\Omega}\times[0,\infty))$ and has a compact support so that*

$$\operatorname{supp} a(\cdot)\subset\Omega(R)\equiv\Omega\cap B_R \quad \textit{for some } R>\rho_0,$$

and further, $a(x)$ satisfies the following properties:

There exists a relatively open set ω in $\overline{\Omega}$ such that

$$\overline{\{x\in\partial\Omega\,;\,x\cdot\boldsymbol{\nu}(x)>0\}}\subset\omega \quad \textit{and} \quad a(x)\geq a_0 \quad \textit{in } \omega$$

for some $a_0>0$, where $\boldsymbol{\nu}(x)$ is the outward-normal vector at $x\in\partial\Omega$. In particular, if $\mathcal{O}$ is star-shaped with respect to the origin, we take $\omega=\emptyset$ and $a(x)\equiv 0$.

Hypothesis B. *$f(u)$ belongs to $C^{2M-1}(\mathbb{R})$, $M = [n/2]+1$, such that*

$$|f^{(2M-1)}(u)| \le k_1 |u|^{\alpha-2M+2} \quad \text{for } u \in \mathbb{R},$$

where α satisfies

$$\alpha > 2M-1 \text{ and } \alpha \ge p-1 \text{ with some } p \in \Big(2, \frac{2(n+1)}{n-1}\Big].$$

Furthermore, we must define the compatibility condition needed in the later arguments. We say the data $\{u_0, u_1\} \in H^{m+1}(\Omega) \times H^m(\Omega)$ satisfies the compatibility condition of order m if $(\partial_t^j u)(\cdot,0) \in H_0^1(\Omega)$ $(j = 0, 1, \dots, m)$ and $(\partial_t^{m+1} u)(\cdot, 0) \in L^2(\Omega)$.

Our arguments are based on the following existence theorem and L^p decay estimates due to Nakao.

Proposition A (Nakao [13]). (i) *Let n be an integer with $n \ge 3$. If n is an even integer, we suppose that $\mathcal{O}$ is convex and $a(x) \equiv 0$. Let us assume that Hypotheses A and B are satisfied. Then there exists a constant $\varepsilon_1 > 0$ such that if the data $\{u_0, u_1\}$ meet*

$$\{u_0, u_1\} \in [H^{2M}(\Omega) \cap W^{2M,1}(\Omega)] \times [H^{2M-1}(\Omega) \cap W^{2M-1,1}(\Omega)],$$

$$I_M \equiv \sum_{i=1}^{2} \big(\|u_0\|_{2M,i} + \|u_1\|_{2M-1,i}\big) \le \varepsilon_1$$

and satisfy the compatibility condition of order $2M-1$, then there exist a unique solution $u \in X_{2M}(\Omega)$ of the problem (P) *and a constant $C > 0$ such that*

$$\|u\|_V + \sup_{t \ge 0} \Big\{ \sum_{j=0}^{2M-1} \|\partial_t^j \nabla u(t)\|_{2M-1-j,2} + \sum_{j=1}^{2M} \|\partial_t^j u(t)\|_2 \Big\} \le C I_M.$$

(ii) *In addition to the assumptions of* (i)*, let us assume Hypotheses A and B with $2M-1$ and $2M$ replaced by $2M$ and $2M+1$, respectively. Then there exists a constant $\varepsilon_2 > 0$ such that if the data meet*

$$\{u_0, u_1\} \in [H^{2M+1}(\Omega) \cap W^{2M+1,1}(\Omega)] \times [H^{2M}(\Omega) \cap W^{2M,1}(\Omega)],$$

$$J_M \equiv \sum_{i=1}^{2} \big(\|u_0\|_{2M+1,i} + \|u_1\|_{2M,i}\big) \le \varepsilon_2$$

and satisfy the compatibility condition of order $2M$*, then there exist a unique solution* $u \in X_{2M+1}(\Omega)$ *of the problem* (P) *and a constant* $C > 0$ *such that*

$$\sup_{t\geq 0}\left\{(1+t)^d\|u_t(t)\|_\infty\right\} \leq CJ_M.$$

The next theorem is concerned with an L^2 bound for u.

Theorem 1. *Let* u *be the solution in Proposition A* (i)*. If we assume further that*

$$\|au_0 + u_1\|_{\frac{2n}{n+2}} < \infty, \tag{2.1}$$

then there exists a constant $C > 0$ *such that*

$$\sup_{t\geq 0}\|u(t)\|_2 \leq \|u_0\|_2 + C\left(\|au_0+u_1\|_{\frac{2n}{n+2}} + I_M^{\alpha+1}\right).$$

The next theorem is concerned with energy nondecay, which will be precisely stated in Section 6.

Theorem 2. *For the solution of the problem (P) with nontrivial data, its energy does not in general decay as t goes to infinity.*

Once we have seen in Theorem 2 that the energy does not in general decay, the scattering problem becomes meaningful. Finally let us state the result concerning the existence of a scattering state. For this, we need a geometrical observation on the shape of $\mathcal{O}$.

Theorem 3. *Let* $\mathcal{O}$ *be a smooth and compact* n*-dimensional manifold in* $\mathbb{R}^n$*, and let its boundary* $\partial\mathcal{O}$ *be an* $(n-1)$*-dimensional smooth and closed submanifold in* $\mathbb{R}^n$*. Let* $\widetilde{\mathcal{O}}$ *be a convex hull of* $\partial\mathcal{O}$*. Then* $\partial\widetilde{\mathcal{O}}$ *is of class* C^1*.*

Remark. Even if $\partial\mathcal{O}$ is smooth, there is a counterexample in which $\partial\widetilde{\mathcal{O}}$ is not of class C^2. For example, let $\mathcal{O}$ be a smooth dumbbell domain in $\mathbb{R}^2$; the boundary $\partial\widetilde{\mathcal{O}}$ of the convex hull of $\partial\mathcal{O}$ is not C^2.

Let $\widetilde{\mathcal{O}}$ be as in Theorem 3. We introduce the initial-boundary-value problem for free-wave equation:

$$\widetilde{(\mathrm{P})}\begin{cases} w_{tt} - \Delta w = 0, & (x,t) \in \widetilde{\Omega}\times(0,\infty),\\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in \widetilde{\Omega},\\ w(x,t) = 0, & (x,t)\in\partial\widetilde{\Omega}\times(0,\infty),\end{cases}$$

where $\widetilde{\Omega} = \mathbb{R}^n \setminus \widetilde{\mathcal{O}}$. Set $\widetilde{\Gamma} = \partial\widetilde{\mathcal{O}} \setminus \partial\mathcal{O}$. In particular, if $\mathcal{O}$ is star-shaped, $\widetilde{\Gamma}$ becomes the empty set and $\widetilde{\Omega} = \Omega$. We list the notation used in Theorem 4: $H_\nabla(\widetilde{\Omega})$ is the completion of $C_0^\infty(\widetilde{\Omega})$ in the Dirichlet norm $\|\nabla\cdot\|_{L^2(\widetilde{\Omega})}$. $\widetilde{E}$ is the energy space of all pairs $\boldsymbol{f} \equiv \{f_1, f_2\}$ of functions belonging to $H_\nabla(\widetilde{\Omega}) \times L^2(\widetilde{\Omega})$ such that

$$\|\boldsymbol{f}\|_{\widetilde{E}}^2 = \|\{f_1, f_2\}\|_{\widetilde{E}}^2 = \frac{1}{2}\int_{\widetilde{\Omega}} (|\nabla f_1|^2 + |f_2|^2)\,dx < \infty.$$

For the solution w of $\widetilde{(\mathrm{P})}$ we simply write $\|w(t)\|_{\widetilde{E}} = \|\{w(t), w_t(t)\}\|_{\widetilde{E}}$.

Our final result reads as follows.

Theorem 4. (i) *Let u be the solution in Proposition A. Then there exists a solution $w^+(t) \in \widetilde{E}$ to the problem $\widetilde{(\mathrm{P})}$ such that*

$$\|u(t) - w^+(t)\|_{\widetilde{E}} = O(t^{-(d-1/2)}) \quad (t \to \infty). \tag{2.2}$$

(ii) *Suppose that $\mathcal{O}$ is convex and $a(x) \equiv 0$ regardless of odd or even dimensions. Let u be the solution in Proposition A. Then there exists a solution $w^+(t) \in X_{2M+1}(\Omega)$ of the problem $\widetilde{(\mathrm{P})}$ with $\widetilde{\Omega} = \Omega$ such that*

$$\|u(t) - w^+(t)\|_E = O(t^{-(L-1)}) \quad (t \to \infty), \tag{2.3}$$

$$\|u(t) - w^+(t)\|_\infty = O(t^{-(d+\varepsilon_0-1)}) \quad (t \to \infty), \tag{2.4}$$

$$\|u(t) - w^+(t)\|_2 = O(t^{-(L'-1)}) \quad (t \to \infty), \tag{2.5}$$

where we set $\varepsilon_0 = (\alpha - 2M + 1)d$,

$$L = \begin{cases} \dfrac{\alpha(n-1)}{2} & \text{if } n \geq 4, \\ \alpha - \delta & \text{if } n = 3, \end{cases} \quad \text{and} \quad L' = \frac{\alpha(n-1)}{2} - \frac{n-1}{n}. \tag{2.6}$$

3. L^2 boundedness

In this section we shall prove Theorem 1. For this, we treat the homogeneous problem:

$$(\mathrm{LP}) \begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases}$$

Lemma 3.1. *Let u be the finite-energy solution of the problem (LP) with data satisfying*

$$\|au_0 + u_1\|_{\frac{2n}{n+2}} < \infty.$$

Then there exists a constant $C > 0$ such that

$$\sup_{t \geq 0} \|u(t)\|_2 \leq \|u_0\|_2 + C\|au_0 + u_1\|_{\frac{2n}{n+2}}. \tag{3.1}$$

Proof. The proof can be done in a way parallel to Ikehata [4] (cf. [5]). For completeness we give it. For the solution u of the problem (LP), we set

$$w(x,t) = \int_0^t u(x,\tau)\,d\tau.$$

Then w belongs, in fact, to $C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega))$, and we can easily see that w is the solution of the following initial-boundary-value problem:

$$\text{(PV)}\ \begin{cases} w_{tt} - \Delta w + a(x)w_t = a(x)u_0 + u_1, & (x,t) \in \Omega \times (0,\infty), \\ w(x,0) = 0, \quad w_t(x,0) = u_0, & x \in \Omega, \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases}$$

Multiplying the equation in (PV) by w_t, we see that

$$\begin{aligned} &\|w_t(t)\|_2^2 + \|\nabla w(t)\|_2^2 + 2\int_0^t \int_\Omega a w_\tau^2\,dx\,d\tau \qquad (3.2) \\ &= \|u_0\|_2^2 + 2\int_0^t \int_\Omega (au_0 + u_1)\,w_\tau(\tau)\,dx\,d\tau = \|u_0\|_2^2 + 2\int_\Omega (au_0+u_1)\,w(t)\,dx \\ &\le \|u_0\|_2^2 + 2\|au_0+u_1\|_{\frac{2n}{n+2}}\,\|w(t)\|_{\frac{2n}{n-2}} \le \|u_0\|_2^2 + C\|au_0+u_1\|_{\frac{2n}{n+2}}\,\|\nabla w(t)\|_2. \end{aligned}$$

The inequality (3.2) and the fact that $w_t(t) = u(t)$ imply that for any ε with $0 < \varepsilon < 1$, there exists a constant C_ε such that

$$\|u(t)\|_2^2 + (1-\varepsilon)\|\nabla w(t)\|_2^2 \le \|u_0\|_2^2 + C_\varepsilon \|au_0+u_1\|_{\frac{2n}{n+2}}^2.$$

Thus we have the desired estimate. □

We denote by $S_0(t;\{u_0,u_1\})$ the solution of the problem (LP). Then it follows from Duhamel's principle that the solution u of the problem (P) can be written as

$$u(t) = S_0(t;\{u_0,u_1\}) + \int_0^t S_0(t-\tau;\{0, f(u(\tau))\})\,d\tau. \qquad (3.3)$$

We apply the L^2 estimate in Lemma 3.1 to equation (3.3) to obtain

$$\|u(t)\|_2 \le \|u_0\|_2 + C\|au_0+u_1\|_{\frac{2n}{n+2}} + C\int_0^t \|f(u(\tau))\|_{\frac{2n}{n+2}}\,d\tau. \qquad (3.4)$$

Using L^p estimates in Proposition A, we see that

$$\|f(u(\tau))\|_{\frac{2n}{n+2}} \le k_1 \|u(\tau)\|_\infty^{\alpha+1-\frac{(n+2)p}{2n}} \|u(\tau)\|_p^{\frac{(n+2)p}{2n}} \le C I_M^{\alpha+1}(1+\tau)^{-L'}, \qquad (3.5)$$

where we set

$$L' = \left(\alpha + 1 - \frac{(n+2)p}{2n}\right) d + \frac{(n+2)p}{2n} \cdot b = \frac{\alpha(n-1)}{2} - \frac{n-1}{n}.$$

By Hypothesis B we can see that $L' > 1$. Hence, it follows from estimate (3.5) that

$$\begin{aligned} \|u(t)\|_2 &\le \|u_0\|_2 + C\|au_0 + u_1\|_{\frac{2n}{n+2}} + C(R) I_M^{\alpha+1} \int_0^t (1+\tau)^{-L'}\, d\tau \\ &\le \|u_0\|_2 + C(R)\left(\|au_0 + u_1\|_{\frac{2n}{n+2}} + I_M^{\alpha+1}\right). \end{aligned}$$

The proof of Theorem 1 is complete. □

4. Some inequalities

In this section we summarize several estimates needed in later arguments. The notation of domains or hypersurfaces ($\widetilde{\Omega}$, $\widetilde{\Gamma}$, etc.) has been described in Section 2.

Lemma 4.1. *Assume that Hypothesis A is satisfied. Then we have the following assertions*:

(i) *For the finite-energy solution w of problem $\widetilde{(\mathrm{P})}$, there exists a constant $C > 0$ such that*

$$\int_0^\infty \int_{\widetilde{\Omega}} a w_t^2\, dx\, dt + \int_0^\infty \int_{\widetilde{\Gamma}} \left(\frac{\partial w}{\partial \widetilde{\boldsymbol{\nu}}}\right)^2 dS\, dt \le C\|w(0)\|_{\widetilde{E}}^2,$$

where $\widetilde{\boldsymbol{\nu}}(x)$ is the outward-normal vector at $x \in \partial\widetilde{\Omega}$. Of course, if $\mathcal{O}$ is star-shaped, then $\widetilde{\Omega} = \Omega$ and $\widetilde{\Gamma} = \emptyset$.

(ii) *Let w be the finite-energy solution of the following Cauchy problem*:

$$(\mathrm{P})_w \begin{cases} w_{tt} - \Delta w = 0, & (x,t) \in \mathbb{R}^n \times [0,\infty), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in \mathbb{R}^n. \end{cases}$$

Then there exists a constant $C > 0$ such that

$$\int_0^\infty \int_\Omega a w_t^2\, dx\, dt \le C\|w(0)\|_{E_0}^2,$$

where we set

$$\|w(0)\|_{E_0}^2 = \frac{1}{2}\int_{\mathbb{R}^n} \left\{ w_t(0)^2 + |\nabla w(0)|^2 \right\} dx.$$

Proof. The space-time estimates of aw_t^2 in parts (i) and (ii) are an immediate consequence of Mochizuki and Nakazawa [11]. For the boundary-integral estimate of $\partial w/\partial\widetilde{\boldsymbol{\nu}}$ over $\widetilde{\Gamma}$, see [8]. □

The proofs of Theorems 2 and 4 are based on the following identities.

Lemma 4.2. (i) *Let w be the finite-energy solution of problem* $\widetilde{(\mathrm{P})}$ *and u the solution in Proposition A. Then we have*

$$\begin{aligned} &2(u(t),w(t))_{\widetilde{E}} + \int_s^t \int_{\widetilde{\Omega}} a u_\tau(\tau) w_\tau(\tau)\,dx\,d\tau \\ &= 2(u(s),w(s))_{\widetilde{E}} + \int_s^t \int_{\widetilde{\Gamma}} u_\tau(\tau)\frac{\partial w}{\partial\widetilde{\boldsymbol{\nu}}}(\tau)\,dS\,d\tau + \int_s^t \int_{\widetilde{\Omega}} f(u(\tau))w_\tau(\tau)\,dx\,d\tau \end{aligned} \tag{4.1}$$

for $0 \le s < t$, where we set

$$(u(t),w(t))_{\widetilde{E}} = \frac{1}{2}\int_{\widetilde{\Omega}} (u_t w_t + \nabla u\cdot\nabla w)\,dx.$$

Of course, if $\mathcal{O}$ is star-shaped, then $\widetilde{\Omega} = \Omega$ and $\widetilde{\Gamma} = \emptyset$.

(ii) *Let w be the finite-energy solution of the problem* $(\mathrm{P})_w$ *in Lemma* 4.1 *with the data having compact supports in $\Omega(R)$, and u the solution in Proposition A. If $n \ge 3$ is odd, then we have*

$$\begin{aligned} &2(u(t),w(t))_E + \int_s^t \int_\Omega a u_\tau(\tau) w_\tau(\tau)\,dx\,d\tau \\ &\qquad = 2(u(s),w(s))_E + \int_s^t \int_\Omega f(u(\tau))w_\tau(\tau)\,dx\,d\tau \end{aligned}$$

for $\sigma_0 < s < t$, where σ_0 is given by $\sigma_0 = \inf\{t\,;\,\mathrm{diam}(\mathcal{O}) + R \le t\}$.

Proof. (i) We note that $u_t = 0$ on $\partial\widetilde{\Omega}\setminus\widetilde{\Gamma}$ and $w_t = 0$ on $\partial\widetilde{\Omega}$. Differentiating $2(u(t),w(t))_{\widetilde{E}}$ with respect to t and using the equations in (P) and $\widetilde{(\mathrm{P})}$, we have

$$\begin{aligned} 2\frac{d}{dt}(u(t),w(t))_{\widetilde{E}} &= -\int_{\widetilde{\Omega}} a u_t w_t\,dx + \int_{\partial\widetilde{\Omega}} \Big(\frac{\partial u}{\partial\widetilde{\boldsymbol{\nu}}} w_t + u_t \frac{\partial w}{\partial\widetilde{\boldsymbol{\nu}}}\Big)\,dS + \int_{\widetilde{\Omega}} f(u) w_t\,dx \\ &= -\int_{\widetilde{\Omega}} a u_t w_t\,dx + \int_{\widetilde{\Gamma}} u_t\frac{\partial w}{\partial\widetilde{\boldsymbol{\nu}}}\,dS + \int_{\widetilde{\Omega}} f(u) w_t\,dx. \end{aligned}$$

Integrate it over (s,t). Then we have (4.1).

(ii) Since n is odd, it follows from Huyghens' principle that

$$w(x,t) = 0 \quad \text{if } |x| + R \le t,$$

and hence $w(x,t)=0$ on $\partial\Omega\times(\sigma_0,\infty)$. Differentiating $2(u(t),w(t))_E$ with respect to t, we have the assertion (ii) by the same argument as in part (i). The proof of Lemma 4.2 is complete. □

Lemma 4.3. *Let u be the solution in Proposition A. Then there exists a constant $C>0$ such that*

$$\int_0^\infty \int_{\widetilde{\Gamma}} u_t^2\, dS\, dt \le C J_M^2.$$

Proof. Since $\widetilde{\Gamma}\subset\Omega$, it follows from the decay estimate of $\|u_t(t)\|_\infty$ that

$$\begin{aligned}\int_0^\infty \int_{\widetilde{\Gamma}} u_t^2\, dS\, dt \le& \mathrm{vol}(\widetilde{\Gamma}) \int_0^\infty \|u_t(t)\|^2_{L^\infty(\widetilde{\Gamma})}\, dt \\ \le& C J_M^2 \mathrm{vol}(\widetilde{\Gamma}) \int_0^\infty (1+t)^{-2d}\, dt \le C J_M^2 \mathrm{vol}(\widetilde{\Gamma}).\end{aligned}$$

This ends the proof of Lemma 4.3. □

The following lemma follows from a routine calculus of Lemma 3.1.

Lemma 4.4. (i) *Let w be the finite-energy solution of the problem $\widetilde{(\mathrm{P})}$ with $w_t(0)\in L^{\frac{2n}{n+2}}(\widetilde{\Omega})$. Then there exists a constant $C>0$ such that*

$$\sup_{t\ge 0}\|w(t)\|_{L^2(\widetilde{\Omega})} \le \|w(0)\|_{L^2(\widetilde{\Omega})} + C\|w_t(0)\|_{L^{\frac{2n}{n+2}}(\widetilde{\Omega})}.$$

(ii) *For the finite-energy solution w of the Cauchy problem $(\mathrm{P})_w$, we have the assertion* (i) *for $\widetilde{\Omega}$ replaced by $\mathbb{R}^n$.*

5. Differentiability of the convex hull

In this section we prove Theorem 3. We must prepare the following facts.

Lemma 5.1. *Let K be a closed, convex set in $\mathbb{R}^n$. For any $p\in\partial K$, there exists a hyperplane through p supporting K.*

Proof. Choose any $\varepsilon>0$ and fix it. Since $p\in\partial K$, there exists $q_\varepsilon\in\mathbb{R}^n\setminus K$ satisfying $d(q_\varepsilon,p)<\varepsilon$. Choose a point $p_\varepsilon\in\partial K$ satisfying $d(p_\varepsilon,q_\varepsilon)=d(K,q_\varepsilon)$. Thus there exists no point of K in the open ball $B(q_\varepsilon,d(K,q_\varepsilon))$ centered at q_ε with radius $d(K,q_\varepsilon)$. It is trivial that

$$d(p_\varepsilon,q_\varepsilon)=d(K,q_\varepsilon)\le d(p,q_\varepsilon)<\varepsilon.$$

Hence by the triangle inequality we see that

$$d(p,p_\varepsilon)\le d(p,q_\varepsilon)+d(q_\varepsilon,p_\varepsilon)<2\varepsilon.$$

Let H_ε denote the hyperplane through p_ε orthogonal to the vector $\overrightarrow{q_\varepsilon p_\varepsilon}$. We will prove that H_ε is a hyperplane supporting K.

Suppose that there exists a point $x \in K$ such that the angle made by the two vectors $\overrightarrow{p_\varepsilon x}$ and $\overrightarrow{q_\varepsilon p_\varepsilon}$ is greater than $\frac{\pi}{2}$. Since K is convex, the line segment $tp_\varepsilon + (1-t)x$ joining p_ε and x lies in K. By the assumption, the point $tp_\varepsilon + (1-t)x$, for any sufficiently small $t > 0$, lies in $B(q_\varepsilon, d(K, q_\varepsilon))$. Since $B(q_\varepsilon, d(K, q_\varepsilon))$ does not contain a point of K, this is a contradiction. Thus, H_ε supports K at p_ε.

Therefore we have proved that for any $\varepsilon > 0$, there exist a point $p_\varepsilon \in K$ with $d(p_\varepsilon, p) < 2\varepsilon$ and a supporting hyperplane H_ε through p_ε. It is easy to prove the existence of a supporting hyperplane through p by the limiting argument. This ends the proof of Lemma 5.1. $\square$

Let K be an n-dimensional, closed, convex set in $\mathbb{R}^n$, and let ∂K be represented locally by $x_n = f(x_1, \ldots, x_{n-1})$ near $p = (\boldsymbol{x}_0, f(\boldsymbol{x}_0)) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Since K is convex, the function f is convex, and hence, by Theorem 10.4 in [17, Rockafeller], f is Lipschitz. For each $\boldsymbol{v} = (v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}$, let $c_{\boldsymbol{v}}(t)$ be a curve lying in ∂K joining $(\boldsymbol{x}_0, f(\boldsymbol{x}_0))$ to $(\boldsymbol{x}_0 + \boldsymbol{v}, f(\boldsymbol{x}_0 + \boldsymbol{v}))$ defined by

$$c_{\boldsymbol{v}}(t) := (\boldsymbol{x}_0 + t\boldsymbol{v}, f(\boldsymbol{x}_0 + t\boldsymbol{v})), \quad 0 \le t \le 1,$$

and its tangent vector given by $(c_{\boldsymbol{v}})'_+(0) = (\boldsymbol{v}, f'_+(\boldsymbol{x}_0; \boldsymbol{v}))$, where

$$f'_+(\boldsymbol{x}_0; \boldsymbol{v}) = \lim_{t \to +0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{v}) - f(\boldsymbol{x}_0)}{t}.$$

Then we have the following lemma:

Lemma 5.2. *Let K be an n-dimensional, closed, convex set in $\mathbb{R}^n$. If $p \in \partial K$ has a unique supporting hyperplane H_p, then $(c_{\boldsymbol{v}})'_+(0) \in H_p$.*

Proof. Let $\boldsymbol{\nu}(p)$ denote the unit-normal vector of H_p such that for any $x \in K \setminus \{p\}$, the angle $\angle(\boldsymbol{\nu}(p), \overrightarrow{px})$ made by $\overrightarrow{px}$ and $\boldsymbol{\nu}(p)$ is not greater than $\frac{\pi}{2}$. It follows from Lemma 5.1 that for each $t > 0$, there exists a hyperplane H_t through $c_{\boldsymbol{v}}(t)$ supporting K. Let $\boldsymbol{\nu}_t$ denote the unit-normal vector of H_t such that for any $x \in K \setminus \{c_{\boldsymbol{v}}(t)\}$, the angle $\angle(\boldsymbol{\nu}_t, \overrightarrow{c_{\boldsymbol{v}}(t)x})$ made by $\overrightarrow{c_{\boldsymbol{v}}(t)x}$ and $\boldsymbol{\nu}_t$ is not greater than $\frac{\pi}{2}$. Since H_p is the unique supporting hyperplane through p, we see that $\lim_{t \to +0} \boldsymbol{\nu}_t = \boldsymbol{\nu}(p)$. Since K is convex, the angle $\angle(\boldsymbol{\nu}(p), \overrightarrow{p\,c_{\boldsymbol{v}}(t)})$ made by $\boldsymbol{\nu}(p)$ and $\overrightarrow{p\,c_{\boldsymbol{v}}(t)}$ does not exceed $\frac{\pi}{2}$. Hence, for each $t > 0$, we have

$$\angle(\boldsymbol{\nu}(p), \boldsymbol{w}(t)) \le \frac{\pi}{2}, \tag{5.1}$$

where

$$\boldsymbol{w}(t) = \frac{1}{\|\overrightarrow{p\,c_{\boldsymbol{v}}(t)}\|}\overrightarrow{p\,c_{\boldsymbol{v}}(t)}.$$

Furthermore, for each $t > 0$, we have

$$\angle(\boldsymbol{\nu}_t, -\boldsymbol{w}(t)) \leq \frac{\pi}{2}. \tag{5.2}$$

Since f is convex, there exists a unique limit vector $\boldsymbol{w}(0) = \lim_{t\to+0}\boldsymbol{w}(t)$. Thus, from (5.1) and (5.2) it follows that

$$\angle(\boldsymbol{\nu}(p), -\boldsymbol{w}(0)) \leq \frac{\pi}{2}, \quad \angle(\boldsymbol{\nu}(p), \boldsymbol{w}(0)) \leq \frac{\pi}{2}.$$

This implies that $\boldsymbol{w}(0)$ is orthogonal to $\boldsymbol{\nu}(p)$, and hence $(c_{\boldsymbol{v}})'_+(0)$ is orthogonal to $\boldsymbol{\nu}(p)$. This ends the proof of Lemma 5.2. □

Lemma 5.3. *Let K be an n-dimensional, closed, convex set in $\mathbb{R}^n$. If $p \in \partial K$ has a unique supporting hyperplane, then ∂K is totally differentiable at p. Furthermore, if each point of ∂K has a unique supporting hyperplane, then ∂K is C^1.*

Proof. Since $p \in \partial K$ has a unique supporting hyperplane H_p, it follows from Lemma 5.2 that the vector $(c_{\boldsymbol{v}})'_+(0) = (\boldsymbol{v}, f'_+(\boldsymbol{x}_0;\boldsymbol{v}))$ is tangent to H_p. In particular, the vector $(c_{\boldsymbol{v}})'_+(0)$ is orthogonal to a normal vector $\boldsymbol{\nu}(p) = (\nu_1(p),\ldots,\nu_n(p))$, $\nu_n(p) \neq 0$, of H_p. Hence we have

$$\sum_{i=1}^{n-1} v_i\nu_i(p) + f'_+(\boldsymbol{x}_0;\boldsymbol{v})\nu_n(p) = 0,$$

or equivalently

$$f'_+(\boldsymbol{x}_0;\boldsymbol{v}) = -\frac{1}{\nu_n(p)}\sum_{i=1}^{n-1} v_i\nu_i(p).$$

Notice that $f'_+(\boldsymbol{x}_0;\boldsymbol{v})$ is a linear functional on $\mathbb{R}^{n-1}$.

For the argument above, we have proved that for each unit vector $\boldsymbol{e} \in \mathbb{R}^{n-1}$,

$$\lim_{t\to+0}\frac{\|f(\boldsymbol{x}_0 + t\boldsymbol{e}) - f(\boldsymbol{x}_0) - f'_+(\boldsymbol{x}_0;t\boldsymbol{e})\|}{t} = 0. \tag{5.3}$$

Choose any $\varepsilon > 0$ and fix it. Since a unit sphere in $\mathbb{R}^{n-1}$ is compact, we may choose finitely many unit vectors $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_{k(\varepsilon)}$ at p such that for any unit vector $\boldsymbol{v}$ at p, $\|\boldsymbol{e}_i - \boldsymbol{v}\| < \varepsilon$ holds for some $i \in \{1, \dots, k(\varepsilon)\}$. Let $\boldsymbol{w}$ be any nonzero vector at p and $\boldsymbol{e}_{i_0}$, one of the vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{k(\varepsilon)}$, satisfying

$$\Big\| \frac{1}{\|\boldsymbol{w}\|}\boldsymbol{w} - \boldsymbol{e}_{i_0} \Big\| < \varepsilon.$$

By the triangle inequality we see that

$$\begin{aligned} &\|f(\boldsymbol{x}_0 + \boldsymbol{w}) - f(\boldsymbol{x}_0) - f'_+(\boldsymbol{x}_0; \boldsymbol{w})\| \le \|f(\boldsymbol{x}_0 + \boldsymbol{w}) - f(\boldsymbol{x}_0 + \|\boldsymbol{w}\|\boldsymbol{e}_{i_0})\| \\ &+ \|f(\boldsymbol{x}_0 + \|\boldsymbol{w}\|\boldsymbol{e}_{i_0}) - f(\boldsymbol{x}_0) - f'_+(\boldsymbol{x}_0; \|\boldsymbol{w}\|\boldsymbol{e}_{i_0})\| \\ &+ \|f'_+(\boldsymbol{x}_0; \boldsymbol{w}) - f'_+(\boldsymbol{x}_0; \|\boldsymbol{w}\|\boldsymbol{e}_{i_0})\|. \end{aligned} \tag{5.4}$$

Since f and $f'_+(\boldsymbol{x}_0; \cdot)$ are Lipschitz, there exists a constant L such that

$$\|f(\boldsymbol{x}_0 + \boldsymbol{w}) - f(\boldsymbol{x}_0 + \|\boldsymbol{w}\|\boldsymbol{e}_{i_0})\| \le L\|\boldsymbol{w} - \|\boldsymbol{w}\|\boldsymbol{e}_{i_0}\| < L\|\boldsymbol{w}\|\varepsilon, \tag{5.5}$$

$$\|f'_+(\boldsymbol{x}_0; \boldsymbol{w}) - f'_+(\boldsymbol{x}_0; \|\boldsymbol{w}\|\boldsymbol{e}_{i_0})\| \le L\|\boldsymbol{w} - \|\boldsymbol{w}\|\boldsymbol{e}_{i_0}\| < L\|\boldsymbol{w}\|\varepsilon. \tag{5.6}$$

From (5.4), (5.5), and (5.6) it follows that

$$\begin{aligned} &\|f(\boldsymbol{x}_0 + \boldsymbol{w}) - f(\boldsymbol{x}_0) - f'_+(\boldsymbol{x}_0; \boldsymbol{w})\| \\ \le &2L\|\boldsymbol{w}\|\varepsilon + \|f(\boldsymbol{x}_0 + \|\boldsymbol{w}\|\boldsymbol{e}_{i_0}) - f(\boldsymbol{x}_0) - f'_+(\boldsymbol{x}_0; \|\boldsymbol{w}\|\boldsymbol{e}_{i_0})\| \end{aligned} \tag{5.7}$$

holds for any nonzero vector $\boldsymbol{w}$. Thus, by (5.3) and (5.7) we have

$$\limsup_{\|\boldsymbol{w}\| \to 0} \frac{\|f(\boldsymbol{x}_0 + \boldsymbol{w}) - f(\boldsymbol{x}_0) - f'_+(\boldsymbol{x}_0; \boldsymbol{w})\|}{\|\boldsymbol{w}\|} \le 2L\varepsilon. \tag{5.8}$$

Since ε is arbitrarily chosen, (5.8) implies that f is totally differentiable at $\boldsymbol{x}_0$ and its partial derivatives at $\boldsymbol{x}_0$ are

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0) = -\frac{\nu_i(p)}{\nu_n(p)}, \quad i = 1, 2, \dots, n-1.$$

Notice that $-\frac{\nu_i(p)}{\nu_n(p)}$, $i = 1, \dots, n-1$, are independent of the choice of a normal vector of H_p. Furthermore, we suppose that each point of ∂K has a unique supporting hyperplane. Let $\{p_i\}$ be any sequence of points of ∂K convergent to p and $\boldsymbol{\nu}(p_j) := (\nu_1(p_j), \dots, \nu_n(p_j))$ a normal vector of the unique supporting hyperplane through p_j. From the argument above it follows that

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}_j) = -\frac{\nu_i(p_j)}{\nu_n(p_j)} \quad \text{for each } i = 1, 2, \dots, n-1,$$

where $p_j = (\boldsymbol{x}_j, f(\boldsymbol{x}_j))$. It follows from the uniqueness of the supporting hyperplane at p that

$$\lim_{j\to\infty} \frac{\nu_i(p_j)}{\nu_n(p_j)} = \frac{\nu_i(p)}{\nu_n(p)} \quad \text{for each } i = 1, 2, \ldots, n-1.$$

Therefore the partial-derivative functions $\frac{\partial f}{\partial x_i}$, $i = 1, \ldots, n-1$, of f are continuous at $\boldsymbol{x}_0$. This ends the proof of Lemma 5.3. □

Proof of Theorem 3. By Lemmas 5.1 and 5.3, it suffices for differentiability of $\partial\widetilde{\mathcal{O}}$ to prove that each point of $\partial\widetilde{\mathcal{O}}$ has a unique supporting hyperplane.

If this is not true, then we can take a point $p \in \partial\widetilde{\mathcal{O}}$ such that there exist two supporting hyperplanes H_1 and H_2 at p. Then we see that $\mathcal{O}$ lies in the intersection of two half spaces determined by the hyperplanes H_1 and H_2 through p. Since $\partial\mathcal{O}$ is smooth, p does not belong to $\partial\mathcal{O}$, and hence we see that $\delta \equiv \operatorname{dist}(p, \partial\mathcal{O}) > 0$, which implies that there exists a point $q_0 \in \partial\mathcal{O}$ with $\operatorname{dist}(p, q_0) = \delta > 0$. Let us take a hyperplane H such that

$$H \text{ is orthogonal to the line segment } [p, q_0] \text{ joining } p \text{ and } q_0,$$
$$\operatorname{dist}(p, H) = \frac{\delta}{n} \text{ for sufficiently large } n \text{ so that } H \cap \mathcal{O} = \emptyset.$$

Let $\mathcal{O}^*$ be the intersection of a half space containing $\mathcal{O}$ determined by H with $\widetilde{\mathcal{O}}$. Then $\mathcal{O}^*$ is the convex subset of $\widetilde{\mathcal{O}}$, which contradicts the minimality of $\widetilde{\mathcal{O}}$. Therefore, each point of $\partial\widetilde{\mathcal{O}}$ has a unique supporting hyperplane. This completes the proof of Theorem 3. □

6. Energy nondecay

In this section we describe Theorem 2 more precisely and prove it. We divide the proof into odd- and even-dimensional cases.

In the case when n is an odd integer with $n \geq 3$, we consider the "Cauchy problem"

$$(\mathrm{P})_w \begin{cases} w_{tt} - \Delta w = 0, & (x,t) \in \mathbb{R}^n \times [0,\infty), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where the data $\{w_0, w_1\}$ satisfy

$$\operatorname{supp} w_0 \cup \operatorname{supp} w_1 \subset B_R. \tag{6.1}$$

In the case when n is an even integer with $n \geq 4$ and $\mathcal{O}$ is convex, we consider the "initial-boundary-value problem" for the free-wave equation

$$(\mathrm{P})_0 \begin{cases} w_{tt} - \Delta w = 0, & (x,t) \in \Omega \times (0,\infty), \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & x \in \Omega, \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases}$$

where the data $\{w_0, w_1\}$ satisfy

$$\operatorname{supp} w_0 \cup \operatorname{supp} w_1 \subset \Omega(R) \quad \text{for some } R \text{ with } R > \rho_0. \tag{6.2}$$

We make the following assumption on the data:

Hypothesis C. *The initial data* $\{w_0, w_1\}$ *satisfy*

$$\{w_0, w_1\} \neq \{0, 0\} \quad \textit{and} \quad \widetilde{I}_M \textit{ is small enough so that } \widetilde{I}_M \leq \varepsilon_1,$$

where

$$\widetilde{I}_M = \begin{cases} \|w_0\|'_{2M,2} + \|w_1\|'_{2M-1,2} + \|w_1\|'_{\frac{2n}{n+2}} & \textit{if } n \textit{ is odd,} \\ \|w_0\|_{2M,2} + \|w_1\|_{2M-1,2} + \|w_1\|_{\frac{2n}{n+2}} & \textit{if } n \textit{ is even.} \end{cases}$$

Odd-dimensional case. It is well known that the Cauchy problem $(\mathrm{P})_w$ has a unique solution w such that if the initial data $\{w_0, w_1\}$ belong to $H^{2M}(\mathbb{R}^n) \times H^{2M-1}(\mathbb{R}^n)$, then

$$w \in \bigcap_{j=0}^{2M} C^j([0,\infty); H^{2M-j}(\mathbb{R}^n))$$

and the energy of w is conserved:

$$\|w(t)\|_{E_0} = \|w(0)\|_{E_0},$$

where we set

$$\|w(t)\|_{E_0}^2 = \|\{w(t), w_t(t)\}\|_{E_0}^2 = \frac{1}{2} \int_{\mathbb{R}^n} \left\{ |\nabla w(t)|^2 + w_t^2(t) \right\} dx.$$

Notice that the supports of the data are compact. Then part (ii) of Lemma 4.4 implies that $\|w(t)\|_{L^2(\mathbb{R}^n)}$ is bounded in t, and hence we have

$$\sup_{t \geq 0} \left\{ \sum_{j=0}^{2M} \|\partial_t^j w(t)\|'_{2M-j,2} \right\} \leq C\widetilde{I}_M. \tag{6.3}$$

Further, w satisfies the finite propagation property:

$$\operatorname{supp} w(\cdot, t) \subset B_{R+t}, \tag{6.4}$$

and Huyghens' principle tells us that

$$w(x,t) = 0 \quad \text{ if } x \in \overline{U} \text{ and } t \geq \sigma_1, \tag{6.5}$$

for some neighborhood U of $\mathcal{O}$ and $\sigma_1 > \sigma_0$, where we set

$$\sigma_0 = \inf \{ t\, ;\, \operatorname{diam}(\mathcal{O}) + R \leq t \} . \tag{6.6}$$

Then we can take $\sigma = \sigma(w_0, w_1) > \max\{1, \sigma_1, R\}$ so that

$$\int_\sigma^\infty \int_\Omega a w_t^2 \, dx \, dt < \frac{\left\{ 2\|w(0)\|_{E_0}^2 - C_1 \|w(0)\|_{E_0} \widetilde{I}_M^{\alpha+1} \right\}^2}{\|w(0)\|_{E_0}^2 + C_0 \widetilde{I}_M^{\alpha+2}} \tag{6.7}$$

holds, which is possible on account of part (ii) of Lemma 4.1, where C_i $(i = 0, 1)$ are constants independent of σ determined later. For this σ and the solution $w(\cdot, \sigma)$ of the problem $(\mathrm{P})_w$ satisfying (6.3), we consider the initial-boundary-value problem

$$(\mathrm{P})_\sigma \begin{cases} u_{tt}^{(\sigma)} - \Delta u^{(\sigma)} + a(x) u_t^{(\sigma)} = f(u^{(\sigma)}), \quad (x,t) \in \Omega \times (0, \infty), \\ u^{(\sigma)}(x, 0) = \sigma^{-\beta} w(x, \sigma), \quad u_t^{(\sigma)}(x, 0) = \sigma^{-\beta} w_t(x, \sigma), \quad x \in \Omega, \\ u^{(\sigma)}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0, \infty), \end{cases}$$

where β is a number with

$$\beta > \frac{n}{2} + \frac{n}{\alpha}. \tag{6.8}$$

Here we note that the initial data $\{\sigma^{-\beta} w(\sigma), \sigma^{-\beta} w_t(\sigma)\}$ satisfy the compatibility condition of order $2M - 1$ on account of (6.5). Moreover, the time σ can be taken, if necessary, large enough. Thus, from the finite-propagation property (6.4), the fact that $\sigma > \max\{1, R\}$, and estimate (6.3), it follows that the initial data $\{\sigma^{-\beta} w(\sigma), \sigma^{-\beta} w_t(\sigma)\}$ satisfy

$$\begin{aligned} &\sum_{i=1}^{2} \left\{ \|\sigma^{-\beta} w(\sigma)\|_{2M,i} + \|\sigma^{-\beta} w_t(\sigma)\|_{2M-1,i} \right\} \qquad (6.9) \\ &\leq C(n) \sigma^{-(\beta - n/2)} \left\{ \|w(\sigma)\|'_{2M,2} + \|w_t(\sigma)\|'_{2M-1,2} \right\} \leq C(n) \sigma^{-(\beta - n/2)} \widetilde{I}_M. \end{aligned}$$

Since $\widetilde{I}_M \leq \varepsilon_1$ and σ can be taken so large that $C(n)\sigma^{-(\beta - n/2)} \leq 1$, we can apply Proposition A to problem $(\mathrm{P})_\sigma$ to obtain

$$\begin{aligned} \|u^{(\sigma)}\|_V + \sup_{t \geq 0} \Big\{ \sum_{j=0}^{2M-1} \|\partial_t^j \nabla u^{(\sigma)}(t)\|_{2M-1-j,2} + \sum_{j=1}^{2M} \|\partial_t^j u^{(\sigma)}(t)\|_2 \Big\} & \\ \leq C(n) \sigma^{-(\beta - n/2)} \widetilde{I}_M, & \qquad (6.10) \end{aligned}$$

and further, using this we can obtain the following estimates:

$$\int_0^\infty \int_\Omega a|u_t^{(\sigma)}(t)|^2\,dx\,dt \le \sigma^{-2\beta}\{\|w(0)\|_{E_0}^2 + C_0\widetilde{I}_M^{\alpha+2}\}, \tag{6.11}$$

$$\int_0^\infty \|f(u^{(\sigma)}(t))\|_2\,dt \le C_1\sigma^{-\beta}\widetilde{I}_M^{\alpha+1} \tag{6.12}$$

for some $C_i > 0$ $(i = 0, 1)$ independent of σ. In fact, multiplying the equation in $(\mathrm{P})_\sigma$ by $u_t^{(\sigma)}$, we see that

$$\|u^{(\sigma)}(t)\|_E^2 + \int_0^t \int_\Omega a|u_\tau^{(\sigma)}(\tau)|^2\,dx\,d\tau = \sigma^{-2\beta}\|w(0)\|_{E_0}^2 + \int_\Omega F(u^{(\sigma)}(\tau))\,dx\,\Big|_{\tau=0}^{\tau=t}, \tag{6.13}$$

where $F(u)$ is the indefinite integral of $f(u)$ and we have used the relation

$$\|u^{(\sigma)}(0)\|_E = \sigma^{-\beta}\|w(\sigma)\|_E = \sigma^{-\beta}\|w(\sigma)\|_{E_0} = \sigma^{-\beta}\|w(0)\|_{E_0}.$$

Here it follows from estimate (6.10) and condition (6.8) on β that

$$\begin{aligned}
&\sup_{t\ge 0}\Big|\int_\Omega F(u^{(\sigma)}(t))\,dx\Big| \le k_2 \sup_{t\ge 0}\|u^{(\sigma)}(t)\|_{\alpha+2}^{\alpha+2} \\
&\le k_2 \sup_{t\ge 0}\{\|u^{(\sigma)}(t)\|_\infty^{\alpha+2-p}\|u^{(\sigma)}(t)\|_p^p\} \le C_0\sigma^{-(\alpha+2)(\beta-n/2)}\widetilde{I}_M^{\alpha+2} \le C_0\sigma^{-2\beta}\widetilde{I}_M^{\alpha+2}
\end{aligned}$$

for some $C_0 > 0$ independent of σ. This and (6.13) imply that

$$\|u^{(\sigma)}(t)\|_E^2 + \int_0^t \int_\Omega a|u_\tau^{(\sigma)}(\tau)|^2\,dx\,d\tau \le \sigma^{-2\beta}\{\|w(0)\|_{E_0}^2 + C_0\widetilde{I}_M^{\alpha+2}\},$$

from which it follows that the estimate (6.11) can be obtained. Similarly, we have

$$\begin{aligned}
&\|f(u^{(\sigma)}(t))\|_2 \le k_1\|u^{(\sigma)}(t)\|_{2(\alpha+1)}^{\alpha+1} \le k_1\|u^{(\sigma)}(t)\|_\infty^{\alpha+1-p/2}\|u^{(\sigma)}(t)\|_p^{p/2} \\
&\le C_1\sigma^{-(\alpha+1)(\beta-n/2)}\widetilde{I}_M^{\alpha+1}(1+t)^{-L} \le C_1\sigma^{-\beta}\widetilde{I}_M^{\alpha+1}(1+t)^{-L}
\end{aligned} \tag{6.14}$$

for some $C_1 > 0$ independent of σ, where $L > 1$ has been defined in (2.6). This implies estimate (6.12).

Now we proceed to the argument that the energy of $u^{(\sigma)}$ never decays. Since $w(t+\sigma) = 0$ on $\partial\Omega \times [0, \infty)$, it follows from part (ii) of Lemma 4.2 that

$$2(u^{(\sigma)}(t), w(t+\sigma))_E + \int_0^t \int_\Omega a u_\tau^{(\sigma)}(\tau) w_\tau(\tau+\sigma)\,dx\,d\tau$$

$$= 2\sigma^{-\beta}\|w(0)\|_{E_0}^2 + \int_0^t \int_\Omega f(u^{(\sigma)}(\tau))w_\tau(\tau+\sigma)\,dx\,d\tau, \quad (6.15)$$

where we have used the relation

$$(u^{(\sigma)}(0), w(\sigma))_E = \sigma^{-\beta}(w(\sigma), w(\sigma))_E = \sigma^{-\beta}(w(\sigma), w(\sigma))_{E_0} = \sigma^{-\beta}\|w(0)\|_{E_0}^2.$$

We suppose that $\|u^{(\sigma)}(t)\|_E$ decays as $t \to \infty$ and proceed to a contradiction. Letting $t \to \infty$ in (6.15), we see that

$$\int_0^\infty \int_\Omega a u_t^{(\sigma)}(t) w_t(t+\sigma)\,dx\,dt$$
$$= 2\sigma^{-\beta}\|w(0)\|_{E_0}^2 + \int_0^\infty \int_\Omega f(u^{(\sigma)}(t))w_t(t+\sigma)\,dx\,dt. \quad (6.16)$$

We apply the Schwarz inequality and estimates (6.11) and (6.12) to identity (6.16) to obtain

$$\sigma^{-\beta}\left\{\|w(0)\|_{E_0}^2 + C_0\widetilde{I}_M^{\alpha+2}\right\}^{1/2}\left(\int_\sigma^\infty \int_\Omega a w_t^2(t)\,dx\,dt\right)^{1/2}$$
$$\geq 2\sigma^{-\beta}\|w(0)\|_{E_0}^2 - C_1\sigma^{-\beta}\|w(0)\|_{E_0}\widetilde{I}_M^{\alpha+1},$$

which implies that

$$\int_\sigma^\infty \int_\Omega a w_t^2\,dx\,dt \geq \frac{\left\{2\|w(0)\|_{E_0}^2 - C_1\|w(0)\|_{E_0}\widetilde{I}_M^{\alpha+1}\right\}^2}{\|w(0)\|_{E_0}^2 + C_0\widetilde{I}_M^{\alpha+2}}.$$

This contradicts (6.7). Therefore, $\|u^{(\sigma)}(t)\|_E$ never decays as $t \to \infty$. Therefore we arrive at the following theorem.

Theorem 6.1. *Let n be an odd integer with $n \geq 3$. Assume that Hypotheses A, B, and C are satisfied. Let w be the solution in the class $X_{2M}(\mathbb{R}^n)$ of the Cauchy problem $(\mathrm{P})_w$ with the initial data $\{w_0, w_1\}$ belonging to $H^{2M}(\mathbb{R}^n) \times H^{2M-1}(\mathbb{R}^n)$ and having the compact supports as in (6.1). Let us take a time $\sigma \equiv \sigma(w_0, w_1) > \max\{1, \sigma_1, R\}$ with $\sigma_1 > \sigma_0$, σ_0 being given in (6.6), so that inequality (6.7) holds. Then the energy $\|u^{(\sigma)}(t)\|_E$ of $u^{(\sigma)}(t)$ never decays as $t \to \infty$, where $u^{(\sigma)}(t)$ is the solution in the sense of Proposition A to the problem $(\mathrm{P})_\sigma$.*

Even-dimensional case. In this case we suppose that $\mathcal{O}$ is convex. Let w be the solution of problem $(\mathrm{P})_0$. Then we can take $\sigma = \sigma(w_0, w_1) >$

$\max\{1, R\}$ so that

$$\int_\sigma^\infty \int_\Omega a w_t^2 \, dx \, dt < \frac{\left\{2\|w(0)\|_E^2 - C_1\|w(0)\|_E \widetilde{I}_M^{\alpha+1}\right\}^2}{\|w(0)\|_E^2 + C_0 \widetilde{I}_M^{\alpha+2}} \tag{6.17}$$

holds, which is possible on account of part (i) of Lemma 4.1. Of course, the constants C_i $(i = 0, 1)$ are independent of σ. By the same argument as in the previous case, part (i) of Lemma 4.4 implies that $\|w(t)\|_2$ is bounded in t, and hence, for this σ and the solution w of problem $(\mathrm{P})_0$, we can consider the initial-boundary-value problem $(\mathrm{P})_\sigma$ with data $\{\sigma^{-\beta} w(\sigma), \sigma^{-\beta} w_t(\sigma)\}$. Here we note that the compact-supports condition on the data is used to derive estimate (6.9). By the part (i) of Lemma 4.2 we have

$$2(u^{(\sigma)}(t), w(t+\sigma))_E + \int_0^t \int_\Omega a u_\tau^{(\sigma)}(\tau) w_\tau(\tau+\sigma) \, dx \, d\tau \\ = 2\sigma^{-\beta}\|w(0)\|_E^2 + \int_0^t \int_\Omega f(u^{(\sigma)}(\tau)) w_\tau(\tau+\sigma) \, dx \, d\tau,$$

and as a result, we can argue the energy nondecay in exactly the same manner as the previous case. Therefore, we may omit the details. Summarizing the above argument, we have the following theorem:

Theorem 6.2. *Let n be an even integer with $n \geq 4$ and $\mathcal{O}$ convex. Assume that Hypotheses A, B, and C are satisfied. Let w be the solution in the class $X_{2M}(\Omega)$ to problem $(\mathrm{P})_0$ with the initial data $\{w_0, w_1\}$ belonging to $H^{2M}(\Omega) \times H^{2M-1}(\Omega)$, having compact supports as in (6.2) and satisfying the compatibility condition of order $2M-1$; $(D_t^j w)(\cdot, 0) \in H_0^1(\Omega)$ $(k = 1, 2, \ldots, 2M-1)$ and $(D_t^{2M} w)(\cdot, 0) \in L^2(\Omega)$. Let us take $\sigma = \sigma(w_0, w_1) > \max\{1, R\}$ so that the inequality (6.17) holds. Then $\|u^{(\sigma)}(t)\|_E$ never decays as $t \to \infty$, where $u^{(\sigma)}(t)$ is the solution in the sense of Proposition A to the problem $(\mathrm{P})_\sigma$.*

7. SCATTERING RATES

In this section we prove Theorem 4.

(i) Let $\widetilde{\mathcal{O}}$ be the convex hull of $\partial\mathcal{O}$, and let $\widetilde{\Omega} = \mathbb{R}^n \setminus \widetilde{\mathcal{O}}$. Then Theorem 3 implies that $\partial\widetilde{\mathcal{O}}$ is of class C^1. We set $\widetilde{\Gamma} = \partial\widetilde{\mathcal{O}} \setminus \partial\mathcal{O}$.

Let $U_0(t)$, $t \in \mathbb{R}$, be the unitary group in the energy space $\widetilde{E}$ which represents the solution w of the problem $\widetilde{(\mathrm{P})}$ with the data $\boldsymbol{f} \equiv \{w_0, w_1\} \in \widetilde{E}$;

$$\{w(t), w_t(t)\} = U_0(t)\boldsymbol{f}.$$

Then it follows from part (i) of Lemma 4.2 that

$$
\begin{aligned}
&(U_0(-t)\boldsymbol{u}(t) - U_0(-s)\boldsymbol{u}(s), \boldsymbol{f})_{\widetilde{E}} \\
&= -\int_s^t \int_{\widetilde{\Omega}} a u_\tau w_\tau \, dx \, d\tau + \int_s^t \int_{\widetilde{\Gamma}} u_\tau \frac{\partial w}{\partial \widetilde{\boldsymbol{\nu}}} \, dS \, d\tau + \int_s^t \int_{\widetilde{\Omega}} f(u) w_\tau \, dx \, d\tau \quad (7.1)
\end{aligned}
$$

for any $0 \le s < t$, where $\boldsymbol{u}(t)$ stands for the pair $\{u(t), u_t(t)\}$. By the Schwarz inequality and part (i) of Lemma 4.1 we have

$$
\begin{aligned}
&|(U_0(-t)\boldsymbol{u}(t) - U_0(-s)\boldsymbol{u}(s), \boldsymbol{f})_{\widetilde{E}}| \\
&\le C\|\boldsymbol{f}\|_{\widetilde{E}} \Big\{ \Big(\int_s^t \int_{\widetilde{\Omega}} a u_\tau^2 \, dx \, d\tau \Big)^{1/2} + \Big(\int_s^t \int_{\widetilde{\Gamma}} u_\tau^2 \, dS \, d\tau \Big)^{1/2} + \int_s^t \|f(u)\|_2 \, d\tau \Big\},
\end{aligned}
$$

which implies from estimate (6.14) in Section 6 and Lemma 4.3 that

$$
\|U_0(-t)\boldsymbol{u}(t) - U_0(-s)\boldsymbol{u}(s)\|_{\widetilde{E}} \to 0 \quad \text{as } s, t \to \infty,
$$

and $U_0(-t)\boldsymbol{u}(t)$ converges in $\widetilde{E}$ as $t \to \infty$. Put

$$
\boldsymbol{f}^+ \equiv \{w_0^+, w_1^+\} = \text{s}-\lim_{t\to\infty} U_0(-t)\boldsymbol{u}(t). \tag{7.2}
$$

Then $\boldsymbol{f}^+ \in \widetilde{E}$ and we have

$$
\|\boldsymbol{u}(t) - U_0(t)\boldsymbol{f}^+\|_{\widetilde{E}} = \|U_0(-t)\boldsymbol{u}(t) - \boldsymbol{f}^+\|_{\widetilde{E}} \to 0 \quad \text{as } t \to \infty, \tag{7.3}
$$

and further,

$$
\begin{aligned}
&\|\boldsymbol{u}(t) - U_0(t)\boldsymbol{f}^+\|_{\widetilde{E}} \\
&\le C\Big\{ \Big(\int_t^\infty \int_{\widetilde{\Omega}} a u_\tau^2 \, dx \, d\tau \Big)^{1/2} + \Big(\int_t^\infty \int_{\widetilde{\Gamma}} u_\tau^2 \, dS \, d\tau \Big)^{1/2} + \int_t^\infty \|f(u)\|_2 \, d\tau \Big\}.
\end{aligned} \tag{7.4}
$$

Note that the support of $a(\cdot)$ is contained in $\Omega(R)$. We see from the decay estimate of $\|u_t(t)\|_\infty$ and estimate (6.14) in Section 6 that

the square of {the first term in the right-hand side of (7.4)}

$$
\le C(R)\int_t^\infty \|u_\tau(\tau)\|_\infty^2 d\tau \le C(R)J_M^2 \int_t^\infty (1+\tau)^{-2d} d\tau \le C(R)J_M^2(1+t)^{-(2d-1)}, \tag{7.5}
$$

the square of {the second term in the right-hand side of (7.4)}

$$
\le \text{vol}(\widetilde{\Gamma})\int_t^\infty \|u_\tau(\tau)\|_{L^\infty(\widetilde{\Gamma})}^2 d\tau \le CJ_M^2 \int_t^\infty (1+\tau)^{-2d} \, d\tau \le CJ_M^2(1+t)^{-(2d-1)}, \tag{7.6}
$$

and

$$
\begin{aligned}
&\{\text{the last term in the right-hand side of (7.4)}\} \\
&\le C I_M^{\alpha+1} \int_t^\infty (1+\tau)^{-L}\, d\tau \le C I_M^{\alpha+1} (1+t)^{-(L-1)},
\end{aligned} \tag{7.7}
$$

which implies that

$$
\|\boldsymbol{u}(t) - U_0(t)\boldsymbol{f}^+\|_{\widetilde{E}} \le C(R)(J_M + I_M^{\alpha+1})(1+t)^{-(d-1/2)}.
$$

The proof of assertion (i) is complete.

(ii) Let us consider the case when $\mathcal{O}$ is convex and $a(x) \equiv 0$. We will prove that $[U_0(t)\boldsymbol{f}^+]_1$ is the solution in the class $X_{2M+1}(\Omega)$ to problem $\widetilde{\text{(P)}}$ with $\widetilde{\Omega} = \Omega$. Here, we denote by $[\{f, g\}]_1 = f$ the first component of $\{f, g\}$. Hereafter, we say problem $\widetilde{\text{(P)}}$ for such a problem. Then we can write problem (P) in terms of $U_0(t)$:

$$
\boldsymbol{u}(t) = U_0(t-s)\boldsymbol{u}(s) + \int_s^t U_0(t-\tau)\{0, f(u(\tau))\}\, d\tau \tag{7.8}
$$

for $0 \le s < t$. It follows from (7.3) and (7.8) that

$$
\Big\| \int_s^t U_0(t-\tau)\{0, f(u(\tau))\}\, d\tau \Big\|_E \to 0 \quad (s, t \to \infty). \tag{7.9}
$$

Setting

$$
\boldsymbol{w}^+(t) = \boldsymbol{u}(t) + \int_t^\infty U_0(t-\tau)\{0, f(u(\tau))\}\, d\tau, \tag{7.10}
$$

we see from identity (7.8) that

$$
\begin{aligned}
\boldsymbol{w}^+(t) &= \boldsymbol{u}(t) + \Big(\int_t^T + \int_T^\infty \Big) U_0(t-\tau)\{0, f(u(\tau))\}\, d\tau \\
&= U_0(t-T)\boldsymbol{u}(T) + \int_T^\infty U_0(t-\tau)\{0, f(u(\tau))\}\, d\tau
\end{aligned} \tag{7.11}
$$

for any fixed $T > 0$. This implies that $[\boldsymbol{w}^+(t)]_1$ is the solution in the class $X_{2M+1}(\Omega)$ to problem $\widetilde{\text{(P)}}$, and we use (7.3), (7.9), and (7.11) to obtain

$$
\begin{aligned}
U_0(t)\boldsymbol{f}^+ &= U_0(t)\, \text{s-}\lim_{T\to\infty} U_0(-T)\boldsymbol{u}(T) \\
&= \text{s-}\lim_{T\to\infty} U_0(t-T)\boldsymbol{u}(T) = \boldsymbol{w}^+(t).
\end{aligned} \tag{7.12}
$$

Thus, $w^+(t) \equiv [U_0(t)\boldsymbol{f}^+]_1$ is the solution in the class $X_{2M+1}(\Omega)$ to problem $\widetilde{(\mathrm{P})}$, and further, it follows from (7.10) and (7.12) that

$$\boldsymbol{u}(t) = U_0(t)\boldsymbol{f}^+ - \int_t^\infty U_0(t-\tau)\{0, f(u(\tau))\}\, d\tau. \tag{7.13}$$

Hence we can deduce from identity (7.13) that

$$\begin{aligned}\|u(t) - w^+(t)\|_E \le &\int_t^\infty \|f(u(\tau))\|_2\, d\tau \\ &\le C(R) I_M^{\alpha+1} \int_t^\infty (1+\tau)^{-L}\, d\tau \le C(R) I_M^{\alpha+1}(1+t)^{-(L-1)},\end{aligned}$$

which implies (2.3).

Noting the nonlinearity condition in Hypothesis B, we have

$$\sum_{i=1}^2 \|f(u(\tau))\|_{2M-1,i} \le C I_M^{\alpha+1}(1+\tau)^{-d-\varepsilon_0}$$

with $\varepsilon_0 = (\alpha - 2M + 1)d > 0$, and hence

$$\begin{aligned}\|u(t) - w^+(t)\|_\infty &\le C \sum_{i=1}^2 \int_t^\infty \|f(u(\tau))\|_{2M-1,i}\, d\tau \\ &\le C(R) I_M^{\alpha+1} \int_t^\infty (1+\tau)^{-d-\varepsilon_0}\, d\tau \le C(R) I_M^{\alpha+1}(1+t)^{-(d+\varepsilon_0-1)},\end{aligned}$$

which implies (2.4).

For the estimate (2.5), we see from (7.13) and (3.5) that

$$\begin{aligned}\|u(t) - w^+(t)\|_2 = \|[\boldsymbol{u}(t) - U_0(t)\boldsymbol{f}^+]_1\|_2 &\le C \int_t^\infty \|f(u(\tau))\|_{\frac{2n}{n+2}}\, d\tau \\ &\le C(R) I_M^{\alpha+1} \int_t^\infty (1+\tau)^{-L'}\, d\tau \le C(R) I_M^{\alpha+1}(1+t)^{-(L'-1)},\end{aligned}$$

which implies (2.5). The proof of Theorem 4 is now finished.

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