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HOMOGENIZATION OF NONLINEAR RANDOM PARABOLIC OPERATORS

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Abstract. We consider the homogenization of nonlinear random parabolic operators. Depending on the ratio between time and spatial scales different homogenization regimes are studied and the homogenization procedure is carried out. The parameter dependent auxiliary problem is investigated and used in the construction of the homogenized operator.

1. Introduction

Let $Q_0 \in \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $Q = (0,T) \times Q_0$. On Q, we consider nonlinear parabolic operators

$$L_\varepsilon u = D_t u - div(a(\frac{x}{\varepsilon^\beta}, \frac{t}{\varepsilon^\alpha}, u, D_x u)) + a_0(\frac{x}{\varepsilon^\beta}, \frac{t}{\varepsilon^\alpha}, u, D_x u).$$

Here the parabolicity means, in particular, that the leading elliptic part,

$$-div(a(\cdot,\cdot,u,D_x u)),$$

is strictly monotone with respect to the gradient $D_x u$ (more precise assumptions are formulated below). Consequently, the entire elliptic term on the right-hand side is pseudo monotone, but *not* monotone in general.

It is assumed that the temporal and spatial heterogeneities have random homogeneous nature which will be described more precisely later. We are interested in the asymptotic behavior of L_ε as $\varepsilon \to 0$. G-convergence theory for parabolic operators guarantees that the limiting operator L^* belongs to the same class of parabolic operators. G-convergence of nonlinear parabolic

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operators has been studied in [17, 19]. To find the form of L^*, some assumptions on the nature of spatial and temporal heterogeneities of a and a_0 need to be imposed. In the periodic setting the homogenization of nonlinear parabolic equations is carried out in [17]. Using the two-scale convergence, the homogenization of nonlinear parabolic equations for some values of α and β is investigated in [14]. In [4], the time homogenization of random nonlinear abstract parabolic equations has been studied. The homogenization of linear parabolic operators with almost periodic and random coefficients has been studied in [23, 22]. We would like also to mention several results on homogenization of nonlinear elliptic operators [2, 5, 10, 11, 15, 16]. We would like to note that general elliptic operators in divergence form are considered in [15, 16], including random homogenization, while articles [2, 5, 10, 11] are devoted to the homogenization of monotone second-order elliptic operators. For general references in the field of homogenization, we refer to [1, 3, 6, 7, 12, 17].

In this paper we consider the homogenization of nonlinear parabolic equations when the fluxes, a and a_0, are random homogeneous fields with respect to temporal and spatial variables. We show that the homogenized operator has the form

$$L^*u = D_t u - div(a^*(x,t,u,D_x u)) + a_0^*(x,t,u,D_x u),$$

where the calculation of a^* and a_0^* depends on the ratio between α and β. As in the case of linear operators depending on the ratio between α and β different regimes are considered: self-similar case ($\alpha = 2\beta$); non self-similar case ($\alpha < 2\beta$); non self-similar case ($\alpha > 2\beta$); spatial case ($\alpha = 0$); temporal case ($\beta = 0$). These regimes yield a different asymptotic behavior of L_ε which is determined by the solution of the auxiliary problem. The auxiliary problem contains a parameter, which is characterized by the ratio between α and β. Depending on the ratio between α and β, the solution of the auxiliary problem has a different nature that determines the homogenized operator. As in [22] the solution of the auxiliary problem does not have independent meaning, and we employ near solutions extensively.

The main idea in carrying out the homogenization procedure is as follows. First we construct a solution for the parabolic equation by rescaling the solution of the corresponding auxiliary problem. In this way the parameter involved in the auxiliary problem is set in terms of some power of ε. Next we study the convergence of the solutions or near solutions of the auxiliary problem as $\varepsilon \to 0$. Further, the results on the convergence of arbitrary solutions for G-converging sequences of operators allow us to calculate the homogenized operator based on a particular solution. This technique has been employed for the periodic case in [17]. In [20] the correctors for periodic homogenization of monotone parabolic operators is also studied.

Since we consider random operators, our main result, Theorem 4.1, is of statistical nature. It states that homogenization takes place almost surely. As in the case of nonlinear elliptic operators, one can deduce from this statistical result the individual homogenization theorem for almost periodic nonlinear parabolic operators. One needs only to consider almost periodic functions as realizations of appropriate random fields (in this case the probability space Ω is the Bohr compactification of $\mathbb{R}^{n+1}$) and follow the proof of Theorem 3.3.1 [17]. We would like to mention that, in the linear case, there is a more general individual homogenization theorem [22] that holds when Ω is a compact topological space and the dynamical system T is strictly ergodic. A nonlinear counterpart of this result is still an open problem even in the case of monotone elliptic operators.

Our motivation for considering homogenization of nonlinear parabolic equations comes from the applications arising in flow in porous media for both saturated and unsaturated media, though one encounters nonlinear parabolic equations in many different applications. Due to uncertainties and the general nature of the heterogeneities in subsurface flows, one no longer can assume periodicity. We employ the results of the present work for the development and analysis of efficient numerical homogenization schemes in another paper [9]. In the porous media applications one is often interested in the gradients of the solutions. In [9] we construct numerical correctors for the solution of nonlinear parabolic equations. The auxiliary problem proposed in this work plays a central role in the calculation of numerical correctors. These correctors further allow us to obtain the convergence of our numerical schemes for the gradients of the solutions. We would like to note that the homogenization results obtained in this work are important in addressing the robustness of the numerical homogenization schemes for more realistic porous media applications.

Finally, we would like to note that the homogenization results and the analysis presented in this paper avoids many details involved in [22], because we study neither the individual homogenization nor the correctors. Moreover, the approach presented in the paper differs from the one in [22]. In this paper we carry out the homogenization using the solution of an auxiliary problem and the theorem on G-convergence of arbitrary solutions.

The paper is organized as follows. In the next section we collect some basic facts that are used later. Section 3 is devoted to the auxiliary problem. In the following section we present the homogenization results.

2. PRELIMINARIES

Let (Ω, Σ, μ) be a probability space and let $L^p(\Omega)$ denote the space of all p-integrable functions. Consider an $(n+1)$-parametric dynamical system on Ω, $T(z) : \Omega \to \Omega$, $z = (x,t) \in \mathbb{R}^{n+1}$ ($t \in \mathbb{R}$, $x \in \mathbb{R}^n$) that satisfies the following conditions:

1) $T(0) = I$, *and* $T(z' + z'') = T(z')T(z'')$;

2) $T(z) : \Omega \to \Omega$ *preserves the measure* μ *on* Ω;

3) *For any measurable function* $f(\omega)$ *on* Ω, *the function* $f(T(z)\omega)$ *defined on* $\mathbb{R}^{n+1} \times \Omega$ *is also measurable.*

The formula $U(z)f(\omega) = f(T(z)\omega)$ defines a strongly continuous $(n+1)$-parametric group of isometries in the space $L^p(\Omega)$. Measurable functions on Ω are considered as *random homogeneous fields*. The function

$$f_\omega(z) = f_\omega(x,t) = f(T(z)\omega)$$

on $\mathbb{R}^{n+1}$ is called a *realization* of the field f. The function f_ω is called a *generic realization* if ω belongs to a subset of measure 1 in Ω. If $f \in L^p(\Omega)$, then its generic realization belongs to $L^p_{loc}(\mathbb{R}^{n+1})$.

Denote by $\langle \cdot \rangle$ the *mean value* over Ω,

$$\langle f \rangle = \int_\Omega f(\omega) d\mu(\omega).$$

For further analysis we will need the Birkhoff ergodic theorem (see, e. g., [8]). Let $f \in L^1_{loc}(\mathbb{R}^{n+1})$. Suppose that for any bounded Lebesgue measurable set $K \in \mathbb{R}^{n+1}$, with nonzero Lebesgue measure $|K|$, there exists the limit

$$M\{f\} = \lim_{\varepsilon \to 0} \frac{1}{|K|} \int_K f(\varepsilon^{-1} z) dz$$

independent of K. In this case we say that f has the *mean value* $M\{f\}$. Assume that the family of functions $f(\frac{z}{\varepsilon})$ is bounded in $L^p_{loc}(\mathbb{R}^{n+1})$, $1 \le p < \infty$. Then f has mean value $M\{f\}$ if and only if $f(z/\varepsilon) \to M\{f\}$ weakly in $L^p_{loc}(\mathbb{R}^{n+1})$ as $\varepsilon \to 0$ [17].

Theorem 2.1 (Birkhoff ergodic theorem). *Let* $f \in L^p(\Omega)$, $1 \le p < \infty$. *Then a generic realization* $f_\omega(z)$ *possesses a mean value. The mean value*

$M\{f_\omega(z)\}$ is an invariant function of $\omega \in \Omega$ and

$$\langle f\rangle = \int_\Omega M\{f_\omega\} d\mu(\omega).$$

If the system T is ergodic, then

$$\langle f\rangle = M\{f_\omega\} \ \textit{a.e. on } \Omega.$$

Remark 2.2. Let $f \in L^p(\Omega)$, $1 \le p < \infty$, and $\alpha > 0$, $\beta > 0$. Then for the mean value $M\{f_\omega\}$ of a generic realization we have that

$$f_\omega(x/\varepsilon^\beta, t/\varepsilon^\alpha) \to M\{f_\omega\}$$

as $\varepsilon \to 0$ weakly in $L^p_{loc}(\mathbb{R}^{n+1})$ (see [17], page 134).

In what follows, we always assume that

(E) *the dynamical system T is ergodic; i.e., any measurable T-invariant function on Ω is constant.*

This assumption is imposed only for the simplicity of the formulations. Our analysis can be carried out without this assumption and the homogenized operators will be ω-dependent, but T-invariant in this case.

Throughout the paper C denotes a generic positive constant, $\|\cdot\|_{p,Q}$ denotes $L^p(Q)$ as well as $L^p(Q)^n$ norms and p' is defined by $1/p + 1/p' = 1$. We use a number of pairs of Banach spaces in duality and $\langle\cdot,\cdot\rangle$ corresponds to duality pairing.

Consider the initial- boundary-value problem

$$\begin{aligned} D_t u_\varepsilon = div\ a_\omega(x/\varepsilon^\beta, t/\varepsilon^\alpha, u_\varepsilon, D_x u_\varepsilon) - \\ -a_{0,\omega}(x/\varepsilon^\beta, t/\varepsilon^\alpha, u_\varepsilon, D_x u_\varepsilon) + f \ \text{ in } Q \\ u_\varepsilon = 0 \text{ on } \ \partial Q_0 \\ u_\varepsilon(t=0) = 0. \end{aligned} \tag{2.1}$$

Here a_ω and $a_{0,\omega}$ are realizations of random homogeneous fields depending also on (η, ξ). We assume that $a(\omega, \eta, \xi)$ and $a_0(\omega, \eta, \xi)$ are Carathéodory functions on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, with values in $\mathbb{R}^n$ and $\mathbb{R}$ respectively, satisfying the following inequalities:

(i) *for any (η, ξ)*

$$|a(\omega,\eta,\xi)|^{p'} + |a_0(\omega,\eta,\xi)|^{p'} \le c_0(|\eta|^p + |\xi|^p) + c(\omega), \ \ \textit{a.e. on } \ \Omega, \tag{2.2}$$

where $p > 1$, $c_0 > 0$ and $c(\omega)$ belongs to $L^1(\Omega)$;

(ii) *for any* (η, ξ) *and* $(\eta, \xi^{'})$

$$(a(\omega, \eta, \xi) - a(\omega, \eta, \xi^{'}), \xi - \xi^{'}) \geq c_1 |\xi - \xi^{'}|^p, \;\; a.e. \;\, on \;\; \Omega, \tag{2.3}$$

where $c_1 > 0$*;*

(iii) *for any* $\chi = (\eta, \xi)$ *and* $\chi' = (\eta^{'}, \xi^{'})$

$$\begin{aligned} |a(\omega, \eta, \xi) - a(\omega, \eta^{'}, \xi^{'})|^{p'} + |a_0(\omega, \eta, \xi) - a_0(\omega, \eta^{'}, \xi^{'})|^{p'} \\ \leq \kappa[(h(\omega) + |\chi|^p + |\chi'|^p)\nu(|\xi - \xi'|) \\ +(h(\omega) + |\chi|^p + |\chi'|^p)^{1-s/p} |\xi - \xi'|^s], \;\; a.e. \;\, on \;\; \Omega, \end{aligned} \tag{2.4}$$

where $\kappa > 0$, $0 < s < \min(p, p')$, $\nu(r)$ *is a continuity modulus (i.e., a nondecreasing continuous function on* $[0, +\infty)$ *such that* $\nu(0) = 0$, $\nu(r) > 0$ *if* $r > 0$, *and* $\nu(r) = 1$ *if* $r > 1$*), and* $h \in L^1(\Omega)$*;*

(iv) $p > 2$.

In what follows, the constants p, c_0, c_1, κ and s, and the functions $c(\omega)$, $c_3(\omega)$, $h(\omega)$ and $\nu(r)$ are independent of ϵ.

Next we briefly review G-convergence results for non-monotone parabolic operators ([17], Section 4.1) that will be used later on. We introduce the following spaces

$$\begin{aligned} &V_0 = L^p(0, T, W_0^{1,p}(Q_0)), \quad \overline{V} = L^p(0, T, W^{1,p}(Q_0)), \\ &W = \{u \in L^p(0, T, W_0^{1,p}(Q_0)) : D_t u \in L^{p'}(0, T, W^{-1,p'}(Q_0))\}, \\ &\overline{W} = \{u \in \overline{V} : D_t u \in L^{p'}(0, T, W^{-1,p'}(Q_0))\}, \quad W_0 = \{u \in W : u(0) = 0\}. \end{aligned} \tag{2.5}$$

Note that the embedding $W_0 \subset L^p(Q) = L^p(0, T, L^p(Q_0))$ is compact (see [13], Section 1.5). By duality, the embedding $L^{p'}(Q) \subset W_0'$ is also compact.

Consider a sequence of general parabolic operators

$$L_k u = D_t u - div(a_k(x, t, u, D_x u)) + a_{0,k}(x, t, u, D_x u)$$

and an operator

$$Lu = D_t u - div(a(x, t, u, D_x u)) + a_0(x, t, u, D_x u).$$

We assume that L_k and L satisfy the corresponding versions of assumptions (2.2)–(2.4) over Q. Next we briefly mention the definition of G-convergence for a sequence of operators L_k to the operator L. For more details we refer to [17] (see Section 4.1 of [17]). Define the operators

$$L_k^1(u, v) = D_t u - div(a_k(x, t, v, D_x u)),$$

$$L^1(u, v) = D_t u - div(a(x, t, v, D_x u))$$

acting on the pairs of functions $(u, v) \in W_0 \times V_0$ and the fluxes

$$\Gamma^k(u,v) = a_k(x,t,v,D_xu), \quad \Gamma^k_0(u,v) = a_{0,k}(x,t,v,D_xu),$$
$$\Gamma(u,v) = a(x,t,v,D_xu), \quad \Gamma_0(u,v) = a_0(x,t,v,D_xu).$$

Given $v \in V_0$, $L^1_k(u,v)$ and $L^1(u,v)$ are strictly monotone parabolic operators with respect to u. Therefore, for any $v \in V_0$ and $f \in W'$, there exist unique solutions $u_k \in W_0$ and $u \in W_0$ of $L^1_k(u_k,v) = f$ and $L^1(u,v) = f$ [18].

Definition (G-convergence) *A sequence L_k G-*converges *to L if for any $v \in V_0$, $u \in W_0$ and $f \in L^{p'}(0,T,W^{-1,p'}(Q_0))$ we have $u_k \to u$ as $k \to \infty$ weakly in W_0, and $\Gamma^k(u_k,v) \to \Gamma(u,v)$, $\Gamma^k_0(u_k,v) \to \Gamma_0(u,v)$ as $k \to \infty$ weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ respectively.*

Remark 2.3. We would like to note that in [17] (where to our best knowledge G-convergence for this class of operators is first introduced) the author calls a G-convergent sequence defined as above "strongly G-convergent sequence".

Next we formulate the theorem on the convergence of arbitrary solutions for a G-convergent sequence of operators ([17], Theorem 4.1.3).

Theorem 2.4. *Assume that L_k G-converges to L, $u_k \in \overline{W}$,*

$$f_k, f \in L^{p'}(0,T,W^{-1,p'}(Q_0)),$$

$L_ku_k = f_k$, $u_k \to u$ weakly in $\overline{W}$, and $f_k \to f$ strongly in W'_0. Then $Lu = f$, and

$$a_k(x,t,u_k,D_xu_k) \to a(x,t,u,D_xu),$$
$$a_{0,k}(x,t,u_k,D_xu_k) \to a_0(x,t,u,D_xu)$$

as $k \to \infty$ weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ respectively.

Following [22] we define spaces of functions on Ω that are similar to $W^{1,p}$. Denote by $\partial_{full} = (\partial_1, \cdots, \partial_{n+1})$ the collection of generators of the group $U(z)$. There is a dense subspace $S \subset L^p(\Omega)$ that is contained in the domains of all operators $\partial^\alpha_{full} = \partial_1^{\alpha_1} \cdots \partial_{n+1}^{\alpha_{n+1}}$, $\alpha \in Z^{n+1}_+$. Next we introduce smoothing operators J^δ. Let $K(z) \in C^\infty_0(\mathbb{R}^{n+1})$ be a non-negative even function such that

$$\int_{\mathbb{R}^{n+1}} K(z)dz = 1,$$

and $K^\delta(z) = \delta^{-(n+1)}K(z/\delta)$. Define the operator J^δ as follows

$$J^\delta f(\omega) = \int_{\mathbb{R}^{n+1}} K^\delta(z) f(T(z)\omega)dz.$$

J^δ is a bounded operator in the space $L^p(\Omega)$ whose norm is not greater than 1. For a generic realization of f we have

$$J^\delta f(T(z)\omega) = \int_{R^{n+1}} K^\delta(z - z_1) f(T(z_1)\omega)dz_1.$$

The latter shows that a generic realization of $J^\delta f$ belongs to $C^\infty(\mathbb{R}^{n+1})$. Thus, for $f \in L^p(\Omega)$ the function $J^\delta f$ belongs to the domain $D(\partial^\alpha, L^p(\Omega))$ for any $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_{n+1}^{\alpha_{n+1}}$; i.e., $J^\delta L^p \subset S$, $\delta > 0$. More information about J^δ can be found in [17] (see Section 3.1.3 of [17]). The following lemma is important for further analysis (see [17], page 139).

Lemma 2.5. *For any $f \in L^p(\Omega)$, we have*

$$\lim_{\delta \to 0} \|J^\delta f - f\|_{L^p(\Omega)} = 0.$$

Further, denote by $\mathcal{V}$ the completion of S with respect to the semi-norm

$$\|f\|_{\mathcal{V}} = \Big(\sum_{i=1}^n \|\partial_i f\|^p_{L_p(\Omega)} \Big)^{1/p}.$$

Note that the completion with respect to a seminorm “cuts off” the kernel of the semi-norm. The operator $\partial = (\partial_1, \ldots, \partial_n) : \mathcal{V} \to L^p(\Omega)^n$ is an isometric embedding. Moreover, the space $\mathcal{V}$ is reflexive, with the dual denoted by $\mathcal{V}'$. By duality we define the operator $\mathbf{div} : L^{p'}(\Omega)^n \to \mathcal{V}'$ by

$$\langle \mathbf{div} u, w \rangle = -\langle u, \partial w \rangle, \ \forall w \in \mathcal{V}. \tag{2.6}$$

We note that the elements of $\mathcal{V}$ in general do not have independent meaning. The space $\mathcal{V}$ contains fields that are not spatially homogeneous. Note that ([22, 17]; page 138 in [17]) the operators ∂_i may be viewed as derivatives along trajectories of the dynamical system $T(z)$

$$(\partial_i f)(T(z)\omega) = \frac{\partial}{\partial z_i} f(T(z)\omega) \tag{2.7}$$

for almost every $\omega \in \Omega$ and $f \in D(\partial_i, L^p(\Omega))$.

We set

$$T_1(t) = T(0, \ldots, 0, t), \quad T_2(x) = T(x_1, \ldots, x_n, 0). \tag{2.8}$$

Let

$$M_t\{f_\omega\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(T_1(\tau)\omega)d\tau, \tag{2.9}$$

$$M_x\{f_\omega\} = \lim_{|K|\to\infty} \frac{1}{|K|}\int_{K} f(T_2(y)\omega)dy. \tag{2.10}$$

These partial mean values are well defined for $f \in L^p(\Omega)$, $1 \le p < \infty$, and for a generic $\omega \in \Omega$.

Now we would like to consider differentiation with respect to time along the trajectories as an unbounded operator in appropriate functional spaces. Define an unbounded operator σ from $\mathcal{V}$ into $\mathcal{V}'$ as follows. $\mathcal{V}_1$, defined as the image of the operator ∂, is a closed subspace of $L^p(\Omega)^n$ and ∂ maps $\mathcal{V}$ onto $\mathcal{V}_1$ isomorphically. The dual space $\mathcal{V}_1'$ can be identified with the factor space $L^{p'}(\Omega)^n/\mathcal{V}_1^{\perp}$, where $\mathcal{V}_1^{\perp} = ker(\mathbf{div})$ is the orthogonal complement to $\mathcal{V}_1$ in $L^{p'}(\Omega)^n$. Then the operator $\mathbf{div}$ can be considered as an operator $\mathbf{div} : \mathcal{V}_1' \to \mathcal{V}'$. Let $S_1 = \partial(S) \subset \mathcal{V}_1$. It is easy to see that $\partial_i(S_1) \subset S_1$. Now we say that $v \in \mathcal{V}_1$ belongs to the domain $D(\sigma_1)$ if there exists $f \in \mathcal{V}_1'$ such that

$$\langle v, \partial_{n+1}\varphi\rangle = -\langle f, \varphi\rangle,\ \forall \varphi \in S_1$$

and set $\sigma_1 v = f$. The (unbounded) operator σ_1 is a well-defined closed linear operator from $\mathcal{V}_1$ into $\mathcal{V}_1'$ and its domain is dense in $\mathcal{V}_1$. Using the mollifiers J^δ, it is easy to verify that $\sigma_1' = -\sigma_1$, where $\sigma_1' : \mathcal{V}_1 \to \mathcal{V}_1'$ is the adjoint operator to σ_1. Now we set

$$\sigma = \mathbf{div} \circ \sigma_1 \circ \partial\,.$$

Then σ is a closed linear operator from $\mathcal{V}$ into $\mathcal{V}'$, with dense domain $\mathcal{W} = D(\sigma)$, and $\sigma' = -\sigma$. The space $\mathcal{W}$ is endowed with the usual graph norm. As a consequence, σ is a maximal monotone operator (see [13], Lemma 1.2 of Chapter 3). Note that in the case $p = 2$ this operator can be defined by means of the spectral decomposition theorem [22].

3. Auxiliary problem

In this section we study an auxiliary problem and near solutions for it. Consider the auxiliary problem

$$\mu\sigma w^\mu - \mathbf{div}\ a(\omega, \eta, \xi + \partial w^\mu) = 0, \tag{3.1}$$

with $\mu > 0$ being a parameter. Define the operator A from $\mathcal{V}$ to $\mathcal{V}^{'}$ by

$$\langle Au, v\rangle = \langle a(\omega, \eta, \xi + \partial u), \partial v\rangle, \tag{3.2}$$

for every $v \in \mathcal{V}$. We note that the operator A depends on (η, ξ). It can be easily verified that A is a strongly monotone; i.e.,

$$\langle Au - Av, u - v\rangle \geq C\|u - v\|_{\mathcal{V}}^p;$$

continuous, and coercive operator from $\mathcal{V}$ to $\mathcal{V}'$. Since σ is a maximal monotone operator whose domain $\mathcal{W}$ is dense in $\mathcal{V}$ it follows from [13], Theorem 1.1 of Chapter 3, (see also [21], Section 32.4) that the solution of (3.1) in $\mathcal{W}$ exists. Uniqueness follows from the fact that $\langle \sigma u, u\rangle = 0$ and A is strongly monotone. Thus we have the following lemma.

Lemma 3.1. *Equation* (3.1) *has a unique solution, $w^\mu \in \mathcal{W}$, and*

$$\|w^\mu\|_{\mathcal{W}} \leq C, \tag{3.3}$$

where $C > 0$ is independent of $\mu > 0$, but depends on (η, ξ).

For the following analysis, we need so-called near solutions that approximate w (the solution of (3.1)) by random fields with smooth realizations. For each element $v \in \mathcal{W}$ we define its near smooth element as follows. Because S is dense in $\mathcal{W}$, we can approximate $v \in \mathcal{W}$ by elements $v_k \in S$, $v_k \to v$ in $\mathcal{W}$. Consider

$$J^\delta v_k = \int_{\mathbb{R}^{n+1}} K^\delta(z - z_1) v_k(T(z_1)\omega) dz_1.$$

Clearly,

$$\|J^\delta v_k\|_{\mathcal{V}} \leq C\|v_k\|_{\mathcal{V}},$$

where C is independent of k. Since σ commutes with J^δ we have

$$\|J^\delta \sigma v_k\|_{\mathcal{V}'} \leq C\|\sigma v_k\|_{\mathcal{V}'}.$$

Consequently, $\|v_k^\delta\|_{\mathcal{W}} \leq C\|v_k\|_{\mathcal{W}}$, where $v_k^\delta = J^\delta v_k$. Hence, we can extend J^δ to $\mathcal{W}$ by continuity, and set $v^\delta = J^\delta v$. For v^δ to be a near solution of (3.1), one needs

$$\|w - v^\delta\|_{\mathcal{V}} \to 0, \quad \|\mu\sigma v^\delta + Av^\delta\|_{\mathcal{V}'} \to 0$$

as $\delta \to 0$. The first limit is true due to the approximation property. The second limit is true due to the fact that J^δ commutes with σ, and A is continuous from $\mathcal{V}$ to $\mathcal{V}'$. Indeed, as $\delta \to 0$,

$$\begin{aligned}\|\mu\sigma v^\delta + Av^\delta\|_{\mathcal{V}'} &= \|\mu\sigma J^\delta w + Av^\delta\|_{\mathcal{V}'} = \|Av^\delta - J^\delta Aw\|_{\mathcal{V}'}\\ &\leq \|Av^\delta - Aw\|_{\mathcal{V}'} + \|Aw - J^\delta Aw\|_{\mathcal{V}'} \to 0, .\end{aligned}$$

For near solutions of auxiliary equation (3.1) we have

$$\mu\sigma w_\delta^\mu + Aw_\delta^\mu = \mathbf{div}\rho_\delta, \tag{3.4}$$

where **div** is defined by (2.6), $\rho_\delta \in L^{p'}(\Omega)$ and

$$\lim_{\delta\to 0} \|\rho_\delta\|^{p'}_{L^{p'}(\Omega)} = \lim_{\delta\to 0}\langle|\rho_\delta|^{p'}\rangle = 0. \tag{3.5}$$

The right-hand side of (3.4) can be written as $\mathbf{div}\rho_\delta$ because it is an element of $\mathcal{V}'$. Being restricted to generic realizations, the auxiliary equation has the form

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} (\mu D_\tau w^\mu_{\delta,\omega}(z)\psi(z) + (a_\omega(z,\eta,\xi + D_y w^\mu_{\delta,\omega}), D_y\psi(z)))dz \\ = \int_{\mathbb{R}^{n+1}} (\rho_{\delta,\omega}(z), D_y\psi(z))dz,\ \forall\psi\in C_0^\infty(\mathbb{R}^{n+1}),\end{aligned} \tag{3.6}$$

where $z = (y,\tau) \in \mathbb{R}^{n+1}$. By the ergodic theorem

$$\int_K |\rho_{\delta,\omega}(x/\varepsilon^\alpha, t/\varepsilon^\beta)|^{p'} dxdt \to |K|\langle|\rho_\delta(\omega)|^{p'}\rangle \tag{3.7}$$

as $\varepsilon \to 0$ for any $\delta > 0$. Furthermore the right-hand side of (3.7) converges to zero as $\delta \to 0$ for each $\varepsilon > 0$.

The following lemma will be used in the analysis.

Lemma 3.2. *Assume $\rho_\delta \in L^p(\Omega)$ and $\langle|\rho|^p\rangle < s(\delta)$, where $s(\delta) \to 0$ as $\delta \to 0$. Then for any sequence $\delta \to 0$ there exists a sequence $\varepsilon_0(\delta)$, such that $\varepsilon_0(\delta) \to 0$ as $\delta \to 0$, and for a generic realization and any $Q \subset \mathbb{R}^{n+1}$*

$$\int_Q |\rho_{\delta,\omega}(x/\varepsilon^\beta, t/\varepsilon^\alpha)|^{p'} dxdt < s(\delta),\ \forall\varepsilon < \varepsilon_0(\delta)\,.$$

Proof. Let $Q_\varepsilon = \{(x,t) : (\varepsilon^\beta x, \varepsilon^\alpha t) \in Q\}$. Then

$$\int_Q |\rho_{\delta,\omega}(x/\varepsilon^\beta, t/\varepsilon^\alpha)|^{p'} dxdt = \varepsilon^{n\beta+\alpha}\int_{Q_\varepsilon} |\rho_{\delta,\omega}(y,\tau)|^{p'} dyd\tau \to |Q|\langle|\rho_\delta|^{p'}\rangle < s(\delta),$$

as $\varepsilon \to 0$. Here we have used the Birkhoff ergodic theorem. From here it follows that there exists a sequence $\varepsilon_0(\delta)$ such that, for all $\varepsilon < \varepsilon_0(\delta)$,

$$\int_Q |\rho_{\delta,\omega}(x/\varepsilon^\beta, t/\varepsilon^\alpha)|^{p'} dxdt < s(\delta). \qquad \square$$

Throughout the paper $s(\delta)$ denotes a generic sequence that converges to zero as $\delta \to 0$.

4. HOMOGENIZATION

The homogenization of a parabolic equation depends on the relation between time and spatial scales [1, 22, 17]. We consider

$$L_\varepsilon u = D_t u - div(a_\omega(x/\varepsilon^\beta, t/\varepsilon^\alpha, u, D_x u)) + a_{0,\omega}(x/\varepsilon^\beta, t/\varepsilon^\alpha, u, D_x u), \quad (4.1)$$

where

$$a_\omega(y, \tau, \eta, \xi) = a(T(z)\omega, \eta, \xi), \quad a_{0,\omega}(y, \tau, \eta, \xi) = a_0(T(z)\omega, \eta, \xi)$$

are realizations of random homogeneous fields a and a_0 and $z = (y, \tau) \in \mathbb{R}^{n+1}$. In what follows, we always work with generic realizations.

Depending on α and β we distinguish the following cases:

- Self-similar case ($\alpha = 2\beta$).
- Non-self-similar case ($\alpha < 2\beta$).
- Non-self-similar case ($\alpha > 2\beta$).
- Spatial case ($\alpha = 0$).
- Temporal case ($\beta = 0$).

The homogenization in each case is presented next. The main idea is as follows. First we construct a solution for the parabolic equation by rescaling the solution of the corresponding auxiliary problem (3.1). After the rescaling, μ in (3.1) may depend on ε. Further, we study the convergence of the solutions obtained as $\varepsilon \to 0$. Employing the results on the convergence of arbitrary solutions for a G-convergent sequence of operators, we calculate the homogenized fluxes a^* and a_0^* using the constructed solution. This technique has been employed for the periodic case in [17]. Note that if $\beta \neq 0$, we can assume that $\beta = 1$, by considering ε^β as a new small parameter. Similarly, the case $\alpha \neq 0$ can be reduced to the case $\alpha = 1$.

Next we formulate our main result.

Theorem 4.1. *Suppose that the dynamical system T is ergodic and L_ε satisfies assumptions* (2.2)–(2.4). *Then the operators L_ε G-converge to L^*, where L^* is given by*

$$L^* u = D_t u - div(a_\omega{}^*(x, t, u, D_x u)) + a_{0,\omega}^*(x, t, u, D_x u). \quad (4.2)$$

The fluxes a^* and a_0^* are defined as follows.

- *In the self-similar case ($\alpha = 2\beta$), a^* and a_0^* are independent of t, x and ω, and*

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w_{\eta,\xi}) \rangle, \quad a_0{}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w_{\eta,\xi}) \rangle,$$

where $w_{\eta,\xi} = w^{\mu=1} \in \mathcal{W}$ *is the unique solution of*

$$\sigma w^{\mu=1} - \mathbf{div}\ a(\omega, \eta, \xi + \partial w^{\mu=1}) = 0. \tag{4.3}$$

- *In the non-self-similar case* $\alpha < 2\beta$, a^* *and* a_0^* *are independent of* t, x *and* ω, *and*

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle, \quad {a_0}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle,$$

where $w_{\eta,\xi} = w^0 \in \mathcal{V}$ *is the unique solution of*

$$-\mathbf{div}\ a(\omega, \eta, \xi + \partial w^0) = 0. \tag{4.4}$$

- *In the non-self-similar case* $\alpha > 2\beta$, a^* *and* a_0^* *are independent of* t, x *and* ω, *and*

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle, \quad {a_0}^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w_{\eta,\xi})\rangle,$$

where $w_{\eta,\xi} = w^\infty \in \mathcal{V}_s$ *is the unique solution of*

$$-\mathbf{div}\ \overline{a}(\omega, \eta, \xi + \partial w^\infty) = 0. \tag{4.5}$$

$\overline{a}$ *and* $\mathcal{V}_s$ *are defined in Section* 4.2.2.

- *In the spatial case* $(\alpha = 0)$,

$$a^*(\omega, \eta, \xi) = M_x\{a(T_2(x)\omega, \eta, \xi + \partial w_{\eta,\xi}(T_2(x)\omega))\},$$
$$a_0^*(\omega, \eta, \xi) = M_x\{a_0(T_2(x)\omega, \eta, \xi + \partial w_{\eta,\xi}(T_2(x)\omega))\},$$

where $w_{\eta,\xi} \in \mathcal{V}$ *is a (unique) solution of*

$$-\mathbf{div}\ a(\omega, \eta, \xi + \partial w_{\eta,\xi}) = 0. \tag{4.6}$$

- *In temporal case* $(\beta = 0)$, *the homogenized fluxes are defined by*

$$\begin{aligned} a^*(\omega, \eta, \xi) &= M_t\{a(T_1(t)\omega, \eta, \xi)\}, \\ a_0^*(\omega, \eta, \xi) &= M_t\{a_0(T_1(t)\omega, \eta, \xi)\}, \end{aligned} \tag{4.7}$$

where P_1 *is defined in* (4.29).

Remark 4.2. In the spatial and temporal cases the homogenized operator is a random operator. However, it is independent of the spatial (respectively, temporal) variable. Roughly speaking, to obtain the homogenized operator in the spatial case it is sufficient to homogenize the elliptic part of the operator, with time being a parameter, while in the temporal case, the homogenized operator is obtained only by means of the time averaging of fluxes.

The theorem on the convergence of arbitrary solutions (Theorem 2.4) for G-convergent sequence of operators allows us to not restrict ourselves to particular boundary or initial conditions. In particular, from Theorem 2.4 and Theorem 4.1, we have the following.

Theorem 4.3. *Under the assumptions of Theorem* 4.1, *let* $u_\varepsilon \in \overline{W}$ *be a solution of*

$$L_\varepsilon u_\varepsilon = f, \qquad f \in L^{p'}(0, T, W^{-1,p'}(Q_0)),$$

such that $\|u_\varepsilon\|_{\overline{W}}$ *is bounded. Then* u_ε *converges to* u *as* $\varepsilon \to 0$ *weakly in* $\overline{W}$ (*up to a subsequence*), *where* u *is a solution of* $L^* u = f$ *and* L^* *is defined in* (4.2).

Remark 4.4. As we mentioned above, the ergodicity assumption is not essential for the proof of Theorem 4.1 and is imposed only to obtain a constant-coefficient homogenized operator in all cases (except spatial and temporal cases). One can carry out the proof for a non-ergodic case essentially in the same manner as that for the ergodic case. The homogenized operators in the non-ergodic case will be invariant with respect to $T(z)$.

Remark 4.5. Note that in the case of spatial and temporal homogenization, the homogenized operator depends on ω. If the operator is random with respect to the time variable, one can apply the results of [4]. However, the results of [4] do not imply convergence of the fluxes.

Remark 4.6. Under an additional regularity assumption on a and a_0 with respect to the time variable, G-convergence results follow from the G-convergence of elliptic parts of parabolic operators (see [17]). However, this additional assumption is too restrictive and not well suited to the case of random operators.

Remark 4.7. In our analysis, for simplicity we assume (2.3) with $p > 2$, though the homogenization results can be obtained under a weaker assumption,

$$(a(\omega,\eta,\xi) - a(\omega,\eta,\xi^{'}), \xi - \xi^{'}) \geq \kappa(h(\omega) + |\xi|^p + |\xi^{'}|^p)^{1-\gamma/p}|\xi - \xi'|^\gamma, \text{ a.e. on } \Omega,$$

with $\gamma \geq \max[p, 2]$, $h \in L^1(\Omega)$ and $p > 2n/(n+2)$ (cf. [17]).

4.1. **Self-similar case ($\alpha = 2\beta$).** Without loss of generality we assume that $\alpha = 2$ and $\beta = 1$. We take $\mu = 1$ in (3.1), and consider near solutions $w_\delta^{\mu=1}$, and set on the level of realizations

$$w_{\varepsilon,\delta}^{\mu=1}(x,t) = \varepsilon w_{\delta,\omega}^{\mu=1}(x/\varepsilon, t/\varepsilon^2). \tag{4.8}$$

(For notational convenience we skip the index ω that should appear on the left to indicate a realization). Then $w^{\mu}_{\varepsilon,\delta}$ satisfies on $\mathbb{R}^{n+1}$ the following equation for almost every ω

$$D_t w^{\mu=1}_{\varepsilon,\delta} - div(a(T(x/\varepsilon, t/\varepsilon^2)\omega, \eta, \xi + D_x w^{\mu=1}_{\varepsilon,\delta})) = div_x \rho_\delta, \tag{4.9}$$

where $\langle|\rho_\delta|^p\rangle \to s(\delta)$, where $s(\delta) \to 0$ as $\delta \to 0$.

Lemma 4.8. *Given Q, for every $\delta > 0$, $w^{\mu=1}_{\varepsilon,\delta} \to 0$ weakly in $\overline{W}$ as $\varepsilon \to 0$.*

Proof. Using the fact that $w^{\mu=1}_{\delta} \in \mathcal{W}$ and (4.8) we obtain that

$$\|w^{\mu=1}_{\varepsilon,\delta}\|_{L^p(0,T,W^{1,p}(Q_0))} \leq C.$$

It follows from (4.8) that $w^{\mu=1}_{\varepsilon,\delta} \to 0$ in $L^p(0, T, L^p(Q_0))$ as $\varepsilon \to 0$ for every $\delta > 0$ (cf. [22], page 238). Next we show that

$$\|D_t w^{\mu=1}_{\varepsilon,\delta}\|_{L^{p'}(0,T,W^{-1,p'}(Q_0))} \leq C.$$

From equation (4.9) we obtain that

$$\|D_t w^{\mu=1}_{\varepsilon,\delta}\|_{L^{p'}(0,T,W^{-1,p'}(Q_0))} \leq C\|w^{\mu=1}_{\varepsilon,\delta}\|_{L^p(0,T,W^{1,p}(Q_0))} + C\|\rho_{\delta,\omega}\|_{p',Q} \leq C.$$

Consequently, $w^{\mu=1}_{\varepsilon,\delta}$ is bounded in $\overline{W}$. Since this family is weakly compact in $\overline{W}$ and converges to zero in $L^p(0, T, L^p(Q_0))$, it converges to zero weakly in $\overline{W}$. □

Define

$$L^* u = D_t u - div(a^*(u, D_x u)) + a^*_0(u, D_x u), \tag{4.10}$$

where $a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w^{\mu=1})\rangle$ and $a^*_0(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w^{\mu=1})\rangle$.

Theorem 4.9. *If $\alpha = 2\beta$, then L_ε G-converges to L^* defined by* (4.10) *for almost every $\omega \in \Omega$.*

Proof. Fix a generic $\omega \in \Omega$ and a sequence $\varepsilon_k \to 0$, and let

$$L_k u = D_t u - div(a_\omega(x/\varepsilon_k, t/\varepsilon_k^2, \eta, \xi + D_x u)) + a_{\omega,0}(x/\varepsilon_k, t/\varepsilon_k^2, \eta, \xi + D_x u).$$

It is known (see [17], Theorem 4.1.1) that up to a subsequence L_k G-converges to a parabolic operator of the form

$$\tilde{L} u = D_t u - div(\tilde{a}(x, t, \eta, \xi + D_x u)) + \tilde{a}_0(x, t, \eta, \xi + D_x u)$$

for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$. We have to show that $\tilde{a} = a^*_\omega$ and $\tilde{a}_0 = a^*_{0,\omega}$ (recall that in this case a^* and a^*_0 are independent of ω).

Equation (4.9) for $w^{\mu=1}_{\varepsilon,\delta}$ can be written as

$$D_t w^{\mu=1}_{\varepsilon,\delta} - div(a_\omega(x/\varepsilon, t/\varepsilon^2, \eta, \xi + D_x w^{\mu=1}_{\varepsilon,\delta}))+$$
$$a_{0,\omega}(x/\varepsilon, t/\varepsilon^2, \eta, \xi + D_x w^{\mu=1}_{\varepsilon,\delta}) = h_{\varepsilon,\delta} + div_x \rho_\delta,$$

where $h_{\varepsilon,\delta} = a_{0,\omega}(x/\varepsilon, t/\varepsilon^2, \eta, \xi + D_x w^{\mu=1}_{\varepsilon,\delta})$.

Next we choose $\delta_k \to 0$ such that $w^{\mu=1}_{\varepsilon_k,\delta_k} \to 0$ weakly in $\overline{W}$, and $\rho_{\delta_k,\omega} \to 0$ in $L^{p'}(Q)^n$ as $k \to \infty$. This is possible because of Lemma 4.8 and Lemma 3.2. Then $w_k = w^{\mu=1}_{\varepsilon_k,\delta_k} \to 0$ weakly in $\overline{W}$, and $\rho_k = \rho_{\delta_k,\omega} \to 0$ in $L^{p'}(Q)^n$ as $k \to \infty$.

Note that $u = w_k$ is a solution of

$$L_k u = h_k + div_x \rho_k,$$

where $h_k = h_{\varepsilon_k,\delta_k}$. Obviously, the sequence h_k is bounded in $L^{p'}(Q)$. Hence, up to a subsequence $h_k \to h$ weakly in $L^{p'}(Q)$ and strongly in W_0' because the embedding $L^{p'}(Q) \subset W_0'$ is compact. Since $w_k \to 0$ weakly in $\overline{W}$, Theorem 2.4 implies that $u = 0$ solves $\tilde{L}u = h$ and

$$\begin{aligned} a_\omega(x/\varepsilon_k, t/\varepsilon_k^2, \eta, \xi + D_x w_k) &\to \tilde{a}(x,t,\eta,\xi) \\ a_{0,\omega}(x/\varepsilon_k, t/\varepsilon_k^2, \eta, \xi + D_x w_k) &\to \tilde{a}_0(x,t,\eta,\xi), \end{aligned} \tag{4.11}$$

as $k \to \infty$ weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$.

On the other hand, using the ergodic theorem (see Remark 2.2) we obtain that

$$\begin{aligned} a_\omega(x/\varepsilon_k, t/\varepsilon_k^2, \eta, \xi + D_x w_k^{\mu=1}) &\to \langle a(\omega, \eta, \xi + \partial w^{\mu=1})\rangle \\ a_{0,\omega}(x/\varepsilon_k, t/\varepsilon_k^2, \eta, \xi + D_x w_k^{\mu=1}) &\to \langle a_0(\omega, \eta, \xi + \partial w^{\mu=1})\rangle, \end{aligned} \tag{4.12}$$

as $k \to \infty$ weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$. Comparing (4.11) and (4.12), we see that

$$\tilde{a}(x,t,\eta,\xi) = \langle a(\omega,\eta,\xi + \partial w^{\mu=1})\rangle, \quad \tilde{a}_0(x,t,\eta,\xi) = \langle a_0(\omega,\eta,\xi + \partial w^{\mu=1})\rangle.$$

This completes the proof. □

4.2. **Non-self-similar cases.**

4.2.1. *Case* $\alpha < 2\beta$. We can assume without loss of generality that $\beta = 1$ and $\alpha < 2$. To construct the homogenized operator for this case we set $\mu = \varepsilon^{2-\alpha}$ in (3.1). First, we study the limit of w^μ as $\mu \to 0$.

Lemma 4.10. *We have that* $w^\mu \to w^0$ *as* $\mu \to 0$ *in* $\mathcal{V}$*, where* $w^0 \in \mathcal{V}$ *is the unique solution of*

$$-\mathbf{div}\ a(\omega, \eta, \xi + \partial w^0) = 0. \tag{4.13}$$

Proof. The existence and uniqueness of the solution of (4.13) can be shown in the same way as that of (3.1) using the fact that A is a strongly monotone operator from $\mathcal{V}$ to $\mathcal{V}'$ (see (3.2)).

To prove the convergence we follow [17], Section 4.2.3. Define $w^{0,k} \in \mathcal{W}$ such that $w^{0,k} \to w^0$ in $\mathcal{V}$. Such a sequence exists since $\mathcal{W}$ is dense in $\mathcal{V}$. Then

$$\begin{aligned}\|w^\mu - w^{0,k}\|_{\mathcal{V}}^p &\le C\langle Aw^\mu - Aw^{0,k}, w^\mu - w^{0,k}\rangle\\ &= C\langle \mu\sigma(w^\mu - w^{0,k}) + Aw^\mu - Aw^{0,k}, w^\mu - w^{0,k}\rangle\\ &\le C\langle -\mu\sigma w^{0,k} - Aw^{0,k}, w^\mu - w^{0,k}\rangle\\ &\le C(\|\mu\sigma w^{0,k}\|_{\mathcal{V}'} + \|Aw^{0,k}\|_{\mathcal{V}'})\|w^\mu - w^{0,k}\|_{\mathcal{V}}.\end{aligned}$$

Here we have used the fact that $\mu\sigma w^\mu - Aw^\mu = 0$. Next, using the fact that $Aw_0 = 0$, we have

$$\|w^\mu - w^{0,k}\|_{\mathcal{V}} \le C(\|\mu\sigma w^{0,k}\|_{\mathcal{V}'} + \|Aw^0 - Aw^{0,k}\|_{\mathcal{V}'}).$$

Thus,

$$\begin{aligned}\|w^\mu - w^0\|_{\mathcal{V}} &\le \|w^\mu - w^{0,k}\|_{\mathcal{V}} + \|w^{0,k} - w^0\|_{\mathcal{V}}\\ &\le C(\mu\|\sigma w^{0,k}\|_{\mathcal{V}'} + \|Aw^0 - Aw^{0,k}\|_{\mathcal{V}'}) + \|w^{0,k} - w^0\|_{\mathcal{V}}.\end{aligned} \tag{4.14}$$

For any $\delta > 0$, we can choose k sufficiently large such that $\|w^{0,k} - w^0\|_{\mathcal{V}} < \delta$ and $\|Aw^0 - Aw^{0,k}\|_{\mathcal{V}'} < \delta$. The latter is possible since A is continuous from $\mathcal{V}$ to $\mathcal{V}'$. Choosing μ sufficiently small, we have that $\mu\|\sigma w^{0,k}\|_{\mathcal{V}'} < \delta$, and hence, by (4.14),

$$\|w^\mu - w^0\|_{\mathcal{V}} < C\delta.$$

The proof is complete. □

Define

$$L^*u = D_t u - div(a^*(u, D_x u)) + a_0^*(u, D_x u), \tag{4.15}$$

where $a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w^0)\rangle$, $a_0^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w^0)\rangle$, and w^0 is the solution of (4.13).

Theorem 4.11. *If* $\alpha < 2\beta$*, then* L_ε *G-converges to* L^* *defined by* (4.15) *for almost every* $\omega \in \Omega$*.*

Proof. Set $\mu = \varepsilon^{2-\alpha}$ in (3.1), consider the near solutions of (3.1), w^μ_δ, and set $w_{\varepsilon,\delta} = \varepsilon w^\mu_\delta(T(x/\varepsilon, t/\varepsilon^\alpha)\omega)$. Furthermore, let $w^0_{\varepsilon,\delta} = \varepsilon w^0_\delta(T(x/\varepsilon, t/\varepsilon^\alpha)\omega)$, where $w^0_{\varepsilon,\delta}$ is the near solution of (4.13). Then $w_{\varepsilon,\delta}$ satisfies on $\mathbb{R}^{n+1}$ for almost every ω the equation

$$D_t w^\mu_{\varepsilon,\delta} - div(a_\omega(x/\varepsilon, t/\varepsilon^2, \eta, \xi + D_x w^\mu_{\varepsilon,\delta})) = div_x \rho_{\delta,\omega}, \tag{4.16}$$

where $\langle|\rho_\delta|^{p'}\rangle < s(\delta)$ and $s(\delta) \to 0$ as $\delta \to 0$. As in the proof of Theorem 4.9, we choose two sequences $\delta_k \to 0$ and $\varepsilon_k = \varepsilon(\delta_k) \to 0$ such that $w_{\varepsilon_k,\delta_k} \to 0$ weakly in $\overline{W}$, and $\rho_k = \rho_{\delta_k,\omega} \to 0$ in $L^{p'}(Q)^n$ as $k \to \infty$. This is possible because of Lemma 4.8 and Lemma 3.2. Then $w_k = w_{\varepsilon_k,\delta_k} \to 0$ weakly in $\overline{W}$ as $k \to \infty$.

As in the proof of Theorem 4.9, the operators L_{ε_k} G-converge to an operator $\tilde{L}$ (up to a subsequence) and using the convergence of arbitrary solutions we have that

$$\begin{aligned} a_\omega(x/\varepsilon_k, t/\varepsilon_k^\alpha, \eta, \xi + D_x w_k) &\to \tilde{a}(x, t, \eta, \xi), \\ a_{0,\omega}(x/\varepsilon_k, t/\varepsilon_k^\alpha, \eta, \xi + D_x w_k) &\to \tilde{a}_0(x, t, \eta, \xi) \end{aligned} \tag{4.17}$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$. Set $w^0_k = w^0_{\varepsilon_k,\delta_k}$. By the ergodic theorem we have

$$\begin{aligned} a_\omega(x/\varepsilon_k, t/\varepsilon_k^\alpha, \eta, \xi + D_x w^0_k) &\to \langle a(\omega, \eta, \xi + \partial w^0)\rangle, \\ a_{0,\omega}(x/\varepsilon_k, t/\varepsilon_k^\alpha, \eta, \xi + D_x w^0_k) &\to \langle a_0(\omega, \eta, \xi + \partial w^0)\rangle \end{aligned} \tag{4.18}$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$.

Since $\|w^\mu - w^0\|_{\mathcal{V}} \to 0$ as $\mu \to 0$ we obtain that $\|D_x w_k - D_x w^0_k\|_{p,Q} \to 0$ as $k \to \infty$. The latter follows from the triangle inequality

$$\begin{aligned} &\|D_x w_k - D_x w^0_k\|_{p,Q} \\ &\le \|D_x w_k - D_x w^\mu\|_{p,Q} + \|D_x w^\mu - D_x w^0\|_{p,Q} + \|D_x w^0 - D_x w^0_k\|_{p,Q} \end{aligned}$$

and Lemma 3.2. Indeed, $\|D_x w_k - D_x w^\mu\|_{p,Q}$ and $\|D_x w^0 - D_x w^0_k\|_{p,Q}$ can be estimated using Lemma 3.2, because they represent the error associated with near solutions, and $\langle|\partial w^\mu_\delta - \partial w^\mu|^p\rangle < s(\delta)$, and $\langle|\partial w^0_\delta - \partial w^0|^p\rangle < s(\delta)$, where $s(\delta) \to 0$ as $\delta \to 0$. Thus, $\|D_x w_k - D_x w^\mu\|_{p,Q}$ and $\|D_x w^0 - D_x w^0_k\|_{p,Q}$ converge to zero as $k \to \infty$. The term $\|D_x w^\mu - D_x w^0\|_{p,Q}$ converges to zero as $\mu \to 0$ or $\varepsilon \to 0$ because of Lemma 4.10. Therefore,

$$\begin{aligned} a_\omega(x/\varepsilon_k, t/\varepsilon_k^\alpha, \eta, \xi + D_x w_k) - a_\omega(x/\varepsilon_k, t/\varepsilon_k^\alpha, \eta, \xi + D_x w^0_k) &\to 0, \\ a_{0,\omega}(x/\varepsilon_k, t/\varepsilon_k^\alpha, \eta, \xi + D_x w_k) - a_{0,\omega}(x/\varepsilon_k, t/\varepsilon_k^\alpha, \eta, \xi + D_x w^0_k) &\to 0 \end{aligned} \tag{4.19}$$

in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$. Combining (4.17), (4.18) and (4.19), we see that L^* defines the homogenized operator. □

4.2.2. *Case* $\alpha > 2\beta$. As before, we take $\beta = 1$. For the analysis of this case we need to consider the asymptotic behavior of w^μ as $\mu \to \infty$. This requires the average of $a(\omega, \eta, \xi)$ over the time variable defined by

$$\overline{a}(\omega, \eta, \xi) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a(T_1(t)\omega, \eta, \xi)dt = M_t\{a(\omega, \eta, \xi)\}$$

(see (2.9)). It is easy to verify that the function $\overline{a}(\omega, \eta, \xi)$ is a Carathéodory function satisfying assumptions (2.2)–(2.4).

Consider the subset of S consisting of functions $f(\omega) = M_t\{f(T_1(t)\omega)\}$. Denote by $\mathcal{V}_s$ the completion of this set with respect to the norm

$$\|f\| = (\sum_{i=1}^{n} \|\partial_i f\|^p_{L_p(\Omega)})^{1/p}.$$

Note that, for $f \in \mathcal{V}_s$, a generic realization of ∂f is independent of the time variable t.

Define the operator $\overline{A} : \mathcal{V}_s \to \mathcal{V}_s'$ by

$$\langle \overline{A}u, v\rangle = \langle \overline{a}(\omega, \eta, \xi + \partial u), \partial v\rangle \quad \forall v \in \mathcal{V}_s.$$

As follows from (2.2)–(2.4), $\overline{A} : \mathcal{V}_s \to \mathcal{V}_s'$ is a bounded, continuous and strongly monotone operator that depends on (η, ξ) as parameters. This implies the existence and uniqueness of $w^\infty \in \mathcal{V}_s$ that solves the equation

$$-\mathbf{div}(\overline{a}(\omega, \eta, \xi + \partial w^\infty)) = 0. \tag{4.20}$$

Denote near solutions of (4.20) by w^∞_δ. Note that w^∞_δ and w^∞ depend on (η, ξ).

Lemma 4.12. $\lim_{\mu\to\infty} \|w^\mu - w^\infty\|_{\mathcal{V}} = 0$.

Proof. Set

$$w^\mu_\delta = w^\infty + \frac{1}{\mu} v_\delta,$$

where v_δ will be defined later. Note that δ does not indicate near solutions here. We will show that w^μ_δ approximates w^μ for large μ. We have

$$\mu\sigma w^\mu_\delta + A w^\mu_\delta = \sigma v_\delta + f_1 + f_{2,\delta},$$

where A is defined as previously by

$$Au = -\mathbf{div}\ a(\omega, \eta, \xi + \partial u)$$

and

$$f_1 = -\mathbf{div}\ a(\omega, \eta, \xi + \partial w^\infty),$$
$$f_2^\delta = \mathbf{div}(a(\omega, \eta, \xi + \partial w_\delta^\mu) - a(\omega, \eta, \xi + \partial w^\infty)).$$

Using the Birkhoff ergodic theorem, we obtain that for every $\varphi \in \mathcal{V}_s$

$$\begin{aligned}\langle f_1, \varphi\rangle &= \langle a(\omega, \eta, \xi + \partial w^\infty)\partial\varphi\rangle = M\{a_\omega(\cdot, \eta, \xi + (\partial w_\omega^\infty))(\partial\varphi)_\omega\} \\ &= \langle M_x\{M_t\{a_\omega(\cdot, \eta, \xi + (\partial w^\infty)_\omega)\}(\partial\varphi)_\omega\}\rangle \\ &= \langle M_x\{\overline{a}_\omega(\cdot, \eta, \xi + (\partial w^\infty)_\omega)(\partial\varphi)_\omega\}\rangle = \langle \overline{a}(\omega, \eta, \xi + \partial w^\infty)\partial\varphi\rangle = 0.\end{aligned}$$

Consider σ as a closed operator from $\mathcal{V}$ to $\mathcal{V}'$. The kernel of σ is $\mathcal{V}_s$. Using the fact that the range of σ is dense in the orthogonal complement of $ker(\sigma^+)$, and the fact that $\sigma^+ = -\sigma$ (σ^+ is the adjoint of σ), we have that there exist $v_\delta \in \mathcal{W}$, $g_\delta \in \mathcal{V}'$ such that $f_1 = -\sigma v_\delta + g_\delta$ and $\|g_\delta\|_{\mathcal{V}'} \le s(\delta)$, where $s(\delta) \to 0$ as $\delta \to 0$. This is how we define v_δ. Then

$$\begin{aligned}C\|w_\delta^\mu - w^\mu\|_{\mathcal{V}}^p &\le \langle \mu\sigma(w_\delta^\mu - w^\mu) + Aw_\delta^\mu - Aw^\mu, w_\delta^\mu - w^\mu\rangle \\ &= \langle \mu\sigma w_\delta^\mu + Aw_\delta^\mu, w_\delta^\mu - w^\mu\rangle = \langle \sigma v_\delta + f_1 + f_{2,\delta}, w_\delta^\mu - w^\mu\rangle \\ &= \langle g_\delta + f_{2,\delta}, w_\delta^\mu - w^\mu\rangle \le (\|g_\delta\|_{\mathcal{V}'} + \|f_{2,\delta}\|_{\mathcal{V}'})\|w_\delta^\mu - w^\mu\|_{\mathcal{V}}.\end{aligned}$$

This implies

$$\|w_\delta^\mu - w^\mu\|_{\mathcal{V}} \le C(\|g_\delta\|_{\mathcal{V}'} + \|f_{2,\delta}\|_{\mathcal{V}'})^{1/(p-1)}.$$

On the other hand using Holder continuity of a we have

$$\|f_{2,\delta}\|_{\mathcal{V}'} \le C\|w_\delta^\mu - w^\infty\|_{\mathcal{V}} = C\mu^{-1}\|v_\delta\|_{\mathcal{V}}.$$

Consequently,

$$\|w_\delta^\mu - w^\mu\|_{\mathcal{V}} \le C(s(\delta) + \mu^{-1}\|v_\delta\|_{\mathcal{V}})^{1/(p-1)}.$$

Furthermore,

$$\|w^\mu - w^\infty\|_{\mathcal{V}} \le \|w_\delta^\mu - w^\infty\|_{\mathcal{V}} + \|w^\mu - w_\delta^\mu\|_{\mathcal{V}} \le \mu^{-1}\|v_\delta\|_{\mathcal{V}} + \|w^\mu - w_\delta^\mu\|_{\mathcal{V}}.$$

Thus, for any $\zeta > 0$ we can choose δ sufficiently small such that for all $\mu > \mu_0$ we have $\|w^\mu - w^\infty\|_{\mathcal{V}} < \zeta$. □

Define

$$L^*u = D_t u - div(a^*(u, D_x u)) + a_0^*(u, D_x u), \tag{4.21}$$

where

$$a^*(\eta, \xi) = \langle a(\omega, \eta, \xi + \partial w^\infty)\rangle, \quad a_0^*(\eta, \xi) = \langle a_0(\omega, \eta, \xi + \partial w^\infty)\rangle \tag{4.22}$$

and w^∞ is the solution of (4.20).

Theorem 4.13. *If $\alpha > 2\beta$, then L_ε G-converges to L^* defined by* (4.21) *for almost every $\omega \in \Omega$.*

Proof. Set $\mu = \varepsilon^{2-\alpha} \to \infty$ as $\varepsilon \to 0$ and

$$w_{\varepsilon,\delta} = \varepsilon w^{\mu}_{\delta,\omega}(x/\varepsilon, t/\varepsilon^{\alpha}), \quad w^{\infty}_{\varepsilon,\delta} = \varepsilon w^{\infty}_{\delta,\omega}(x/\varepsilon, t/\varepsilon^{\alpha}),$$

where w^{μ}_{δ} are near solutions of (3.1) and w^{∞}_{δ} are near solutions of (4.20). Then $w_{\varepsilon,\delta}$ satisfies on $\mathbb{R}^{n+1}$ for almost every ω

$$D_t w_{\varepsilon,\delta} - div(a_\omega(x/\varepsilon, t/\varepsilon^{\alpha}, \eta, \xi + D_x w_{\varepsilon,\delta})) = div_x \rho_{\delta,\omega},$$

where $\langle |\rho_\delta|^{p'} \rangle \to 0$ as $\delta \to 0$. As in the proof of Theorem 4.9, we choose two sequences $\delta_k \to 0$ and $\varepsilon_k = \varepsilon(\delta_k) \to 0$ such that $w_k = w_{\varepsilon_k,\delta_k} \to 0$ weakly in $\overline{W}$, and $\rho_k = \rho_{\delta_k} \to 0$ in $L^{p'}(Q)^n$ as $k \to \infty$. This is possible because of Lemma 4.8 and Lemma 3.2. Using the convergence of arbitrary solutions for G-convergent sequences of operators as in the proof of Theorem 4.9, we obtain that for almost every ω

$$\begin{aligned} a_\omega(x/\varepsilon_k, t/\varepsilon^{\alpha}, \eta, \xi + D_x w_k) &\to \tilde{a}(x, t, \eta, \xi) \\ a_{0,\omega}(x/\varepsilon_k, t/\varepsilon^{\alpha}, \eta, \xi + D_x w_k) &\to \tilde{a}_{0,\omega}(x, t, \eta, \xi) \end{aligned} \tag{4.23}$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$. On the other hand, using the ergodic theorem, we have that

$$\begin{aligned} a_\omega(x/\varepsilon_k, t/\varepsilon^{\alpha}, \eta, \xi + D_x w^{\infty}_k) &\to \langle a(\omega, \eta, \xi + \partial w^{\infty}) \rangle, \\ a_{0,\omega}(x/\varepsilon_k, t/\varepsilon^{\alpha}, \eta, \xi + D_x w^{\infty}_k) &\to \langle a_0(\omega, \eta, \xi + \partial w^{\infty}) \rangle \end{aligned} \tag{4.24}$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$. Since

$$\lim_{\mu \to \infty} \|w^{\mu} - w^{\infty}\|_{\mathcal{V}} = 0,$$

as in the proof of Theorem 4.11, we obtain that

$$\|D_x w_k - D_x w^{\infty}_k\|_{p,Q} \to 0$$

as $k \to \infty$. Consequently,

$$\begin{aligned} a_\omega(x/\varepsilon_k, t/\varepsilon^{\alpha}, \eta, \xi + D_x w_k) - a_\omega(x/\varepsilon_k, t/\varepsilon^{\alpha}, \eta, \xi + D_x w^{\infty}_k) &\to 0, \\ a_{0,\omega}(x/\varepsilon_k, t/\varepsilon^{\alpha}, \eta, \xi + D_x w_k) - a_{0,\omega}(x/\varepsilon_k, t/\varepsilon^{\alpha}, \eta, \xi + D_x w^{\infty}_k) &\to 0 \end{aligned} \tag{4.25}$$

in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$. Combining (4.23), (4.24) and (4.25), we see that L^* defines the homogenized operator. $\square$

4.3. **Spatial homogenization.** As before, we take $\beta = 1$. Note that in this case the homogenized fluxes a^* and a_0^* depend on ω, but their generic realizations are independent of the spatial variable x.

Denote by $w = w_{\eta,\xi} \in \mathcal{V}$ the solution of auxiliary equation (4.13); i.e.

$$-\mathbf{div}\ a(\omega, \eta, \xi + \partial w) = 0. \tag{4.26}$$

The existence and uniqueness of this equation is discussed previously (see Lemma 4.10).

We show that the homogenized operator has the form

$$L^* u = D_t u - div(a^*_\omega(t, u, D_x u)) + a^*_{0,\omega}(t, u, D_x u). \tag{4.27}$$

Theorem 4.14. *The operator L_ε G-converges to L^* defined by* (4.27) *for almost every $\omega \in \Omega$.*

Proof. Set $\mu = \varepsilon^2$ and $w_{\varepsilon,\delta} = \varepsilon w^\mu_{\delta,\omega}(x/\varepsilon, t)$, where w^μ_δ are near solutions of (3.1). Then $w_{\varepsilon,\delta}$ satisfies in $\mathbb{R}^{n+1}$

$$D_t w_{\varepsilon,\delta} - div(a_\omega(x/\varepsilon, t, \eta, \xi + D_x w_{\varepsilon,\delta})) = div_x \rho_{\delta,\omega},$$

where $\langle |\rho_\delta|^{p'} \rangle \to 0$ as $\delta \to 0$. As in the proof of Theorem 4.9, we choose two sequences $\delta_k \to 0$ and $\varepsilon_k = \varepsilon(\delta_k) \to 0$ such that $w_k = w_{\varepsilon_k,\delta_k} \to 0$ weakly in $\overline{W}$, and $\rho_k = \rho_{\delta_k} \to 0$ in $L^{p'}(Q)^n$ as $k \to \infty$. This is possible for any sequence $\delta \to 0$ because of Lemma 4.8 and Lemma 3.2.

Using the convergence of arbitrary solutions for a G-convergent sequence of operators as in the proof of Theorem 4.9, we obtain that

$$a_\omega(x/\varepsilon_k, t, \eta, \xi + D_x w_k) \to \tilde{a}(x, t, \eta, \xi),$$
$$a_{0,\omega}(x/\varepsilon_k, t, \eta, \xi + D_x w_k) \to \tilde{a}_0(x, t, \eta, \xi)$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$.

Set $w_{0,k} = w_\omega(x/\varepsilon_k, t)$. Using the ergodic theorem and the argument as in [22] (page 228) we obtain

$$\begin{aligned} a_\omega(x/\varepsilon_k, t, \eta, \xi + D_x w_{x,k}) = a(T_1(t)T_2(x/\varepsilon_k)\omega, \eta, \xi + D_x w_{x,k}) \to \\ \to M_x\{a(T_1(t)T_2(x)\omega, \eta, \xi + \partial w_x)\} \\ a_{0,\omega}(x/\varepsilon_k, t, \eta, \xi + D_x w_{x,k}) = a_0(T_1(t)T_2(x/\varepsilon_k)\omega, \eta, \xi + D_x w_{x,k}) \to \\ \to M_x\{a_0(T_1(t)T_2(x)\omega, \eta, \xi + \partial w_x)\} \end{aligned}$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$.

Since $\|w^\mu - w_x\|_{\mathcal{V}} \to 0$ as $\mu \to 0$, as in Theorem 4.11, we obtain

$$\|D_x w_k - D_x w_{0,k}\|_{p,Q} \to 0$$

as $k \to \infty$. Consequently,

$$\begin{aligned} a_\omega(x/\varepsilon_k, t, \eta, \xi + D_x w_k) - a_\omega(x/\varepsilon_k, t, \eta, \xi + D_x w_{0,k}) \to 0 \\ a_{0,\omega}(x/\varepsilon_k, t, \eta, \xi + D_x w_k) - a_{0,\omega}(x/\varepsilon_k, t, \eta, \xi + D_x w_{0,k}) \to 0 \end{aligned} \tag{4.28}$$

in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$. Thus,

$$\tilde{a}(x, t, \eta, \xi) = a^*_\omega(t, \eta, \xi), \quad \tilde{a}_0(x, t, \eta, \xi) = a^*_{0,\omega}(t, \eta, \xi).$$

The proof is complete. □

4.4. **Time homogenization.** Here we take $\alpha = 1$. Following [22] we introduce the orthogonal projection operator P_1 as follows

$$P_1 f = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(T_1(\tau)\omega) d\tau. \tag{4.29}$$

Consider

$$a^*(\omega, \eta, \xi) = P_1 a(\omega, \eta, \xi), \quad a^*_0(\omega, \eta, \xi) = P_1 a_0(\omega, \eta, \xi). \tag{4.30}$$

Generic realizations of these fluxes are independent of t. We show that the homogenized operator is given by

$$L^* u = D_t u - {a_\omega}^*(x, u, Du) - a^*_{0,\omega}(x, u, Du). \tag{4.31}$$

Theorem 4.15. *The operator L_ε G-converges to L^* defined by* (4.31) *for almost every $\omega \in \Omega$.*

Proof. Consider

$$F = P_1 a(\omega, \eta, \xi) - a(\omega, \eta, \xi), \quad f = \mathbf{div} F,$$

where **div** is defined by (2.6). Since $\langle f, \varphi\rangle = \langle F, \partial\varphi\rangle = 0$ for any $\varphi \in \mathcal{V}_s$, we have, as in the proof of Lemma 4.12, that there exist $w_\zeta \in \mathcal{W}$, $g_\zeta \in \mathcal{V}'$ such that $f = -\sigma w_\zeta + g_\zeta$ and $\|g_\zeta\|_{\mathcal{V}'} \le s(\zeta)$, where $s(\zeta) \to 0$ as $\zeta \to 0$.

We employ the theorem on the convergence of arbitrary solution for w_ζ. Since $w_\zeta \in \mathcal{W}$, we need near solutions. Set $w_{\varepsilon,\delta,\zeta} = \varepsilon w_{\delta,\zeta}(T(x, t/\varepsilon)\omega)$, where $w_{\delta,\zeta}$ is an approximation of w_ζ that has smooth realizations, and

$$\|w_{\delta,\zeta} - w_\zeta\|_{\mathcal{V}} \le s(\delta),$$

where $s(\delta) \to 0$ as $\delta \to 0$. As in the proof of Theorem 4.9, it can be shown that $w_{\varepsilon,\delta,\zeta} \to 0$ as $\varepsilon \to 0$ weakly in $\overline{W}$ and strongly in $\overline{V}$ for any $\delta > 0$ and $\zeta > 0$. Since any realization can be considered as a function of t with values in corresponding local functional space on $\mathbb{R}^n_x$, this follows from the following equations:

$$\|w_{\delta,\zeta}(T(x,t/\varepsilon)\omega)\|^p_{\overline{V}} = \int_0^t \int_{Q_0} |D_x w_{\delta,\zeta}(T(x,\tau/\varepsilon)\omega)|^p dx d\tau =$$

$$= \varepsilon \int_0^{t/\varepsilon} \int_{Q_0} |D_x w_{\delta,\zeta}(T(x,\tau)\omega)|^p dx d\tau \to \|w_{\delta,\zeta}\|_{\mathcal{V}},$$

$$\|D_t w_{\delta,\zeta}(T(x,t/\varepsilon)\omega)\|^{p'}_{\overline{V}'} = \int_0^t \|D_\tau w_{\delta,\zeta}(T(x,\tau/\varepsilon)\omega)\|^{p'}_{W^{-1,p'}(Q_0)} d\tau =$$

$$= \varepsilon \int_0^{t/\varepsilon} \|D_\tau w_{\delta,\zeta}(T(x,\tau)\omega)\|^{p'}_{W^{-1,p'}(Q_0)} d\tau \to \|\sigma w_{\delta,\zeta}\|_{\mathcal{V}'}$$

as $\epsilon \to 0$.

Next we show that for any sequence $\zeta \to 0$ and $\delta \to 0$ there exists a sequence $\varepsilon = \varepsilon(\delta,\zeta) \to$ such that $w_{\varepsilon,\delta,\zeta} \to 0$ weakly in $\overline{W}$ and strongly in $\overline{V}$ as well as

$$g_{\varepsilon,\zeta} = g_\zeta(T(x,t/\varepsilon)\omega) \to 0$$

in $\overline{V}'$. (Clearly the same holds also for $w_{\varepsilon,\delta,\zeta}$). This can be shown using Lemma 3.2.

Consequently, given a sequence $\zeta \to 0$, there exists a sequence $\varepsilon(\zeta) \to 0$ such that $g_{\varepsilon,\zeta} \to 0$ in $\overline{V}'$. Next we choose sequences $\delta_k \to 0$, $\zeta_k \to 0$ and $\varepsilon_k = \varepsilon_k(\delta_k,\zeta_k) \to 0$ such that $w_{\varepsilon_k,\delta_k,\zeta_k} \to 0$ weakly in $\overline{W}$ and strongly in $\overline{V}$ as well as

$$g_k = g_{\varepsilon_k,\zeta_k} = g_{\zeta_k}(T(x,t/\varepsilon_k)\omega) \to 0$$

in $\overline{V}'$. Then w_k satisfies on $\mathbb{R}^{n+1}$

$$D_t w_k - div(a_\omega(x,t/\varepsilon_k),\eta,\xi + D_x w_k)) + a_{0,\omega}(x,t/\varepsilon_k,\eta,\xi + D_x w_k)$$
$$= g_k - div(a^*(T_2(x)\omega,\eta,\xi)) + \varphi_k + \psi_k,$$

where

$$\varphi_k = -div(a_\omega(x,t/\varepsilon_k,\eta,\xi + D_x w_k) - a_\omega(x,t/\varepsilon_k,\eta,\xi)),$$
$$\psi_k = a_{0,\omega}(x,t/\varepsilon_k,\eta,\xi + D_x w_k).$$

Because of the Holder continuity of a we obtain that $\varphi_k \to 0$ in $\overline{V}'$. Similarly

$$\psi_k \to \psi = a_0(T_2(x)\omega,\eta,\xi)$$

weakly in $L^{p'}(Q)$ for almost every ω. Using the theorem on the convergence of arbitrary solutions, we have that L_k G-converges to some

$$\tilde{L}u = D_t u - div(\tilde{a}(x,t,u,D_x u)) + \tilde{a}_0(x,t,u,D_x u)$$

for almost every ω. Here

$$a_\omega(x,t/\varepsilon_k,\eta,\xi + D_x w_k) \to \tilde{a}(x,t,\eta,\xi),$$
$$a_{0,\omega}(x,t/\varepsilon_k,\eta,\xi + D_x w_k) \to \tilde{a}_0(x,t,\eta,\xi)$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$ respectively, and $u = 0$ is a solution of

$$\tilde{L}u = -div(a^*(T_2(x)\omega,\eta,\xi)) + a_0^*(T_2(x)\omega,\eta,\xi).$$

On the other hand, using the ergodic theorem and the argument as in [22] (page 228), we obtain

$$a_\omega(x,t/\varepsilon_k,\eta,\xi) \to a^*(T_2(x)\omega,\eta,\xi),$$
$$a_{0,\omega}(x,t/\varepsilon_k,\eta,\xi) \to a_0^*(T_2(x)\omega,\eta,\xi)$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$. Here a^* and a_0^* are defined by (4.30).

Since $w_k \to 0$ strongly in $\overline{V}$, we have that

$$a_\omega(x,t/\varepsilon_k,\eta,\xi + D_x w_k) - a_\omega(x,t/\varepsilon_k,\eta,\xi) \to 0,$$
$$a_{0,\omega}(x,t/\varepsilon_k,\eta,\xi + D_x w_k) - a_{0,\omega}(x,t/\varepsilon_k,\eta,\xi) \to 0$$

strongly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$ as $k \to \infty$, and we conclude. □

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REFERENCES

[1] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, "Asymptotic Analysis for Periodic Structures," North Holland, Amsterdam, 1978.

[2] A. Braides, V. Chiadò Piat , and A. Defranceschi, *Homogenization of almost periodic monotone operators*, Ann. Inst. H. Poincaré, Anal. non Lin., 9 (1992), 399–432.

[3] A. Braides and A. Defranceschi, "Homogenization of Multiple Integrals," Clarendon Press, Oxford Univ. Press, New York, 1998.

[4] G. Bruno, A. Pankov, and T. Pankova, *Time averaging for random nonlinear abstract parabolic equations*, Abstr. Appl. Anal., 5 (2000), 1–11.

[5] V. Chiadò Piat, G. Dal Maso, and A. Defranceschi, *G-convergence of monotone operators*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 7 (1990), 123–160.

[6] D. Cioranescu and P. Donato, "An Introduction to Homogenization," Clarendon Press, Oxford Univ. Press, New York, 1999.

[7] G. Dal Maso, "An Introduction to Γ-convergence," Birkhäuser, Boston, 1993.
[8] N. Dunford and J. T. Schwartz, "Linear Operators, Vol. 1," Wiley, New York, 1957.
[9] Y. Efendiev and A. Pankov, *Numerical homogenization of nonlinear random parabolic operators*, SIAM Multiscale Model. Simul., 2 (2004), 237–268.
[10] N. Fusco and G. Moscariello, *Further results on the homogenization of quasilinear operators*, Ric Mat., 35 (1986), 231–246.
[11] N. Fusco and G. Moscariello, *On the homogenization of quasilinear divergence structure operators*, Ann. Mat. Pura. Appl., Vol. 146 (1987), 1–13.
[12] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, "Homogenization of Differential Operators and Integral Functionals," Springer-Verlag, New York, 1994.
[13] J. L. Lions, "Quelques Méthodes de Résoluton des Problèmes aux Limites Non Linéairies," Dunod, Paris, 1969.
[14] A. K. Nandakumaran and M. Rajesh, *Homogenization of a nonlinear degenerate parabolic differential equation*, Electron. J. Differential Equations, 17 (2001), 1–19 (electronic).
[15] A. Pankov, *Homogenization and G-convergence of nonlinear elliptic operators in divergence form*, Dokl. AN SSSR, 278 (1984), 37–41 (in Russian).
[16] A. Pankov, *Strong G-convergence of nonlinear elliptic operators and homogenization*, Constanin Carathéodory — An International Tribute (ed. Th. Rassias), World Sci. Publ., Singapore, 1991, 1075–1099.
[17] A. Pankov, "G-convergence and Homogenization of Nonlinear Partial Differential Operators," Kluwer Academic Publishers, Dordrecht, 1997.
[18] R. E. Showalter, "Monotone Operators in Banach space and Nonlinear Partial Differential Equations," Vol. 49 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1997.
[19] N. Svanstedt, *G-convergence of parabolic operators*, Nonlinear Anal., 36 (1999), 807–842.
[20] N. Svanstedt, *Correctors for the homogenization of monotone parabolic operators*, J. Nonlinear Math. Phys., 7 (2000), 268–283.
[21] E. Zeidler, "Nonlinear Functional Analysis and its Applications, Vols IIA and IIB", Springer, New York, 1990.
[22] V. V. Zhikov, S. M. Kozlov, and O. A. Oleĭ nik, *Averaging of parabolic operators*, Trudy Moskov. Mat. Obshch., 45 (1982), 182–236.
[23] V. V. Zhikov, S. M. Kozlov, and O. A. Oleĭ nik,*Averaging of parabolic operators with almost periodic coefficients*, Mat. Sb. (N.S.), 117(159) (1982), 69–85.