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QUASILINEAR PARABOLIC EQUATIONS WITH LOCALIZED REACTION

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Abstract. In this paper, we study a nonnegative blow-up solution of the Dirichlet problem for a quasilinear parabolic equation $(u^\alpha)_t = \Delta u + f(u) + g(u(x_0(t),t))$ in $B(R)$, where $B(R) = \{x \in \mathbf{R}^\mathbf{N}\,;\, |x| < R\}$, $0 < \alpha \le 1$, $x_0(t) \in C^\infty([0,\infty); B(R))$ satisfies $x_0(t) \neq 0$, and $f(\xi)$ and $g(\xi)$ satisfy some blow-up conditions. We consider radial blow-up solutions $u(r,t)$ $(r = |x|)$ which are non-increasing in r, and give the classification between total blow-up and single point blow-up according to the growth orders of f and g.

Especially in the case $\alpha = 1$ we completely classify blow-up phenomena except for $f \sim g$ as follows. (I) When $g = o(f)$, any blow-up solution blows up only at the origin (single point blow-up); (II) When $f = o(g)$, (i) a single point blow-up solution exists (ii) there exists an initial data such that the solution blows up in the whole domain $B(R)$ (total blow-up) (iii) there are no other blow-up phenomena except total blow-up and single point blow-up.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^\mathbf{N}$ with smooth boundary $\partial\Omega$. In this paper we shall consider the initial boundary-value problem of the quasilinear parabolic equation with both localized (nonlocal) reaction $g(u(x_0(t),t))$ and local reaction $f(u(x,t))$,

$$(u^\alpha)_t = \Delta u + f(u) + g(u(x_0(t),t)), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $u_t = \partial u/\partial t$, Δ is the $N-$dimensional Laplacian, $0 < \alpha \le 1$, $f(\xi)$ and $g(\xi)$ are nonnegative functions, and $x_0(t) \in C^\infty([0,\infty)\,;\,\Omega)$. Throughout this paper we assume

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(A1) $f(\xi)$, $g(\xi) \in C^\infty(\mathbf{R}_+) \cap C(\bar{\mathbf{R}}_+)$, where $\mathbf{R}_+ = (0,\infty)$ and $\bar{\mathbf{R}}_+ = [0,\infty)$; $f(\xi)$, $g(\xi) > 0$ for $\xi > 0$; $f(\xi^{1/\alpha})$ and $g(\xi^{1/\alpha})$ are locally Lipschitz continuous in $[0,\infty)$,

(A2) $u_0(x) \geq 0, \in C(\bar{\Omega})$ and $u_0(x) = 0$, $x \in \partial\Omega$,

(A3) $g(\xi)$ is nondecreasing in $\xi \geq 0$.

We shall only consider nonnegative solutions $u = u(x,t)$.

Under conditions (A1)(A2), a nonnegative weak solution of (1.1)-(1.3) exists locally in time (See [7] and [20] for $\alpha = 1$ and Theorem 2.2 for $\alpha \leq 1$). When $\alpha = 1$, nonnegative solutions of (1.1)-(1.3) are unique (See [6] and [20]). When $0 < \alpha < 1$, we do not know whether or not nonnegative solutions of (1.1)-(1.3) are unique. But, assuming condition (A3), we can obtain a comparison theorem (See Proposition 2.7).

Moreover, we shall assume the following blow-up conditions (A4) on f and (A5) on g.

(A4) For any $\lambda > 0$, there exists a continuous function $h_\lambda(\xi)$ and $\xi_\lambda > 0$ such that

$$h_\lambda(\xi) \leq f(\xi) - \lambda\xi \text{ for } \xi \geq 0; \tag{1.4}$$

$$h_\lambda(\xi) > 0, \quad \int_\xi^\infty \frac{\alpha\eta^{\alpha-1}}{h_\lambda(\eta)}\, d\eta < \infty \quad \text{if } \xi > \xi_\lambda; \tag{1.5}$$

$$h_\lambda(\xi) \text{ is nondecreasing for } \xi > \xi_\lambda \text{ and } \Gamma(\xi) = h_\lambda(\xi^{1/\alpha}) \text{ is convex in } (0,\infty). \tag{1.6}$$

(A5) $g(\xi)$ is nondecreasing for $\xi \geq 0$ and for any $\lambda > 0$, there exists a continuous function $\tilde{h}_\lambda(\xi)$ and $\tilde{\xi}_\lambda > 0$ such that

$$\tilde{h}_\lambda(\xi) \leq g(\xi) - \lambda\xi \text{ for } \xi \geq 0; \tag{1.7}$$

$$\tilde{h}_\lambda(\xi) > 0, \quad \int_\xi^\infty \frac{\alpha\eta^{\alpha-1}}{\tilde{h}_\lambda(\eta)}\, d\eta < \infty \quad \text{if } \xi > \tilde{\xi}_\lambda; \tag{1.8}$$

$$\tilde{h}_\lambda(\xi) \text{ is nondecreasing in } \xi > \tilde{\xi}_\lambda. \tag{1.9}$$

When (A4) or (A5) holds, we see that the solution $u(x,t)$ blows up in finite time for large initial data (see [6] and [20] for $\alpha = 1$ and see [14], Theorem 3.1 and Theorem 3.5 for $0 < \alpha < 1$). Namely, for some $T \in (0,\infty)$,

$$\lim_{t \uparrow T} ||u(\cdot,t)||_{L^\infty(\Omega)} = \infty. \tag{1.10}$$

We say this time T is the blow-up time.

Remark 1.1. We note that condition (A5) is equivalent to the following condition, which is assumed in [20] when $\alpha = 1$,

$$(A5)' \qquad \lim_{\xi \to \infty} \frac{g(\xi)}{\xi} = \infty, \tag{1.11}$$

there exists a $\tilde{\xi}_1 > 0$ such that

$$g(\xi) > 0 \text{ and } \int_{\xi}^{\infty} \frac{\alpha \eta^{\alpha-1}}{g(\eta)}\, d\eta < \infty \quad \text{ for } \xi > \tilde{\xi}_1, \tag{1.12}$$

and

$$g(\xi) \text{ is nondecreasing in } \xi \geq 0. \tag{1.13}$$

When $f(\xi) \equiv 0$ and (A5) (or equivalently (A5)') hold, we further see that the blow-up solution of (1.1)-(1.3) blows up in the whole domain Ω (See [2, 6, 20] for $\alpha = 1$ and Theorem 3.8 in §3 for $0 < \alpha < 1$). Namely, if we define

$$\begin{aligned} S = \{x \in \bar{\Omega}\,;\ & \text{there exists a sequence } (x_m, t_m) \in \Omega \times (0, T) \\ & \text{such that } x_m \to x, \quad t_m \uparrow T \text{ and } u(x_m, t_m) \to \infty \text{ as } m \to \infty\}, \end{aligned} \tag{1.14}$$

then

$$S = \bar{\Omega}. \tag{1.15}$$

S is called "the blow-up set" of u and each x of S is called "a blow-up point" of u. This blow-up phenomenon is called "total blow-up".

On the other hand, when $g(\xi) \equiv 0$, we assume condition (A6) which is a stronger condition than (A4):

(A6) There exists a function $\Phi(\xi)$ such that

$$\Phi(\xi) > 0,\ \Phi'(\xi) > 0, \text{ and } \Phi''(\xi) \geq 0; \tag{1.16}$$

$$\int_1^{\infty} \frac{d\xi}{\Phi(\xi)} < \infty; \tag{1.17}$$

there are constants $\varepsilon_0 > 0$, $c > 0$ and $\xi_2 > 0$ such that

$$f'(\xi)\Phi(\xi) - (1+\varepsilon_0) f(\xi)\Phi'(\xi) \geq c\Phi(\xi)\Phi'(\xi) \text{ for } \xi > \xi_2. \tag{1.18}$$

Moreover, we assume $\Omega = B(R) \equiv \{x \in \mathbf{R}^{\mathbf{N}} \,|\, |x| < R\}$ and assume that $u_0(x)$ satisfies the following condition:

(A7) $u_0(x) = u_0(r)$ $(r = |x|)$ is a radially symmetric function in $x \in \mathbf{R}^{\mathbf{N}}$ and a nonincreasing function in $r \geq 0$, and $u_0(R) = 0$.

Then, it is well known that if $u(x,t)$ is nondecreasing in $t \geq 0$ (when $\alpha = 1$, this assumption is not required), the blow-up solution blows up only at the origin (see [23, 11, 12, 8] for $\alpha = 1$ and [16] for $0 < \alpha < 1$). Namely,

$$S = \{0\}. \tag{1.19}$$

We call this blow-up phenomena "single point blow-up".

We note that in [12] (see also [8]) and [16] the above results were obtained under the weaker condition (A6) with $\varepsilon_0 = 0$ (See condition (A6)' in §6). But, in our results we use the stronger condition (A6) with $\varepsilon_0 > 0$.

So, we consider the initial boundary-value problem (1.1)-(1.3) with $\Omega = B(R)$ assuming both conditions (A4) and (A6).

Furthermore, when $\alpha \in (0,1)$, to guarantee that $u(x,t)$ is nondecreasing in $t \geq 0$ (Lemma 2.3), we assume

$$(A8) \qquad \frac{d}{dt}|x_0(t)| \leq 0 \quad \text{ for } t \geq 0 \tag{1.20}$$

and assume

$$(A9) \qquad \Delta u_0 + f(u_0) + g(u_0(0), 0) \geq 0 \quad \text{ in } B(R) \tag{1.21}$$

in the following sense: For any domain $D \subset B(R)$ with smooth boundary ∂D and nonnegative $\varphi(x) \in C^2(\bar{D})$ which vanishes on the boundary ∂D,

$$\int_D \{u_0 \Delta\varphi + f(u_0)\varphi + g(u_0(x_0(0)))\varphi\}\, dx - \int_{\partial D} u_0 \partial_n \varphi \, dS \geq 0, \tag{1.22}$$

where n denotes the outer unit normal to the boundary.

Then, we are interested in whether total blow-up or single point blow-up occurs according to the relations of the growth orders between $f(\xi)$ and $g(\xi)$.

About this problem, when (1.1) is

$$u_t = \Delta u + u^p + u^q(x_0,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.23}$$

namely, when $\alpha = 1$, $f(u) = u^p$, $g(u) = u^q$ (p, $q > 1$) and $x_0(t) \equiv x_0 \in B(R)$ in (1.1), Okada-Fukuda [17] recently obtained the results:

When $x_0 = 0$,

(I) If $p > q + 1$, then under the additional condition on u_0, the blow-up solution of (1.1)-(1.3) blows up only at the origin, namely, $S = \{0\}$;

(II) If $p \leq q+1$ (in this case the assumption (A9) is not needed) then the blow-up solution blows up in the whole domain $B(R)$, namely, $S = \overline{B(R)}$;

When $x_0 \neq 0$,

(I) If $p > q + 1$ then under additional condition on $u_0(x)$, the blow-up solution of (1.1)-(1.3) blows up only at the origin, namely, $S = \{0\}$;

(II) If $p < q$, then there exists a blow-up solution such that it blows up in the whole domain $B(R)$, namely, $S = \overline{B(R)}$.

Thus, in the case $x_0 = 0$, they showed that $p = q+1$ is the cut off "straight line" between the cases where single point blow-up occurs and single point blow-up does not occur. But, in the case $x_0 \neq 0$, they could not see clearly which blow-up phenomena occur or not, especially, when $p-1 \leq q \leq p$.

More recently, Fukuda-Suzuki [13] solved these problems when $x_0 \neq 0$ as follows:
When $x_0 \neq 0$,

(I) If $p > q$ or $p = q\,(> 2)$ then the blow-up solution of (1.1)-(1.3) blows up only at the origin, namely, $S = \{0\}$;

(II) If $p < q$, then the following results hold:

(i) there exist initial data $u_0(x)$ such that solutions of (1.1)-(1.3) blow up in whole domain $B(R)$, namely, $S = \overline{B(R)}$.

(ii) there exist initial data $u_0(x)$ such that solutions of (1.1)-(1.3) blow up only at the origin, namely, $S = \{0\}$.

(iii) there are no other blow-up phenomena except total blow-up and single point blow-up, namely, if the solution blows up in finite time then the blow-up set S is $\{0\}$ or $\overline{B(R)}$.

Thus, we see that in the case $x_0 \neq 0$, the line $p = q$ is the cut off "straight line" between the cases where total blow-up occurs and total blow-up does not occurs.

The main purpose of this paper is to extend these results when $x_0 \neq 0$ to the more general equation (1.1) with $0 < \alpha \leq 1$ and moving $x_0(t)$.

For this purpose, we shall classify the blow-up phenomena by the following two conditions:

(A10) (the case where $g = o(f)$), $g(\xi) = o(f(\xi))$ as $\xi \to \infty$.

(A11) (the case where $f = o(g)$), $g(\xi)$ is a nondecreasing function in large $\xi \geq 0$ and there exists $\varepsilon_1 \in (0,1)$ such that $f(\xi) = o(g(\varepsilon_1 \xi))$ as $\xi \to \infty$.

Here, for two functions $f(t)$ and $g(t)$, we say that $f(t) = o(g(t))$ as $t \to \infty$ if $|f(t)/g(t)| \to 0$ as $t \to \infty$. Condition (A10) is the case where $f(\xi)$ grows up more rapidly than $g(\xi)$ and Condition (A11) is the case where $f(\xi)$ grows up more slowly than $g(\xi)$.

Remark 1.2. Equation

$$(u^{1/m})_t = \Delta u + u^{p/m} + u^{q/m}(x_0(t), t) \quad (m \geq 1), \quad (x,t) \in \Omega \times (0,T) \quad (1.24)$$

satisfies (A1) if $p \geq 1$ and $q \geq 1$, satisfies (A4) (or (A5)) if $p > m$ (or $q > m$), satisfies (A1), (A6) and (A10) if $p > q \geq 1$ and $p > m$, and satisfies (A1), (A5), and (A11) if $q > p \geq 1$ and $q > m$. We note that if $\max\{p, q\} < m$ then the solution of (1.24)(1.2)(1.3) does not blow up in finite time (see Remark 1.3 below).

Remark 1.3. If $f(\xi) = o(\xi)$ and $g(\xi) = o(\xi)$ as $\xi \to \infty$ (namely, $p, q < m$ in case (1.24)) then the solution of (1.1)-(1.3) exists globally in time (see Theorem 2.14).

We shall show that the shapes of the blow-up set of the solution of (1.1)-(1.3) with $\Omega = B(R)$ will be different from each other in the above two cases when $x_0(t) \neq 0$ for $t \geq 0$. We summarize our results as follows:

Theorem 1.4 (the case $0 < \alpha < 1$)**.** *Let* $\Omega = B(R)$ *and* $0 < \alpha < 1$. *Assume* (A1)-(A3), (A5)-(A8) *and* $x_0(t) \neq 0$ *for* $t \geq 0$. *Then, the following results hold:*

(*I*) (*the case where* $g = o(f)$). *Assume* (*A*10). *Then, the blow-up solution of* (1.1)-(1.3) *which is nondecreasing in* $t \geq 0$ *for each* $x \in B(R)$ *blows up only at the origin; that is,* $S = \{0\}$*;*

(*II*) (*the case where* $f = o(g)$). *Assume* (*A*11).

(*i*) *Let* $T_1 > 0$ *and let* $R' \in (0, R)$ *satisfy* $\max_{t\in[0,T_1]} |x_0(t)| < R'$. *Then, there exists* $M > 0$ *such that if* $u_0(0) \geq M$ *and*

$$u_0(R') > \varepsilon_1 u_0(0), \tag{1.25}$$

then the solution of (1.1)-(1.3) *blows up in the whole domain* $B(R)$ *at the blow-up time* $T < T_1$, *namely,* $S = \overline{B(R)}$, *where* ε_1 *is as in condition* (*A*11).

(*ii*) *There exists an initial data* u_0 *satisfying* (*A*9) *such that the maximal solution of* (1.1)-(1.3) *blows up only at the origin; that is,* $S = \{0\}$, *where the maximal solution is greater than any solution which has the same initial data* $u_0(x)$ (*see Definition* 2.1(*V*) *and Proposition* 2.11).

Remark 1.5. Under assumptions (A7)-(A9), we see that the maximal weak solution $u(x, t)$ of (1.1)-(1.3) is nondecreasing in $t \geq 0$ for each $x \in B(R)$ (See Lemma 2.13).

When $\alpha = 1$ (the semilinear case), we can classify the blow-up phenomena more completely under weaker conditions, except for the case where $f \sim g$.

Theorem 1.6. (*the case* $\alpha = 1$) *Let* $\Omega = B(R)$ *and* $\alpha = 1$. *Assume* (A1)-(A3), (A5)-(A7) *and* $x_0(t) \neq 0$ *for* $t \geq 0$. *Then, the following results hold.*

(*I*) (*the case where* $g = o(f)$). *Assume* (*A*10). *Then, the blow-up solution of* (1.1)-(1.3) *blows up only at the origin; that is,* $S = \{0\}$*;*

(*II*) (*the case where* $f = o(g)$). *Assume* (*A*11).

(*i*) *Let* $T_1 > 0$ *and let* $R' \in (0, R)$ *satisfy* $\max_{t\in[0,T_1]} |x_0(t)| < R'$. *Then, there exists* $M > 0$ *such that if* $u_0(0) \geq M$ *then the solution of* (1.1)-(1.3) *blows up in the whole domain* $B(R)$ *at the blow-up time* $T < T_1$, *namely,* $S = \overline{B(R)}$, *where* ε_1 *is as in condition* (*A*11).

(ii) There exists an initial data u_0 such that the solution of (1.1)-(1.3) *blows up only at the origin; that is, $S = \{0\}$.*

(iii) Assume that for each $K > 0$

$$(A12) \qquad \limsup_{\xi\to\infty} \max_{0\le\eta\le K} \frac{(\log\Phi(\xi-\eta))_\xi}{(\log\Phi(\xi))_\xi} \le 1+\varepsilon_0, \tag{1.26}$$

where Φ and ε_0 appear in condition $(A6)$, *then there are no other blow-up phenomena except total blow-up and single point blow-up, namely, if the solution blows up in finite time then the blow-up set S is $\{0\}$ or $\overline{B(R)}$.*

Remark 1.7. In Theorem 1.6, we do not require assumption (A8) and the assumption that the solution $u(x,t)$ is nondecreasing in $t \ge 0$ for each $x \in B(R)$, which is assumed in Theorem 1.4.

In the next corollaries, we state our results for the two exact cases.

Corollary 1.8 (the case where the equation is (1.24))**.** *Let $\Omega = B(R)$. Assume* $(A2)$ *and* $(A7)$, *p, $q > m$ and $x_0(t) \ne 0$ for $t \ge 0$.*

When $m = 1$, the following results hold (see Fukuda-Suzuki [13] *in the case where $x_0(t) \equiv x_0 \in B(R)$ (constant)).*

(I) If $p > q$, then the blow-up solution of (1.24), (1.2), (1.3) *blows up only at the origin;*

(II) If $p < q$, then the following results hold:

(i) a single point blow-up solution of (1.24), (1.2), (1.3) *exists. (ii) a total blow-up solution of* (1.24), (1.2), (1.3) *exists.*

(iii) there are no other blow-up phenomena except total blow-up and single point blow-up.

When $m > 1$, we further assume $(A8)$. *Then, the above results (I), and (i) and (ii) of (II), hold, where in (I) the blow-up solution $u(x,t)$ is assumed to be nondecreasing in $t \ge 0$ for each $x \in B(R)$.*

Namely, in the case where $m = 1$ and $x_0(t) \ne 0$ in $t \ge 0$, we see that $p = q$ is the cut off "straight line" between the cases where single point blow-up occurs and single point blow-up does not occur. Thus, we see that the cut off "straight lines" are different in the cases $x_0(t) \ne 0$ and $x_0(t) \equiv 0$, when $m = 1$.

Corollary 1.9 (the case where $f(\xi) = e^{\beta\xi}$ and $g(\xi) = e^{\gamma\xi}$)**.** *Let $\Omega = B(R)$ and let $f(\xi) = e^{\beta\xi}$ and $g(\xi) = e^{\gamma\xi}$ $(\beta, \gamma > 0)$. Assume* $(A2)$, $(A7)$, *and $x_0(t) \ne 0$ for $t \ge 0$.*

When $\alpha = 1$, the following results hold.

(I) If $\beta > \gamma$, then the blow-up solution of (1.1)-(1.3) *blows up only at the origin;*

(II) If $\beta < \gamma$ then the following results hold. (i) a single point blow-up solution of (1.1)-(1.3) *exists. (ii) a total blow-up solution of* (1.1)-(1.3) *exists. (iii) there are no other blow-up phenomena except total blow-up and single point blow-up.*

When $0 < \alpha < 1$, we further assume (A8). Then, the above results (I), and (i) and (ii) of (II), hold, where in (I) the blow-up solution $u(x,t)$ is assumed to be nondecreasing in $t \geq 0$ for each $x \in B(R)$.

Here, we note that when $\alpha = 1$ (semilinear case), the result (II)(iii) in Corollary 1.8 and 1.9 was already proved in [2].

Proof. It is not difficult to see that $f(\xi) = e^{\beta\xi}$ or $g(\xi) = e^{\gamma\xi}$ $(\beta, \gamma > 0)$ satisfies (A6)(A12) or (A5), respectively (see e.g. Friedman-Mcleod [11]). We also see that if $\beta > \gamma$ then $f(\xi)$ and $g(\xi)$ satisfy (A10), and if $\gamma > \beta$, then $f(\xi)$ and $g(\xi)$ satisfy (A11). Thus, we obtain the corollary by Theorem 1.4 and 1.6. □

Remark 1.10. Let $\Omega = B(R)$ and assume (A2) and (A7). For the case where $f(\xi) = e^{\beta\xi}$ and $g(\xi) = e^{\gamma\xi}$ $(\beta, \gamma > 0)$ in (1.1) with $\alpha = 1$, it was open in [17] whether total blow-up occurs or single point blow-up occurs and $x_0(t) \equiv x_0 \neq 0$. As in Corollary 1.9 above, we solve this problem when $\beta \neq \gamma$. Namely, the line $\beta = \gamma$ is the cut off line.

For the case where $f(\xi) = e^{\xi}$ and $g(\xi) = \xi^q$ in (1.1) with $\alpha = 1$, the problem has not been solved (see [17]). Applying Theorem 1.6 we also solve this case. Namely, when $f(\xi) = e^{\xi}$ and $g(\xi) = \xi^q$, the blow-up solution of (1.1)-(1.3) blows up only at the origin.

Remark 1.11. We can not get any result in the case where $f(\xi) \sim g(\xi)$ as $\xi \to \infty$.

We note that even if $\alpha = 1$ (semilinear case), the results about the existence of single point blow-up solutions in the case where $f = o(g)$ (which are also obtained in Fukuda-Suzuki [13] for the special case (1.24) with $m = 1$) are new. It seems to be interesting that the problem has both a single point blow-up solution and a total blow-up solution by only choosing initial data u_0 as in (II), since we have never known that other problems have such phenomena (see below).

Our methods are similar to those of Fukuda-Suzuki [13]. Namely, the methods which are based on the maximum principle are essentially due to Chen [8] (see also Fujita-Chen [12]) for the proof of (I) of Theorem 1.4 (or Theorem 1.6), due to Okada-Fukuda [17] for the proof of (i) of (II) in Theorem 1.4 (or Theorem 1.6) and due to Fukuda-Suzuki [13] for the proof of (ii) of (II) in Theorem 1.4 (or Theorem 1.6). But, we can not apply their methods directly, since our equation is quasilinear and degenerate at $u = 0$.

For semilinear parabolic equations, similar problems having both a local source term and a nonlocal source term appear in the gaseous ignition models whose nonlocal source terms are space integral sources, and are treated by Bebernes-Bressan-Lacey [2], Bricher-Akdoğu [4] and Bricher [5]. They consider the following forms of equations

$$u_t = \Delta u + f(u) + g(t), \tag{1.27}$$

in $B(R)$, where $f(u) = u^p$ $(p > 1)$ or e^u and

$$g(t) = g_1(t) \equiv \frac{K}{\mathrm{Vol}.B(R)} \int_{B(R)} u_t(y,t)\, dy \quad \text{or}$$

$$g_2(t) \equiv \frac{K}{\mathrm{Vol}.B(R)} \int_{B(R)} e^{u(y,t)}\, dy$$

$(0 < K < 1)$, and they obtain the following results for the radially symmetric solutions $u(x,t) = u(r,t)$ $(r = |x|)$ which blow up in finite time and are nondecreasing functions in $t \geq 0$ and nonincreasing functions in $r \geq 0$:

When $f(u) = e^u$ and $g(t) = g_1(t)$, the solutions blow up only at the origin ([2, 4]), and when $f(u) = u^p$ and $g(t) = g_1(t)$, the solutions blow up in the whole domain $B(R)$ if $1 < p < 1 + 2/N$, and only at the origin if $p > 1 + 2/N$ and K is sufficiently small ([2]). When $f(u) = e^u$ and $g(t) = g_2(t)$, the solutions blow up in the whole domain $B(R)$ if $N \leq 2$ and only at the origin if $N \geq 3$ and K is sufficiently small ([5]). Bricher-Akdoğu and Bricher also obtain the precise L^∞-estimates for the blow-up solutions near the blow-up time $T < \infty$.

Thus, in the gaseous ignition models, each problem only has one type of blow-up (single-point or total) for all radially symmetric solutions satisfying $u_t \geq 0$ and $u_r \leq 0$.

Now, we mention the existence, uniqueness and (total) blow-up of local solutions of (1.1)-(1.3) in a general domain Ω, in more detail.

The existence of local solutions of (1.1)-(1.3) was shown by Cannon-Yin [7] when $\alpha = 1$ and Souplet [20] extended their results to more general forms which contain various nonlocal problems. But, in the quasilinear case $0 < \alpha < 1$ there are few papers studying these problems and we show the existence of local solutions of (1.1)-(1.3) in the quasilinear case (Theorem 2.2).

The uniqueness of local solutions was studied by Chadam-Peirce-Yin [6] and Souplet [20] (for more general equations) when $\alpha = 1$. When $0 < \alpha < 1$

and $g(\xi) \equiv 0$, the uniqueness of local solutions was studied by Aronson-Crandall-Peletier [1]. When $0 < \alpha < 1$ and $g(\xi) \not\equiv 0$, however, there are few papers studying the uniqueness of solutions and we can not show it. So, our discussions in our paper are carried out without using the uniqueness of local solutions of (1.1) when $0 < \alpha < 1$, and for this reason we must consider several kinds of weak solutions (see §2).

The existence of blow-up solutions of (1.1)-(1.3) was shown also by [6] and [20] in the case $\alpha = 1$. When $x_0(t) = x_0 \in \Omega$ in $t \geq 0$, under assumption (A5) with $f(\xi)$ convex, it was shown in [6] that if initial data $u_0(x)$ is large enough in the neighborhood of fixed x_0, then the solution of (1.1)-(1.3) blows up in finite time. Their results were extended in [20] to moving $x_0(t)$. We note that [20] does not require the assumptions that initial data $u_0(x)$ is large enough in the neighborhood of x_0 and $f(\xi)$ is convex. Our blow-up results are the extensions of [20] to $0 < \alpha < 1$, however, to prove our results, we can not use the method of [20] which is based on constructing suitable blow-up subsolutions. Our method is essentially the same as that of [6] which is a comparison with the blow-up solutions of the corresponding radially symmetric problems.

Further, when $\alpha = 1$ and $f(u) \equiv 0$, it was also shown in [2, 6] (see also [21]) that the blow-up solution of (1.1)-(1.3) blows up in the whole domain Ω. We extend their results to the quasilinear case $0 < \alpha < 1$.

Namely, the second purpose of this paper is to give theorems for the existence of local solutions of (1.1)-(1.3) and give several blow-up and total blow-up conditions.

The rest of the paper is organized as follows. In the next section §2, we define several kinds of weak solutions of (1.1)-(1.3) and show the existence and some comparison theorems for those solutions of (1.1)-(1.3). Moreover, we show several preliminary propositions and lemmas. We also show in §2 the global existence of solutions when $g(\xi) = o(\xi)$ and $f(\xi) = o(\xi)$ as $\xi \to \infty$. In §3 we give several blow-up and total blow-up conditions. Some of these results are used in the proof of Theorem 1.4 and Theorem 1.6. (i) of (II) in Theorem 1.4 (or Theorem 1.6) is shown in §4, (I) in Theorem 1.4 (or Theorem 1.6) is shown in §5, and (ii) of (II) in Theorem 1.4 (or Theorem 1.6) is shown in §6. Finally, (iii) of (II) in Theorem 1.4 is shown in §7.

2. Definitions and preliminary

In this section, we define several kinds of weak solutions of (1.1)-(1.3) and show the existence of them. Moreover, we state some comparison theorems and some preliminary propositions and lemmas. We note that we do not know whether or not the uniqueness of the usual weak solutions of (1.1)-(1.3) holds and so we must consider several kinds of weak solutions for the proof of Theorem 1.4 and 1.6. But we can obtain some comparison theorems for the usual solutions and the uniqueness theorem for maximal weak solutions.

We shall use the following notation frequently: For $x_0 \in \mathbf{R^N}$ and $d_0 > 0$,

$$B(x_0; d_0) \equiv \{x \in \mathbf{R^N}\,;\, |x - x_0| < d_0\}. \tag{2.1}$$

We begin with definitions of several weak solutions. Let Ω be a bounded domain with smooth boundary $\partial\Omega$ and $x_0(t) \in C^\infty([0,\infty)\,;\,\Omega)$.

Definition 2.1. (I) A function u in $\Omega \times (0,T)$ is called a *weak solution of* (1.1) *in* $\Omega \times (0,T)$, if it satisfies

i) $u(x,t) \geq 0$ in $\bar\Omega \times [0,T)$ and $\in C(\bar\Omega \times [0,T))$,

ii) For any domain $D \subset \Omega$ with smooth boundary ∂D, $0 < \tau < T$ and non-negative function $\varphi(x,t) \in C^2(\bar D \times [0,T))$ which vanishes on the boundary ∂D,

$$\int_D u^\alpha(x,\tau)\varphi(x,\tau)\,dx - \int_D u^\alpha(x,0)\varphi(x,0)\,dx \tag{2.2}$$
$$= \int_0^\tau \int_D \{u^\alpha \varphi_t + u\Delta\varphi + f(u)\varphi + g(u(x_0(t),t))\varphi\}dxdt - \int_0^\tau \int_{\partial D} u\partial_n\varphi\,dSdt,$$

where n denotes the outer unit normal to the boundary.

A supersolution [or subsolution] is similarly defined with equality of (2.2) replaced by $\geq$ [or $\leq$].

(II) A function u in $\Omega \times (0,T)$ is called a *weak solution of* (1.1)-(1.3) *in* $\Omega \times (0,T)$, if u is a weak solution of (1.1) in $\Omega \times (0,T)$ and satisfies (1.2) and (1.3).

(III) A function u in $\Omega \times (0,T]$ with $T < \infty$ is called a *weak solution of* (1.1)-(1.3) *in* $\Omega \times (0,T]$, if there exists a weak solution $v(x,t)$ of (1.1)-(1.3) in $\Omega \times (0,T')$ with $T < T'$ such that $u(x,t) = v(x,t)$ in $\Omega \times (0,T]$.

(IV) A function u in $\Omega \times (0,T)$ is called a *weak solution of* (1.1)-(1.3) *in* $\Omega \times (0,T)$ *with the maximum existence time* T if u is a weak solution of (1.1)-(1.3) in $\Omega \times (0,T)$ and u is not a weak solution of (1.1) in $\Omega \times (0,T]$.

(V) A function u in $\Omega \times (0,T)$ is called the *maximal weak solution of* (1.1) *in* $\Omega \times (0,T)$ if it is a weak solution of (1.1)-(1.3) in $\Omega \times (0,T)$ and for any weak solution of (1.1)-(1.3) in $\Omega \times (0,T')$, $v(x,t) \leq u(x,t)$ in $\Omega \times (0,\min\{T,T'\})$.

We first show the existence of weak solutions of (1.1)-(1.3) with the maximum existence time $T > 0$.

Theorem 2.2. *Assume* (A1) *and* (A2). *Then, there exists a local* (*in time*) *weak solution* $u(x,t)$ *of* (1.1)-(1.3) *with the maximum existence time* $T > 0$.

We need the following

Proposition 2.3. *Assume (A1) and (A2). There exists a positive nonincreasing function $T(\xi) > 0$ ($\xi > 0$) such that if*

$$(0 \leq) \max_{x \in \bar{\Omega}} u_0(x) \leq m_0 \tag{2.3}$$

for some $m_0 \geq 0$, then a weak solution of (1.1)-(1.3) *exists in $\Omega \times (0, T(m_0)]$ and*

$$u(x,t) \leq m_0 + 1 \quad \text{in } \Omega \times (0, T(m_0)]. \tag{2.4}$$

Proof. Let $m_0 > 0$ and assume (2.3). Set

$$\tilde{f}(\xi) = \begin{cases} f(\xi) & \text{if } 0 \leq \xi \leq m_0 + 1, \\ f(m_0 + 1) & \text{if } \xi > m_0 + 1, \end{cases}$$

$$\tilde{g}(\xi) = \begin{cases} g(\xi) & \text{if } 0 \leq \xi \leq m_0 + 1, \\ g(m_0 + 1) & \text{if } \xi > m_0 + 1. \end{cases}$$

We consider the following problem:

$$(u^\alpha)_t = \Delta u + \tilde{f}(u) + \tilde{g}(u(x_0(t), t)), \quad (x,t) \in \Omega \times (0, T), \tag{2.5}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0, T), \tag{2.6}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.7}$$

Put

$$M(m_0) = \sup_{\xi \geq 0} \{\tilde{f}(\xi) + \tilde{g}(\xi)\}. \tag{2.8}$$

We need the following lemma.

Lemma 2.4. *There exists a weak solution $u(x,t)$ of* (2.5)-(2.7) *in $\Omega \times (0, \infty)$ satisfying*

$$u(x,t) \leq h(t) \equiv (M(m_0)t + m_0^\alpha)^{1/\alpha} \quad \text{in } \Omega \times (0, \infty). \tag{2.9}$$

Proof. We define continuous functions $u_n(x,t)$ in $\bar{\Omega} \times [0, \infty)$ ($n \geq 1$) by induction as follows. Let $u_1(x,t) = u_0(x)$ and let u_{n+1} in $\bar{\Omega} \times [0, \infty)$ be the solution of the initial boundary-value problem

$$(u_{n+1}^\alpha)_t - \Delta u_{n+1} = \tilde{f}(u_n) + \tilde{g}(u_n(x_0(t), t)), \quad (x,t) \in \Omega \times (0, \infty), \tag{2.10}$$

$$u_{n+1}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0, \infty), \tag{2.11}$$

$$u_{n+1}(x,0) = u_0(x), \quad x \in \Omega. \tag{2.12}$$

Then, by the usual existence and uniqueness theorems for the quasilinear degenerate parabolic equations (See [1, 3, 9, 10, 18]), it is not difficult to see that there exists a unique continuous nonnegative solution $u_n(x,t)$ of (2.10)-(2.12) and the solution satisfies

$$u_n(x,t) \le h(t) \equiv (M(m_0)t + m_0^\alpha)^{1/\alpha} \quad \text{in } \Omega \times (0,\infty). \tag{2.13}$$

Hence, the sequence $\{u_n(x,t)\}$ is uniformly bounded in $\bar{\Omega} \times [0,T]$ $(0 < T < \infty)$, and then it is equicontinuous in $\bar{\Omega}\times[0,T]$ (see [9, 10]). Thus, it is possible to find a subsequence (still denoted u_n) and a function $u(x,t) \in C(\bar{\Omega}\times[0,\infty))$ such that

$$u_n \to u \quad (\ge 0) \quad \text{as } n \to \infty \tag{2.14}$$

locally uniformly in $\bar{\Omega} \times [0,\infty)$. By (2.13), we see that u satisfies (2.9). By passing to the limit in the integral identity satisfied by u_n, such as (2.2), we also see that $u(x,t)$ is a weak solution of (2.5)-(2.7) in $\Omega \times (0,\infty)$. □

Proof of Proposition 2.3 (continued). Let $u(x,t)$ be a weak solution of (2.5)-(2.7) as in Lemma 2.4. Put

$$T(\xi) = \inf_{0<\eta\le\xi} \frac{1}{2} \cdot \frac{(\eta+1)^\alpha - \eta^\alpha}{M(\eta)} \Big(= \frac{1}{2} \cdot \frac{(\xi+1)^\alpha - \xi^\alpha}{M(\xi)} > 0 \Big). \tag{2.15}$$

Then, $T(\xi)$ is a nonincreasing function in $\xi \ge 1$. Further, by (2.9) we have $u(x,t) < m_0 + 1$ in $\Omega \times (0, T(m_0)]$. Hence $\tilde{f}(u) = f(u)$ and $\tilde{g}(u(x_0(t),t)) = g(u(x_0(t),t))$ in $\Omega \times (0, T(m_0)]$. The proof is complete. □

Proof of Theorem 2.2. We construct a sequence $\{t_n\} \subset (0,\infty)$ and a weak solution $u_n(x,t)$ of (1.1) in $\Omega \times (0, t_n]$ for each $n \ge 1$ by induction, in the following way. Let

$$m_1 = \max_{x\in\bar{\Omega}} u_0(x) \tag{2.16}$$

and

$$t_1 = T(m_1) \tag{2.17}$$

where $T(m_1)$ is as in Proposition 2.3 with m_0 replaced by m_1. Then, by Proposition 2.3 there exists a weak solution $u_1(x,t)$ of (1.1)-(1.3) in $\Omega\times(0,t_1]$ with initial data u_0. Next, let

$$m_2 = \max_{x\in\bar{\Omega}} u_1(x,t_1) \tag{2.18}$$

and

$$t_2 = T(m_2) \tag{2.19}$$

where $T(m_2)$ is as in Proposition 2.3 with m_0 replaced by m_2. Then, there exists a weak solution $u_2(x,t)$ of (1.1)-(1.3) in $\Omega \times (0, t_2]$ with initial data $u_1(x,t_1)$.

. . . .

Let

$$m_n = \max_{x \in \bar{\Omega}} u_{n-1}(x, t_{n-1}) \tag{2.20}$$

and

$$t_n = T(m_n) \tag{2.21}$$

where $T(m_n)$ is as in Proposition 2.3 with m_0 replaced by m_n. Then, there exists a weak solution $u_n(x,t)$ of (1.1)-(1.3) in $\Omega \times (0, t_n]$ with initial data $u_{n-1}(x, t_{n-1})$.

....

Thus, we define

$$T_0 = 0, \quad T_n = \Sigma_{k=1}^{n} t_k \quad \text{for } n \geq 1 \quad \text{and} \quad T = \lim_{n\to\infty} T_n, \tag{2.22}$$

and for each $n \geq 1$,

$$u(x,t) = u_n(x, t - T_{n-1}) \quad \text{if } T_{n-1} \leq t < T_n. \tag{2.23}$$

Then, it is not difficult to see that $u(x,t)$ is a weak solution of (1.1)-(1.3) in $\Omega \times (0, T)$.

If $T = \infty$, T is the maximum existence time. So, let $T < \infty$. We shall show

$$\lim_{n\to\infty} ||u(x, T_n)||_{L^\infty(\Omega)} = \lim_{n\to\infty} ||u_n(x, t_n)||_{L^\infty(\Omega)} = \infty. \tag{2.24}$$

Assume to the contrary that there exist $\ell \in (0, \infty)$ and a subsequence $\{t_{n_j}\} \subset \{t_n\}$ such that

$$\lim_{j\to\infty} ||u_{n_j}(x, t_{n_j})||_{L^\infty(\Omega)} < \ell. \tag{2.25}$$

Then, since $t_{n_j} = T(m_{n_j}) \geq T(\ell) > 0$, we have

$$T \geq \lim_{j\to\infty} jT(\ell) = \infty. \tag{2.26}$$

This is a contradiction to $T < \infty$ and so we get (2.24). Hence, we see that T is the maximum existence time of u. The proof is complete. □

Next, we show the comparison theorem for solutions of (1.1)-(1.3) (see Proposition 2.7), which is often used later. By this comparison theorem, we can show some properties of solutions with the maximum existence time and show the existence of maximal solutions of (1.1)-(1.3). As was mentioned before, we note that we do not know whether or not the uniqueness of solutions of (1.1)-(1.3) holds when $0 < \alpha < 1$ (In the case of $\alpha = 1$, the uniqueness of solutions of (1.1)-(1.3) holds [20]).

To show the comparison theorem we consider the initial boundary-value problem

$$(u^\alpha)_t = \Delta u + f(u) + g(t), \quad (x,t) \in \Omega \times (0, T), \tag{2.27}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.28}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2.29}$$

where $g(t)(\geq 0) \in C[0,T)$ is a continuous given function.

Lemma 2.5. *Assume* (*A*1) *and* (*A*2), *and let* $u(x,t)$ *be a weak solution of* (2.27)-(2.29). *If* $g(0) > 0$ *then* $u(x,t) > 0$ *in* $\Omega \times [0,T)$.

Proof. We note that there exist $h_0 > 0$ and $t_0 > 0$ such that $g(t) > h_0$ for $0 \leq t < t_0$, since $g(0) > 0$. Let $x_0 \in \Omega$ and let $d_0 > 0$ satisfy

$$B(x_0; d_0) = \{x \in \mathbf{R}^\mathbf{N}\,;\, |x - x_0| < d_0\} \subset \Omega$$

and let v be a weak solution of the problem

$$(v^\alpha)_t = \Delta v + g(t), \quad (x,t) \in B(x_0; d_0) \times (0,T), \tag{2.30}$$

$$v(x,t) = 0, \quad (x,t) \in \partial B(x_0; d_0) \times (0,T), \tag{2.31}$$

$$v(x,0) = 0, \quad x \in B(x_0; d_0). \tag{2.32}$$

Put $w(x,t) = v(x+x_0,t)$. Then, since initial data $w(x,0) = w(r,0)$ $(r = |x|)$ $\equiv 0$ is radially symmetric in x and nonincreasing in $r \geq 0$, by the usual comparison theorem for (2.30) (see [1, 3]) $w(x,t) = w(r,t)$ $(r = |x|)$ is radially symmetric in x and nonincreasing in $r \geq 0$ for each $t \geq 0$, and so $\sup_{x \in B(x_0;d_0)} v(x,t) = v(x_0,t)$. Put

$$t_1 = \sup\{s \in [0, t_0)\,;\, v(x_0,t) = 0 \text{ for } 0 \leq t \leq s\}. \tag{2.33}$$

We shall show that $t_1 = 0$.

Let $\varphi(x) \in C_0^\infty(B(x_0; d_0))$ satisfy $\varphi(x) \geq 0$ and $\int_{B(x_0;d_0)} \varphi(x)\, dx = 1$.

Assume to the contrary that $t_1 > 0$. Then, choosing $\varphi(x)$ as a test function in the integral identity satisfied by $v(x,t)$ (see (2.2)), we get

$$0 = \int_{B(x_0;d_0)} v^\alpha(x,\tau)\varphi(x)\, dx \geq \int_0^\tau \int_{B(x_0;d_0)} g(t)\varphi(x)\, dxdt \geq h_0\tau \tag{2.34}$$

for $0 \leq \tau \leq t_1$, since $v(x,t) = 0$ in $B(x_0; d_0) \times [0, t_1]$. This is a contradiction and so we get $t_1 = 0$.

Therefore, there exists a sequence $\{t_n\} \subset (0, t_0)$ $(n \geq 2)$ such that $t_n \downarrow 0$ as $n \to \infty$ and $v(x_0, t_n) > 0$. Noting the positivity of solutions of (2.30) (see e.g. Lemma 2.1 of [16]), we have $v(x_0,t) > 0$ for $t \in (t_n, T)$ $(n \geq 2)$, namely, $v(x_0,t) > 0$ for $t \in (0,T)$.

Thus, the usual comparison theorem implies that $u(x_0,t) \geq v(x_0,t) > 0$ for $t \in (0,T)$. Since x_0 can be chosen arbitrarily, we see that $u(x,t) > 0$ in $\Omega \times (0,T)$. The proof is complete. □

This lemma leads to the positivity of solutions of (1.1)-(1.3) under the condition $u_0(x_0(0)) > 0$ as follows.

Proposition 2.6. *Assume (A1), (A2) and* $u_0(x_0(0)) > 0$*. Let* $u(x,t)$ *be a weak solution of* (1.1)-(1.3)*. Then,* $u(x,t) > 0$ *in* $\Omega \times (0,T)$*, hence* $u \in C^\infty(\Omega \times (0,T))$*.*

Proof. If we put $g(t) = g(u(x_0(t),t))$, we see that g(t) is continuous and $g(0) = g(u_0(x_0(0))) > 0$. It follows from Lemma 2.5 that $u(x,t) > 0$ in $\Omega \times (0,T)$. Using the usual parabolic regularization method (see e.g. Ladyzenskaja et al. [15]), we get $u \in C^\infty(\Omega \times (0,T))$. □

Proposition 2.7. *Assume (A1)–(A3). Let u be a weak subsolution of* (1.1) *in* $\Omega \times (0,T)$ *and v be a weak supersolution of* (1.1) *in* $\Omega \times (0,T)$*. Then, if* $u(x,t) \le v(x,t)$ *on the parabolic boundary of* $\Omega \times (0,T)$ *and* $u(x_0(0),0) < v(x_0(0),0)$, $u(x,t) < v(x,t)$ *in the whole* $\Omega \times (0,T)$*. Further, if* $u(x,t)$, $v(x,t) > 0$ *in* $\Omega \times (0,T]$, u, $v \in C^\infty(\Omega \times (0,T]$ *and*

$$(u^\alpha)_t - \Delta u \le (v^\alpha)_t - \Delta v, \quad (x,t) \in \Omega \times (0,T], \tag{2.35}$$

then $u(x,T) < v(x,T)$ *in* Ω*.*

We need the following lemma.

Lemma 2.8. *Assume (A1)–(A3). Let u be a weak subsolution of* (1.1) *in* $\Omega \times (0,T)$ *and v be a weak solution of* (1.1) *in* $\Omega \times (0,T)$*. Then, if* $u(x,t) \le v(x,t)$ *on the parabolic boundary of* $\Omega \times (0,T)$ *and* $u(x_0(0),0) < v(x_0(0),0)$, $u(x,t) < v(x,t)$ *in the whole* $\Omega \times (0,T)$*.*

Proof. Assume that $u(x,t) \le v(x,t)$ on the parabolic boundary of $\Omega \times (0,T)$ and $u(x_0(0),0) < v(x_0(0),0)$. Since $g(v(x_0(0),0)) > 0$, we note by Proposition 2.6 that $v(x,t) > 0$ in $\Omega \times (0,T)$ and $v \in C^\infty(\Omega \times (0,T))$.

Let $w(x,t)$ be a weak solution of the problem

$$(w^\alpha)_t = \Delta w + f(w) + g(v(x_0(t),t)) \quad \text{in } \Omega \times (0,T), \tag{2.36}$$

$$w(x,t) = u(x,t) \qquad \text{on the parabolic boundary of } \Omega \times (0,T). \tag{2.37}$$

Then, it is not difficult to see that $w > 0$ in $\Omega\times(0,T)$ and $w \in C^\infty(\Omega\times(0,T))$. Moreover, by the usual comparison theorem (see [1]), $w(x,t) \le v(x,t)$ in $\Omega \times (0,T)$.

We first show that $w(x,t) < v(x,t)$ in $(0,T)$. Assume to the contrary that $w(x_1,t_1) = v(x_1,t_1)$ for some $(x_1,t_1) \in \Omega \times (0,T)$. Put $z = w - v$. Then

$$\alpha w^{\alpha-1} z_t + \alpha\{\tilde{w}^{\alpha-1}\} v_t z - \Delta z - \tilde{f} z \le 0 \quad \text{in } \Omega \times (0,T), \tag{2.38}$$

where

$$\tilde{f} = \int_0^1 f'(\theta w + (1-\theta)v)\, d\theta$$

and $\{\tilde{w}^{\alpha-1}\}$ is similarly defined.
Putting $\tilde{z}(x,t) = w(x,t)e^{-\gamma t}$ ($\gamma > 0$) we have

$$\tilde{z}_t - \frac{1}{\alpha w^{\alpha-1}}\Delta\tilde{z} \le (C_1(x,t) - \gamma)\tilde{z} \text{ in } \Omega \times (0,T), \tag{2.39}$$

where $C_1(x,t) = \{\tilde{f} - \alpha\{\tilde{w}^{\alpha-1}\}v_t\}/(\alpha w^{\alpha-1})$.

Let $d > 0$ satisfy $\overline{B(x_1;d) \times (t_1 - d, t_1 + d)} \subset \Omega \times (0,T)$ and choose $\gamma > 0$ to satisfy $\gamma > C_1(x,t)$ in $B(x_1;d) \times (t_1 - d, t_1 + d)$. Note that $\tilde{z} \le 0$ in $\Omega \times (0,T)$ and $\tilde{z}(x_1,t_1) = 0$. Then, the strong maximum principle (Theorem 5 of [19], page 173) leads to $\tilde{z} = 0$ in $B(x_1;d) \times (t_1 - d, t_1]$. Thus, in a similar way, we get $\tilde{z} = 0$ in $\Omega \times [0,t_1]$, that is, $w(x,t) = v(x,t)$ in $\Omega \times [0,t_1]$. This is a contradiction to $w(x_0(0),0) = u(x_0(0),0) < v(x_0(0),0)$, and hence we obtain $w(x,t) < v(x,t)$ in $\Omega \times (0,T)$.

Next, we show that $u(x_0(t),t) < v(x_0(t),t)$ in $(0,T)$. Put

$$t_0 = \sup\{T' \in [0,T)\,;\, u(x_0(t),t) < v(x_0(t),t) \text{ in } [0,T']\}. \tag{2.40}$$

We note that, by the assumption, $t_0 > 0$. Assume to the contrary that $t_0 < T$. Then $u(x_0(t),t) < v(x_0(t),t)$ in $[0,t_0)$ and $u(x_0(t_0),t_0) = v(x_0(t_0),t_0)$. By assumption (A3), $g(u(x_0(t),t)) \le g(v(x_0(t),t))$ in $[0,t_0]$. Hence, by the usual comparison theorem (see [1]), we have $u(x,t) \le w(x,t)$ in $\Omega \times [0,t_0]$ and so $u(x,t) \le w(x,t) < v(x,t)$ in $\Omega \times [0,t_0]$. This is a contradiction to $u(x_0(t_0),t_0) = v(x_0(t_0),t_0)$, and so we get $u(x_0(t),t) < v(x_0(t),t)$ in $(0,T)$.

Thus, similarly to above, we have $u(x,t) \le w(x,t) < v(x,t)$ in $\Omega \times (0,T)$. The proof is complete. □

Proof of Proposition 2.7. Assume that $u(x,t) \le v(x,t)$ on the parabolic boundary of $\Omega \times (0,T)$ and $u(x_0(0),0) < v(x_0(0),0)$. Let $h(x,t)$ be a continuous nonnegative function on the parabolic boundary of $\Omega \times (0,\infty)$ satisfying $u(x,t) \le h(x,t) \le v(x,t)$ on the parabolic boundary of $\Omega \times (0,T)$ and $u(x_0(0),0) < h(x_0(0),0) < v(x_0(0),0)$. Let $w(x,t)$ be a weak solution of (1.1) in $\Omega \times (0,T')$ with the maximum existence time $T' > 0$ satisfying $w(x,t) = h(x,t)$ on the parabolic boundary of $\Omega \times (0,\infty)$. Then, it is not difficult to see that $w(x,t) > 0$ in $\Omega \times (0,T')$ and $w \in C^\infty(\Omega \times (0,T'))$. Further, by the method as in the proof of Lemma 2.8, we see that $T' \ge T$ and $w(x,t) < v(x,t)$ in $\Omega \times (0,T)$. It also follows from Lemma 2.8 that $u(x,t) < w(x,t)$ in $\Omega \times (0,T)$ and so $u(x,t) < v(x,t)$ in $\Omega \times (0,T)$.

Next, we further assume that $u(x,t), v(x,t) > 0$ in $\Omega \times (0,T]$, $u, v \in C^\infty(\Omega \times (0,T])$, and (2.35). Then, similarly to the proof of Lemma 2.8, we can show $u(x,T) < v(x,T)$ in Ω. The proof is complete. □

Next, we state the property of a solution (1.1)-(1.3) with the maximum existence time and the existence of the maximal solution of (1.1)-(1.3). We use the following lemma.

Lemma 2.9. *Assume* $(A1)$, $(A2)$. *Let* u *be a weak solution of* (1.1)-(1.3) *in* $\Omega \times (0,T)$ *with* $T < \infty$. *If*

$$\sup_{\Omega\times(0,T)} u(x,t) < \infty, \tag{2.41}$$

then u *is a weak solution of* (1.1)-(1.3) *in* $\Omega \times (0,T]$, *namely, there exists a weak solution* $v(x,t)$ *of* (1.1)-(1.3) *in* $\Omega\times(0,T')$ *with the maximum existence time* $T' > T$ *such that* $u(x,t) = v(x,t)$ *in* $\Omega \times (0,T)$.

Proof. (i) When $\sup_{\Omega\times(0,T)} u(x,t) < \infty$, we see by [10] that u is uniformly continuous in $\Omega\times(0,T)$ and hence can be extended continuously to $\Omega\times(0,T]$. By Theorem 2.2, there exists a weak solution w of the following problem with the maximum existence time $\tilde{T} > 0$.

$$(w^{\alpha})_t = \Delta w + f(w) + g(w(x_0(t+T),t)), \quad (x,t) \in \Omega \times (0,\tilde{T}), \tag{2.42}$$

$$w(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\tilde{T}), \tag{2.43}$$

$$w(x,0) = u(x,T), \quad x \in \Omega. \tag{2.44}$$

Put

$$v(x,t) = \begin{cases} u(x,t) & \text{in } \bar{\Omega} \times [0,T), \\ w(x,t-T) & \text{in } \bar{\Omega} \times [T, T+\tilde{T}). \end{cases} \tag{2.45}$$

Then, it is not difficult to see that v is a weak solution of (1.1)-(1.3) in $\Omega \times (0, T+\tilde{T})$ with the maximum existence time $T+\tilde{T}$. The proof is complete. □

Proposition 2.10. *Assume* $(A1)$–$(A3)$.

(i) *Let* u *be a weak solution of* (1.1)-(1.3). *Then, there exists a weak solution* $v(x,t)$ *in* $\Omega \times (0,T')$ *with the maximum existence time* T' $(\geq T)$ *such that* $u(x,t) = v(x,t)$ *in* $\Omega \times (0,T)$.

(ii) *Let* u *be a weak solution of* (1.1)-(1.3) *with the maximum existence time* $T \in (0,\infty)$. *Then,*

$$\lim_{t\uparrow T} ||u(\cdot,t)||_{L^{\infty}(\Omega)} = \infty. \tag{2.46}$$

Proof. (i) When $T = \infty$, clearly T is the maximum existence time of u. When $T < \infty$ and $\sup_{\Omega\times(0,T)} u(x,t) = \infty$, T is the maximum existence time of u. When $T < \infty$ and $\sup_{\Omega\times(0,T)} u(x,t) < \infty$, we see from Lemma 2.9 that there exists a weak solution $v(x,t)$ of (1.1)-(1.3) in $\Omega\times(0,T')$ with the maximum existence time $T' > T$ such that $u(x,t) = v(x,t)$ in $\Omega \times (0,T)$.

(ii) Assume to the contrary that there exist a sequence $\{t_j\} \subset (0,T)$ and $M > 0$ such that $t_j \uparrow T$ as $j \to \infty$ and $\sup_j ||u(\cdot,t_j)||_{L^\infty(\Omega)} < M$. We consider the following problem with an ordinary differential equation:

$$(\xi^\alpha)_t = f(\xi) + g(\xi) \quad \text{in } t > 0, \tag{2.47}$$

$$\xi(0) = M. \tag{2.48}$$

Then, there exists a solution $\xi(t)$ of (2.47)(2.48) in $(0,t_0)$ for some $t_0 > 0$. We note that $\xi(t)$ and $u(x,t+t_j)$ are weak solutions of the equation

$$(u^\alpha)_t = \Delta u + f(u) + g(u(x_0(t+t_j),t)), \quad (x,t) \in \Omega \times (0,t_0). \tag{2.49}$$

Hence, it follows from Proposition 2.7 that $u(x,t+t_j) < \xi(t) \le \xi(t_0/2) \equiv M'$ in $\Omega \times (0,t_0/2)$ for $j \ge 1$. Choose $j_0 \ge 1$ to satisfy $0 < T - t_{j_0} \le t_0/2$. Then, we have $u(x,t) \le M'$ in $\Omega \times (t_{j_0},T)$. Thus, similarly to the proof of (i), we see that T is not the maximum existence time of u. This is a contradiction of the assumption and so we get (2.46). The proof is complete. □

Proposition 2.11. *Assume (A1)–(A3). Then, there exists a unique maximal weak solution u of (1.1)-(1.3) in $\Omega \times (0,T)$. If v is a subsolution of (1.1) in $\Omega \times (0,T)$ satisfying $v(x,t) = 0$ on $\partial\Omega \times (0,T)$ and $v(x,0) \le u_0(x)$ in Ω then $v(x,t) \le u(x,t)$ in the whole domain $\Omega \times (0,T)$. Hence, if u is the maximal weak solution of (1.1)-(1.3) in $\Omega \times (0,T)$ with the maximum existence time T and v is a weak solution of (1.1)-(1.3) in $\Omega \times (0,T')$ with the maximum existence time T', then $T \le T'$.*

Proof. For each integer $n \ge 1$, let $u_n(x,t)$ be a weak solution of the following problem with the maximum existence time $T_n > 0$.

$$(u^\alpha)_t = \Delta u + f(u) + g(u(x_0(t),t)), \quad (x,t) \in \Omega \times (0,T), \tag{2.50}$$

$$u(x,t) = \frac{1}{n}, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.51}$$

$$u(x,0) = u_0(x) + \frac{1}{n}, \quad x \in \Omega. \tag{2.52}$$

Then, by Proposition 2.7 and Lemma 2.10(ii) we see that $T_{n+1} \ge T_n$ and $u_{n+1}(x,t) \le u_n(x,t)$ in $\Omega \times (0,T_n)$ for each $n \ge 1$. Put

$$T = \lim_{n\to\infty} T_n, \tag{2.53}$$

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{2.54}$$

Then, because of [10], u is a weak solution of (1.1)-(1.3). Further, T is the maximum existence time. In fact, otherwise, $\sup_{\Omega\times(0,T)} u(x,t) < \infty$ and hence similarly to the proof of Proposition 2.10, we see that $T_n > T$ for large $n \geq 1$. This is a contradiction to $T_n \leq T$ and so we see that T is the maximum existence time. The uniqueness of maximal solutions is clear.

Let v be a subsolution of (1.1) in $\Omega \times (0, T)$ satisfying $v(x,t) = 0$ on $\partial\Omega \times (0,T)$ and $v(x,0) \leq u_0(x)$ in Ω. Then, it follows from Proposition 2.7 that $v(x,t) \leq u_n(x,t)$ in $\Omega \times (0, T_n)$ and so $v(x,t) \leq u(x,t)$ in $\Omega \times (0,T)$. The rest of the assertions of Proposition 2.11 also follow from Proposition 2.10(ii). The proof is complete. □

Next, we assume (A7)(A9) and we shall show the monotonicity of solutions $u(x,t)$ of (1.1)-(1.3) with $\Omega = B(R)$.

Lemma 2.12. *Let $\Omega = B(R)$. Assume (A1), (A2), (A7). Let $u(x,t)$ be a weak solution of (1.1)-(1.3). Then, for each $t \geq 0$, $u(x,t) = u(r,t)$ $(r = |x|)$ is radially symmetric in $x \in B(R)$ and nonincreasing in $r \geq 0$. Furthermore, if $u(r_0,t_0) > 0$ for some $(r_0,t_0) \in (0,R) \times (0,T)$, then u is a C^∞−function in the neighborhood of (r_0,t_0) and*

$$\frac{\partial u}{\partial r}(r_0,t_0) < 0. \tag{2.55}$$

Proof. Let $u(x,t)$ be a weak solution of (1.1)-(1.3). We first note that if $u(x_1,t_1) > 0$ for some $(x_1,t_1) \in \Omega \times (0,T)$, then u is a C^∞−function in the neighborhood of (x_1,t_1) (See the proof of Proposition 2.6).

Let $w(x,t)$ be a unique weak solution of the following problem.

$$(w^\alpha)_t = \Delta w + f(w) + g(u(x_0(t),t)), \quad (x,t) \in B(R) \times (0,T), \tag{2.56}$$

$$w(x,t) = 0, \quad (x,t) \in \partial B(R) \times (0,T), \tag{2.57}$$

$$w(x,0) = u_0(x), \quad x \in B(R). \tag{2.58}$$

Then, the uniqueness of solutions ([1]) implies $u(x,t) = w(x,t)$ in $\Omega \times (0,T)$. Since equation (2.56) is invariant under the reflection in x, as in [11] and [16], the usual comparison theorem ([1]) for (2.56) implies that for each $t \geq 0$ $u(x,t) = w(x,t) = w(r,t)$ $(r = |x|)$ is a radially symmetric function in $x \in \Omega$ and nonincreasing in $r \geq 0$. Further, we see

$$\frac{\partial u}{\partial r}(r_0,t_0) = \frac{\partial w}{\partial r}(r_0,t_0) < 0 \tag{2.59}$$

if $w(r_0,t_0) > 0$ for some $(r_0,t_0) \in (0,R) \times (0,T)$. The proof is complete. □

Lemma 2.13. *Let $\Omega = B(R)$. Assume (A1)–(A3), (A7)–(A9). Let $u(x,t)$ be the maximal weak solution of (1.1)-(1.3). Then, u is nondecreasing in t for each $x \in \Omega$. If $u(x,t) > 0$ for some $(x,t) \in \Omega \times (0,T)$, then $\partial u(x,t)/\partial t \geq 0$.*

Proof. The method of the proof is the same as that of Bricher [5].

By (A3), we see that for any bounded domain $D \subset \Omega$ with smooth boundary ∂D, $0 < \tau < T$ and nonnegative $\varphi(x,t) \in C^2(\bar{D} \times [0,T))$ which vanishes on the boundary ∂D,

$$\begin{aligned}
&\int_0^\tau \int_D \{u_0 \Delta\varphi + f(u_0)\varphi + g(u_0(x_0(t)))\varphi\}\, dxdt - \int_0^\tau \int_{\partial D} u_0 \partial_n \varphi \, dSdt \\
&\geq \int_0^\tau \int_D \{u_0 \Delta\varphi + f(u_0)\varphi + g(u_0(x_0(0)))\varphi\}\, dxdt - \int_0^\tau \int_{\partial D} u_0 \partial_n \varphi \, dSdt \geq 0
\end{aligned} \tag{2.60}$$

where n denotes the outer unit normal to the boundary, which implies that $u_0(x)$ is a subsolution of (1.1).

Let $u(x,t)$ be a weak maximal solution of (1.1)-(1.3). Let $t_1 > 0$ and put

$$v(x,t) = \begin{cases} u_0(x) & \text{in } \bar{\Omega} \times [0, t_1), \\ u(x, t - t_1) & \text{in } \bar{\Omega} \times [t_1, t_1 + T). \end{cases} \tag{2.61}$$

Then, it is not difficult to see that $v(x,t)$ is also a subsolution of (1.1) in $\Omega \times (0, t_1 + T)$. Hence, because of Proposition 2.11 $v(x,t) \leq u(x,t)$ in $\Omega \times (0,T)$, whence $u(x,t) \leq u(x, t+t_1)$ in $\Omega \times (0, T - t_1)$. Therefore, since $t_1 \in (0,T)$ can be chosen arbitrarily, u is nondecreasing in t for each $x \in \Omega$. We can also show that if $u(x,t) > 0$ for some $(x,t) \in \Omega \times (0,T)$, then $\partial u(x,t)/\partial t \geq 0$. The proof is complete. □

Finally, we assume again that Ω is a bounded domain in $\mathbf{R}^\mathbf{N}$ with smooth boundary $\partial\Omega$ and we shall show the global existence of any solution of (1.1)-(1.3) when $f(\xi) = o(\xi)$ and $g(\xi) = o(\xi)$ as $\xi \to \infty$.

Let D be a star shaped domain with origin $a \in \mathbf{R}^\mathbf{N}$ with smooth boundary ∂D satisfying $\Omega \subset D$. Let w_0 be a unique solution of the boundary-value problem

$$\begin{cases} -\Delta w = 1 & \text{in } D \\ w = 0 & \text{in } \partial D. \end{cases} \tag{2.62}$$

Then, we note $w_0 > 0$ in D by the maximum principle. Put

$$\mu = \frac{1}{||w_0||_{L^\infty(D)}} \, (> 0). \tag{2.63}$$

We assume

(A13) There exist $\gamma \in (0, \mu)$ and $C > 0$ such that

$$\sup_{0 \leq \xi' \leq \xi} f(\xi') + \sup_{0 \leq \xi' \leq \xi} g(\xi') \leq \gamma\xi + C \qquad \text{for } \xi \geq 0. \tag{2.64}$$

Theorem 2.14. *Assume (A1)–(A3) and (A13). If u is a weak solution of* (1.1)-(1.3) *with the maximum existence time $T > 0$, then $T = \infty$ and u is uniformly bounded in $\Omega \times (0, \infty)$.*

Proof. The method of the proof is similar to that of [16] (see also [6]).

Without loss of generality we can assume $a = 0$. Put, for $k \in (0, 1)$, $w_k(x) = w_0(kx)$ and

$$D_k = \{x\,;\, kx \in D\}. \tag{2.65}$$

Then, $\Omega \subset D \subset \bar{D} \subset D_k$, $||w_k||_{L^\infty(D_k)} = ||w_0||_{L^\infty(D)} = 1/\mu$ and w_k is a unique solution of the boundary value problem

$$\begin{cases} -\Delta w = k^2 & \text{in } D_k, \\ w = 0 & \text{in } \partial D_k. \end{cases} \tag{2.66}$$

Choose $k \in (0, 1)$ to satisfy $\gamma = \mu k^2$ and put

$$v_h(x) = h w_k(x) - \frac{C}{\gamma} \quad \text{for } h > 0. \tag{2.67}$$

If h is sufficiently large, we have

$$v_h(x) > u_0(x) \quad \text{in } \Omega. \tag{2.68}$$

Furthermore, it follows from (A13) that

$$-\Delta v_h = hk^2 = \gamma\Big(\frac{h}{\mu} - \frac{C}{\gamma}\Big) + C = \gamma ||v_h||_{L^\infty(D_k)} + C \geq f(v_h) + g(v_h(x_0(t))). \tag{2.69}$$

Thus, v_h is a supersolution of (1.1).

Let u be a solution of (1.1)-(1.3) with the maximum existence time $T > 0$. Note (2.68). Applying Proposition 2.7 to v_h and u and noting Proposition 2.10(ii), we see $T = \infty$ and

$$v_h(x) \geq u(x, t) \quad \text{in } \Omega \quad \text{for } t \geq 0. \tag{2.70}$$

The proof is complete. □

3. Blow-up and total blow-up conditions

In this section, we assume (A1)-(A3) and (A5) (or (A4)), and discuss conditions for the blow-up and total blow-up of solutions of (1.1)-(1.3). Especially, under condition (A5) we show that if

$$\int_0^T g(u(x_0(t), t))\, dt = \infty, \tag{3.1}$$

then the blow-up solution $u(x, t)$ of (1.1)-(1.3) blows up in the whole domain Ω, which immediately implies that when $f(u) \equiv 0$, the blow-up solution

$u(x,t)$ of (1.1)-(1.3) blows up in the whole domain Ω (see Souplet [21] when $\alpha = 1$). These results are important and used often later.

We first show the following blow-up theorem assuming (A5).

Theorem 3.1. *Assume (A1)–(A3) and (A5). Let $u(x,t)$ be a weak solution of (1.1)-(1.3) and let G be a subset of Ω with $L^N(G) \neq 0$, where $L^N(G)$ is the Lebesgue measure of the set $G \subset \mathbf{R^N}$. Then, there exists large $h > 0$ such that if*

$$\inf_{x\in G} u_0(x) \geq h, \tag{3.2}$$

then $u(x,t)$ blows up in finite time.

This theorem will follow from the following proposition which will also be used in subsequent sections.

Proposition 3.2. *Assume (A1)–(A3) and (A5), and let $u(x,t)$ be a weak solution of (1.1)-(1.3) with the maximum existence time $T > 0$. Let $0 < \tilde{T} < \infty$ and set*

$$K = \{x_0(t)\,;\, t \in [0, \tilde{T}]\}. \tag{3.3}$$

Further, let D be a domain in $\mathbf{R^N}$ satisfying $K \subset D \subset \Omega$. Then, for any $\varepsilon \in (0, \tilde{T})$ there exists $h_0 > \tilde{\xi}_\lambda (> 0)$ such that if

$$\inf_{x\in D} u_0(x) \geq h \tag{3.4}$$

for some $h \geq h_0$, then $T < \varepsilon$ and

$$\inf_{x\in K} u(x,t) \geq h \quad \text{ in } (0,T). \tag{3.5}$$

So, we shall show Proposition 3.2. As is mentioned in the introduction, the method of the proof is similar to that of [6], however, we do not require that $g(\xi)$ be convex, which is assumed in [6]. We need several lemmas to prove this proposition. Put

$$d_0 = \operatorname{dist}(K, \partial D)\,(> 0). \tag{3.6}$$

Let λ be the eigenvalue of $-\Delta$ in $B(d_0)$ with zero Dirichlet boundary condition and $s(x)$ the corresponding eigenfunction satisfying

$$\int_{B(d_0)} s(x)\,dx = 1. \tag{3.7}$$

Here we define $B(d_0) = B(0; d_0)$ and

$$B(x_1; r) = \{x \in \mathbf{R^N}\,;\, |x - x_1| < r\}$$

for $r > 0$ and $x_1 \in \mathbf{R^N}$, as in §2. We consider the following problem:

$$(v^\alpha)_t = \Delta v + g(v(0,t)), \quad (x,t) \in B(d_0) \times (0,T), \tag{3.8}$$

$$v(x,t) = 0, \quad (x,t) \in \partial B(d_0) \times (0,T), \tag{3.9}$$
$$v(x,0) = v_0(x), \quad x \in B(d_0), \tag{3.10}$$

where $v_0(x) = v_0(r) \in C(\overline{B(d_0)})$ $(r = |x|)$ is a radially symmetric function in x and a nonincreasing function in $r \geq 0$, and $v_0(r) > 0$ for $r \in (0, d_0)$ and $v_0(d_0) = 0$. Let $\varepsilon > 0$. We choose $\xi_1 > \tilde{\xi}_\lambda$ so large that

$$\int_{\xi_1}^{\infty} \frac{\alpha \eta^{\alpha-1}}{\tilde{h}_\lambda(\eta)} \, d\eta < \varepsilon, \tag{3.11}$$

where $\tilde{\xi}_\lambda$ and $\tilde{h}_\lambda(\xi)$ are as in condition (A5).

Let $v(x,t)$ be the maximal weak solution of (3.8)-(3.10) with the maximum existence time $T_v > 0$ and put

$$J(t) = \int_{B(d_0)} v^\alpha(x,t) s(x) \, dx. \tag{3.12}$$

We first show the following lemma.

Lemma 3.3. *Assume* (A1), (A3) *and* (A5). *Then, if* $J(0) \geq \xi_1^\alpha$, *then* $T_v \leq \varepsilon$ *and*

$$v(0,t) \geq \xi_1 \quad \text{for } t \in (0, T_v). \tag{3.13}$$

Proof. The method of the proof is similar to that of Imai-Mochizuki [14].

We note that for each $t \geq 0$ $v(x,t) = v(r,t)$ $(r = |x|)$ is also a radially symmetric function in x and a nonincreasing function in $r \geq 0$. Choosing $s(x)$ as a test function in the integral identity satisfied by $v(x,t)$ (see (2.2)), we get

$$\begin{aligned} J(t) &\geq J(0) + \int_0^t \int_{B(d_0)} \{-\lambda v + g(v(0,t))\} s(x) \, dx dt \qquad (3.14) \\ &\geq J(0) + \int_0^t \tilde{h}_\lambda(v(0,t)) \, dt. \end{aligned}$$

Thus, we have

$$v^\alpha(0,t) \geq J(t) \geq J(0) + \int_0^t \tilde{h}_\lambda(v(0,t)) \, dt \equiv \beta(t) \quad \text{for} \quad t \in (0, T_v). \tag{3.15}$$

Note here $\tilde{h}_\lambda(\rho) > 0$ in $\rho \geq \xi_1 \geq \tilde{\xi}_\lambda$ and $v(0,0) \geq \{J(0)\}^{1/\alpha} \geq \xi_1$. Hence, since $v(0,t)$ is continuous in $t \geq 0$, we see that $v^\alpha(0,t) \geq \beta(t) \geq J(0) \geq \xi_1^\alpha$ and $\beta(t)$ is increasing in $t > 0$. Therefore, since $\tilde{h}_\lambda(\rho)$ is nondecreasing in $\rho > \xi_1$, we obtain

$$\tilde{h}_\lambda(v(0,t)) \geq \tilde{h}_\lambda(\beta^{1/\alpha}(t)) \quad \text{for } t \in (0, T_v), \tag{3.16}$$

or equivalently

$$1 \le \frac{\tilde{h}_\lambda(v(0,t))}{\tilde{h}_\lambda(\beta^{1/\alpha}(t))}. \tag{3.17}$$

Integrate both sides of (3.17) over $(0, t)$ and noting $d\,\beta(t) = \tilde{h}_\lambda(v(0,t))\,dt$, we have

$$t \le \int_0^t \frac{\tilde{h}_\lambda(v(0,t))}{\tilde{h}_\lambda(\beta^{1/\alpha}(t))}\,dt \le \int_{J(0)}^{\beta(t)} \frac{d\rho}{\tilde{h}_\lambda(\rho^{1/\alpha})} \le \int_{\xi_1}^{\infty} \frac{\alpha\xi^{\alpha-1}}{\tilde{h}_\lambda(\xi)}\,d\xi < \varepsilon \quad \text{for } t \in (0, T_v) \tag{3.18}$$

giving us

$$T_v \le \varepsilon. \tag{3.19}$$

Thus, we obtain the first part of the assertions of the lemma. Since $\beta(t) \ge \xi_1^\alpha$ as above, by (3.15) we get (3.13). The proof is complete. □

Lemma 3.4. *Assume* (*A*1)–(*A*3) *and* (*A*5), *and let* $u(x,t)$ *be a weak solution of* (1.1)-(1.3) *with the maximum existence time* T. *Let* $\tilde{T}$ *and* K *be as in Proposition* 3.2. *Let* $0 < \varepsilon < \tilde{T}$ *and let* $\xi_1 > \tilde{\xi}_\lambda$ *satisfy* (3.11). *Then, if* $J(0) \ge \xi_1^\alpha$ *and for each* $x' \in K$

$$u_0(x) > v_0(x - x') = v(x - x', 0) \qquad \text{in } x \in B(x'\,;\, d_0), \tag{3.20}$$

where $v(x,t)$ *is as in Lemma* 3.3, *then* $T \le \varepsilon$ *and*

$$u(x,t) \ge \xi_1 \qquad \text{for } (x,t) \in K \times (0,T). \tag{3.21}$$

Proof. Let $v(x,t)$ be as in Lemma 3.3. Put $T_1 = \min\{T_v, T\}$. Then, by Lemma 3.3, $T_1 \le T_v \le \varepsilon < \tilde{T}$. We shall show for each $x \in K$,

$$u(x,t) > v(0,t) \qquad \text{for } t \in (0, T_1). \tag{3.22}$$

Assume to the contrary that (3.22) does not hold. Then, putting

$$t_1 = \sup\{T' \in [0, T_1]\,;\, u(x,t) > v(0,t) \text{ for } (x,t) \in K \times (0, T')\}, \tag{3.23}$$

we have $0 < t_1 < T_1$. Hence,

$$u(x,t) > v(0,t) \text{ for } (x,t) \in K \times (0, t_1) \tag{3.24}$$

and there exists $x_1 \in K$ such that

$$u(x_1, t_1) = v(0, t_1). \tag{3.25}$$

Now, putting $\tilde{v}(x,t) = v(x - x_1, t)$ we compare $u(x,t)$ and $\tilde{v}(x,t)$ in $B(x_1, d_0) \times (0, t_1]$. By (3.24), we have

$$u(x_0(t), t) > v(0,t) = \tilde{v}(x_1, t) \qquad \text{for } t \in (0, t_1). \tag{3.26}$$

Since $g(\xi)$ is nondecreasing in $\xi \ge 0$ by (A5), we have

$$(\tilde{v}^\alpha)_t - \Delta\tilde{v} = g(\tilde{v}(x_1, t)) \le g(u(x_0(t), t)) \tag{3.27}$$

$$\leq (u^\alpha)_t - \Delta u \quad (x,t) \in B(x_1\,;\,d_0) \times (0, t_1].$$

Noting (3.20) and $\tilde{v} \leq u$ on the parabolic boundary of $B(x_1\,;\,d_0) \times (0, t_1]$, and applying Proposition 2.7 to $\tilde{v}$ and u, we see that $\tilde{v}(x,t) < u(x,t)$ in $B(x_1; d_0) \times (0, t_1)$ and $\tilde{v}(x, t_1) < u(x, t_1)$ in $B(x_1\,;\,d_0)$. Hence $v(0, t_1) = \tilde{v}(x_1, t_1) < u(x_1, t_1)$. This is a contradiction to (3.25) and so we get (3.22).

Therefore, from (ii) of Proposition 2.10, $T = T_1 \leq T_v \leq \varepsilon \leq \tilde{T}$ and so by Lemma 3.3, $u(x,t) > v(0,t) \geq \xi_1$ in $K \times (0,T)$. The proof is complete. □

Proof of Proposition 3.2. Let $0 < \varepsilon < \tilde{T}$. We choose $\xi_1 > \tilde{\xi}_\lambda$ to satisfy (3.11) and choose $h_0 > 0$ satisfying $h_0 > \xi_1$.

Let $h \geq h_0$ and let $\{v_{0,n}\}$ be a sequence of functions such that for each $n \geq 1$, $v_{0,n}(x) = v_{0,n}(r) \in C(\bar{B}(d_0))$ $(r = |x|)$ is radially symmetric in $x \in B(d_0)$ and nonincreasing in $r \geq 0$, $v_{0,n}(r) > 0$ in $(0, d_0)$, $v_{0,n}(d_0) = 0$ and $v_{0,n} \uparrow h$ uniformly in each compact set in $B(d_0)$ as $n \to \infty$. Then, for large n,

$$J_n(0) = \int_{B(d_0)} v_{0,n}^\alpha(x) s(x)\, dx \xi_1^\alpha \; (> \tilde{\xi}_\lambda^\alpha) \tag{3.28}$$

and

$$J_n(0) \uparrow h^\alpha \qquad \text{as } n \to \infty, \tag{3.29}$$

where $B(d_0)$ and $s(x)$ appear in (3.6) and (3.7). Hence, for large n,

$$\int_{\{J_n(0)\}^{1/\alpha}}^\infty \frac{\alpha \eta^{\alpha-1}}{\tilde{h}_\lambda(\eta)}\, d\eta \leq \int_{\xi_1}^\infty \frac{\alpha \eta^{\alpha-1}}{\tilde{h}_\lambda(\eta)}\, d\eta < \varepsilon. \tag{3.30}$$

Assume (3.4). Then, since $u_0(x) \geq h > v_{0,n}(x - x')$ for $x' \in K$ and $x \in B(x'\,;\,d_0)$. Hence, Lemma 3.4 with $\xi_1 = \{J_n(0)\}^{1/\alpha}$ implies that $T \leq \varepsilon$ and $u(x,t) \geq \{J_n(0)\}^{1/\alpha}$ in $K \times (0,T)$ for $n \geq 1$. Thus, by (3.29) we obtain $u(x,t) \geq h$ in $K \times (0,T)$. The proof is complete. □

Here, we mention that similar results hold under assumption (A4) instead of (A5) as follows; these are also used in the next sections.

Proposition 3.5. *Assume* (A1)–(A3) *and* (A4), *and let* $u(x,t)$ *be a weak solution of* (1.1)-(1.3) *with the maximum existence time* $T > 0$. *Let* K *be a compact set in* Ω *and let* D *be a domain in* $\mathbf{R}^\mathbf{N}$ *satisfying* $K \subset D \subset \Omega$. *Then, for any* $\varepsilon > 0$ *there exists* $h_0 > \xi_\lambda > 0$ *such that if* $h \geq h_0$ *and*

$$\inf_{x \in D} u_0(x) \geq h, \tag{3.31}$$

then $T < \varepsilon$ *and*

$$\inf_{x \in K} u(x,t) \geq h \qquad \text{in } (0,T). \tag{3.32}$$

Proof. By using the method of the proof of Theorem 1 of Imai-Mochizuki [14], the method of the proof is similar to that of Proposition 3.2 and so we omit it. □

We are now in a position to show Theorem 3.1, if the following lemma holds:

Lemma 3.6. *Let $v(x,t)$ be the solution of the problem*

$$\begin{cases} (v^\alpha)_t = \Delta v, & (x,t) \in \Omega \times (0,T), \\ v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ v(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.33}$$

let G be a subset of Ω with $L^N(G) \neq 0$, where $L^N(G)$ is the Lebesgue measure of $G \subset \mathbf{R^N}$, and let D be a subset of Ω satisfying $\bar{D} \subset \Omega$. Then, when $\alpha = 1$, for any $t_0 > 0$ there exists $C = C(G, D, t_0) > 0$ such that if for some $h' > 0$

$$\inf_{x\in G} u_0(x) \geq h', \tag{3.34}$$

then

$$\inf_{x\in D} v(x,t_0) \geq Ch', \tag{3.35}$$

and when $0 < \alpha < 1$, there exist $C = C(G,D) > 0$ and $\tau_0 = \tau_0(G,D)$ such that if (3.34) holds for some $h' > 0$, then

$$\inf_{x\in D} v(x, h'^{-(1-\alpha)}\tau_0) \geq Ch'. \tag{3.36}$$

Proof. These results are shown in the proof of Lemma 3.6 of R. Suzuki [22]. □

Proof of Theorem 3.1. Let $u(x,t)$ be a weak solution of (1.1)-(1.3) with the maximum existence time $T > 0$ and let G be a subset of Ω with $L^N(G) \neq 0$, where $L^N(G)$ is the Lebesgue measure of $G \subset \mathbf{R^N}$. Let $\tilde{T}$, K and D be as in Proposition 3.2. Let $t_0 \in (0, \tilde{T}/2)$ and consider $u(x,t_0)$ as the initial data u_0. Then, by Proposition 3.2, for $0 < \varepsilon < \tilde{T}/2$ there exists $h_0 > 0$ such that if

$$\inf_{x\in D} u(x,t_0) \geq h \tag{3.37}$$

for some $h \geq h_0$ then $T < t_0 + \varepsilon$ $(< \tilde{T})$.

On the other hand, let v be as in Lemma 3.6. Then, by Lemma 3.6, for large $h > 0$ there exist $0 < t_0 < \tilde{T}/2$ and $\tilde{h} > 0$ such that

$$\inf_{x\in G} u_0(x) \geq \tilde{h}\,(> 0) \tag{3.38}$$

implies

$$\inf_{x\in D} v(x,t_0) \geq h. \tag{3.39}$$

Thus, since $u(x,t) \geq v(x,t)$ in $\Omega \times (0,T)$ by the usual comparison theorem, we see that (3.38) leads to $T < t_0 + \varepsilon$ $(< \tilde{T})$. The proof is complete. □

Finally, we give the sufficient conditions for the total blow-up, which are shown by [2] (Theorem 4.1 of [2]) and [21] (Theorem 4.1 of [21]) in the case $\alpha = 1$. But, in the proof, we can not use their methods, since their methods strongly depend on the semilinearity of the equation. We note that we do not get the precise estimate for solutions near the blow-up time which is obtained in [21].

We shall consider the following Dirichlet problem:

$$(u^\alpha)_t = \Delta u + f(x,t) + g(t), \quad (x,t) \in \Omega \times (0,T), \tag{3.40}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.41}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{3.42}$$

where $f(x,t)$ and $g(t)$ are nonnegative continuous functions for $(x,t) \in \bar{\Omega} \times [0,T)$ and $t \in [0,\infty)$, respectively.

Proposition 3.7. *Assume* (A2). *Let* $u(x,t)$ *be a weak solution of* (3.40)-(3.42). *Then,*

$$\int_0^T g(t)\,dt = \infty \tag{3.43}$$

implies

$$\sup_{(x,t)\in\Omega\times(0,T)} u(x,t) = \infty. \tag{3.44}$$

Furthermore, we see that for each compact set $K \subset \Omega$,

$$\lim_{t\uparrow T} \inf_{x\in K} u(x,t) = \infty. \tag{3.45}$$

Especially, in the case where $\sup_{(x,t)\in\Omega\times(0,T)} f(x,t) < \infty$, (3.44) *holds if and only if* (3.43) *holds.*

The next theorem immediately follows from this proposition.

Theorem 3.8. *Assume* (A1)–(A3). *Let* $u(x,t)$ *be a weak solution of the problem*

$$(u^\alpha)_t = \Delta u + g(u(x_0(t),t)), \quad (x,t) \in \Omega \times (0,T), \tag{3.46}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.47}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{3.48}$$

and let $T \in (0,\infty]$ be the maximum existence time of $u(x,t)$. If $0 < T < \infty$, then $u(x,t)$ blows up in the whole domain Ω, namely, for each compact set $K \subset \Omega$

$$\lim_{t \uparrow T} \inf_{x \in K} u(x,t) = \infty. \tag{3.49}$$

Proof. Let $u(x,t)$ be a weak solution of (3.46) with the maximum existence time $T \in (0,\infty)$. Then, by Proposition 2.10,

$$\sup_{(x,t) \in \Omega \times (0,T)} u(x,t) = \infty. \tag{3.50}$$

Hence, using Proposition 3.7 with $f(x,t) \equiv 0$ and $g(t) = g(u(x_0(t),t))$ we get (3.43) and so (3.49) for each compact set $K \subset \Omega$. □

Now, we shall show Proposition 3.7. Let $v(x,t)$ be the weak solution of the problem

$$(v^\alpha)_t = \Delta v + g(t), \quad (x,t) \in B(d) \times (0,T), \tag{3.51}$$

$$v(x,t) = 0, \quad (x,t) \in \partial B(d) \times (0,T), \tag{3.52}$$

$$v(x,0) = 0, \quad x \in B(d), \tag{3.53}$$

where $d > 0$ and $B(d) = B(0;d)$. Then, as in the proof of Lemma 2.12, we see that for each $t > 0$, $v(x,t) = v(r,t)$ $(r = |x|)$ is radially symmetric in x and nonincreasing in $r \geq 0$. Hence, $\max_{x \in B(d)} v(x,t) = v(0,t)$ for $t \geq 0$. We need the following lemma:

Lemma 3.9. *Assume* (3.43). *Let $v(x,t)$ be the weak solution of* (3.51)-(3.53). *Then,*

$$\sup_{B(d) \times (0,T)} v(x,t) = \infty. \tag{3.54}$$

Namely, there exists a sequence $\{t_j\} \subset (0,T)$ such that $t_j \uparrow T$ as $j \to \infty$ and

$$\lim_{j \to \infty} v(0,t_j) = \infty. \tag{3.55}$$

Proof. Assume to the contrary that

$$\sup_{B(d) \times (0,T)} v(x,t) = C < \infty. \tag{3.56}$$

Let λ be the first eigenvalue of $-\Delta$ in $B(d)$ with zero Dirichlet boundary condition and $s(x)$ the corresponding eigenfunction satisfying

$$\int_{B(d)} s(x)\, dx = 1. \tag{3.57}$$

As in the proof of Lemma 3.3, choosing $s(x)$ as a test function in the integral identity satisfied by $v(x,t)$ (see (2.2)) we get

$$\begin{aligned}\infty > C^\alpha &\geq \int_{B(d)} v^\alpha(x,t)s(x)\,dx \qquad (3.58)\\ &\geq -\lambda \int_0^t \int_{B(d)} v(x,t)s(x)\,dxdt + \int_0^t \int_{B(d)} g(t)s(x)\,dxdt\\ &\geq -\lambda Ct + \int_0^t g(t)\,dt \quad \text{for } 0 \leq t < T.\end{aligned}$$

This is a contradiction to (3.43) and so we get (3.54). (3.55) follows from (3.54). □

Proof of Proposition 3.7. We first show that (3.43) leads to (3.45). Assume (3.43) and let $u(x,t)$ be the weak solution of (3.40)-(3.42) and $v(x,t)$ the solution of (3.51)-(3.53). Let $D \subset \Omega$ be a domain satisfying $\bar{D} \subset \Omega$. We choose $d > 0$ such that for each $x' \in D$, $B(x';d) \subset \Omega$.

Putting $v_{x'}(x,t) = v(x-x',t)$, we compare $u(x,t)$ and $v_{x'}(x,t)$ in $B(x';d) \times (0,T)$. Since $v_{x'}(x,t)$ is also a weak solution of (3.51), by the usual comparison theorem we have for each $x' \in D$,

$$v_{x'}(x,t) \leq u(x,t) \quad \text{in } B(x';d) \times (0,T). \qquad (3.59)$$

Hence,

$$u(x',t) \geq v_{x'}(x',t) = v(0,t) \quad \text{in } (0,T). \qquad (3.60)$$

Thus, because of Lemma 3.9, there exists a sequence $\{t_j\} \uparrow T$ such that for each $x' \in D$,

$$u(x',t_j) \geq v(0,t_j) \quad \text{as } j \to \infty. \qquad (3.61)$$

Therefore, as in the proof of Theorem 3.2 of [16] (see [8] in case $\alpha = 1$), we see that for each compact set $K \subset D$,

$$\lim_{t \uparrow T} \inf_{x \in K} u(x,t) = \infty.$$

These results lead to (3.45) for each compact set $K \subset \Omega$ and so (3.44).

Finally, assume

$$\sup_{(x,t)\in\Omega\times(0,T)} f(x,t) < \infty. \qquad (3.62)$$

We shall show that (3.44) leads to (3.43). Assume (3.44). Assume to the contrary that

$$\int_0^T g(t)\,dt < \infty. \qquad (3.63)$$

Put

$$h(t) = \left\{ \int_0^t g(t)\,dt + C_1 t + C_2 \right\}^{1/\alpha} \qquad \text{in } 0 < t < T, \tag{3.64}$$

where $C_1 = \sup_{(x,t)\in\Omega\times(0,T)} f(x,t)$ and $C_2 = \sup_{x\in\Omega} u_0^\alpha(x)$. Since $h(t)$ is the supersolution of (3.40) and $u_0(x) \le h(0)$ in Ω, by the usual comparison theorem and (3.63) we have

$$u(x,t) \le h(t) \le \sup_{0<t<T} h(t) < \infty \qquad \text{in } \Omega \times (0,T). \tag{3.65}$$

This is a contradiction to (3.44) and so we see that (3.44) leads to (3.43). The proof is complete. □

4. Total blow-up

In this and the sequel sections we restrict ourselves to $\Omega = B(R)$ and initial data $u_0(x)$ satisfying (A7). Then, we note from Lemma 2.12 that for each $t \ge 0$ $u(x,t) = u(r,t)$ $(r = |x|)$ is a radially symmetric function in x and a nonincreasing function in $r \ge 0$.

In this section, we consider the case $f(\xi) = o(g(\xi))$ and prove (i) of (II) in Theorem 1.4 and (i) of (II) in Theorem 1.6. These results follow from the following theorem.

Theorem 4.1. *Let $\Omega = B(R)$. Assume (A1)–(A3), (A5), (A7), (A11). Let $T_1 > 0$ and let $R' \in (0,R)$ satisfy $\max_{t\in[0,T_1]} |x_0(t)| < R'$. Then, there exists $M > 0$ such that if $u_0(0) \ge M$ and*

$$u_0(R') > \varepsilon_1 u_0(0), \tag{4.1}$$

then the solution of (1.1)-(1.3) blows up in the whole domain $B(R)$ at the blow-up time $T < T_1$, namely, $S = \overline{B(R)}$, where ε_1 is as in condition (A11).

Remark 4.2. We note that the assumption $x_0(t) \ne 0$ is not required in this theorem.

Proof. The method of the proof is similar to that of Okada-Fukuda [17] and Fukuda-Suzuki [13].

We choose $R_1 \in (0,R)$ to satisfy $\max_{t\in[0,T_1]} |x_0(t)| < R_1$. Let $R_1 < R' < R$ and let $u(x,t)$ be a weak solution of (1.1)-(1.3) with the maximum existence time $T > 0$.

We first note by Proposition 2.6 that $u(x,t) > 0$ in $B(R) \times (0,T)$ and $u \in C^\infty(B(R) \times (0,T))$. We also note from Proposition 3.2 that if for large $h > 0$

$$\inf_{x\in B(R_1)} u_0(x) \ge h, \tag{4.2}$$

then $T < T_1$ and

$$u(0,t) \geq u(x_0(t),t) \geq h \quad \text{for } t \in (0,T). \tag{4.3}$$

Now, we choose $\varphi(x) \in C^2(\bar{B}(R'))$ to satisfy $\varphi(x) = \varphi(r)$ ($r = |x|$) is radially symmetric in x, nonincreasing in $r \geq 0$ and for some $\mu > 0$,

$$\begin{cases} -\Delta\varphi(x) \leq \mu\varphi(x) & \text{in } B(R'), \\ \varphi > 0 & \text{in } B(R'), \\ \varphi(x) = 1 & \text{in } B(R_1), \\ \varphi = 0 & \text{on } \partial B(R'). \end{cases} \tag{4.4}$$

Put

$$J(x,t) = u(x,t) - \varepsilon_1\varphi(x)u(0,t) \quad \text{in } B(R'), \tag{4.5}$$

where ε_1 is as in (A11). We need several lemmas.

Lemma 4.3. *Assume*

$$J(x,0) = u_0(x) - \varepsilon_1\varphi(x)u_0(0) > 0 \quad \textit{in } \overline{B(R')} \tag{4.6}$$

and (4.2) *for some* $h > 0$. *If* h *is large enough then*

$$J(x,t) = u(x,t) - \varepsilon_1\varphi(x)u(0,t) > 0 \quad \textit{in } \overline{B(R')} \times [0,T). \tag{4.7}$$

Proof. Put

$$T_2 = \sup\{\tilde{T}\,;\, 0 \leq \tilde{T} \leq T \text{ and } J(x,t) > 0 \text{ in } \overline{B(R')} \times [0,\tilde{T})\}. \tag{4.8}$$

We first note $T_2 > 0$. Assume to the contrary that $T_2 < T(< T_1)$.

We compute

$$\begin{aligned} &\alpha u^{\alpha-1}J_t - \Delta J \qquad (4.9)\\ &\geq (u^\alpha)_t - \alpha u^{\alpha-1}\varepsilon_1\varphi(x)u_t(0,t) - \Delta u - \mu\varepsilon_1 u(0,t)\varphi(x) \\ &\geq f(u) + g(u(x_0(t),t)) - \frac{\alpha u^{\alpha-1}\varepsilon_1\varphi(x)}{\alpha u^{\alpha-1}(0,t)}\{f(u(0,t)) + g(u(x_0(t),t))\} \\ &\qquad - \mu\varepsilon_1 u(0,t)\varphi(x). \end{aligned}$$

Here, we use (4.4) and $\Delta u(0,t) \leq 0$.

Since $J(x,t) = u(x,t) - \varepsilon_1\varphi(x)u(0,t) \geq 0$ in $\overline{B(R')} \times [0,T_2]$, we see that $u(x_0(t),t) \geq \varepsilon_1\varphi(x_0(t))u(0,t) \geq \varepsilon_1 u(0,t)$ in $[0,T_2]$ and so $g(u(x_0(t),t)) \geq g(\varepsilon_1 u(0,t))$ in $[0,T_2]$.

Hence, it follows from (4.3) and conditions (A5) and (A11) that if $h > 0$ in (4.3) is large enough, then

$$\alpha u^{\alpha-1}J_t - \Delta J \geq f(u) + g(u(x_0(t),t)) \tag{4.10}$$

$$- \frac{\alpha(\varepsilon_1\varphi(x)u(0,t))^{\alpha-1}\varepsilon_1\varphi(x)}{\alpha u^{\alpha-1}(0,t)}\{f(u(0,t)) + g(u(x_0(t),t))\} - \mu\varepsilon_1 u(0,t)\varphi(x)$$
$$\geq (1-\varepsilon_1^\alpha)g(\varepsilon_1 u(0,t)) - \varepsilon_1^\alpha f(u(0,t)) - \mu\varepsilon_1 u(0,t) > 0 \quad \text{in } \overline{B(R')} \times (0,T_2].$$

Now, by the definition of T_2 (> 0) we see that $J(x,t) \geq 0$ in $\overline{B(R')} \times (0,T_2]$ as above-mentioned and there exists $x_1 \in \overline{B(R')}$ such that $J(x_1,T_2) = 0$. Then, we note $x_1 \in B(R')$, since $J(x,t) = u(x,t) > 0$ on $\partial B(R') \times (0,T)$.

Hence, $J_t(x_1,T_2) \leq 0$ and $\Delta J(x_1,T_2) \geq 0$, and so

$$\alpha\{u(x_1,t_2)\}^{\alpha-1}J_t(x_1,T_2) - \Delta J(x_1,T_2) \leq 0.$$

This is a contradiction to (4.10) and so we get $T_2 = T$. The proof is complete. □

Lemma 4.4. *Let u be as in Lemma 4.3. If h in (4.2) is large enough, then*

$$\lim_{t\uparrow T} u(x,t) = \infty \quad \text{for each } x \in B(R'). \tag{4.11}$$

Proof. Since T is the maximum existence time of u, we see by Proposition 2.10(ii) that

$$\lim_{t\uparrow T} u(0,t) = \lim_{t\uparrow T} ||u(\cdot,t)||_{L^\infty(B(R))} = \infty. \tag{4.12}$$

Hence, by (4.7) we get (4.11). □

Lemma 4.5. *Let u be as in Lemma 4.3. If h in (4.2) is large enough, then*

$$\int_0^T g(u(x_0(t),t))\,dt = \infty. \tag{4.13}$$

Proof. Assume to the contrary that

$$\int_0^T g(u(x_0(t),t))\,dt < \infty. \tag{4.14}$$

Let $R_2 \in (R_1,R')$ and let $\varphi(x) \in C_0^\infty(B(R_2)\backslash\overline{B(R_1)})$ satisfy $\varphi(x) \geq 0$ in $B(R_2)\backslash\overline{B(R_1)}$ and $\int_{B(R_2)\backslash\overline{B(R_1)}} \varphi(x)\,dx = 1$. Considering $\varphi(x)$ as a test function in the integral identity (2.2) with $D = B(R_2)\backslash\overline{B(R_1)}$, we have

$$\int_{B(R_2)\backslash\overline{B(R_1)}} u^\alpha(x,t)\varphi(x)\,dx - \int_{B(R_2)\backslash\overline{B(R_1)}} u^\alpha(x,0)\varphi(x)\,dx \tag{4.15}$$
$$= \int_0^t \int_{B(R_2)\backslash\overline{B(R_1)}} \{u\Delta\varphi + f(u)\varphi + g(u(x_0(t),t))\varphi\}\,dxdt.$$

Here, we note by (A5), (A11) and Lemma 2.12 that for some constant $C_1 > 0$,

$$u(x,t) \leq g(u(x_0(t),t)) + C_1 \quad \text{for } (x,t) \in B(R_2)\backslash\overline{B(R_1)} \times (0,T),$$

$$f(u(x,t)) \leq g(u(x_0(t),t)) + C_1 \quad \text{for } (x,t) \in B(R_2)\backslash\overline{B(R_1)} \times (0,T).$$

Hence, by (4.14), (4.15) and the monotonicity of $u(r,t)$ with respect to r, we have

$$\begin{aligned} u^\alpha(R_2,t) &\leq \int_{B(R_2)\backslash\overline{B(R_1)}} u^\alpha(x,t)\varphi(x)\,dx \qquad (4.16)\\ &\leq 2\int_0^t \{g(u(x_0(t),t)) + C_1\}dt \int_{B(R_2)\backslash\overline{B(R_1)}} \{|\Delta\varphi| + \varphi\}\,dx \\ &\quad + \int_{B(R_2)\backslash\overline{B(R_1)}} u^\alpha(x,0)\varphi(x)\,dx < \infty \quad \text{for } t \in (0,T). \end{aligned}$$

This is a contradiction to (4.11) if h is large enough in (4.2), and so we get (4.13). The proof is complete. □

Proof of Theorem 4.1 (continued). Assume that the initial data u_0 satisfies $u_0(0) \geq M\ (> 0)$ and (4.1). Then, clearly, u_0 satisfies (4.6) and

$$\inf_{x\in B(R_1)} u_0(x) \geq \varepsilon_1 u_0(0). \qquad (4.17)$$

Hence, by Lemma 4.5, if M is large enough then $T < T_1$ and (4.13) holds. Therefore, by Proposition 3.7 we have $S = \overline{B(R)}$. The proof is complete. □

5. Single point blow-up I

In this section, we consider the case $g(\xi) = o(f(\xi))$ and prove (I) in Theorem 1.4 and (I) in Theorem 1.6. These results follow from the following theorem:

Theorem 5.1. *Let $\Omega = B(R)$. Assume $(A1)$–$(A3)$, $(A6), (A7), (A10)$ and $x_0(t) \neq 0$ for $t \geq 0$. Let $u(x,t)$ be a blow-up weak solution of* (1.1)-(1.3) *with the maximum existence time $T > 0$, namely, $T < \infty$. Further, in the case $0 < \alpha < 1$, assume that $u(x,t)$ is nondecreasing in $t \geq 0$ for each $x \in B(R)$. Then, $u(x,t)$ blows up only at the origin; that is, $S = \{0\}$.*

The method of the proof is similar to that of Fukuda-Suzuki [13]. The following lemma plays an essential role in proving Theorem 5.1.

Lemma 5.2. *Let $\Omega = B(R)$. Assume $(A1)$–$(A3)$, $(A6), (A7), (A10)$ and $x_0(t) \neq 0$ for $t \geq 0$. Let $u(x,t)$ be a blow-up weak solution of* (1.1)-(1.3) *with the maximum existence time $T \in (0,\infty)$. Let $D \subset B(R_1)$ be a domain where $R_1 \in (0,R)$ satisfies $R_1 < \inf_{t\in[0,T]} |x_0(t)|$, and assume that $u_0(x) > 0$ in $B(R_1)$. Moreover, in the case $0 < \alpha < 1$, assume*

$$\partial_t u(x,t) \geq 0 \quad \text{in } D \times (0,T). \qquad (5.1)$$

If there exist $\nu \in \mathbf{S}^{N-1}$ *and* $\delta > 0$ *such that*

$$\nu \cdot \nabla u(x,t) \le -\delta |\nabla u(x,t)| < 0 \quad \text{in } D \times (0,T), \tag{5.2}$$

then u *does not uniformly blow up in* D*:*

$$\inf_{x \in D} u(x,t) \le M < \infty \quad \text{in } t \in (0,T). \tag{5.3}$$

Proof. The method of the proof is similar to that of K. Mochizuki-R. Suzuki [16] (see also [12, 8]). We show this lemma by contradiction assuming

$$\sup_{t \in (0,T)} \inf_{x \in D} u(x,t) = \infty. \tag{5.4}$$

Then, there exists a sequence $\{t_n\} \subset (0,T)$ such that $t_n \uparrow T$ and

$$\lim_{n \to \infty} \inf_{x \in D} u(x,t_n) = \infty. \tag{5.5}$$

Hence, for any compact set $K \subset D$,

$$\lim_{t \to \infty} \inf_{x \in K} u(x,t) = \infty. \tag{5.6}$$

In fact, this follows from assumption (5.1) in the case $0 < \alpha < 1$ and the method of [8] in the case $\alpha = 1$ (see the proof of Lemma 4.3 of [8]).

For $a = (a_1, \ldots, a_N) \in D$ and $\gamma > 0$, put

$$\omega(\gamma) = \omega(a;\gamma) \equiv \{x = (x_1 \ldots, x_N) \in \mathbf{R^N}\,;\, a_j < x_j < a_j + \gamma,\ j = 1, \ldots, N\}. \tag{5.7}$$

We choose $\gamma > 0$ so small that the closure of the rectangular region $\omega(\gamma)$ is included in D: $\overline{\omega(\gamma)} \subset D$. We note by Proposition 2.6 and Lemma 2.12 that $u(x,t) > 0$ in $B(R_1) \times (0,T)$, for each $t \ge 0$ $u(x,t) = u(r,t)$ $(r = |x|)$ is radially symmetric in $x \in B(R)$ and nonincreasing in $r \ge 0$, $\partial u / \partial r < 0$ in $(0,R) \times (0,T)$, and u is a C^∞ classical solution of (1.1) in $D \times (0,T)$.

Without loss of generality we can choose $\nu = (-1, 0, \ldots, 0)$ in (5.2). It then follows that

$$\delta^{-1} \partial_1 u \ge |\nabla u| > 0, \text{ where } \partial_1 = \partial / \partial x_1 \quad \text{in } \overline{\omega(\gamma)} \times (0,T). \tag{5.8}$$

We put $\beta(u) = u^\alpha$ and

$$J(x,t) = \partial_1 u(x,t) - \rho(x) \Phi(u(x,t)), \tag{5.9}$$

where

$$\rho(x) = \rho_{a,\gamma,\varepsilon}(x) = \varepsilon \prod_{k=1}^{N} \sin \pi (x_k - a_k)/\gamma \tag{5.10}$$

and Φ is as given in (A6). By a direct calculation, J is shown to satisfy the equation

$$(\beta' J)_t - \Delta J = \partial_1 \{f(u) + g(u(x_0(t),t))\} - \rho \Phi' \{f(u) + g(u(x_0(t),t))\}$$

$$
\begin{aligned}
&- \rho\beta'' u_t\Phi - N(\pi/\gamma)^2\rho\Phi + 2\Phi'\nabla\rho\cdot\nabla u + \rho\Phi''|\nabla u|^2 \qquad (5.11)\\
&= \rho\{f'\Phi - \Phi' f - \Phi' g(u(x_0(t),t)) - N(\pi/\gamma)^2\Phi\} + f'J\\
&\qquad - \rho\beta'' u_t\Phi + 2\Phi'\nabla\rho\cdot\nabla u + \rho\Phi''|\nabla u|^2
\end{aligned}
$$

in $\omega(\gamma)\times(\tau,T)$ $(0<\tau<T)$. Since $\beta'' u_t \le 0$ by (5.1) (in the case $\alpha = 1$, $\beta''\equiv 0$) and

$$
|\nabla\rho\cdot\nabla u| \le \frac{\varepsilon N\pi}{\gamma\delta}\partial_1 u = \frac{\varepsilon N\pi}{\gamma\delta}(J+\rho\Phi) \quad \text{in } \overline{\omega(\gamma)}\times(0,T), \tag{5.12}
$$

by (5.8), we have

$$
\begin{aligned}
&(\beta' J)_t - \Delta J - \Big(f' - \frac{2\varepsilon N\pi}{\gamma\delta}\Phi'\Big)J \qquad (5.13)\\
&\ge \rho\Big\{f'\Phi - (1+\varepsilon_0)\Phi' f + \Phi'\{\varepsilon_0 f - \varepsilon_0 f(u(x_0(t),t))\}\\
&\quad + \frac{\varepsilon_0 f(u(x_0(t),t)) - g(u(x_0(t),t))}{\Phi}\Phi'\Phi - \frac{N(\pi/\gamma)^2}{\Phi'}\Phi\Phi' - \frac{2\varepsilon N\pi}{\gamma\delta}\Phi'\Phi\Big\},
\end{aligned}
$$

where $\varepsilon_0 > 0$ is as in (A6).

By (A10), we have for some constant $C>0$,

$$
\varepsilon_0 f(u(x_0(t),t)) - g(u(x_0(t),t)) \ge -C \quad \text{in } \overline{\omega(\gamma)}\times(0,T). \tag{5.14}
$$

We note that $\Phi(\xi)\to\infty$, $\Phi'(\xi)\to\infty$ and $f'(\xi)\to\infty$ as $\xi\to\infty$ by (A6). Hence, choosing τ very close to T, we have

$$
\frac{\varepsilon_0 f(u(x_0(t),t)) - g(u(x_0(t),t))}{\Phi} - \frac{N(\pi/\gamma)^2}{\Phi'} \ge -\frac{c}{2} \quad \text{in } \omega(\gamma)\times(\tau,T) \tag{5.15}
$$

with c given as in (A6) and we have by (A6),

$$
f(u(x,t)) - f(u(x_0(t),t)) \ge 0 \quad \text{in } \omega(\gamma)\times(\tau,T) \tag{5.16}
$$

since $u(x,t)\ge u(x_0(t),t)$ in $\overline{\omega(\gamma)}\times(0,T)$.

We fix such a τ. Thus, from (5.15), (5.16) and (A6), we obtain for small $\varepsilon>0$,

$$
(\beta' J)_t - \Delta J - \Big(f' - \frac{2\varepsilon N\pi}{\gamma\delta}\Phi'\Big)J \ge \rho\Big(c - \frac{c}{2} - \frac{2\varepsilon N\pi}{\gamma\delta}\Big)\Phi\Phi' \ge 0 \quad \text{in } \omega(\gamma)\times(\tau,T). \tag{5.17}
$$

We can apply the maximum principle to (5.17) to obtain

$$
J(x,t) > 0 \quad \text{in } \omega(\gamma)\times(\tau,T). \tag{5.18}
$$

In fact, by (5.8) the inequality

$$
J(x,\tau) = \partial_1 u(x,\tau) - \rho(x)\Phi(u(x,\tau)) > 0 \quad \text{on } \omega(\gamma) \tag{5.19}
$$

holds for small $\varepsilon > 0$. Moreover, since $\rho(x) = 0$ on $\partial\omega(\gamma)$, we have

$$J(x,t) = \partial_1 u(x,t) > 0 \quad \text{on } \partial\omega(\gamma) \times (\tau, T). \tag{5.20}$$

We rewrite (5.18) as

$$\frac{\partial_1 u(x,t)}{\Phi(u)} > \rho(x) \quad \text{in } \omega(\gamma) \times (\tau, T) \tag{5.21}$$

and integrate both sides by x_1 over the interval $(a_1, a_1 + \gamma)$. Then, we have for any $t \in (\tau, T)$,

$$\begin{aligned} \int_{u(a_1,x',t)}^{u(a_1+\gamma,x',t)} \frac{du}{\Phi(u)} \varepsilon \prod_{k=2}^{N} \sin \frac{\pi}{\gamma}(x_k - a_k) \int_0^\gamma \sin \frac{\pi}{\gamma} x_1 \, dx_1 \\ = \frac{2\gamma\varepsilon}{\pi} \prod_{k=2}^{N} \sin \frac{\pi}{\gamma}(x_k - a_k) > 0, \end{aligned} \tag{5.22}$$

where $x' = (x_2, \cdots, x_N)$ and $a_j < x_j < a_j + \gamma$ $(j = 2, \cdots, N)$. By (1.17) and (5.6), the left side decays to 0 as $t \uparrow T$. Hence, a contradiction occurs and the lemma is proved. □

Proof of Theorem 5.1. Let $\Omega = B(R)$. Let $u(x,t)$ be a weak solution of (1.1)-(1.3) with the maximum existence time $T \in (0, \infty)$. Then, clearly $u_0(x) \not\equiv 0$. Since for each $t \geq 0$ $u(x,t) = u(r,t)$ $(r = |x|)$ is radially symmetric in $x \in B(R)$ and nonincreasing in $r \geq 0$, we see $u_0(0) > 0$.

Assume to the contrary that $S \neq \{0\}$. Then, there exists $\tilde{x}_1 (\neq 0) \in \overline{B(R)}$ such that $\tilde{x}_1 \in S$. Namely, there exists a sequence $(\tilde{x}_n, t_n) \in B(R) \times (0, T)$ satisfying $\tilde{x}_n \to \tilde{x}_1$, $t_n \uparrow T$ and $u(\tilde{x}_n, t_n) \to \infty$. Hence, letting $0 < r_1 < |\tilde{x}_1|$ we have

$$\lim_{n\to\infty} \inf_{|x|<r_1} u(x, t_n) = \infty. \tag{5.23}$$

Let $R_1 \in (0, R)$ be as in Lemma 5.2 and satisfy $u_0(x) > 0$ in $B(R_1)$. We note that $u > 0$ in $B(R_1) \times (0, T)$ and $u \in C^\infty(B(R_1) \times (0, T))$. Let $0 < r_2 < R_2 = \min\{r_1, R_1\}$, put $D(\gamma) = \{x = (x_1, \ldots, x_N) \in \mathbf{R^N} \,;\, -r_2 - \gamma < x_1 < -r_2,\ -\gamma < x_j < \gamma,\ j = 2, \ldots, N\}$ and choose $\gamma > 0$ small enough to satisfy $D(\gamma) \subset B(R_2)$. Then, by (5.23),

$$\lim_{n\to\infty} \inf_{x \in D(\gamma)} u(x, t_n) = \infty. \tag{5.24}$$

Further, it follows from Lemma 2.12 that if $\nu = (-1, 0, \cdots, 0)$, then

$$\nu \cdot \nabla u(x,t) = -\frac{\partial u}{\partial x_1} = \frac{x_1}{r}|\nabla u| \leq -\frac{r_2}{R_1}|\nabla u| = \frac{r_2}{R_1}\frac{\partial u}{\partial r} < 0 \text{ in } D(\gamma) \times (0, T). \tag{5.25}$$

We note that in the case $0 < \alpha < 1$, $\partial_t u(x,t) \ge 0$ in $D(\gamma) \times (0,T)$ by the assumption.

Thus, applying Lemma 5.2 with $D = D(\gamma)$, we get

$$\inf_{x \in D(\gamma)} u(x,t) \le M < \infty \quad \text{in } t \in (0,T). \tag{5.26}$$

This is a contradiction of (5.24) and we obtain $S = \{0\}$. The proof is complete. □

6. Single point blow-up II

In this section, we prove (ii) of (II) in Theorem 1.4 (or Theorem 1.6). For this aim, we assume

(A6)' There exists a function $\Phi(\xi)$ such that

$$\Phi(\xi) > 0,\ \Phi'(\xi) > 0, \text{ and } \Phi''(\xi) \ge 0 \quad \text{for } \xi > 0; \tag{6.1}$$

$$\int_1^\infty \frac{d\xi}{\Phi(\xi)} < \infty; \tag{6.2}$$

there are constants $c > 0$ and $\xi_2 > 0$ such that

$$f'(\xi)\Phi(\xi) - f(\xi)\Phi'(\xi) \ge c\Phi(\xi)\Phi'(\xi) \text{ for } \xi > \xi_2. \tag{6.3}$$

This condition is a weaker condition than (A6) and a stronger condition than (A4). Under this condition we construct a single point blow-up solution when $x_0(t) \ne 0$ for $t \ge 0$. Here, we do not require the assumption (A11).

(ii) of (II) in Theorem 1.4 (or Theorem 1.6) follows from the next theorem:

Theorem 6.1. *Let $\Omega = B(R)$. Assume (A1)–(A3), (A6)$'$, (A7) and assume $x_0(t) \ne 0$ for $t \ge 0$. When $0 < \alpha < 1$, assume (A8). Then, there exists an initial data u_0 satisfying (A9) such that the maximal solution of* (1.1)-(1.3) *blows up only at the origin; that is, $S = \{0\}$.*

Remark 6.2. When $\alpha = 1$, by the uniqueness of solutions [20], we see that any solution of (1.1)-(1.3) coincides with the maximal solution of (1.1)-(1.3).

The method of the proof is similar to that of Fukuda-Suzuki [13]. We shall use the following auxiliary function for the solution u of (1.1)-(1.3), as in the proof of Lemma 5.2.

$$J(x,t) = \partial_1 u(x,t) - \rho(x)\Phi(u(x,t)), \tag{6.4}$$

where $\rho(x) = \rho_{a,\gamma,\varepsilon}(x)$ is as in (5.10) and Φ is as given in (A6)'.

We first have a lemma which is similar to Lemma 5.2:

Lemma 6.3. *Let $\Omega = B(R)$. Assume (A1)–(A3), (A6)$'$, (A7) and $x_0(t) \neq 0$ for $t \geq 0$. Let $T > 0$, let $R_1 \in (0, R)$ satisfy $R_1 < \inf_{t\in[0,T]} |x_0(t)|$ and let $\omega(\gamma) \subset B(R_1)$, where $\omega(\gamma) = \omega(a; \gamma)$ is defined by (5.7). Assume $u_0(x) > 0$ in $B(R_1)$ and let u be a weak solution of (1.1)-(1.3) in $\overline{B(R)} \times (0, T)$ satisfying (5.8). When $0 < \alpha < 1$, assume (5.1) with $D = \overline{\omega(\gamma)}$. If the inequality*

$$g(u(x_0(t), t)) \leq K \qquad \text{in } t \in [0, T) \tag{6.5}$$

holds for some $K > 0$, then there exists a constant $\eta_0 = \eta_0(\gamma, K) > 0$ such that if $u \geq \eta_0$ in $\overline{\omega(\gamma)} \times [0, T)$,

$$(\beta' J)_t - \Delta J - \Big(f' - \frac{2\varepsilon N\pi}{\gamma\delta}\Phi'\Big)J \geq 0 \ \text{ in } \omega(\gamma) \times (0, T) \qquad \text{for } 0 < \varepsilon < \frac{\gamma\delta c}{4N\pi}, \tag{6.6}$$

where c is as given in (A6)$'$ and $\beta(u) = u^\alpha$.

Proof. By the assumption (6.5), we have in a similar manner as in the proof of Lemma 5.2,

$$\begin{aligned}
&(\beta' J)_t - \Delta J - \left(f' - \frac{2\varepsilon N\pi}{\gamma\delta}\Phi'\right) J \qquad\qquad (6.7)\\
&\geq \rho\left\{f'\Phi - \Phi' f - \frac{g(u(x_0(t), t))}{\Phi}\Phi'\Phi - \frac{N(\pi/\gamma)^2}{\Phi'}\Phi\Phi' - \frac{2\varepsilon N\pi}{\gamma\delta}\Phi'\Phi\right\}\\
&\geq \rho\left\{f'\Phi - \Phi' f - \left(\frac{K}{\Phi} + \frac{N(\pi/\gamma)^2}{\Phi'} + \frac{c}{2}\right)\Phi'\Phi\right\} \qquad \text{in } \omega(\gamma) \times (0, T)\\
&\qquad \text{for } 0 < \varepsilon < \frac{\gamma\delta c}{4N\pi}.
\end{aligned}$$

Since $\Phi'(\xi) \to \infty$ and $\Phi(\xi) \to \infty$ as $\xi \to \infty$ as in the proof of Lemma 5.2, there exists a constant $\eta_0 > \xi_2$ where ξ_2 is as given in (6.3) such that if $\xi \geq \eta_0$, then

$$\frac{K}{\Phi(\xi)} + \frac{N(\pi/\gamma)^2}{\Phi'(\xi)} \leq \frac{c}{2}. \tag{6.8}$$

Thus, we get (6.6) by (6.3) and (6.7) if $u(x, t) \geq \eta_0$ in $\overline{\omega(\gamma)} \times [0, T)$. The proof is complete. □

We now prove Theorem 6.1. Let $\Omega = B(R)$ and assume (A1), (A6)' and $x_0(t) \neq 0$ for $t \geq 0$.

Let $T_1 > 0$ and let R_1, R_2, $R_3 \in (0, R)$ satisfy $0 < R_3 < R_2 < R_1 < \inf_{t\in[0,T_1]} |x_0(t)|$. Put $R_0 = R$ and let $h_i(r) \in C^2([0, \infty))$ ($i = 1$, 2, 3) satisfy $h_i(r) = 1$ for $0 \leq r \leq R_i$, $h_i(r) = 0$ for $r \geq R_{i-1}$, $h_i'(r) < 0$ for $R_i < r < R_{i-1}$

and for some $\delta_i \in (0, R_{i-1} - R_i)$,

$$h_i''(r) + \frac{N-1}{r} h_i'(r) \geq 0 \quad \text{in } R_{i-1} - \delta_i \leq r \leq R_{i-1}. \tag{6.9}$$

We shall consider the following function $v_{k,\ell,m}(x)$ (k, ℓ, $m > 0$) as the initial data u_0 and show that the solution of (1.1)-(1.3) blows up only at the origin for large k, ℓ, m:

$$v_{k,\ell,m}(x) = kh_1(x) + \ell h_2(x) + mh_3(x) \quad \text{in } \mathbf{R}^{\mathbf{N}}, \tag{6.10}$$

where $h_i(x) = h_i(r)$ ($r = |x|$) for each $i = 1, 2. 3$. We first show the following lemma.

Lemma 6.4. *There exists a constant $k_0 > 1$ such that for any $k \geq k_0$, $\ell \geq 1$ and $m \geq 1$,*

$$\Delta v_{k,\ell,m} + f(v_{k,\ell,m}) \geq 0 \quad \text{in } \mathbf{R}^{\mathbf{N}}. \tag{6.11}$$

Proof. We have

$$\Delta v_{k,\ell,m} + f(v_{k,\ell,m}) = k\Delta h_1 + f(kh_1) = k\Big\{h_{1,rr} + \frac{N-1}{r} h_{1,r}\Big\} + f(kh_1) \geq 0 \tag{6.12}$$

$$\text{for } R_0 - \delta_1 \leq |x| \leq R_0.$$

We have for large $k > 1$,

$$\begin{aligned} \Delta v_{k,\ell,m} + f(v_{k,\ell,m}) &= k\Delta h_1 + f(kh_1) \qquad (6.13)\\ &= k\left\{\Delta h_1 + h_1(x) \int_0^1 f'(\theta k h_1(x))\, d\theta + \frac{f(0)}{k}\right\} \geq 0 \\ &\qquad \text{for } R_1 \leq |x| \leq \tilde{r}_1 = R_0 - \delta_1. \end{aligned}$$

In fact, since $\theta k h_1(x) \geq (1/2)\tilde{h}_1 k$ for $R_1 \leq |x| \leq R_0 - \delta_1$ and $\theta \geq 1/2$ where $\tilde{h}_1 = \min_{R_1 \leq |x| \leq R_0 - \delta_1} h_1(x) > 0$, $f'(\xi) \geq -C$ ($C > 0$) for $\xi \geq 0$ and $f'(\xi) \to \infty$ as $\xi \to \infty$ by $(A6)'$, we have for $R_1 \leq |x| \leq \tilde{r}_1$,

$$\begin{aligned} &h_1(x) \int_0^1 f'(\theta k h_1(x))\, d\theta \qquad (6.14)\\ &\quad \geq -\frac{C}{2} h_1(x) + \tilde{h}_1 \int_{1/2}^1 f'(\theta k h_1(x))\, d\theta \to \infty \quad \text{as } k \to \infty, \end{aligned}$$

which implies (6.13).

By a way similar to above, we have

$$\Delta v_{k,\ell,m} + f(v_{k,\ell,m}) \geq \ell \left\{h_{2,rr} + \frac{N-1}{r} h_{2,r}\right\} \geq 0 \text{ for } R_1 - \delta_2 \leq |x| \leq R_1, \tag{6.15}$$

$$\Delta v_{k,\ell,m} + f(v_{k,\ell,m}) = \ell\Delta h_2 + f(k+\ell h_2) \tag{6.16}$$
$$\geq \ell\left\{\Delta h_2 + h_2(x)\int_0^1 f'(\theta(k+\ell h_2))\,d\theta + h_2(x)\frac{f(0)}{k+\ell h_2(x)}\right\} \geq 0$$
$$\text{for } R_2 \leq |x| \leq R_1-\delta_2 \text{ and large } k>1,$$

$$\Delta v_{k,\ell,m} + f(v_{k,\ell,m}) \geq m\left\{h_{3,rr} + \frac{N-1}{r}h_{3,r}\right\} \geq 0 \text{ for } R_2-\delta_3 \leq |x| \leq R_2, \tag{6.17}$$

$$\Delta v_{k,\ell,m} + f(v_{k,\ell,m}) = m\Delta h_3 + f(k+\ell+mh_3) \tag{6.18}$$
$$\geq m\left\{\Delta h_3 + h_3(x)\int_0^1 f'(\theta(k+\ell+mh_3))\,d\theta + h_3(x)\frac{f(0)}{k+\ell+mh_3(x)}\right\} \geq 0$$
$$\text{for } |x| \leq R_2-\delta_3 \text{ and large } k>1.$$

Combining (6.12)-(6.18), we obtain (6.11). □

So, we shall consider a weak solution u of (1.1)-(1.3) with the initial data $u_0(x) = v_{k_0,\ell,m}(x)$, where $k_0>0$ is as in Lemma 6.4.

Lemma 6.5. *When $0<\alpha<1$, assume (A8). Put $k=k_0$. Let $u(x,t)$ be the maximal weak solution of (1.1)-(1.3) with the initial data $u_0(x) = v_{k_0,\ell,m}(x)$ and let $T>0$ be the maximum existence time of u. Then, for any $K>1$, there exists a constant $\ell_0 = \ell_0(K,T_1) \geq 1$ independent of $m \geq 1$ such that the following result holds: For any $\ell \geq \ell_0$, we have $T<T_1$ and there exists a constant $M = M(K,\ell)>0$ independent of $m\geq 1$ such that if*

$$g(u(x_0(t),t)) \leq K \text{ for } 0 \leq t < T' \tag{6.19}$$

for some $0<T'\leq T$, then

$$u(x,t) \leq M \qquad \text{on } |x| = R_1,\ 0 \leq t < T'. \tag{6.20}$$

Furthermore, if $T'=T$, then the maximal solution u blows up only at the origin.

Proof. Put

$$\tilde{r} = \frac{R_1+R_2}{2}. \tag{6.21}$$

Let $a = (-\tilde{r},0,\cdots,0) \in \mathbf{R^N}$, $\omega(\gamma) = \omega(a;\gamma)$ and $\rho(x) = \rho_{a,\gamma,\varepsilon}(x)$, where $\omega(a;\gamma)$ and $\rho(x) = \rho_{a,\gamma,\varepsilon}(x)$ are as in Lemma 6.3 and in (6.4) respectively. We choose $\gamma>0$ small enough to satisfy $\overline{\omega(\gamma)} \subset B(R_1)\backslash\overline{B(R_2)}$. Let $u(x,t)$

be the maximal weak solution of (1.1)-(1.3) with $u_0(x) = v_{k_0,\ell,m}(x)$ and let $T > 0$ be the maximum existence time of u. Then, as in (5.25), we have

$$\frac{\partial u}{\partial x_1}(x,t) \geq \frac{R_2}{R_1}|\nabla u| > 0 \text{ in } \overline{\omega(\gamma)} \times [0,T), \tag{6.22}$$

and so (5.8) holds with $\delta = R_2/R_1$. We note that $u_0 = v_{k_0,\ell,m}$ satisfies (A2),(A7) and (A9) by Lemma 6.4, and so we note by Lemma 2.13 that $\partial_t u \geq 0$ in $B(R) \times (0,T)$ when $0 < \alpha < 1$.

Put

$$J(x,t) = \frac{\partial}{\partial x_1}u(x,t) - \rho(x)\Phi(u(x,t)),$$

where Φ is as given in $(A6)'$. Assume

$$g(u(x_0(t),t)) \leq K \quad \text{for } 0 \leq t \leq T'$$

for some $K > 1$ and $0 < T' \leq \min\{T, T_1\}$.

By virtue of Lemma 6.3, there exists $\eta_0 = \eta_0(\gamma, K) > 0$ such that if $u \geq \eta_0$ in $\overline{\omega(\gamma)} \times [0,T')$ and $0 < \varepsilon < \gamma\delta c/4N\pi$ with $\delta = R_2/R_1$, then

$$(\beta' J)_t - \Delta J - \left(f' - \frac{2\varepsilon N\pi}{\gamma\delta}\Phi'\right) J \geq 0 \quad \text{in } \omega(\gamma) \times (0,T'), \tag{6.23}$$

where c is as given in $(A6)'$ and $\beta(u) = u^\alpha$.

Indeed, by Proposition 3.5, there exists $\ell_0 = \ell_0(\gamma, \eta_0, T_1) > 0$ such that if $\ell \geq \ell_0$ then $T < T_1$ and

$$\inf_{x \in \omega(\gamma)} u(x,t) \geq \eta_0 \quad \text{for } 0 < t < T. \tag{6.24}$$

Thus, we see that (6.23) holds for $\ell \geq \ell_0$ and $\varepsilon \in (0, \gamma\delta c/4N\pi)$.

Let $\ell \geq \ell_0$ be fixed. If $\varepsilon \in (0, \gamma\delta c/4N\pi)$ is small enough, then

$$\begin{aligned} J(x,0) &= \frac{\partial}{\partial x_1}u_0(x) - \rho(x)\Phi(u_0(x)) \qquad (6.25)\\ &= \ell\frac{\partial}{\partial x_1}h_2(x) - \rho(x)\Phi(k_0 + \ell h_2(x))0 \quad \text{in } \overline{\omega(\gamma)}. \end{aligned}$$

Moreover, since $\rho(x) = 0$ on $\partial\omega(\gamma)$, we have by (6.22),

$$J(x,t) = \frac{\partial}{\partial x_1}u(x,t) > 0 \quad \text{in } \partial\omega(\gamma) \times (0,T'). \tag{6.26}$$

Thus, because of (6.23), the maximum principle leads to

$$J(x,t) > 0 \quad \text{in } \omega(\gamma) \times (0,T'), \tag{6.27}$$

that is,

$$\frac{1}{\Phi}\frac{\partial}{\partial x_1}u(x,t) > \rho(x). \tag{6.28}$$

So, putting

$$G(u) = \int_u^\infty \frac{1}{\Phi(\xi)}\, d\xi \,(< \infty) \tag{6.29}$$

and integrating both sides of (6.28) by x_1 over the interval $(-\tilde{r}, -\tilde{r}+\gamma)$ we have, as in (5.22),

$$-\{G(u(-\tilde{r}+\gamma, x', t)) - G(u(-\tilde{r}, x', t))\} > \frac{2\gamma\varepsilon}{\pi} \prod_{k=2}^{N} \sin \frac{\pi}{\gamma} x_k \text{ in } (0, T'), \tag{6.30}$$

where $x' = (x_2, \ldots, x_N)$ and $0 < x_j < \gamma$ $(j = 2, \cdots, N)$.

Thus, putting $x' = (\gamma/2, \cdots, \gamma/2)$ we have

$$u(x,t) \le u(-\tilde{r}, x', t) < G^{-1}(2\gamma\varepsilon/\pi) \quad \text{on } |x| = R_1,\ t \in [0, T'), \tag{6.31}$$

where $G^{-1}(\eta)$ is the inverse function of $\eta = G(u)$.

When $T' = T$, similarly to the proof of Theorem 5.1, we see by Lemma 6.3 and Lemma 2.13 that the maximal solution u blows up only at the origin. The proof is complete. □

Let $(k, \ell) = (k_0, \ell_0)$ and let $w_0(x) = w_0(r) \in C([R_1, R])$ $(r = |x|)$ be a radially symmetric function in x satisfying that

$$w_0(r) \begin{cases} = M + 1 & \text{on } r = R_1, \\ \ge k_0 h_1(r) + 1\, (= v_{k_0,\ell_0,m}(r) + 1) & \text{in } R_1 < r < |x_0(0)|, \\ = k_0 h_1(r) + 1\, (= v_{k_0,\ell_0,m}(r) + 1) & \text{in } |x_0(0)| \le r \le R, \end{cases} \tag{6.32}$$

where $M = M(K, \ell_0)$ is as in Lemma 6.5. We shall compare u with the solution w of the initial boundary-value problem

$$(w^\alpha)_t = \Delta w + f(w) + g(w(x_0(t), t)), \quad \text{in } \{R_1 < |x| < R\} \times (0, \tilde{T}), \tag{6.33}$$

$$w(x,t) = M + 1, \quad \text{on } |x| = R_1,\ 0 \le t < \tilde{T}, \tag{6.34}$$

$$w(x,t) = 1, \quad \text{on } |x| = R,\ 0 \le t < \tilde{T}, \tag{6.35}$$

$$w(x,0) = w_0(x), \quad \text{in } \{R_1 < |x| < R\}. \tag{6.36}$$

Lemma 6.6. *There exists a solution $w(x,t)$ of (6.33)-(6.36) in $\{R_1 < |x| < R\} \times (0, T_w)$ for some $T_w > 0$ such that*

$$g(w(x_0(t), t)) \le g(k_0 h_1(x_0(0)) + 1) + \frac{1}{2} \quad \text{in } [0, T_w). \tag{6.37}$$

Proof. Similarly, as in the proof of Proposition 2.3, we can see the existence of a local solution w of (6.33)-(6.36) in $\{R_1 < |x| < R\} \times (0, \tilde{T})$. Since $w_0(x_0(0)) = k_0 h_1(x_0(0)) + 1$, by the continuity of w we also see

$$g(w(x_0(t), t)) \le g(k_0 h_1(x_0(0)) + 1) + \frac{1}{2} \quad \text{in } [0, T_w)$$

for small $T_w > 0$. The proof is complete. □

Proof of Theorem 6.1. Put

$$K \equiv g(k_0 h_1(x_0(0)) + 1) + 1. \tag{6.38}$$

Let $k_0 \geq 1$ be as in Lemma 6.4 and $\ell_0 = \ell_0(K, T_1) \geq 1$ be as in Lemma 6.5 for this constant $K > 0$. Let u be the maximal weak solution of (1.1)-(1.2) with the initial data $u_0(x) = v_{k_0,\ell_0,m}(x)$ and the maximum existence time $T > 0$. We note that $u_0(x)$ satisfies (A2), (A7) and (A9).

Since (A6)' leads to (A4), it follows from Proposition 3.5 that if $m \geq 1$ is large enough then $T < \min\{T_1, T_w\}$, where T_1 appears in the above and T_w is as in Lemma 6.6. Then, we shall show

$$g(u(x_0(t), t)) < g(u_0(x_0(0)) + 1) + 1 = K \text{ for } 0 \leq t < T. \tag{6.39}$$

Here, we note $u_0(x_0(0)) = v_{k_0,\ell_0,m}(x_0(0)) = k_0 h_1(x_0(0))$.

Assume to the contrary that (6.39) does not hold. Then, there exists a $T' \in (0, T)$ such that

$$g(u(x_0(t), t)) < K \quad \text{for } 0 \leq t < T' \tag{6.40}$$

and

$$g(u(x_0(T'), T')) = K. \tag{6.41}$$

Then, by Lemma 6.5, there exists $M = M(K, \ell_0) > 0$ such that

$$u(x, t) \leq M \quad \text{on } |x| = R_1,\ 0 \leq t < T'.$$

Hence, letting w be as in Lemma 6.6, by the comparison theorem (Proposition 2.7) we obtain $u \leq w$ in $\{R_1 \leq |x| \leq R\} \times [0, T')$, and so we get by Lemma 6.6, $g(u(x_0(t), t)) \leq g(w(x_0(t), t)) \leq g(k_0 h_1(x_0(0)) + 1) + 1/2$ in $[0, T')$, whence $g(u(x_0(T'), T')) \leq g(k_0 h_1(x_0(0)) + 1) + 1/2$. This is a contradiction of (6.41). Thus, we get (6.39).

Therefore, since T is the maximal existence time of u, Lemma 6.5 implies that the maximal solution u blows up only at the origin. The proof is complete. □

7. Proof of (iii) of (II) in Theorem 1.6

In this section, we prove (iii) of (II) in Theorem 1.6 when $\alpha = 1$. For this aim, we use the next condition.

(A14) For any $K > 0$, there exists a function $\Phi_K(\xi)$ such that

$$\Phi_K(\xi) > 0,\ \Phi'_K(\xi) > 0, \text{ and } \Phi''_K(\xi) \geq 0 \quad \text{for } \xi > 0; \tag{7.1}$$

$$\int_1^\infty \frac{d\xi}{\Phi_K(\xi)} < \infty; \tag{7.2}$$

there are constants $c > 0$ and $\xi_3 > K$ such that

$$f'(\xi)\Phi_K(\xi-\eta) - f(\xi)\Phi'_K(\xi-\eta) \geq c\Phi_K(\xi-\eta)\Phi'_K(\xi-\eta) \quad \text{for } \xi > \xi_3 \text{ and } \eta \in [0,K]. \tag{7.3}$$

The condition similar to (A14) is originally introduced by Bebernes-Bressan-Lacey (Theorem 4.6 of [2]), and has been formulated with necessary refinements here.

(iii) of (II) in Theorem 1.6 follows from the following theorem, which is first shown by [2] under stronger assumptions on f.

Theorem 7.1. *Let $\alpha = 1$ and $\Omega = B(R)$. Assume* (A1), (A2), (A7), (A14). *Let $u(x,t)$ be the blow-up weak solution of* (1.1)-(1.3) *with the maximum existence time $T < \infty$. Put*

$$G(t) = \int_0^t g(u(x_0(t),t))\,dt. \tag{7.4}$$

Then

(i) if $G(T) < \infty$ then $u(x,t)$ blows up only at the origin; namely, $S = \{0\}$;

(ii) if $G(T) = \infty$ then $u(x,t)$ blows up in the whole domain $B(R)$, namely, $S = \overline{B(R)}$.

Proof. Part (ii) was already shown by [2]. The method of the proof of (i) is similar to that of [2], [12] and [17]. So, we omit the proof. □

Remark 7.2. In Theorem 7.1, we do not need the assumption that $x_0(t) \neq 0$ in $t \geq 0$, which is assumed in Theorem 1.6.

Proof of (iii) of (II) in Theorem 1.6. We note that conditions (A6) and (A12) lead to condition (A14). Hence, (iii)(II) of Theorem 1.6 follows from Theorem 7.1. □

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