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THE CRITICAL NEUMANN PROBLEM FOR SEMILINEAR ELLIPTIC EQUATIONS WITH THE HARDY POTENTIAL

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Abstract. We investigate the solvability of the Neumann problem (1.1) involving the critical Sobolev nonlinearity with an indefinite weight function and the Hardy potential. We prove that there exists $\lambda^* > 0$ such that for $\lambda \in (0, \lambda^*)$, problem (1.1) admits at least two distinct solutions.

1. Introduction

In this paper we investigate the solvability of the nonlinear Neumann problem

$$\begin{cases} -\Delta u - \lambda \frac{u}{|x|^2} & = Q(x)|u|^{2^*-2}u \ \text{ in } \ \Omega, \\ \frac{\partial u}{\partial \nu} & = 0 \ \text{ on } \ \partial\Omega, \ u > 0 \ \text{ on } \Omega, \end{cases} \tag{1.1}$$

where $2^* = \frac{2N}{N-2}$, $N \geq 3$ is the critical Sobolev exponent, $\lambda > 0$ is a parameter and $\Omega \subset \mathbb{R}^N$ is an open bounded set with a smooth boundary $\partial\Omega$. Throughout this paper we assume that $Q(x)$ is a continuous function on $\bar{\Omega}$ changing sign and such that

$$(\mathbf{Q}) \qquad \int_\Omega Q(x)\, dx < 0.$$

We always assume, with the exception of Section 4, that $0 \in \partial\Omega$ and that $|\{x \in \bar{\Omega};\, Q(x) = 0\}| = 0$. Here $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$. We look for solutions of problem (1.1) as critical points of the variational functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega \big(|Du|^2 - \lambda \frac{u^2}{|x|^2}\big)\, dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\, dx$$

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for $u \in H^1(\Omega)$. Here $H^1(\Omega)$ is the Sobolev space equipped with the norm

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 + u^2\right) dx.$$

We point out here that if $Q(x) > 0$ on Ω, then problem (1.1) does not have a solution. Indeed, testing (1.1) with constant functions we get

$$-\lambda \int_\Omega \frac{u}{|x|^2}\, dx = \int_\Omega Q(x) u^{2^*-1}\, dx,$$

which is a contradiction. On the other hand, if (1.1) has a solution $u \in H^1(\Omega)$, then testing this equation with $\left(u^{2^*-1}+\epsilon\right)^{-1}$ we get

$$-(2^*-1)\int_\Omega \frac{|\nabla u|^2 u^{2^*-2}}{\left(u^{2^*-1}+\epsilon\right)^2} dx - \lambda \int_\Omega \frac{u^2}{|x|^2(u^{2^*-1}+\epsilon)}\, dx = \int_\Omega Q(x) \frac{u^{2^*-1}}{u^{2^*-1}+\epsilon} dx.$$

Letting $\epsilon \to 0$, we get

$$\int_\Omega Q(x)\, dx < 0.$$

These two remarks justify our assumption (**Q**). If Q changes sign and satisfies (**Q**), then the variational functional J_λ has a mountain - pass structure. In this case we are able to show the existence of at least two distinct solutions. However, if $Q(x) < 0$ on $\bar{\Omega}$, then J_λ is coercive. In this case we can show the existence of a solution which is a global minimizer. This will be shown in Section 5. The term $\frac{u}{|x|^2}$ is related to the Hardy inequality: if $0 \in \Omega$, then

$$\int_\Omega \frac{u^2}{|x|^2}\, dx \leq \frac{1}{\bar{\mu}} \int_\Omega |\nabla u|^2\, dx$$

for every $u \in H^1_\circ(\Omega)$, where $\bar{\mu} = \frac{(N-2)^2}{4}$ is an optimal constant. In this paper we need the $H^1(\Omega)$-version of this inequality, which will be formulated in Section 2.

Problems involving the Hardy potential and the critical nonlinearity have an extensive literature in the case of the Dirichlet problem. The seminal paper by Brezis and Nirenberg [11] has inspired research on elliptic equations with the critical Sobolev exponents. It appears that [31] has been the first paper dealing simultaneously with both the Hardy potential and the critical Sobolev exponent. This paper extends the results of [11] to the Dirichlet problem

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} & = u^{2^*-1} + \lambda u \ \text{ in } \ \Omega, \\ u & = 0 \ \text{ on } \ \partial\Omega, \ u > 0 \ \text{ on } \ \Omega. \end{cases}$$

In particular, if $0 < \lambda_1(\mu)$ denotes the first eigenvalue of the operator $-\Delta - \frac{\mu}{|x|^2}$, then this problem has a solution in $H^1_\circ(\Omega)$ if $\mu \leq \bar{\mu} - 1$ and $0 < \lambda < \lambda_1(\mu)$. Since then a number of results have been obtained, in particular for the p-Laplacian [2], [3], [41] [13], and problems involving the Sobolev - Hardy potentials [1], [15], [16], [26], [27], [25],[32], [34].

Problem (1.1) with $\lambda = 0$ and under the assumption (**Q**) was considered in [17], where the existence of a solution was established. In the case $-\lambda > 0$ problem (1.1) was studied in [18], however with $0 \in \Omega$. In this paper the existence of a solution was proved under the assumption $Q(x) >$ on $\bar{\Omega}$. An analogous problem to (1.1), however without the Hardy potential and with $Q(x) > 0$ on $\bar{\Omega}$, that is,

$$\begin{cases} -\Delta u - \lambda u = Q(x)|u|^{2^*-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

was studied in [19]. The authors of [19], using a topological linking, proved the existence of a sign changing solution.

Throughout this paper we denote strong convergence by " $\to$ " and weak convergence by " $\rightharpoonup$ ". The norms in the Lebesgue space $L^p(\Omega)$, $1 \leq p < \infty$, are denoted by $\|\cdot\|_p$.

Let $\phi : X \to \mathbb{R}$ be a C^1 functional on a Banach space X. We recall that a sequence $\{x_n\} \subset X$ is a Palais- Smale sequence for ϕ at level c (a $(PS)_c$ sequence for short) if $\phi(x_n) \to c$ and $\phi'(x_n) \to 0$ in X^*. Finally, we say that the functional ϕ satisfies the Palais -Smale condition at level c ($(PS)_c$ condition for short) if each $(PS)_c$ sequence is relatively compact in X.

The paper is organized as follows. In Section 2 we obtain the existence of the first solution through a local minimization, the Ekeland variational principle and P.L. Lions concentration - compactness principle. Section 3 is devoted to the proof of the existence of a second solution. We find the energy level of a variational functional below which the Palais - Smale condition holds. Our approach is based on P.L. Lions' concentration-compactness principle [35]. This is used to find a second solution through the mountain-pass principle. Proposition 3.1 shows the important role of the shape of the coefficient $Q(x)$ and the mean curvature of the boundary $\partial\Omega$ in the verification of the Palais-Smale condition. In Section 4 we discuss the case $0 \in \Omega$. The important ingredients here are local estimates of a positive solution around the singular point 0 presented in Lemma 5. Theorems 1, 2, 3 and 4 are the main results of this paper.

2. Local minimization

The first solution will be obtained as a local minimizer of the functional J_λ. In this paper we shall use the following version of the Hardy inequality (see [12]): for every $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that

$$\int_\Omega \frac{u^2}{|x|^2}\,dx \leq \Big(\frac{1}{\bar{\mu}} + \epsilon\Big)\int_\Omega |\nabla u|^2\,dx + C(\epsilon)\int_\Omega u^2\,dx \tag{2.1}$$

for every $u \in H^1(\Omega)$.

Lemma 2.1. *Let $0 < \lambda < \bar{\mu}$. Suppose that $\{u_n\} \subset H^1(\Omega)$ is a $(PS)_c$ sequence for the functional J_λ. Then $\{u_n\}$ is bounded in $H^1(\Omega)$.*

Proof. Arguing by contradiction, assume that $\|u_n\|$ is unbounded and we may assume that $\|u_n\| \to \infty$. We put $v_n = \frac{u_n}{\|u_n\|}$. We may assume that $v_n \rightharpoonup v$ in $H^1(\Omega)$ and by the Sobolev embedding theorem $v_n \to v$ in $L^p(\Omega)$ for $2 \leq p < 2^*$. Since $J'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$, we have

$$\int_\Omega \Big(\nabla v_n \nabla\phi - \lambda\frac{v_n\phi}{|x|^2}\Big)\,dx = \|u_n\|^{2^*-2}\int_\Omega Q(x)|v_n|^{2^*-2}v_n\phi\,dx + o(1)$$

for every $\phi \in H^1(\Omega)$. From this we deduce that

$$0 = \lim_{n\to\infty}\int_\Omega Q(x)|v_n|^{2^*-2}v_n\phi\,dx = \int_\Omega Q(x)|v|^{2^*-2}v\phi\,dx$$

for every $\phi \in H^1(\Omega)$. Since $Q(x) = 0$ on a set of measure 0, we get that $v = 0$ a.e. on Ω. Using the fact that $\{u_n\}$ is a $(PS)_c$ sequence, we can write the following two relations:

$$\frac{1}{2}\int_\Omega \Big(|\nabla v_n|^2 - \lambda\frac{v_n^2}{|x|^2}\Big)\,dx = \frac{\|u_n\|^{2^*-2}}{2^*}\int_\Omega Q(x)|v_n|^{2^*}\,dx + o(1)$$

and

$$\int_\Omega \Big(|\nabla v_n|^2 - \lambda\frac{v_n^2}{|x|^2}\Big)\,dx = \|u_n\|^{2^*-2}\int_\Omega Q(x)|v_n|^{2^*}\,dx + o(1).$$

This yields

$$\lim_{n\to\infty}\int_\Omega \Big(|\nabla v_n|^2 - \lambda\frac{v_n^2}{|x|^2}\Big)\,dx = 0 \tag{2.2}$$

and

$$\lim_{n\to\infty}\|u_n\|^{2^*-2}\int_\Omega Q(x)|v_n|^{2^*}\,dx = 0.$$

Since $\frac{\lambda}{\bar{\mu}} < 1$, we can choose $\epsilon > 0$ so that $\lambda\big(\frac{1}{\bar{\mu}} + \epsilon\big) < 1$. Then from (2.1) and (2.2) we get

$$\begin{aligned}\int_\Omega |\nabla v_n|^2\,dx &= \lambda\int_\Omega \frac{v_n^2}{|x|^2}\,dx + o(1) \le \big(\frac{\lambda}{\bar{\mu}} + \lambda\epsilon\big)\int_\Omega |\nabla v_n|^2\,dx \\ &+ \lambda C(\epsilon)\int_\Omega v_n^2\,dx + o(1) = \big(\frac{\lambda}{\bar{\mu}} + \lambda\epsilon\big)\int_\Omega |\nabla v_n|^2\,dx + o(1).\end{aligned}$$

From this we derive that

$$\lim_{n\to\infty}\int_\Omega |\nabla v_n|^2\,dx = 0.$$

Consequently, $v_n \to 0$ in $H^1(\Omega)$ and this contradicts the fact that $\|v_n\| = 1$ for each n. □

We shall use the decomposition $H^1(\Omega) =$ span $1 \times V$, where $V = \{v \in H^1(\Omega);\ \int_\Omega v\,dx = 0\}$. Having this decomposition we can define an equivalent norm on $H^1(\Omega)$ by

$$\|u\|_V^2 = \|\nabla v\|_2^2 + t^2.$$

We need the following qualitative statement (see [9]):

(S) There exists a number $\eta > 0$ such that for each $t \in \mathbb{R}$ and $v \in V$ the inequality

$$\Big(\int_\Omega |\nabla v|^2\,dx\Big)^{\frac{1}{2}} \le \eta|t|$$

implies

$$\int_\Omega Q(x)|t + v(x)|^{2^*}\,dx \le \frac{|t|^{2^*}}{2}\int_\Omega Q(x)\,dx.$$

This follows from the continuity of the embedding of V into $L^{2^*}(\Omega)$.

Lemma 2.2. *There exist constants $\lambda^* > 0$, $\beta > 0$ and $\rho > 0$ such that for every $0 < \lambda < \lambda^*$ we have*

$$J_\lambda(u) \ge \beta \ \text{ for } \|u\|_V = \rho \ \text{ and } \inf_{\|u\|_V \le \rho} J_\lambda(u) < 0.$$

Proof. We distinguish two cases: (i) $\|\nabla v\|_2 \le \eta|t|$ and (ii) $\|\nabla v\|_2 > \eta|t|$, where $\eta > 0$ is a constant from statement (S). Let

$$\alpha = -\frac{1}{2}\int_\Omega Q(x)\,dx.$$

Then these two cases lead to the following estimate

$$J_\lambda(u) \geq \min\Big(\frac{\eta^2\rho^2}{4(1+\eta^2)^2}, \frac{\alpha\rho^{2^*}}{(1+\eta^2)^{\frac{2^*}{2}}}\Big) - \lambda\int_\Omega \frac{u^2}{|x|^2}\,dx$$

for $\|u\|_V = \rho$ (for details see [20] or [9]). By the Hardy inequality (see (2.1)) we get

$$J_\lambda(u) \geq \min\Big(\frac{\eta^2\rho^2}{4(1+\eta^2)^2}, \frac{\alpha\rho^{2^*}}{(1+\eta^2)^{\frac{2^*}{2}}}\Big) - \lambda C\rho^2$$

for $\|u\|_V = \rho$ and some constant $C > 0$ independent of ρ. We now choose a constant $\lambda^* > 0$ such that

$$J_\lambda(u) \geq \frac{1}{2}\min\Big(\frac{\eta^2\rho^2}{4(1+\eta^2)^2}, \frac{\alpha\rho^{2^*}}{(1+\eta^2)^{\frac{2^*}{2}}}\Big)$$

for $\|u\|_V = \rho$ and $0 < \lambda < \lambda^*$. To prove the second assertion we note that for $t > 0$ sufficiently small we have

$$J_\lambda(t) = -\frac{t^2}{2}\int_\Omega \frac{dx}{|x|^2}\,dx - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)\,dx < 0.$$

Hence, $a_\lambda := \inf_{\|u\|_V \leq \rho} J_\lambda(u) < 0$. □

From now on, we always assume that $0 < \lambda < \lambda^* \leq \bar{\mu}$. We now establish the existence of a minimizer of J_λ over the ball $\{u; \|u\|_V \leq \rho\}$. In the proof we shall use P.L. Lions' concentration - compactness principle which involves the best Sobolev constant S and the Hardy - Sobolev constant $S_{-\lambda}$. We recall that the constant S is defined by

$$S = \inf\{\int_{\mathbb{R}^N} |\nabla u|^2\,dx;\, u \in D^{1,2}(\mathbb{R}^N),\, \int_{\mathbb{R}^N} |u|^{2^*}\,dx = 1\},$$

where $D^{1,2}(\mathbb{R}^N)$ is the Sobolev space obtained as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$

The best Sobolev constant S is achieved by

$$U(x) = \frac{c_N}{\big(1+|x|^2\big)^{\frac{N-2}{2}}},$$

where $c_N > 0$ is a constant depending on N. The function U, called an instanton, satisfies the equation

$$-\Delta U = U^{2^*-1} \text{ in } \mathbb{R}^N.$$

We also have

$$\int_{\mathbb{R}^N} |\nabla U|^2\,dx = \int_{\mathbb{R}^N} |U|^{2^*}\,dx = S^{\frac{N}{2}}.$$

The constant $S_{-\lambda}$ is defined by

$$S_{-\lambda} = \inf_{u\in D^{1,2}(\mathbb{R}^N)-\{0\}} \frac{\int_{\mathbb{R}^N}\big(|\nabla u|^2 - \frac{\lambda u^2}{|x|^2}\big)\,dx}{\Big(\int_{\mathbb{R}^N}|u|^{2^*}\,dx\Big)^{\frac{2}{2^*}}}$$

for $0<\lambda<\bar{\mu}$. It is known [39] that $S_{-\lambda}$ is attained by the function

$$V_\lambda(x) = \frac{\big(4N(\bar{\mu}-\lambda)(N-2)\big)^{\frac{N-2}{4}}}{\Big(|x|^{\frac{\lambda'}{\sqrt{\bar{\mu}}}} + |x|^{\frac{\bar{\lambda}}{\sqrt{\bar{\mu}}}}\Big)^{\sqrt{\bar{\mu}}}},$$

where $\bar{\lambda} = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu}-\lambda}$ and $\lambda' = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\lambda}$. For $\epsilon>0$ we put

$$V_{\lambda,\epsilon}(x) = \frac{\big(4N(\bar{\mu}-\lambda)(N-2)\big)^{\frac{N-2}{4}}\epsilon^{\frac{N-2}{2}}}{\Big(\epsilon^2|x|^{\frac{\lambda'}{\sqrt{\bar{\mu}}}} + |x|^{\frac{\bar{\lambda}}{\sqrt{\bar{\mu}}}}\Big)^{\sqrt{\bar{\mu}}}}.$$

The function $V_{\lambda,\epsilon}$ is a solution of the problem

$$\begin{cases} -\Delta u - \lambda\frac{u}{|x|^2} = |u|^{2^*-2}u & \text{in } \mathbb{R}^N-\{0\},\\ u(x)\to 0 & \text{as } |x|\to\infty. \end{cases}$$

In paper [39] it was shown that

$$S_{-\lambda} = S\big(1 - \frac{4\lambda}{(N-2)^2}\big)^{\frac{N-1}{N}}.$$

Theorem 2.3. *For every $0<\lambda<\lambda^*$, problem (1.1) admits a solution.*

Proof. By the Ekeland variational principle [24] there exists $\{u_n\}\subset H^1(\Omega)$ such that $J_\lambda(u_n)\to a_\lambda$ and $J'_\lambda(u_n)\to 0$ in $H^{-1}(\Omega)$. Since $\{u_n\}$ is a bounded sequence in $H^1(\Omega)$ we may assume up to a subsequence that $u_n \rightharpoonup u_\lambda$ in $H^1(\Omega)$. By the P.L. Lions' concentration - compactness principle [35] there exist at most countable set J, a set of distinct points $\{x_j\}\subset\bar{\Omega}-\{0\}$, $j\in J$, and positive numbers ν_j, μ_j, $j\in J$, ν_0, μ_0 and λ_0 such that

$$|u_n|^{2^*}\,dx \rightharpoonup d\nu = |u|^{2^*}\,dx + \sum_{j\in J}\nu_j\delta_{x_j} + \nu_0\delta_0,$$

$$|\nabla u_n|^2\,dx \rightharpoonup d\mu \ge |\nabla u|^2\,dx + \sum_{j\in J}\mu_j\delta_{x_j} + \mu_0\delta_0$$

and
$$\frac{u_n^2}{|x|^2}\,dx \rightharpoonup d\tilde{\lambda} = \frac{u^2}{|x|^2}\,dx + \lambda_0\delta_0$$
in the sense of measure. Moreover, numbers ν_j, μ_j, μ_0, ν_0 and λ_0 satisfy the following inequalities
$$S\nu_j^{\frac{2}{2^*}} \le \mu_j \quad \text{if } x_j \in \Omega, \tag{2.3}$$
$$\frac{S}{2^{\frac{2}{N}}}\nu_j^{\frac{2}{2^*}} \le \mu_j \quad \text{if } x_j \in \partial\Omega \tag{2.4}$$
and
$$\frac{S_{-\lambda}}{2^{\frac{2}{N}}}\nu_0^{\frac{2}{2^*}} \le \mu_0 - \lambda\lambda_0. \tag{2.5}$$
Testing $J'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$ with a family of functions φ_δ, $\delta > 0$, concentrating at x_j (or 0) as $\delta \to 0$, we derive
$$\mu_j \le Q(x_j)\nu_j, \quad j \in J \tag{2.6}$$
and
$$\mu_0 \le Q(0)\nu_0 + \lambda\lambda_0. \tag{2.7}$$
Inequality (2.6) implies that $\{u_n\}$ can only concentrate at points x_j where $Q(x_j) > 0$. If $\nu_j > 0$, then in the case $x_j \in \partial\Omega$, (2.4) and (2.6) imply that
$$\frac{S^{\frac{N}{2}}}{2Q(x_j)^{\frac{N}{2}}} \le \nu_j. \tag{2.8}$$
If $x_j \in \Omega$, then (2.3) and (2.6) yield
$$\frac{S^{\frac{N}{2}}}{Q(x_j)^{\frac{N}{2}}} \le \nu_j. \tag{2.9}$$
Similarly, if $Q(0) > 0$ and $\nu_0 > 0$, then by (2.5) and (2.7) we obtain
$$\frac{S_{-\lambda}^{\frac{N}{2}}}{2Q(0)^{\frac{N}{2}}} \le \nu_0. \tag{2.10}$$
If $Q(0) \le 0$, then (2.7) yields $\mu_0 \le \lambda\lambda_0$. Hence by (2.5) we have $\nu_0 = 0$. So (2.5) and (2.7) imply that $\mu_0 = \lambda\lambda_0$. On the other hand by Lemma 2.1 we have $\lambda_0 \le \frac{\mu_0}{\bar{\mu}}$. Hence, if $\lambda_0 > 0$, then $\bar{\mu} \le \lambda$ which is a contradiction. Therefore $\lambda_0 = \mu_0 = 0$ and there is no concentration at 0 if $Q(0) \le 0$. We now consider
$$a_\lambda = \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty}\left[J_\lambda(u_n) - \frac{1}{2}\langle J'_\lambda(u_n), u_n\rangle\right] \tag{2.11}$$

$$= \lim_{n\to\infty} \frac{1}{N}\int_\Omega Q(x)|u_n|^{2^*}\,dx = \frac{1}{N}\int_\Omega Q(x)|u|^{2^*}\,dx$$
$$+ \sum_{j\in J, Q(x_j)>0} Q(x_j)\nu_j + \nu_0 Q(0).$$

Since $u_n \rightharpoonup u$ in $H^1(\Omega)$, we have $u \in \{v;\ \|v\|_V \le r\}$. On the other hand, u satisfies equation (1.1), so

$$J_\lambda(u) = \frac{1}{N}\int_\Omega Q(x)|u|^{2^*}\,dx = \frac{1}{N}\int_\Omega \big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\big)\,dx.$$

Thus, equality (2.11) implies that

$$a_\lambda \ge a_\lambda + \sum_{j\in J, Q(x_j)>0} Q(x_j)\nu_j + Q(0)\nu_0.$$

This obviously implies that $\nu_j = 0$ for each $j \in J$ and $\nu_0 = 0$. In a similar way we show that $\mu_j = 0$ for $j \in J$ and $\mu_0 = 0$. Finally, from (2.5) we deduce that $\lambda_0 = 0$. This shows that up to a subsequence, $u_n \to u$ in $H^1(\Omega)$. Since $|u|$ is also a minimizer, we may assume by the strong maximum principle that $u > 0$ on Ω. □

3. Existence of a second solution

We obtain a second solution to problem (1.1) using the mountain - pass principle [7]. If Q is positive somewhere on $\partial\Omega$, then we can define

$$0 < Q_M = \max_{x\in\bar\Omega} Q(x) \quad\text{and}\quad 0 < Q_m = \max_{x\in\partial\Omega} Q(x).$$

From now on the positive solution obtained in Theorem 2.3 will be denoted by u_λ. The Palais - Smale condition will be expressed in terms of Q_m and Q_M.

Proposition 3.1. *Suppose that Q is positive somewhere on $\partial\Omega$ and let $0 < \lambda < \lambda^*$. If $u = 0$ and $u = u_\lambda$ are the unique critical points of J_λ, then the $(PS)_c$ condition holds for*

$$c < \bar c := J_\lambda(u_\lambda) + \min\Big(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}\Big).$$

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ - sequence. By Lemma 2.1 $\{u_n\}$ is bounded in $H^1(\Omega)$. We may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $L^2(\Omega, \frac{1}{|x|^2}\,dx)$

and moreover $u_n \to u$ in $L^p(\Omega)$ for $2 \le p < 2^*$. By the concentration - compactness principle there exist points $x_j \in \bar{\Omega}$, $j \in J$, and positive numbers μ_j, ν_j, $j \in J$, λ_0, μ_0 and ν_0 such that relations (2.3) – (2.7) are satisfied. Moreover, the index set J is at most countable. At points with possible concentration inequalities (2.8), (2.9) and (2.10) hold and there is no concentration at points belonging to the set $\{x \in \bar{\Omega};\, Q(x) \le 0\}$. We put $v_n = u_n - u$. By the Brezis - Lieb lemma [10] we have

$$\int_\Omega Q(x)|u_n|^{2^*}\,dx = \int_\Omega Q(x)|u|^{2^*}\,dx + \int_\Omega |v_n|^{2^*}\,dx + o(1). \tag{3.1}$$

By the weak convergence of $\{u_n\}$ in $H^1(\Omega)$ and $L^2(\Omega, \frac{1}{|x|^2}\,dx)$ we also have

$$\int_\Omega \frac{u_n^2}{|x|^2}\,dx = \int_\Omega \frac{u^2}{|x|^2}\,dx + \int_\Omega \frac{v_n^2}{|x|^2}\,dx + o(1) \tag{3.2}$$

and

$$\int_\Omega |\nabla u_n|^2\,dx = \int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla v_n|^2\,dx + o(1). \tag{3.3}$$

Relations $J_\lambda(u_n) = c + o(1)$ and $J'_\lambda(u_n) = o(1)$ combined with (3.1), (3.2) and (3.3) yield

$$J_\lambda(u) + \frac{1}{2}\int_\Omega \big(|\nabla v_n|^2 - \lambda\frac{v_n^2}{|x|^2}\big)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|v_n|^{2^*}\,dx = c + o(1) \tag{3.4}$$

and

$$\int_\Omega \big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2} - Q(x)|u|^{2^*}\big)\,dx + \int_\Omega \big(|\nabla v_n|^2 - \lambda\frac{v_n^2}{|x|^2} - Q(x)|v_n|^{2^*}\big)\,dx = o(1).$$

Since $\langle J'_\lambda(u), u\rangle = 0$, we deduce from the last equation that

$$\int_\Omega \big(|\nabla v_n|^2 - \lambda\frac{v_n^2}{|x|^2}\big)\,dx = \int_\Omega Q(x)|v_n|^{2^*}\,dx + o(1).$$

It then follows from (3.4) that

$$J_\lambda(u) + \frac{1}{N}\int_\Omega Q(x)|v_n|^{2^*}\,dx = c + o(1).$$

Since 0 and u_λ are the only critical points of J_λ we must have either $u = 0$ or $u = u_\lambda$. At points of $\bar{\Omega}$, where a concentration occur, we have either (2.8) or (2.9) or (2.10). We then get either

$$\min\Big(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}\Big) \le c$$

or

$$J_\lambda(u_\lambda) + \min\Big(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}\Big) \le c.$$

Both cases imply that $\lambda_0 = \mu_0 = \nu_0 = 0$ and $\mu_j = \nu_j = 0$ for $j \in J$. □

We set

$$S_\infty = \min\Big(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}\Big).$$

We now establish the existence of a second solution in three cases.

Case I: $S_\infty = \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}$. Let $Q_m = Q(y)$ for some $y \in \partial\Omega$, $y \neq 0$. We define a function $\varphi \in C^1(\mathbb{R}^N)$ such that $\varphi = 1$ on $B(y, \frac{r}{2})$, $\varphi = 0$ on $\mathbb{R}^N - B(y, r)$ and $0 \le \varphi \le 1$ on $\mathbb{R}^N$. The radius r is chosen so that $Q(x) > 0$ on $B(y,r) \cap \bar{\Omega}$ and $0 \notin B(y,r)$. We put $U_{\epsilon,y} = \epsilon^{-\frac{(N-2)}{2}} U\big(\frac{x-y}{\epsilon}\big)$ and $v_\epsilon = \varphi U_{\epsilon,y}$. If $H(y)$ denotes the mean curvature of the boundary $\partial\Omega$ at y, then the following estimates hold (see [4], [5], [40])

$$\frac{\int_\Omega |\nabla U_{\epsilon,y}|^2\,dx}{\Big(\int_\Omega U_{\epsilon,y}^{2^*}\,dx\Big)^{\frac{2}{2^*}}} \le \begin{cases} \frac{S}{2^{\frac{2}{N}}} - A_N H(y)\epsilon \log\frac{1}{\epsilon} + O(\epsilon) & \text{if } N = 3, \\ \frac{S}{2^{\frac{2}{N}}} - A_N H(y)\epsilon + O\big(\epsilon^2 \log\frac{1}{\epsilon}\big) & \text{if } N = 4, \\ \frac{S}{2^{\frac{2}{N}}} - A_N H(y)\epsilon + O\big(\epsilon^2\big) & \text{if } N \ge 5, \end{cases} \tag{3.5}$$

where $A_N > 0$ is a constant depending on N. Since

$$\int_{(|x-y|\ge r)\cap\Omega} U_{\epsilon,y}^{2^*}\,dx = O\big(\epsilon^N\big) \quad \text{and} \quad \int_{(|x-y|\ge r)\cap\Omega} |\nabla U_{\epsilon,y}|^2\,dx = O\big(\epsilon^{N-2}\big),$$

it is clear that (3.5) holds with $U_{\epsilon,y}$ replaced by v_ϵ.

Proposition 3.2. *Let $0 < \lambda < \lambda^*$ and $N \ge 3$. Suppose that Q is positive somewhere on $\partial\Omega$, $Q(y) = Q_m$, with $y \neq 0$, $H(y) > 0$ and that*

$$|Q(x) - Q(y)| = o(|x-y|) \quad \textit{for } x \textit{ near } y. \tag{3.6}$$

If $S_\infty = \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}$, then

$$\max_{t\ge 0} J_\lambda(u_\lambda + tv_\epsilon) < J_\lambda(u_\lambda) + \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}$$

for $\epsilon > 0$ sufficiently small.

Proof. We use some ideas from the proof of Lemma 2.5 from paper [36]. To estimate $J_\lambda(u_\lambda + tv_\epsilon)$ we use the following inequality: given $q > 2$ and $\kappa \in (1, q-1)$ we can find a constant $C > 0$ such that

$$(s+t)^q \geq s^q + t^q + qs^{q-1}t + qst^{q-1} - Ct^\kappa s^{q-\kappa} \tag{3.7}$$

for every $s, t \geq 0$. Applying this to the term $\int_\Omega Q^+|u_\lambda + tv_\epsilon|^{2^*}\,dx$ with $q = 2^*$ and $\kappa = \frac{N+1}{N-2}$, we obtain

$$\begin{aligned}
J_\lambda(u_\lambda + tv_\epsilon) \leq{}& \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2\,dx + \frac{t^2}{2}\int_\Omega |\nabla v_\epsilon|^2 + t\int_\Omega \nabla v_\epsilon \nabla u_\lambda\,dx \\
&- \frac{\lambda}{2}\int_\Omega \frac{u_\lambda^2}{|x|^2}\,dx - \frac{\lambda t^2}{2}\int_\Omega \frac{v_\epsilon^2}{|x|^2}\,dx - t\lambda\int_\Omega \frac{u_\lambda v_\epsilon}{|x|^2}\,dx \\
&- \frac{t^{2^*}}{2^*}\int_\Omega Q^+ v_\epsilon^{2^*}\,dx - \frac{1}{2^*}\int_\Omega Q^+ u_\lambda^{2^*}\,dx - t\int_\Omega Q^+ u_\lambda^{2^*-1} v_\epsilon\,dx \\
&+ Ct^{\frac{N+1}{N-2}}\int_\Omega Q^+ u_\lambda^{\frac{N-1}{N-2}} v_\epsilon^{\frac{N+1}{N-2}}\,dx \\
&- t^{2^*-1}\int_\Omega Q^+ u_\lambda v_\epsilon^{2^*-1}\,dx + \frac{1}{2^*}\int_\Omega Q^- u_\lambda^{2^*}\,dx.
\end{aligned}$$

Since u_λ is a weak solution of (1.1) this inequality can be simplified to

$$\begin{aligned}
J_\lambda(u_\lambda + tv_\epsilon) \leq J_\lambda(u_\lambda) &+ \frac{t^2}{2}\int_\Omega |\nabla v_\epsilon|^2\,dx - \frac{t^{2^*}}{2^*}\int_\Omega Q v_\epsilon^{2^*}\,dx \\
&+ Ct^{\frac{N+1}{N-2}}\int_\Omega Q^+ u_\lambda^{\frac{N-1}{N-2}} v_\epsilon^{\frac{N+1}{N-2}}\,dx.
\end{aligned}$$

By straightforward calculations we get

$$\int_\Omega Q^+ u_\lambda^{\frac{N-1}{N-2}} v_\epsilon^{\frac{N+1}{N-2}}\,dx \leq C\int_{|x-y|\leq r} \frac{\epsilon^{\frac{N+1}{2}}}{\left(\epsilon^2 + |x-y|^2\right)^{\frac{N+1}{2}}}\,dx \leq C\epsilon^{\frac{N-1}{2}}.$$

So we get the following estimate

$$\begin{aligned}
J_\lambda(u_\lambda + tv_\epsilon) &\leq J_\lambda(u_\lambda) + \frac{t^2}{2}\int_\Omega |\nabla v_\epsilon|^2\,dx - \frac{t^{2^*}}{2^*}\int_\Omega Q v_\epsilon^{2^*}\,dx + Ct^{\frac{N+1}{N-2}}\epsilon^{\frac{N-1}{2}} \\
&:= J_\lambda(u_\lambda) + \Psi_\epsilon(t).
\end{aligned} \tag{3.8}$$

It is clear that $\lim_{t\to 0}\Psi_\epsilon(t) = 0$ and $\lim_{t\to\infty}\Psi_\epsilon(t) = -\infty$. Therefore there exists $t_\epsilon > 0$ such that

$$\Psi_\epsilon(t_\epsilon) = \max_{t\geq 0}\Psi_\epsilon(t).$$

We also have by Lemma 2.2 that

$$0 < \beta \leq \max_{t \geq 0} J_\lambda(u_\lambda + tv_\epsilon) \leq J_\lambda(u_\lambda) + \Psi_\epsilon(t_\epsilon) < \Psi_\epsilon(t_\epsilon). \tag{3.9}$$

We now show that there exists $0 < T_1 < T_2$ such that $T_1 \leq t_\epsilon \leq T_2$ for every $\epsilon > 0$ small enough. Inequality (3.9) shows that $t_\epsilon \not\to 0$ as $\epsilon \to 0$. So we have $T_1 \leq t_\epsilon$ for some $T_1 > 0$. Assume that $t_{\epsilon_n} \to \infty$ for some sequence $\epsilon_n \to 0$. Then $\Psi'_{\epsilon_n}(t_{\epsilon_n}) = 0$, that is,

$$t_{\epsilon_n} \int_\Omega |\nabla v_\epsilon|^2 \, dx - t_\epsilon^{2^*-1} \int_\Omega Q v_{\epsilon_n}^{2^*} \, dx + C \frac{N+1}{N-2} t_{\epsilon_n}^{\frac{3}{N-2}} \epsilon_n^{\frac{N-1}{2}} = 0.$$

From this we derive

$$\int_\Omega |\nabla v_{\epsilon_n}|^2 \, dx = t_{\epsilon_n}^{\frac{4}{N-2}} \int_\Omega Q v_{\epsilon_n}^{2^*} \, dx + C \frac{N+1}{N-2} t_{\epsilon_n}^{\frac{5-N}{N-2}} \epsilon_n^{\frac{N-1}{2}}.$$

Since

$$\lim_{n \to \infty} \int_\Omega Q v_{\epsilon_n}^{2^*} \, dx > 0 \quad \text{and} \quad \lim_{n \to \infty} \int_\Omega |\nabla v_{\epsilon_n}|^2 \, dx > 0,$$

the above identity cannot be satisfied for large n. Therefore, there exists a constant $T_2 > 0$ such that $t_\epsilon \leq T_2$ for small $\epsilon > 0$. Thus, we have

$$\begin{aligned} \max_{t \geq 0} J_\lambda(u_\lambda + tv_\epsilon) &\leq J_\lambda(u_\lambda) + \Psi_\epsilon(t_\epsilon) \leq J_\lambda(u_\lambda) \\ &+ \frac{1}{N} \frac{\left(\int_\Omega |\nabla v_\epsilon|^2 \, dx \right)^{\frac{N}{2}}}{\left(\int_\Omega Q(x) v_\epsilon^{2^*} \, dx \right)^{\frac{N-2}{2}}} + C T_2^{\frac{3}{N-2}} \epsilon^{\frac{N-1}{2}}. \end{aligned}$$

By assumption (3.6) we have expansion

$$\int_\Omega Q v_\epsilon^{2^*} \, dx = Q_m \int_\Omega v_\epsilon^{2^*} \, dx + o(\epsilon).$$

The last two relations combined with (3.5) give

$$\max_{t \geq 0} J_\lambda(u_\lambda + tv_\epsilon) < J_\lambda(u_\lambda) + \frac{S^{\frac{N}{2}}}{2N Q_m^{\frac{N-2}{2}}}$$

for $\epsilon > 0$ small enough. □

To apply the mountain-pass principle, we choose $w \in H^1(\Omega)$ such that $\|w\| > \rho$ and $J_\lambda(w) < 0$ and let $\Gamma = \{\gamma \in C([0,1], H^1(\Omega)) : \gamma(0) = u_\lambda, \gamma(1) = w\}$ and put

$$\beta \leq c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)).$$

Lemma 2.2 and Propositions 3.1, 3.2 allow us to formulate the following existence result:

Theorem 3.3. *Let* $0 < \lambda < \lambda^*$, $N \geq 3$. *Suppose that* Q *is positive somewhere on* $\partial\Omega$ *and that* $Q_m = Q(y)$ *with* $y \in \partial\Omega$, $y \neq 0$, $H(y) > 0$. *Moreover, we assume that* (3.6) *holds. If* $S_\infty = \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}$, *then for every* $0 < \lambda < \lambda^*$ *problem* (1.1) *has at least two distinct solutions.*

Proof. Arguing by contradiction, assume that 0 and u_λ are the unique critical points of J_λ. Applying the mountain-pass principle, we obtain a critical point v different than 0 and u_λ, which gives a contradiction. The fact that we can take v to be positive on Ω, follows from Theorem 10 in [9]. □

Case II: $S_\infty = \frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}$**.** In this case we obviously assume that $Q(0) > 0$. Let $\varphi \in C^1(\mathbb{R}^N)$ with $\varphi(x) = 1$ on $B(0, \frac{r}{2})$ and $\varphi(x) = 0$ on $\mathbb{R}^N - B(0, r)$. The radius is chosen so that $Q(x) > 0$ for $x \in B(0, r) \cap \bar{\Omega}$. We use the family of functions $V_{\lambda,\epsilon}$ and put $w_\epsilon = \varphi(x)V_{\lambda,\epsilon}(x)$. We put

$$E_\lambda(u) = \frac{\int_\Omega \left(|\nabla u|^2 - \lambda \frac{u^2}{|x|^2}\right) dx}{\left(\int_\Omega |u|^{2^*}\, dx\right)^{\frac{2}{2^*}}}.$$

It is known that

$$E_\lambda(V_{\lambda,\epsilon}) < \frac{S_{-\lambda}}{2^{\frac{2}{N}}} - C_1 \epsilon^{\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu - \lambda}}} + o\left(\epsilon^{\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu-\lambda}}}\right) \tag{3.10}$$

for $N \geq 5$, $\lambda < \bar\mu - 1$ and $\epsilon > 0$ sufficiently small. For computational details we refer the reader to paper [30]. We need the following estimate.

Lemma 3.4. *Let* $N \geq 4$ *and* $1 < q < \frac{N}{3}$. *Then*

$$\left(\int_\Omega w_\epsilon^{\frac{4q}{N-2}}\, dx\right)^{\frac{1}{q}} \leq C\left(\epsilon^{\left(\frac{N}{q}-2\right)\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu-\lambda}}} + \epsilon^2\right) \tag{3.11}$$

for $0 < \lambda < \bar\mu$, *where* $C > 0$ *is a constant independent of* ϵ.

Proof. We put $r(\epsilon) = \frac{r}{\epsilon^{\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu-\lambda}}}}$. Changing variables $x = y\epsilon^{\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu-\lambda}}}$, we get

$$\int_\Omega w_\epsilon^{\frac{4q}{N-2}}\, dx \leq c_N \int_{|y|<r(\epsilon)} \frac{\epsilon^{2q + N\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu-\lambda}}}}{\left(\epsilon^{2+\frac{\lambda'}{\sqrt{\bar\mu-\lambda}}}|y|^{\frac{\lambda'}{\sqrt{\bar\mu}}} + \epsilon^{\frac{\bar\lambda}{\sqrt{\bar\mu-\lambda}}}|y|^{\frac{\bar\lambda}{\sqrt{\bar\mu}}}\right)^{2q}}$$

$$= c_N \epsilon^{(N-2q)\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda}} \int_{|y|<r(\epsilon)} \frac{dy}{\Big(|y|^{\frac{\lambda'}{\sqrt{\bar\mu}}} + |y|^{\frac{\bar\lambda}{\sqrt{\bar\mu}}}\Big)^{2q}}$$

$$= c_N \epsilon^{(N-2q)\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda}} \Big[\int_{|y|<r_\circ} \frac{dy}{\big(|y|^{\frac{\lambda'}{\sqrt{\bar\mu}}} + |y|^{\frac{\bar\lambda}{\sqrt{\bar\mu}}}\big)^{2q}} + \int_{r_\circ<|y|<r(\epsilon)} \frac{dy}{\big(|y|^{\frac{\lambda'}{\sqrt{\bar\mu}}} + |y|^{\frac{\bar\lambda}{\sqrt{\bar\mu}}}\big)^{2q}} \Big]$$

$$= c_N \epsilon^{(N-2q)\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda}} \big(I_1 + I_2\big).$$

Here $r_\circ > 0$ is a fixed small number. Since $q < \frac{N}{3} < \frac{N\sqrt{\bar\mu}}{2\lambda'}$, we have

$$I_1 \le \omega_N \int_0^{r_\circ} \frac{s^{N-1}}{s^{\frac{2q\lambda'}{\sqrt{\bar\mu}}}}\, ds = \omega_N \frac{r_\circ^{N-\frac{2q\lambda'}{\sqrt{\bar\mu}}}}{N - \frac{2q\lambda'}{\sqrt{\bar\mu}}} := a.$$

We now estimate the integral I_2:

$$I_2 \le \omega_N \int_{r_\circ}^{r(\epsilon)} \frac{s^{N-1}}{s^{\frac{2q\bar\lambda}{\sqrt{\bar\mu}}}}\, ds \le \omega_N \frac{r^{N-\frac{2q\bar\lambda}{\sqrt{\bar\mu}}} \epsilon^{-(N-\frac{2q\bar\lambda}{\sqrt{\bar\mu}})\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda}}}{N - \frac{2q\bar\lambda}{\sqrt{\bar\mu}}} := b\epsilon^{-(N-\frac{2q\bar\lambda}{\sqrt{\bar\mu}})\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda}}.$$

Hence,

$$\int_\Omega w_\epsilon^{\frac{4q}{N-2}}\, dx \le c_N \epsilon^{(N-2q)\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda}} \Big(a + b\epsilon^{-(N-\frac{2q\bar\lambda}{\sqrt{\bar\mu}})\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda}}\Big) = ac_N \epsilon^{(N-2q)\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda}} + b\epsilon^{2q}$$

and estimate (3.11) readily follows. □

We aim to show that the values of the integral $J_\lambda(u_\lambda + tw_\epsilon)$, $t \ge 0$, are below the level $\frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}$. In a similar situation in Proposition 3.2, the instanton $U_{\epsilon,y}$ was centered at point $y \ne 0$ and we used the fact that u_λ was bounded in a small ball $B(y,r)$. In the present situation we only know that $C^1(\bar\Omega) - \{0\}$ and u_λ is possibly unbounded around 0. To overcome this difficulty, we prove the higher integrability property of u_λ.

Lemma 3.5. *If*

$$(*) \qquad \frac{\lambda(p+2)}{4}\Big(\frac{1}{\bar\mu} + 1\Big) < 1,$$

then $u_\lambda \in L^p(\Omega)$.

Proof. Let $L > 0$ and $p > 2$ and put $u = u_\lambda$ and $u_L = \min(u, L)$. We test equation (1.1) with $\varphi = uu_L^{p-2}$. We get

$$\int_\Omega |\nabla u|^2 u_L^{p-2}\, dx + (p-2)\int_\Omega \nabla u \nabla u_L u_L^{p-3} u\, dx$$

$$= \int_\Omega Q(x)|u|^{2^*-2}u^2u_L^{p-2}\,dx + \lambda\int_\Omega \frac{u^2u_L^{p-2}}{|x|^2}\,dx$$
$$\le Q_M\int_\Omega |u|^{2^*-2}u^2u_L^{p-2}\,dx + \lambda\int_\Omega \frac{u^2u_L^{p-2}}{|x|^2}\,dx.$$

We rewrite this inequality as

$$\int_\Omega |\nabla u|^2u_L^{p-2}\,dx + (p-2)\int_\Omega |\nabla u_L|^2u_L^{p-2}\,dx \tag{3.12}$$
$$\le Q_M\int_\Omega |u|^{2^*-2}u^2u_L^{p-2}\,dx + \lambda\int_\Omega \frac{u^2u_L^{p-2}}{|x|^2}\,dx.$$

We now observe that

$$\int_\Omega |\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx = \int_\Omega |\nabla u|^2u_L^{p-2}\,dx + \frac{(p-2)^2}{4}\int_\Omega |\nabla u|^2u_L^{p-4}u^2\,dx$$
$$+ (p-2)\int_\Omega \nabla u\nabla u_Lu_L^{p-3}u\,dx \tag{3.13}$$
$$= \int_\Omega |\nabla u|^2u_L^{p-2}\,dx + \frac{p^2-4}{4}\int_\Omega |\nabla u_L|^2u_L^{p-2}\,dx.$$

From (3.12) we deduce that

$$\int_\Omega |\nabla u|^2u_L^{p-2}\,dx + \frac{p^2-4}{4}\int_\Omega |\nabla u_L|^2u_L^{p-2}\,dx$$
$$\le \frac{p+2}{4}Q_M\int_\Omega u^{2^*-2}u^2u_L^{p-2}\,dx + \frac{\lambda(p+2)}{4}\int_\Omega \frac{u^2u_L^{p-2}}{|x|^2}\,dx.$$

This combined with (3.13), gives

$$\int_\Omega \nabla(uu_L^{\frac{p}{2}-1})|^2\,dx \le \frac{(p+2)}{4}Q_M\int_\Omega u^{2^*-2}u^2u_L^{p-2}\,dx + \frac{\lambda(p+2)}{4}\int_\Omega \frac{u^2u_L^{p-2}}{|x|^2}\,dx.$$

Applying Lemma 2.1 with $\epsilon = 1$, we obtain

$$\int_\Omega |\nabla(uu_L^{\frac{p}{2}-1})|^2\,dx \le \frac{p+2}{4}Q_M\int_\Omega u^{2^*-2}u^2u_L^{p-2}\,dx$$
$$+ \frac{\lambda(p+2)}{4}\big(\frac{1}{\bar\mu}+1\big)\int_\Omega |\nabla(uu_L^{\frac{p}{2}-1})|^2\,dx + \frac{\lambda(p+2)}{4}C(1)\int_\Omega u^2u_L^{p-2}\,dx.$$

Let $d = \big(1 - \frac{\lambda(p+2)}{4}\big(\frac{1}{\bar\mu}+1\big)\big)$. Then we have

$$d\Big(\int_\Omega |\nabla\big(uu_L^{\frac{p}{2}-1}\big)|^2\,dx + \int_\Omega u^2u_L^{p-2}\,dx\Big) \tag{3.14}$$

$$\leq \frac{p+2}{4} Q_M \int_\Omega u^{2^*-2} u^2 u_L^{p-2}\, dx + \big(\frac{\lambda(p+2)}{4} C(1) + 1\big) \int_\Omega u^2 u_L^{p-2}\, dx.$$

We choose a constant $K > 0$ such that

$$\frac{p+2}{4} Q_M \Big(\int_{u \geq K} u^{2^*}\, dx \Big)^{\frac{2}{N}} \leq \frac{dS}{2}.$$

Using the Sobolev and Hölder inequalities, we derive from (3.14) that

$$\frac{dS}{2} \Big(\int_\Omega u^{\frac{2^* p}{2}}\, dx \Big)^{\frac{2}{2^*}} \leq \Big(\frac{p+2}{4} Q_M K^{2^*-2} + \frac{\lambda(p+2)}{4} C(1) + 1 \Big) \int_\Omega u^2 u_L^{p-2}\, dx.$$

Letting $L \to \infty$ and then iterating the resulting inequality, starting from $p = 2^*$, the assertion follows as long as λ and p satisfy (*). □

A similar result for the Dirichlet problem can be found in paper [37].

Proposition 3.6. *Let* $0 < \lambda < \lambda^*$ *and* $N > 5$. *Suppose that*

$$|Q(x) - Q(0)| = o\Big(|x|^{\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu - \lambda}}}\Big) \tag{3.15}$$

for x *close to* 0. *If* $S_\infty = \frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}$, *then*

$$\max_{t \geq 0} J_\lambda(u_\lambda + t w_\epsilon) < J_\lambda(u_\lambda) + \frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}.$$

Proof. We apply inequality (3.7) with $q = 2^*$ and $\kappa = 2$. Applying this to the integral $\int_\Omega Q^+ |u_\lambda + t w_\epsilon|^{2^*}\, dx$, we obtain

$$\begin{aligned}
J_\lambda(u_\lambda + t w_\epsilon) &= \frac{1}{2} \int_\Omega |\nabla u_\lambda|^2\, dx + \frac{t^2}{2} \int_\Omega |\nabla w_\epsilon|^2\, dx + t \int_\Omega \nabla u_\lambda \nabla w_\epsilon\, dx \\
&- \frac{\lambda}{2} \int_\Omega \frac{u_\lambda^2}{|x|^2}\, dx - \frac{\lambda t^2}{2} \int_\Omega \frac{w_\epsilon^2}{|x|^2}\, dx - \lambda t \int_\Omega \frac{u_\lambda w_\epsilon}{|x|^2}\, dx \qquad (3.16) \\
&- \frac{1}{2^*} \int_\Omega Q^+ u_\lambda^{2^*}\, dx - \frac{t^{2^*}}{2^*} \int_\Omega w_\epsilon^{2^*}\, dx - t \int_\Omega Q^+ u_\lambda^{2^*-1} w_\epsilon\, dx \\
&- t^{2^*-1} \int_\Omega Q^+ u_\lambda w_\epsilon^{2^*-1}\, dx + C t^{\frac{4}{N-2}} \int_\Omega Q^+ u_\lambda^2 w_\epsilon^{\frac{4}{N-2}}\, dx + \frac{1}{2^*} \int_\Omega Q^- u_\lambda^{2^*}\, dx \\
&\leq J_\lambda(u_\lambda) + \frac{t^2}{2} \int_\Omega |\nabla w_\epsilon|^2\, dx - \frac{t^2}{2} \int_\Omega \frac{w_\epsilon^2}{|x|^2}\, dx - \frac{t^{2^*}}{2^*} \int_\Omega Q w_\epsilon^{2^*}\, dx \\
&+ C t^{\frac{4}{N-2}} \int_\Omega Q^+ u_\lambda^2 w_\epsilon^{\frac{4}{N-2}}\, dx = J_\lambda(u_\lambda) + \tilde\Psi_\epsilon(t).
\end{aligned}$$

Since $\lim_{t\to 0}\tilde{\Psi}_\epsilon(t) = 0$ and $\lim_{t\to\infty}\tilde{\Psi}_\epsilon(t) = -\infty$, there exists $t_\epsilon > 0$ such that $\tilde{\Psi}_\epsilon(t_\epsilon) = \max_{t\geq 0}\tilde{\Psi}_\epsilon(t)$. As in the proof of Proposition 3.2 we show that there exist constants $0 < T_1 < T_2$ such that $T_1 < t_\epsilon < T_2$ for ϵ small. We now fix $1 < q < \frac{N}{3}$. By taking λ^*, smaller if necessary, it then follows from Lemma 3.5 that $u_\lambda \in L^{\frac{2q}{q-1}}(\Omega)$. Using Lemmas 3.5 and 5 we get

$$\begin{aligned}\int_\Omega Q u_\lambda^2 w_\epsilon^{\frac{4}{N-2}}\,dx &\leq Q_M\Big(\int_\Omega u_\lambda^{\frac{2q}{q-1}}\,dx\Big)^{\frac{q-1}{q}}\Big(\int_\Omega w_\epsilon^{\frac{4q}{N-2}}\,dx\Big)^{\frac{1}{q}} \\ &\leq C_1\Big(\epsilon^{\left(\frac{N}{q}-2\right)\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu-\lambda}}} + \epsilon^2\Big).\end{aligned} \tag{3.17}$$

We may assume that $\lambda^* \leq \frac{3}{4}\bar\mu$ by taking λ^* smaller if necessary. This allows us to use estimate (3.10). Indeed, we have $\frac{N}{q} - 2 > 1$ and $2 > \frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu-\lambda}}$. The desired estimate follows from (3.15), (3.17) and (3.10). □

Theorem 3.7. *Suppose that assumptions of Proposition* 3.6 *hold. Then problem* (1.1) *admits two distinct solutions.*

As in Theorem 3.3 the second solution is obtained using the mountain - pass principle.

Case III: $S_\infty = \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$**.** Let $x_M \in \Omega$ be such that $Q_M = Q(x_M)$. We choose $r > 0$ so that $Q(x) > 0$ on $\bar{B}(x_M, r) \subset \Omega$. Let a function $\varphi \in C^1(\mathbb{R}^N)$ be such that $\varphi(x) = 1$ on $B(x_M, \frac{r}{2})$, $\varphi(x) = 0$ on $\mathbb{R}^N - B(x_M, r)$ and $0 \leq \varphi(x) \leq 1$ on $\mathbb{R}^N$. We put $z_\epsilon = \varphi U_{\epsilon, x_M}$.

Theorem 3.8. *Let* $0 < \lambda < \lambda^*$ *and* $N > 5$. *Suppose that*

$$|Q(x) - Q(x_M)| = o\big(|x - x_M|^2\big) \tag{3.18}$$

for x *close to* x_M. *If* $S_\infty = \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$, *then problem* (1.1) *admits two distinct solutions.*

Proof. To apply the mountain - pass principle we must show that

$$\max_{t\geq 0} J_\lambda(u_\lambda + t z_\epsilon) < J_\lambda(u_\lambda) + \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}. \tag{3.19}$$

Applying inequality (3.7) with $q = 2^*$ and $\kappa = \frac{N+1}{N-2}$, we obtain the following inequality

$$J_\lambda(u_\lambda + tz_\epsilon) \le J_\lambda(u_\lambda) + \frac{t^2}{2}\int_\Omega |\nabla z_\epsilon|^2\,dx - \frac{\lambda t^2}{2}\int_\Omega \frac{z_\epsilon^2}{|x|^2}\,dx - \frac{t^{2^*}}{2^*}\int_\Omega Qz_\epsilon^{2^*}\,dx$$
$$+ Ct^{\frac{N+1}{N-2}}\int_\Omega Qu_\lambda^{\frac{N-1}{N-2}} z_\epsilon^{\frac{N+1}{N-2}}\,dx := J_\lambda(u_\lambda) + \bar{\Psi}_\epsilon(t). \tag{3.20}$$

Since $\lim_{t\to 0}\bar{\Psi}_\epsilon(t) = 0$ and $\lim_{t\to\infty}\bar{\Psi}_\epsilon(t) = -\infty$, there exists $t_\epsilon > 0$ such that $\bar{\Psi}_\epsilon(t_\epsilon) = \max_{t\ge 0}\bar{\Psi}_\epsilon(t)$. As in the proof of Proposition 3.2 we show that there exist constants $0 < T_1 < T_2$ such that $T_1 < t_\epsilon < T_2$ for small ϵ. For the integral $\int_\Omega Qu_\lambda^{\frac{N-1}{N-2}} z_\epsilon^{\frac{N+1}{N-2}}\,dx$, we have the estimate

$$\int_\Omega Qu_\lambda^{\frac{N-1}{N-2}} z_\epsilon^{\frac{N+1}{N-2}}\,dx \le C\epsilon^{\frac{N-1}{2}}, \tag{3.21}$$

where $C > 0$ is a constant. Since

$$\int_\Omega \frac{z_\epsilon^2}{|x|^2}\,dx \ge c_1\epsilon^2 \tag{3.22}$$

for $N \ge 5$, where $c_1 > 0$ is a constant independent of ϵ, we derive from (3.20), (3.21) and (3.22) that

$$J_\lambda(u_\lambda + tz_\epsilon) \le J_\lambda(u_\lambda) + \frac{1}{N}\frac{\left(\int_\Omega |\nabla z_\epsilon|^2\,dx\right)^{\frac{N}{2}}}{\left(\int_\Omega Qz_\epsilon^{2^*}\,dx\right)^{\frac{N-2}{2}}} + CT_2^{\frac{N+1}{N-2}}\epsilon^{\frac{N-1}{2}} - c_1\epsilon^2.$$

Since

$$\int_\Omega |\nabla z_\epsilon|^2\,dx = S^{\frac{N}{2}} + O\big(\epsilon^{N-2}\big), \quad \int_\Omega z_\epsilon^{2^*}\,dx = S^{\frac{N}{2}} + O\big(\epsilon^N\big)$$

and

$$\int_\Omega Qz_\epsilon^{2^*}\,dx = Q_M\int_\Omega z_\epsilon^{2^*}\,dx + o(\epsilon^2),$$

estimate (3.19) follows. □

Remark 3.9. If $Q_m = Q(0)$ and 0 is the only point on the boundary $\partial\Omega$ where Q_m is attained, then

$$S_\infty \quad = \quad S_\infty = \min\left(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}\right)$$

$$= \quad \min\Big(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S_{-\lambda}^{\frac{N}{2}}}{2NQ(0)^{\frac{N-2}{2}}}\Big).$$

Therefore, for the existence of a second solution we only consider cases II and III.

4. Existence of solutions in the case $0 \in \Omega$

We now consider problem (1.1) when $0 \in \Omega$. In this case, inequality (2.1) (see Lemma 2.1 in [18]) as well as Lemmas 2.1, 2.2, continues to hold. Repeating the argument of Theorem 2.3 we can show the existence of a local minimizer of J_λ, which is again denoted by u_λ. Inspection of the proof of Proposition 3.1, leads to the following form of the Palais - Smale condition:

Proposition 4.1. *Suppose that Q is positive somewhere on $\partial\Omega$. If $u = 0$ and $u = u_\lambda$ are the only critical points of J_λ, the the $(PS)_c$ condition holds for*

$$c < c^* = J_\lambda(u_\lambda) + \min\Big(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S_{-\lambda}^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}\Big).$$

We put

$$S_\infty^* := \min\Big(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S_{-\lambda}^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}\Big).$$

We restrict ourselves to the case

$$S_\infty^* = \frac{S_{-\lambda}^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}.$$

The remaining cases are the same as cases I and II (with $Q_M = Q(x_M)$, $x_M \neq 0$) of Section 3.

Let ϕ be a cut - off function with supp $\phi \subset \overline{B(0,r)} \subset \Omega$ from Section 3. We put $w_\epsilon = \phi V_{\lambda,\epsilon}$. We assume that $Q(x) > 0$ for $x \in \overline{B(0,r)}$.

Lemma 4.2. *Let $0 \in \Omega$. If $u \in H^1(\Omega)$ is a positive solution of (1.1), then there exists a ball $B(0,\rho_\circ) \subset \Omega$ such that*

$$a|x|^{-\lambda'} \leq u(x) \leq b|x|^{-\lambda'}$$

for $x \in B(0,\rho_\circ) - \{0\}$, where $a > 0$ and $b > 0$ are constants.

The proof of this estimate is analogous to that of Theorem 1.1 in [29] (see also [18]). We give an outline of the proof in Section 5.

Lemma 4.3. *There exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\int_\Omega u_\lambda w_\epsilon^{2^*-1}\,dx \geq c_1\epsilon^{\frac{N-2}{2}} \tag{4.1}$$

and

$$\int_\Omega u_\lambda^{\frac{N-1}{N-2}} w_\epsilon^{\frac{N+1}{N-2}}\,dx \leq c_2\epsilon^{\frac{N-1}{2}} \tag{4.2}$$

for small $\epsilon > 0$.

Proof. It follows from Lemma 4.2 that

$$\int_\Omega u_\lambda w_\epsilon^{2^*-1}\,dx \geq ac_N\int_{|x|\leq r}\frac{\epsilon^{\frac{N+2}{2}}}{|x|^{-\lambda'}\Big(\epsilon^2|x|^{\frac{\lambda'}{\sqrt{\bar\mu}}}+|x|^{\frac{\bar\lambda}{\sqrt{\bar\mu}}}\Big)^{\frac{N+2}{2}}}\,dx.$$

By the change of variables $x = y\epsilon^{\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda}}$ we obtain

$$\int_\Omega u_\lambda w_\epsilon^{2^*-1}\,dx \geq ac_N\int_{|x|\leq r(\epsilon)}\frac{\epsilon^{k(N)}}{|y|^{\lambda'}\Big(|y|^{\frac{\lambda'}{\sqrt{\bar\mu}}}+|y|^{\frac{\bar\lambda}{\sqrt{\bar\mu}}}\Big)^{\frac{N+2}{2}}}\,dy,$$

where

$$k(N) = \frac{N+2}{2} - \lambda'\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda} + N\frac{\sqrt{\bar\mu}}{\sqrt{\bar\mu}-\lambda} - \frac{\bar\lambda}{\sqrt{\bar\mu}-\lambda}\frac{N+2}{2} = \frac{N-2}{2}$$

and estimate (4.1) follows. Similarly, we have

$$\begin{aligned}
\int_\Omega u_\lambda^{\frac{N-1}{N-2}} w_\epsilon^{\frac{N+1}{N-2}}\,dx &\leq bc_N\int_{|x|\leq r(\epsilon)}\frac{\epsilon^{\frac{N-1}{2}}}{|x|^{\lambda'\frac{N-1}{N-2}}\Big(|x|^{\frac{\lambda'}{\sqrt{\bar\mu}}}+|x|^{\frac{\bar\lambda}{\sqrt{\bar\mu}}}\Big)^{\frac{N+1}{2}}}\,dx \\
&= bc_N\epsilon^{\frac{N-1}{2}}\Big(\int_{|x|\leq r_\circ}\frac{1}{|x|^{\lambda'\frac{N-1}{N-2}}\Big(|x|^{\frac{\lambda'}{\sqrt{\bar\mu}}}+|x|^{\frac{\bar\lambda}{\sqrt{\bar\mu}}}\Big)^{\frac{N+1}{2}}}\,dx \\
&+ \int_{r_\circ\leq|x|\leq r(\epsilon)}\frac{1}{|x|^{\lambda'\frac{N-1}{N-2}}\Big(|x|^{\frac{\lambda'}{\sqrt{\bar\mu}}}+|x|^{\frac{\bar\lambda}{\sqrt{\bar\mu}}}\Big)^{\frac{N+1}{2}}}\,dx\Big) \\
&= bc_2c_N\big(I_1+I_2\big),
\end{aligned}$$

where $r_\circ > 0$ is fixed. We note that $\lim_{\epsilon\to 0} r(\epsilon) = \infty$. We now estimate I_1 and I_2. We have

$$I_1 \le \omega_N \int_0^{r_\circ} \frac{s^{N-1}}{s^{\lambda'\frac{N-1}{N-2}}\left(s^{\frac{\lambda'}{\mu}} + s^{\frac{\bar\lambda}{\sqrt{\mu}}}\right)^{\frac{N+1}{2}}}\, ds \le \omega_N \int_0^{r_\circ} s^{N-1-\lambda'\frac{N-1}{N-2}-\frac{\lambda'}{\sqrt{\mu}}\frac{N+1}{2}}\, ds.$$

Since $N - \lambda'\frac{N-1}{N-2} - \frac{\lambda'}{\sqrt{\mu}}\frac{N+1}{2} = N - \lambda'\frac{2N}{N-2} > 0$, we see that $I_2 < \infty$. For the integral, we have the following estimate

$$I_2 \le \omega_N \int_{r_\circ}^{r(\epsilon)} s^{N-1-\lambda'\frac{N-1}{N-2}-\bar\lambda\frac{N+1}{N-2}}\, ds.$$

We now observe that $N - \lambda'\frac{N-1}{N-2} - \bar\lambda\frac{N+1}{N-2} < 0$, so $I_2 < \infty$ and estimate (4.2) follows. □

Theorem 4.4. *Let $0 < \lambda < \lambda^*$ and $N > 5$. Suppose that*

$$|Q(x) - Q(0)| = o\big(|x|^{\frac{N-1}{2}}\big) \tag{4.3}$$

for x close to 0. If $\bar S_\infty = \frac{S^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}$, then problem (1.1) admits two distinct solutions.

Proof. We start by proving that

$$\max_{t\ge 0} J_\lambda(u_\lambda + tw_\epsilon) < J_\lambda(u_\lambda) + \frac{S^{\frac{N}{2}}}{NQ(0)^{\frac{N-2}{2}}}. \tag{4.4}$$

We apply inequality (3.7) with $q = 2^*$ and $\kappa = \frac{N+1}{N-2}$. Using Lemma 4.3 we then obtain

$$\begin{aligned} J_\lambda(u_\lambda + tw_\epsilon) \le J_\lambda(u_\lambda) &+ \frac{t^2}{2}\int_\Omega |\nabla w_\epsilon|^2\, dx - \frac{\lambda t^2}{2}\int_\Omega \frac{w_\epsilon^2}{|x|^2}\, dx - \frac{t^{2^*}}{2^*}\int_\Omega Q w_\epsilon^{2^*}\, dx \\ &- c_1 t^{2^*-1}\epsilon^{\frac{N-2}{2}} + c_2 t^{\frac{N+1}{N-2}}\epsilon^{\frac{N-1}{2}} = J_\lambda(u_\lambda) + \tilde\Phi(t). \end{aligned}$$

Let $\tilde\Phi(t_\epsilon) = \max_{t\ge 0}\tilde\Phi(t)$, where $t_\epsilon > 0$. As in the proof of Proposition 3.2 we can show that there exist constants $0 < T_1 < T_2$ such that $T_1 < t_\epsilon < T_2$ for small $\epsilon > 0$. Hence, we have

$$J_\lambda(u_\lambda + tw_\epsilon) \le J_\lambda(u_\lambda) + \frac{1}{N}\frac{\left(\int_\Omega |\nabla w_\epsilon|^2\, dx\right)^{\frac{N}{2}}}{\left(\int_\Omega Q w_\epsilon^{2^*}\, dx\right)^{\frac{N-2}{2}}} - c_1 T_1^{2^*-1}\epsilon^{\frac{N-2}{2}} + c_2 T_2^{\frac{N+1}{N-2}}\epsilon^{\frac{N-1}{2}}.$$

Since

$$\int_\Omega |\nabla w_\epsilon|^2\,dx = S_{-\lambda}^{\frac{N}{2}} + O\big(\epsilon^{N-2}\big),\ \ \int_\Omega w_\epsilon^{2^*}\,dx = S_{-\lambda}^{\frac{N}{2}} + O\big(\epsilon^N\big)$$

and

$$\int_\Omega Q w_\epsilon^{2^*}\,dx = Q(0)\int_\Omega w_\epsilon^{2^*}\,dx + o\big(\epsilon^{\frac{N-1}{2}}\big)$$

estimate (4.4) follows. An application of the mountain - pass principle completes the proof. □

5. Final remarks

1. We give here some comments about the proof of Lemma 4.2. First we observe that Lemma 3.4 remains valid if $0 \in \bar{\Omega}$ and for a more general problem

$$\begin{cases} -\Delta u - \lambda \frac{u}{|x|^2} = f(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\lambda > 0$ and $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$|f(x,t)| \le C\big(|t| + |t|^{2^*-1}\big) \tag{5.2}$$

for all $(x,t) \in \bar{\Omega} \times \mathbb{R}$, where $C > 0$ is a constant. If (*) holds and $u \in H^1(\Omega)$ is a solution of problem (5.1), then $u \in L^p(\Omega)$. Obviously u does not have to be positive on Ω. Repeating the argument from [29], we can show that if $u \in H^1(\Omega)$ is a solution of (5.1), then there exist a small ball $B(0,\rho) \subset \Omega$ and a constant $b > 0$ such that

$$|u(x)| \le b|x|^{-(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\lambda})} \tag{5.3}$$

for $x \in B(0,\rho) - \{0\}$. This estimate was established for a Dirichlet problem in [29] for equation (5.1) with f satisfying (5.2). Since (5.3) is a local property of a solution around the interior point 0 of Ω, the approach presented in [29] also remains true for solutions of problem (5.1). We only need the higher integrability property of solutions of (5.1), which is guaranteed by the first comment. If $u \in H^1(\Omega)$ is a positive solution, then lower local estimate of u follows from Proposition 2.2 in [23].

2. The existence results for problem (1.1) remain true, under assumption (**Q**) for the case without the Hardy potential, that is,

$$\begin{cases} -\Delta u - \lambda u = Q(x)|u|^{2^*-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,\ u > 0 \text{ on } \Omega, \end{cases}$$

where $\lambda > 0$. In this situation, the Palais - Smale condition for the corresponding variational functional holds for

$$c < \min\Big(\frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}\Big) := S_\infty.$$

Solutions of this problem are continuous on $\bar{\Omega}$, so we do not need Lemmas 3.4, 3.5. For the existence of a second solution we need only consider two cases: $S_\infty = \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$ and $S_\infty = \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}$.

3. Suppose that $Q(x) < 0$ on $\bar{\Omega}$ and put $Q^* = \inf_{\bar{\Omega}} -Q(x) > 0$. Then the functional J_λ is coercive on $H^1(\Omega)$. Indeed, by inequality (2.1) with $\epsilon = 1$, we get

$$J_\lambda(u) \geq \frac{1}{2}\big(1 - \lambda(\frac{1}{\bar{\mu}} + 1)\big) \int_\Omega |\nabla u|^2\, dx - \lambda C(1) \int_\Omega u^2 + Q^* \int_\Omega |u|^{2^*}\, dx.$$

Applying the Young inequality, we get for $\delta > 0$

$$\begin{aligned} J_\lambda(u) \geq{}& \frac{1}{2}\big(1 - \lambda(\frac{1}{\bar{\mu}} + 1)\big) \int_\Omega |\nabla u|^2\, dx \\ &+ \big(Q^* - \lambda\delta C(1)\big) \int_\Omega |u|^{2^*}\, dx - \lambda C(1)\delta^{-\frac{2^*}{2^*-2}}|\Omega|. \end{aligned}$$

We now choose λ^* and δ so that

$$a_1 := 1 - \lambda\big(\frac{1}{\bar{\mu}} + 1\big) > 0 \quad \text{and} \quad b_1 := Q^* - \lambda\delta C(1) > 0$$

for $0 < \lambda < \lambda^*$. Then

$$J_\lambda(u) \geq \frac{a_1}{2} \int_\Omega |\nabla u|^2\, dx + b_1 \int_\Omega |u|^{2^*}\, dx - C_1(\delta, \lambda).$$

We now observe that $J_\lambda(t) < 0$ for $t > 0$ small. So $A_\lambda := \inf_{u \in H^1(\Omega)} J_\lambda(u) < 0$. By the Ekeland variational principle, there exists $\{u_n\} \subset H^1(\Omega)$ such that $J_\lambda(u_n) \to A_\lambda$ and $J'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$. This sequence is bounded and we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$. Since $Q(x) < 0$ on Ω there are no concentration points of $\{u_n\}$, so $u_n \to u$ in $H^1(\Omega)$. Therefore problem (1.1) has a solution which is a global minimizer of J_λ.

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