

SPECTRAL ANALYSIS OF STATIONARY SOLUTIONS OF THE CAHN–HILLIARD EQUATION

PETER HOWARD

Department of Mathematics, Texas A&M University
College Station, TX 77843

(Submitted by: Gui-Qiang Chen)

Abstract. For the Cahn–Hilliard equation on \mathbb{R} , there are precisely three types of bounded non-constant stationary solutions: periodic solutions, pulse-type *reversal* solutions, and monotonic transition waves. We study the spectrum of the linear operator obtained upon linearization about each of these waves, establishing linear stability for all transition waves, linear instability for all reversal waves, and linear instability for a representative class of periodic waves. For the case of transitions, the author has shown in previous work that linear stability implies nonlinear stability, and so nonlinear (phase-asymptotic) stability is established for such waves.

1. INTRODUCTION

We consider the Cahn–Hilliard equation on \mathbb{R} ,

$$u_t = (M(u)(F'(u) - \kappa u_{xx})_x)_x, \quad (1.1)$$

where $\kappa > 0$ is assumed constant, and throughout the analysis we will make the following assumptions:

(H0) $M \in C^2(\mathbb{R})$ and $F \in C^4(\mathbb{R})$.

(H1) F has a double-well form: there exist real numbers $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$ so that F is strictly decreasing on $(-\infty, \alpha_1)$ and (α_3, α_5) and strictly increasing on (α_1, α_3) and $(\alpha_5, +\infty)$, and additionally F is concave up on $(-\infty, \alpha_2) \cup (\alpha_4, +\infty)$ and concave down on (α_2, α_4) . The interval (α_2, α_4) is typically referred to as the *spinodal region*.

We note that, for each F satisfying assumptions (H0)–(H1), there exists a unique pair of values u_1 and u_2 (the *binodal* values) so that

$$F'(u_1) = \frac{F(u_2) - F(u_1)}{u_2 - u_1} = F'(u_2)$$

Accepted for publication: September 2008.

AMS Subject Classifications: 35B10, 35B35.

and the line passing through $(u_1, F(u_1))$ and $(u_2, F(u_2))$ lies entirely on or below F .

(H2) For $u \in [u_1, u_2]$, $M(u) \geq m_0 > 0$.

For equation (1.1) under conditions (H0)–(H1), and for $M(u) \geq m_0 > 0$ for all $u \in \mathbb{R}$, there exist precisely three types of bounded non-constant stationary solutions $\bar{u}(x)$: periodic solutions, pulse-type *reversal* solutions for which $\bar{u}(\pm\infty) = u_{\pm}$, $u_- = u_+$, and monotonic transition waves for which $\bar{u}(\pm\infty) = u_{\pm}$, $u_- \neq u_+$ (see [3] and Theorems 1.1, 1.3, and 1.4 here). We study the spectrum of the linear operator obtained upon linearization about each of these, establishing linear stability for all transition waves, linear instability for all reversal waves, and linear instability for a representative class of periodic waves. For the case of transitions, the author has shown in [20] that linear stability implies nonlinear stability, and so nonlinear (phase-asymptotic) stability is established for such waves.

The Cahn–Hilliard equation—often augmented by a driving or reaction term—arises in the study of several phenomena, including phase separation [8], growth and dispersal of biological populations [10], chemical reaction kinetics [28], image inpainting [4], and as the modulation equation for a viscous incompressible fluid under the action of an external force on an infinite strip [34, 44]. Our analysis is particularly motivated by the study of spinodal decomposition, a phenomenon in which rapid cooling of a homogeneously mixed binary alloy causes separation to occur, resolving the mixture into regions of different crystalline structure in which one component or another is dominant, with these regions separated by steep transition layers. (For a general reference on spinodal decomposition, the reader is referred to J. W. Cahn’s 1967 Institute of Metals Lecture, printed as [8].) More precisely, this phase separation is typically considered to take place in two stages: First, a relatively fast process occurs in which the homogeneous mixture quickly begins to separate (this is the stage of spinodal decomposition), followed by a second *coarsening* process (sometimes referred to as *ripening*) during which the regions continue to broaden on a relatively slower timescale, and the maximum difference in concentrations of the two components continues to increase (though in practice this latter increase can be extremely small). (For particularly accessible accounts of experimental studies of this process, the reader is referred to [25, 26, 27, 31].)

In this context, u typically denotes one component of the binary alloy (or a convenient affine transformation of this concentration), and the Cahn–Hilliard equation arises from the conservation law

$$u_t + \nabla \cdot \left\{ -M(u) \nabla \frac{\delta E}{\delta u} \right\} = 0,$$

where $M(u)$ denotes the mobility associated with the component u and is typically assumed positive, and E denotes the total free energy associated with u . The Cahn–Hilliard equation as considered here arises from these considerations and a form of $E(u)$ proposed in 1958 by Cahn and Hilliard, who were considering particularly the interfacial energy between two components of a binary compound [9]. Taking $F(u)$ to denote the bulk free energy density associated with a homogeneously arranged alloy with concentration u , Cahn and Hilliard posed the energy functional

$$E(u) = \int_{\Omega} F(u) + \frac{\kappa}{2} |\nabla u|^2 dx, \quad (1.2)$$

where in the setting of [9] Ω denotes a bounded open subset of \mathbb{R}^3 , and the term $\frac{\kappa}{2} |\nabla u|^2$ arises as a first-order correction that accounts for interfacial energy. (In fact, the functional $E(u)$ was originally proposed by van der Waals in [45] as an appropriate energy for a two-phase system.) In the case of relatively high temperatures, F generally has a quadratic form, expressing the fact that free energy is minimized by configurations that maximize entropy, and such configurations correspond with values of u for which the components of the alloy are evenly mixed to the point allowed by mass conservation. On the other hand, as temperature drops, free energy increases at a rate proportional to entropy, and F can take on the double-well form of (H1). In this way, the original (high-temperature) homogeneous configuration $u = u_h$ can become unstable at the lower temperature, and consequently small perturbations from u_h do not dissipate, but rather propagate with quite complicated dynamics. (See, for example, the numerical calculations [5, 7, 12, 13, 32, 33].) Our goal in the current analysis is to determine the stability or instability of stationary solutions $\bar{u}(x)$ to (1.1), and to interpret our results in the context of this nonlinear evolution. We note at the outset of our investigation that a closely related result has already been established by Alexiades and Aifantis in [1]. Namely, the authors establish that reversal and periodic solutions to (1.1) are *not* minimizers of the energy functional (1.2), and that transition waves *are* minimizers of this functional. (In fact, the authors work in a more general setting than the one described here, taking $\kappa = \kappa(u)$ in (1.2), which is consistent with the derivation of Cahn and

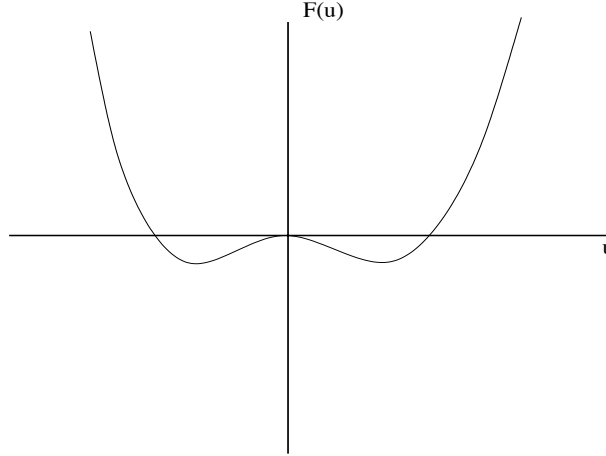


FIGURE 1. Bulk free energy in standard form.

Hilliard.) In the case of transition waves, we combine the methods of [1] with a straightforward Evans function argument to establish our spectral result. Our instability results, however, both for the case of reversals and for the case of periodic solutions, are based entirely on Evans function techniques.

We observe at the outset that for any linear function $G(u) = Au + B$, we can replace $F(u)$ in (1.1) with $H(u) = F(u) - G(u)$. If we take

$$G(u) = \frac{F(u_2) - F(u_1)}{u_2 - u_1}(u - u_h) + F(u_h),$$

where u_h is the unique value for which both $F''(u_h) < 0$ and $F'(u_h) = (F(u_2) - F(u_1))/(u_2 - u_1)$, then $H(u)$ has a local maximum $H = 0$ and local minima at the binodal values $H(u_1) = H(u_2)$. Finally, replacing u with $u + u_h$, we can shift H so that the local maximum is located at $u = 0$ (see Figure 1). We will refer to a double well function $F(u)$ for which the local maximum occurs at $u = 0$ and with equivalent local minima as *standard form*.

Upon linearization of (1.1) about the stationary solution $\bar{u}(x)$ (of any type), we obtain the linear equation

$$v_t = (M(\bar{u}(x))(F''(\bar{u})v - \kappa v_{xx})_x)_x, \quad (1.3)$$

where we have used the observation that $\bar{u}(x)$ satisfies

$$F'(\bar{u}) - \kappa \bar{u}_{xx} = C = \text{constant}.$$

The eigenvalue problem associated with (1.3) has the form

$$L\phi = \lambda\phi, \quad (1.4)$$

with

$$L\phi := (M(\bar{u}(x))(F''(\bar{u}(x))\phi - \kappa\phi_{xx})_x)_x. \quad (1.5)$$

In the cases of transition and reversal stationary solutions, we have well-defined asymptotic endstates for $\bar{u}(x)$, and it is natural to work in reference to the constant coefficient ODE that arises upon taking limits $x \rightarrow \pm\infty$ of (1.4). In particular, we can immediately deduce from these asymptotic considerations that the essential spectrum associated with either transition waves or reversal waves must be the negative real axis. It is also clear from these asymptotic problems that (1.4) has two solutions that decay at $-\infty$ and two solutions that decay at $+\infty$. We denote the first pair $\phi_1^-(x; \lambda)$, $\phi_2^-(x; \lambda)$ and the second $\phi_1^+(x; \lambda)$, $\phi_2^+(x; \lambda)$, and observe that away from essential spectrum all other solutions must grow at an exponential rate. It follows that any L^2 eigenfunction of (1.4) must be a linear combination of ϕ_1^- and ϕ_2^- (to ensure decay at $-\infty$) and of ϕ_1^+ and ϕ_2^+ (to ensure decay at $+\infty$). Consequently, away from essential spectrum, we have an L^2 eigenfunction if and only if $W(\phi_1^-, \phi_2^-, \phi_1^+, \phi_2^+) = 0$, where W is a standard Wronskian

$$W(\phi_1^-, \phi_2^-, \phi_1^+, \phi_2^+) = \det \begin{pmatrix} \phi_1^- & \phi_2^- & \phi_1^+ & \phi_2^+ \\ \phi_1^{-'} & \phi_2^{-'} & \phi_1^{+'} & \phi_2^{+'} \\ \phi_1^{-''} & \phi_2^{-''} & \phi_1^{+''} & \phi_2^{+''} \\ \phi_1^{-'''} & \phi_2^{-'''} & \phi_1^{+'''} & \phi_2^{+'''} \end{pmatrix}.$$

Loosely following [2, 11, 29], we define the Evans function (for transitions and reversals) as

$$D(\lambda) = W(\phi_1^-, \phi_2^-, \phi_1^+, \phi_2^+) \Big|_{x=0}. \quad (1.6)$$

In the current setting $D(\lambda)$ is not analytic in a neighborhood of the critical point $\lambda = 0$, but it can be defined analytically as a function of $\rho = \sqrt{\lambda}$, and we set

$$\bar{D}(\rho) := D(\lambda).$$

The function $\bar{u}_x(x)$ is an eigenfunction of L associated with the eigenvalue $\lambda = 0$, and so we know immediately that $D(0) = 0$. Our condition for spectral (and linear) stability requires that this be the only zero of D with non-negative real part. Following [20], we will say that a transition or reversal wave is spectrally stable if the following condition is satisfied:

(\mathcal{D}): The Evans function $D(\lambda)$ has precisely one zero in $\{\operatorname{Re} \lambda \leq 0\}$, necessarily at 0, and $\bar{D}_{\rho\rho}(0) \neq 0$.

We are now in a position to state the following theorems on spectral and nonlinear stability for transition waves, and on spectral instability for reversals.

Theorem 1.1 (Spectral stability for transition waves.). *For equation (1.1), under conditions (H0)–(H2), there exist two transition wave solutions $\bar{u}(x)$ and $\bar{u}(-x)$, both of which are strictly monotonic, and both of which approach the binodal values $u_1 < u_2$:*

$$\lim_{x \rightarrow -\infty} \bar{u}(x) = u_1; \quad \lim_{x \rightarrow +\infty} \bar{u}(x) = u_2.$$

Under the additional condition $M(u) \geq m_0 > 0$ for all $u \in \mathbb{R}$, these are the only transition wave solutions to (1.1) (up to translation). Moreover, each of these transition waves is spectrally stable in the sense that (\mathcal{D}) is satisfied.

Combined with the nonlinear analysis of [20], we have the following theorem on stability for transition wave solutions to the Cahn–Hilliard equation.

Theorem 1.2 (Nonlinear stability for transition waves). *Under the conditions of Theorem 1.1, the transition waves $\bar{u}(x)$ and $\bar{u}(-x)$ described there are nonlinearly stable as follows: For Hölder continuous initial perturbations $u(0, x) - \bar{u} \in C^{0+\gamma}(\mathbb{R})$, $\gamma > 0$, with*

$$|u(0, x) - \bar{u}(x)| \leq E_0(1 + |x|)^{-\frac{3}{2}},$$

for some E_0 sufficiently small, there exists some local tracking function $\delta(t)$ so that

$$|u(t, x + \delta(t)) - \bar{u}(x)| \leq CE_0(1 + |x| + \sqrt{t})^{-\frac{3}{2}},$$

with

$$\delta(t) = -\frac{1}{u_+ - u_-} \int_{-\infty}^{+\infty} (u(0, x) - \bar{u}(x)) dx + \mathbf{O}(t^{-\frac{1}{4}}).$$

Remark 1.1. For a more complete statement of Theorem 1.2, and for details regarding the local tracking function $\delta(t)$ —which is not a Dirac delta function—the reader is referred to [20].

For reversal solutions we have the following theorem on spectral (and linear) instability.

Theorem 1.3 (Spectral instability for reversal waves.). *For equation (1.1), under conditions (H0)–(H2), let u_3 and u_4 denote real numbers fixed between*

the binodal values such that $u_1 < u_3 < u_4 < u_2$, and suppose that one of the following holds:

$$F'(u_3) = \frac{F(u_4) - F(u_3)}{u_4 - u_3} > F'(u_4); \text{ or } F'(u_3) > \frac{F(u_4) - F(u_3)}{u_4 - u_3} = F'(u_4).$$

In the first case, there exists a unique (up to translation) reversal solution satisfying

$$\lim_{x \rightarrow \pm\infty} \bar{u}(x) = u_3; \quad \text{and} \quad \max_{x \in \mathbb{R}} \bar{u}(x) = u_4,$$

(here, u_3 must lie outside the spinodal region), while for the second

$$\lim_{x \rightarrow \pm\infty} \bar{u}(x) = u_4; \quad \text{and} \quad \min_{x \in \mathbb{R}} \bar{u}(x) = u_3$$

(here, u_4 must lie outside the spinodal region). Under the additional assumption $M(u) \geq m_0 > 0$ for all $u \in \mathbb{R}$, this categorizes all possible reversal solutions to (1.1). Moreover, each such solution is linearly unstable with a positive real eigenvalue.

In the case of periodic stationary solutions, we loosely follow the approach of Gardner [16, 17, 18], with additional aspects of the analysis taken from Oh and Zumbrun [35, 36], Schneider [41, 42, 43], and Papanicolaou [37, 39]. Letting $\bar{u}(x)$ be a periodic solution to (1.1) with period X , we proceed by searching for eigenfunctions of (1.4) with the particular form $\phi(x) = e^{i\xi x} p(x)$, where $\xi \in \mathbb{R}$ and $p(x)$ has period X . (A detailed explanation of the nature of this choice can be found, for example, on page 171 of [38].) Upon substitution of this ansatz into (1.4), we arrive at an eigenvalue problem for p on a bounded domain $x \in [0, X]$,

$$L_\xi p = \lambda p; \quad p^{(k)}(0) = p^{(k)}(X), k = 0, 1, 2, 3, \quad (1.7)$$

where

$$L_\xi := e^{-i\xi x} L e^{i\xi x}. \quad (1.8)$$

We proceed now by constructing solutions to (1.7) in terms of a basis of solutions to (1.4) $\{\phi_j\}_{j=1}^4$ initialized by $\phi_j^{(k-1)}(0; \lambda) = \delta_j^k$ for $k = 1, 3, 4$, with $(b\phi_j)'(0; \lambda) = \delta_2^j$, where $b(x) := F''(\bar{u}(x))$ and δ_j^k denotes a standard Kronecker delta function. In particular, we create a basis of solutions for (1.7) $\{p_j\}_{j=1}^4$ through the relations $p_j(x; \lambda) = e^{-i\xi x} \phi_j(x; \lambda)$. Looking for solutions

$$p(x) = \sum_{j=1}^4 V_j(\lambda) p_j(x; \lambda)$$

that satisfy the boundary conditions of (1.7), we conclude that there exists a solution to (1.7) if and only if there exists $\xi \in \mathbb{R}$ such that $e^{i\xi X}$ is an eigenvalue of the eigenvalue problem

$$M(\lambda; X)V = e^{i\xi X}V; \quad V = (V_1, V_2, V_3, V_4)^{\text{tr}} \quad (1.9)$$

where $M(\lambda; X)$ is the monodromy or Floquet matrix

$$M(\lambda; X) = \begin{pmatrix} \phi_1(X; \lambda) & \phi_2(X; \lambda) & \phi_3(X; \lambda) & \phi_4(X; \lambda) \\ (b\phi_1)'(X; \lambda) & (b\phi_2)'(X; \lambda) & (b\phi_3)'(X; \lambda) & (b\phi_4)'(X; \lambda) \\ \phi_1''(X; \lambda) & \phi_2''(X; \lambda) & \phi_3''(X; \lambda) & \phi_4''(X; \lambda) \\ \phi_1'''(X; \lambda) & \phi_2'''(X; \lambda) & \phi_3'''(X; \lambda) & \phi_4'''(X; \lambda) \end{pmatrix}. \quad (1.10)$$

Accordingly, we define the Evans function for periodic waves as

$$E(\lambda, \xi) := \det(M(\lambda, X) - e^{i\xi X}I). \quad (1.11)$$

In most Evans function literature, this quantity is denoted D , and we only use E here to distinguish our Evans function for periodic waves from our Evans function for transition and reversal waves. It is clear from the discussion leading up to (1.11) that, for periodic waves, L has an eigenvalue λ_* whenever there exists $\xi \in \mathbb{R}$ so that $E(\lambda_*, \xi) = 0$.

Theorem 1.4 (Existence of periodic waves.). *For equation (1.1), under conditions (H0)–(H2), let u_3 and u_4 denote real numbers fixed between the binodal values such that $u_1 < u_3 < u_4 < u_2$, and such that*

$$F'(u_3) > \frac{F(u_4) - F(u_3)}{u_4 - u_3} > F'(u_4).$$

Then there exists a non-constant periodic solution to (1.1) with minimum value u_3 and maximum value u_4 . Under the additional assumption $M(u) \geq m_0 > 0$ for all $u \in \mathbb{R}$, this categorizes all possible non-constant periodic solutions to (1.1).

We now state an instability result for a restricted class of periodic solutions. In proving this result, we will state a general condition for instability of all periodic waves from Theorem 1.4, and the condition can readily be checked numerically on a case-by-case basis. In Theorem 1.5, we address an important class of periodic waves for which the condition can be verified analytically.

Theorem 1.5 (Spectral instability for periodic waves.). *For equation (1.1), under conditions (H0)–(H2), suppose additionally that $M(u) \equiv 1$ and that $F(u)$ in standard form is an even function. Then for each $u_* \in (0, u_2)$ there*

exists a unique (up to translation) periodic solution $\bar{u}(x)$ to (1.1) with minimum $-u_*$ and maximum $+u_*$. Moreover, this periodic solution is unstable with a positive real eigenvalue, provided $F'''(u) \geq 0$ for $u \in [0, u_*]$.

We note that the condition on $F'''(u)$ is by no means necessary for instability, and we only specify it because it can easily be checked, and because it is valid for the forms of F most frequently studied (see, for example, [1]). In particular, it is valid for polynomials of the form $F(u) = au^4 - bu^2$, where a and b are both positive constants.

Roughly speaking, our results can be interpreted in the context of spinodal decomposition as follows. When the homogeneously mixed binary alloy is cooled, the constant solution $u(t, x) = u_h$ becomes unstable, and spinodal decomposition begins. If the natural perturbation to this homogeneous configuration is fairly random, then we might heuristically expect it to evolve toward small-amplitude periodic waves. (While this is only heuristic in the current setting, Grant has verified this picture in the case of bounded domains [14, 15].) Each of these periodic solutions is unstable, and so the solution moves away from them, coarsening further toward the stable transition wave. Though we do not make any effort to verify it here, numerical experiments suggest that the periodic solutions become less stable as their amplitudes near the binodal values (the endstates of the stable transition wave). In Figure 2, we plot a number of different stationary solutions for the case of (1.1) with $M \equiv 1$, $\kappa = 1$, and $F(u) = \frac{1}{8}u^4 - \frac{1}{4}u^2$, while in Figure 3, we plot the leading eigenvalue associated with these waves as a function of amplitude (these eigenvalues were obtained numerically). For small amplitudes, the eigenvalues are relatively large, and this corresponds with the relatively fast spinodal decomposition phase. As the amplitude grows, however, the leading eigenvalues decrease, and we enter the relatively slow coarsening phase. In this way, Figure 2 can almost be viewed as a time evolution from the constant homogeneous state $u_h = 0$ to the final transition front from -1 to $+1$. Along these lines, it is interesting to compare Figure 2 with Figure 2 of [33], which depicts a numerical evolution of a perturbation of $u_h = 0$ for (1.1) with $M \equiv 1$, $\kappa = .001$, and $F(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$.

2. INSTABILITY OF PERIODIC WAVES

In this section, we develop a general condition for instability of the periodic solutions described in Theorem 1.4. Our approach will be through Taylor expansion of the Evans function $E(\lambda, \xi)$. We note at the outset that the

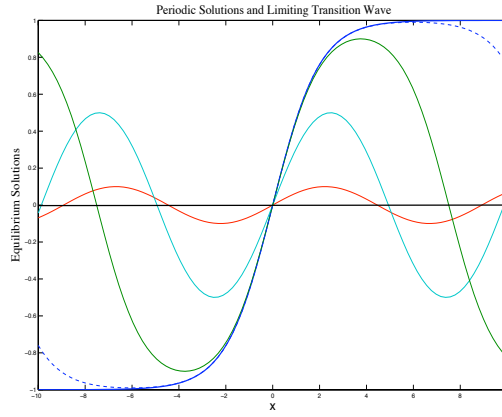


FIGURE 2. Stationary solutions of the Cahn–Hilliard equation.

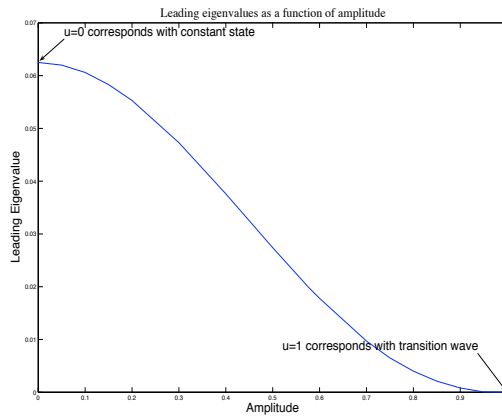


FIGURE 3. Leading eigenvalues plotted as a function of magnitude.

periodic waves $\bar{u}(x)$ of Theorem 1.4 satisfy the ODE

$$\kappa \bar{u}_{xx} = F'(\bar{u}) - \frac{F(u_4) - F(u_3)}{u_4 - u_3}, \quad (2.1)$$

and that without loss of generality we will shift $\bar{u}(x)$ so that $\bar{u}_{xx}(0) = 0$.

Our starting point will be the relation

$$E(\lambda, \xi) = \det \begin{pmatrix} [\phi_1] - [e^{i\xi x}] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(b\phi_1)'] & [(b\phi_2)'] - [e^{i\xi x}] & [(b\phi_3)'] & [(b\phi_4)'] \\ [\phi_1''] & [\phi_2''] & [\phi_3''] - [e^{i\xi x}] & [\phi_4''] \\ [\phi_1'''] & [\phi_2'''] & [\phi_3'''] & [\phi_4'''] - [e^{i\xi x}] \end{pmatrix},$$

where $[\phi_k] := \phi_k(X; \lambda) - \phi_k(0; \lambda)$, and we recall the notation $b(x) = F''(\bar{u}(x))$. As an initial simplification, we observe that by integrating (1.4), we can obtain the general relationship

$$[\phi_k'''] = \frac{1}{\kappa} [(b\phi_k)'] - \frac{\lambda}{\kappa M(\bar{u}(X))} \int_0^X \phi_k(x; \lambda) dx. \quad (2.2)$$

Upon substitution of (2.2) into $E(\lambda, \xi)$, followed by the algebraic operation of dividing the second row in the matrix of $E(\lambda, \xi)$ by κ and subtracting the result from row 4, we find that $E(\lambda, \xi)$ can be written as the determinant of

$$\begin{pmatrix} [\phi_1] - [e^{i\xi x}] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(b\phi_1)'] & [(b\phi_2)'] - [e^{i\xi x}] & [(b\phi_3)'] & [(b\phi_4)'] \\ [\phi_1''] & [\phi_2''] & [\phi_3''] - [e^{i\xi x}] & [\phi_4''] \\ -\frac{\lambda}{\kappa M_0} \int \phi_1 & -\frac{\lambda}{\kappa M_0} \int \phi_2 + \frac{[e^{i\xi x}]}{\kappa} & -\frac{\lambda}{\kappa M_0} \int \phi_3 & -\frac{\lambda}{\kappa M_0} \int \phi_4 - [e^{i\xi x}] \end{pmatrix}, \quad (2.3)$$

where for notational brevity we have taken $M_0 = M(\bar{u}(X))$, and following the convention of [35, 36] we have written

$$\int \phi_k := \int_0^X \phi_k(x; \lambda) dx.$$

In our expansion, we will first consider the case $\xi = 0$, which corresponds with perturbations with period X . In this case,

$$E(\lambda, 0) = -\frac{\lambda}{\kappa M_0} \det \begin{pmatrix} [\phi_1] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(b\phi_1)'] & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ [\phi_1''] & [\phi_2''] & [\phi_3''] & [\phi_4''] \\ \int \phi_1 & \int \phi_2 & \int \phi_3 & \int \phi_4 \end{pmatrix},$$

and we see immediately that $E(0, 0) = 0$, while

$$E_\lambda(0, 0) = -\frac{1}{\kappa M_0} \det \begin{pmatrix} [\phi_1] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(b\phi_1)'] & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ [\phi_1''] & [\phi_2''] & [\phi_3''] & [\phi_4''] \\ \int \phi_1 & \int \phi_2 & \int \phi_3 & \int \phi_4 \end{pmatrix} \Big|_{\lambda=0}. \quad (2.4)$$

We turn our attention, then, to the evaluation of the ϕ_k at $\lambda = 0$. Upon setting $\lambda = 0$ in (1.4) and integrating twice, we find that, for each $k = 1, 2, 3, 4$,

$$\kappa\phi_k'' - b(x)\phi_k = -M_0\left((b\phi_k)'(0) - \kappa\phi_k'''(0)\right) \int_0^x \frac{dy}{M(\bar{u}(y))} + \kappa\phi_k''(0) - b(0)\phi_k(0).$$

In this way, we see that each of the $\phi_k(x; 0)$ can be obtained by the solution of a second order ODE:

$$\begin{aligned} \kappa\phi_1'' - b(x)\phi_1 &= -b(0); & \phi_1(0) &= 1, (b\phi_1)'(0) = 0 \\ \kappa\phi_2'' - b(x)\phi_2 &= -M_0 \int_0^x \frac{dy}{M(\bar{u}(y))}; & \phi_2(0) &= 0, (b\phi_2)'(0) = 1 \\ \kappa\phi_3'' - b(x)\phi_3 &= \kappa; & \phi_3(0) &= 0, (b\phi_3)'(0) = 0 \\ \kappa\phi_4'' - b(x)\phi_4 &= \kappa M_0 \int_0^x \frac{dy}{M(\bar{u}(y))}; & \phi_4(0) &= 0, (b\phi_4)'(0) = 0. \end{aligned} \tag{2.5}$$

In our analysis, we will also make use of two important linear combinations, $m(x) := \kappa\phi_1(x; 0) + b(0)\phi_3(x; 0)$ and $w(x) = \kappa\phi_2(x; 0) + \phi_4(x; 0)$, which respectively satisfy

$$\begin{aligned} \kappa m'' - b(x)m &= 0; & m(0) &= \kappa, m'(0) = -\kappa \frac{b'(0)}{b(0)} \\ \kappa w'' - b(x)w &= 0; & w(0) &= 0, w'(0) = \frac{\kappa}{b(0)}. \end{aligned} \tag{2.6}$$

By a standard variation of parameters representation, we can now understand each of the ϕ_k in terms of two linearly independent solutions to the homogeneous problem

$$\kappa\phi'' - b(x)\phi = 0. \tag{2.7}$$

As is clear from (2.1), one solution to this equation is $\bar{u}_x(x)$, while the second can be written in terms of $\bar{u}_x(x)$ by reduction of order:

$$\psi(x) = \begin{cases} \bar{u}_x(x) \int_0^x \frac{dy}{\bar{u}_y^2} & 0 \leq x \leq x_1 \\ \bar{u}_x(x) \int_{2x_1}^x \frac{dy}{\bar{u}_y^2} + \frac{2K_1}{\bar{u}_{xx}(x_1)} \bar{u}_x(x) & x_1 \leq x \leq x_2 \\ -\bar{u}_x(x) \int_x^X \frac{dy}{\bar{u}_y^2} + 2\bar{u}_x(x) \left(\frac{\bar{u}_{xx}(x_1)K_2 - \bar{u}_{xx}(x_2)K_1}{\bar{u}_{xx}(x_1)\bar{u}_{xx}(x_2)} \right) & x_2 \leq x \leq X, \end{cases} \tag{2.8}$$

where $x_1 < x_2$ denote the two places for which $\bar{u}_x(x_k) = 0$, and

$$K_1 := \lim_{x \rightarrow x_1^-} \left(\bar{u}_{xx}(x) \int_0^x \frac{dy}{\bar{u}_y^2} + \frac{1}{\bar{u}_x(x)} \right)$$

$$K_2 := \lim_{x \rightarrow x_2^-} \left(\bar{u}_{xx}(x) \int_{2x_1}^x \frac{dy}{\bar{u}_y^2} + \frac{1}{\bar{u}_x(x)} \right) + 2K_1 \frac{\bar{u}_{xx}(x_2)}{\bar{u}_{xx}(x_1)},$$

both of which are well defined. We note, in particular, that $\bar{u}_x(x)$ and $\psi(x)$ are the solutions of (2.7) with initial conditions $\bar{u}_x(0) > 0$, $\bar{u}_{xx}(0) = 0$ (the second by our choice of shift) and $\psi(0) = 0$, $\psi'(0) = 1/\bar{u}_x(0) > 0$, and consequently $W(\bar{u}_x(x), \psi(x)) \equiv 1$, where W denotes a standard Wronskian.

We now have enough notation in place to state a result on the derivatives of $E(\lambda, \xi)$.

Lemma 2.1. *Under the assumptions of Theorem 1.4, and with $\bar{u}(x)$ as described there, we have that $E(\lambda, \xi)$ is analytic in a neighborhood of $(0, 0)$, and that the following relations hold:*

$$E(0, 0) = E_\lambda(0, 0) = E_{\lambda\xi}(0, 0) = 0,$$

$$E_\xi(0, 0) = E_{\xi\xi}(0, 0) = E_{\xi\xi\xi}(0, 0) = 0; \quad E_{\xi\xi\xi\xi}(0, 0) = 24X^4,$$

$$E_{\lambda\lambda}(0, 0) = 2 \frac{b(X)\psi'(X)}{\bar{u}_x(0)\kappa^3} \det \begin{pmatrix} \int \phi_3 & \int w \\ [\phi_3] & [w] \end{pmatrix}$$

$$\times \left[\left(\int_0^X \frac{\bar{u}(x) - \bar{u}(0)}{M(\bar{u}(x))} dx \right)^2 - \int_0^X \frac{dx}{M(\bar{u}(x))} \int_0^X \frac{(\bar{u}(x) - \bar{u}(0))^2}{M(\bar{u}(x))} dx \right]$$

$$E_{\lambda\xi\xi}(0, 0) = 2 \frac{X^2}{\kappa\bar{u}_x(X)} \left(b(X)[w]h'(X) - [\phi_3] \int_0^X \frac{\bar{u}(x) - \bar{u}(0)}{M(\bar{u}(x))} dx \right)$$

$$+ 2 \frac{X^2}{\kappa} \left[\frac{b(X)[\phi_2']}{M_0} \int_0^X w(x) dx - \int_0^X \frac{dx}{M(\bar{u}(x))} \int_0^X \phi_3(x; 0) dx \right].$$

Proof. First, by variation of parameters, we have

$$\phi_3(x; 0) = -\bar{u}_x(x) \int_0^x \psi(y) dy + \psi(x)(\bar{u}(x) - \bar{u}(0)), \quad (2.9)$$

from which differentiation reveals the useful relation $[\phi_3'] = 0$. Likewise, we can show

$$[\phi_2'] = \frac{\psi'(X)M_0}{\kappa} \int_0^X \frac{\bar{u}(x) - \bar{u}(0)}{M(\bar{u}(x))} dx. \quad (2.10)$$

Additionally, we observe that $w(x) = \frac{\kappa\bar{u}_x(0)}{b(0)}\psi(x)$, from which we find $[w'] = 0$.

Since $m(x)$ and $w(x)$ constitute a complete basis for (2.7), we have

$$\begin{aligned}\bar{u}_x(x) &= \frac{\bar{u}_x(0)}{\kappa}m(x) + \frac{b'(0)\bar{u}_x(0)}{\kappa}w(x) \\ &= \bar{u}_x(0)\phi_1(x; 0) + b'(0)\bar{u}_x(0)\phi_2(x; 0) \\ &\quad + \frac{b(0)}{\kappa}\bar{u}_x(0)\phi_3(x; 0) + \frac{b'(0)\bar{u}_x(0)}{\kappa}\phi_4(x; 0).\end{aligned}$$

We can immediately conclude the linear dependencies

$$\begin{aligned}\int \phi_1 + b'(0) \int \phi_2 + \frac{b(0)}{\kappa} \int \phi_3 + \frac{b'(0)}{\kappa} \int \phi_4 &= 0 \\ [\phi_1^{(k)}] + b'(0)[\phi_2^{(k)}] + \frac{b(0)}{\kappa}[\phi_3^{(k)}] + \frac{b'(0)}{\kappa}[\phi_4^{(k)}] &= 0,\end{aligned}\tag{2.11}$$

for differentiation up to any order $k = 0, 1, 2, \dots$. Returning to (2.4), this linear dependence clearly gives $E_\lambda(0, 0) = 0$.

We proceed next with the calculation of $E_{\lambda\lambda}(0, 0)$. In this case, we obtain a sum of four determinants, each with a λ -derivative on a different row. We combine these determinants by eliminating ϕ_1 from each (with (2.11)), so that they are all in terms of the same three variables. Finally, we perform an elementary row operation so that the columns are ordered in Wronskian fashion. The calculation is tedious but straightforward, and leads eventually to

$$\frac{1}{2}E_{\lambda\lambda}(0, 0) = \frac{1}{\kappa\bar{u}_x(0)M_0} \det \begin{pmatrix} \int h & \int \phi_2 & \int \phi_3 & \int \phi_4 \\ [h] & [\phi_2] & [\phi_3] & [\phi_4] \\ [(bh)'] & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ [h''] & [\phi_2''] & [\phi_3''] & [\phi_4''] \end{pmatrix} \Big|_{\lambda=0},$$

where

$$\begin{aligned}h(x) &:= \bar{u}_x(0)(\partial_\lambda\phi_1)(x; 0) + \bar{u}_x(0)b'(0)(\partial_\lambda\phi_2)(x; 0) \\ &\quad + \bar{u}_x(0)\frac{b(0)}{\kappa}(\partial_\lambda\phi_3)(x; 0) + \bar{u}_x(0)\frac{b'(0)}{\kappa}(\partial_\lambda\phi_4)(x; 0),\end{aligned}\tag{2.12}$$

and consequently

$$\left(M(\bar{u}(x))(b(x)h - \kappa h_{xx}) \right)_x = \bar{u}_x(x); \quad h^{(j)}(0) = 0, \quad j = 0, 1, 2, 3.$$

We note that the initial conditions on $h(x)$ are clear from (2.12) and the fact that the initial conditions on the ϕ_k do not vary with λ . Upon integrating this last equation twice, and using variation of parameters, we can derive

the useful expression

$$h'(X) = \frac{\psi'(X)}{\kappa} \int_0^X \frac{(\bar{u}(x) - \bar{u}(0))^2}{M(\bar{u}(x))} dx. \quad (2.13)$$

According to (2.5) and (2.12), we have

$$\begin{aligned} [\phi_2''] &= \frac{1}{\kappa} b(X) [\phi_2] - \frac{M_0}{\kappa} \int_0^X \frac{dx}{M(\bar{u}(x))}; \\ [\phi_3''] &= \frac{1}{\kappa} b(X) [\phi_3]; \\ [\phi_4''] &= \frac{1}{\kappa} b(X) [\phi_4] + M_0 \int_0^X \frac{dx}{M(\bar{u}(x))}; \\ h''(X) &= \frac{1}{\kappa} b(X) h - \frac{1}{\kappa} \int_0^X \frac{\bar{u}(x) - \bar{u}(0)}{M(\bar{u}(x))} dx. \end{aligned} \quad (2.14)$$

Upon substitution of (2.14) into our determinant, we arrive at

$$\begin{aligned} \frac{1}{2} E_{\lambda\lambda}(0, 0) &= \frac{1}{\kappa \bar{u}_x(0) M_0} \\ &\times \det \begin{pmatrix} \int h & \int \phi_2 & \int \phi_3 & \int \phi_4 \\ h & [\phi_2] & [\phi_3] & [\phi_4] \\ (bh)' & [(b\phi_2)'] & [(b\phi_3)'] & [(b\phi_4)'] \\ -\frac{1}{\kappa} \int_0^X \frac{\bar{u}(x) - \bar{u}(0)}{M(\bar{u}(x))} dx & -\frac{M_0}{\kappa} \int_0^X \frac{dx}{M(\bar{u}(x))} & 0 & M_0 \int_0^X \frac{dx}{M(\bar{u}(x))} \end{pmatrix}, \end{aligned}$$

where the determinant is evaluated at $\lambda = 0$. Combining this relation with the observations just following (2.9), we conclude

$$\begin{aligned} \frac{1}{2} E_{\lambda\lambda}(0, 0) &= \frac{b(X)}{\bar{u}_x(0) \kappa^2} \left[\frac{[\phi_2']}{M_0} \int_0^X \frac{\bar{u}(x) - \bar{u}(0)}{M(\bar{u}(x))} dx \right. \\ &\quad \left. - h'(X) \int_0^X \frac{dx}{M(\bar{u}(x))} \right] \det \begin{pmatrix} \int \phi_3 & \int w \\ [\phi_3] & [w] \end{pmatrix}. \end{aligned}$$

Finally, we use (2.10) and (2.13) to eliminate $[\phi_2']$ and $h'(X)$, and we conclude

$$\begin{aligned} \frac{1}{2} E_{\lambda\lambda}(0, 0) &= \frac{b(X) \psi'(X)}{\bar{u}_x(0) \kappa^3} \det \begin{pmatrix} \int \phi_3 & \int w \\ [\phi_3] & [w] \end{pmatrix} \\ &\times \left[\left(\int_0^X \frac{\bar{u}(x) - \bar{u}(0)}{M(\bar{u}(x))} dx \right)^2 - \int_0^X \frac{dx}{M(\bar{u}(x))} \int_0^X \frac{(\bar{u}(x) - \bar{u}(0))^2}{M(\bar{u}(x))} dx \right]. \end{aligned}$$

Here, the expression in square brackets is strictly negative by Cauchy-Schwartz.

We proceed next with the expansion of $E(\lambda, \xi)$ in ξ . We begin with (2.3), computing $E(0, \xi)$ and then expanding $(e^{i\xi X} - 1)$ in powers of ξ . In this expansion, which is straightforward and omitted, the first non-zero coefficient is the coefficient of ξ^4 , and we find

$$E(0, \xi) = X^4 \xi^4 + \mathbf{O}(|\xi|^5).$$

We next consider the case $E_\lambda(0, \xi)$, for which we already have $E_\lambda(0, 0) = 0$. In addition to this we find that $E_{\lambda\xi}(0, 0) = 0$, and so we focus on the coefficient of ξ^2 . First, upon expansion of $(e^{i\xi X} - 1)$ in (2.3), we observe two ways in which we can get terms of order ξ^2 : we can multiply the expression $\frac{1}{2}X^2\xi^2$ by a term that does not involve ξ or we can have a product of $i\xi X$ with itself. We observe, however, that our condition $E_{\lambda\xi}(0, 0) = 0$ ensures that the former terms must sum to 0. (Precisely the same cancellation that leads to this identity annihilates those terms.) For the product terms, our approach will be to simplify terms prior to differentiating in λ . If we write

$$E(\lambda, \xi) = A_0(\lambda) + A_1(\lambda)\xi + A_2(\lambda)\xi^2 + \dots, \quad (2.15)$$

then

$$\begin{aligned} A_2(\lambda) = & -X^2 \left[\det \begin{pmatrix} [\phi_1] & [\phi_2 + \frac{1}{\kappa}\phi_4] \\ b(X)[\phi_1'] & b(X)[(\phi_2 + \frac{1}{\kappa}\phi_4)'] \end{pmatrix} + \det \begin{pmatrix} [\phi_1] & [\phi_3] \\ [\phi_1''] & [\phi_3''] \end{pmatrix} \right. \\ & \left. + \det \begin{pmatrix} [(b(\phi_2 + \frac{1}{\kappa}\phi_4))'] & [(b\phi_3)'] \\ [(\phi_2 + \frac{1}{\kappa}\phi_4)'] & [\phi_3'] \end{pmatrix} \right] \\ & + \frac{\lambda X^2}{\kappa M_0} \left[\det \begin{pmatrix} [\phi_1] & [\phi_4] \\ \int \phi_1 & \int \phi_4 \end{pmatrix} + \det \begin{pmatrix} [(b\phi_2)'] & [(b\phi_4)'] \\ \int \phi_2 & \int \phi_4 \end{pmatrix} + \det \begin{pmatrix} [\phi_3''] & [\phi_4''] \\ \int \phi_3 & \int \phi_4 \end{pmatrix} \right]. \end{aligned}$$

We take a λ -derivative of this expression and set $\lambda = 0$ in the result to obtain $A_2'(0) = P + Q$, where

$$\begin{aligned} P = & -X^2 \left[\det \begin{pmatrix} \partial_\lambda \phi_1 & [\phi_2 + \frac{1}{\kappa}\phi_4] \\ b(X)(\partial_\lambda \phi_1)' & b(X)[(\phi_2 + \frac{1}{\kappa}\phi_4)'] \end{pmatrix} \right. \\ & + \det \begin{pmatrix} [\phi_1] & \partial_\lambda(\phi_2 + \frac{1}{\kappa}\phi_4) \\ b(X)[\phi_1'] & b(X)(\partial_\lambda(\phi_2 + \frac{1}{\kappa}\phi_4))' \end{pmatrix} \\ & + \det \begin{pmatrix} \partial_\lambda \phi_1 & [\phi_3] \\ (\partial_\lambda \phi_1)'' & [\phi_3''] \end{pmatrix} + \det \begin{pmatrix} [\phi_1] & \partial_\lambda \phi_3 \\ [\phi_1''] & (\partial_\lambda \phi_3)'' \end{pmatrix} \\ & \left. + \det \begin{pmatrix} (b\partial_\lambda(\phi_2 + \frac{1}{\kappa}\phi_4))' & [(b\phi_3)'] \\ (\partial_\lambda(\phi_2 + \frac{1}{\kappa}\phi_4))'' & [\phi_3''] \end{pmatrix} + \det \begin{pmatrix} [(b(\phi_2 + \frac{1}{\kappa}\phi_4))'] & (b\partial_\lambda \phi_3)' \\ [(\phi_2 + \frac{1}{\kappa}\phi_4)'] & (\partial_\lambda \phi_3)'' \end{pmatrix} \right] \end{aligned}$$

and

$$Q = \frac{X^2}{\kappa M_0} \left[\det \begin{pmatrix} [\phi_1] & [\phi_4] \\ \int \phi_1 & \int \phi_4 \end{pmatrix} + \det \begin{pmatrix} [(b\phi_2)'] & [(b\phi_4)'] \\ \int \phi_2 & \int \phi_4 \end{pmatrix} + \det \begin{pmatrix} [\phi_3''] & [\phi_4''] \\ \int \phi_3 & \int \phi_4 \end{pmatrix} \right].$$

In the calculation that follows, it will be convenient to focus separately on the terms that contain differentiation in λ (specified as P) and the terms that don't (specified as Q). Upon elimination of $[\phi_1]$ by the substitution $[\phi_1] = -\frac{b'(X)}{\kappa}[w] - \frac{b(X)}{\kappa}[\phi_3]$, we find that

$$P = \frac{X^2}{\kappa \bar{u}_x(X)} \left(b(X)[w]h'(X) - [\phi_3] \int_0^X \frac{\bar{u}(x) - \bar{u}(0)}{M(\bar{u}(x))} dx \right).$$

For Q , we proceed by using (2.14) to eliminate $[\phi_3'']$ and $[\phi_4'']$. A short calculation leads to the expression

$$Q = \frac{X^2}{\kappa} \left[\frac{b(X)[\phi_2']}{M_0} \int_0^X w(x) dx - \int_0^X \frac{dx}{M(\bar{u}(x))} \int_0^X \phi_3(x; 0) dx \right].$$

Combining these observations, we conclude

$$\begin{aligned} A_2'(0) &= \frac{X^2}{\kappa \bar{u}_x(X)} \left(b(X)[w]h'(X) - [\phi_3] \int_0^X \frac{\bar{u}(x) - \bar{u}(0)}{M(\bar{u}(x))} dx \right) \\ &\quad + \frac{X^2}{\kappa} \left[\frac{b(X)[\phi_2']}{M_0} \int_0^X w(x) dx - \int_0^X \frac{dx}{M(\bar{u}(x))} \int_0^X \phi_3(x; 0) dx \right]. \end{aligned}$$

This concludes the proof of Lemma 2.1. \square

We now have the following general condition for instability of periodic waves $\bar{u}(x)$ as described in Theorem 1.4.

Lemma 2.2. *Under the assumptions of Theorem 1.4, and with $\bar{u}(x)$ as described there, instability of $\bar{u}(x)$ is implied by the condition $E_{\lambda\xi\xi}(0, 0) < 0$.*

Proof. Observing that the leading order of the eigenvalue problem (1.7) is independent of ξ , we can apply a standard perturbation argument to establish that there exists a continuous curve $\lambda_*(\xi)$ such that $\lambda_*(0) = 0$ and $E(\lambda_*(\xi), \xi) = 0$. (See [38], Sections 8 and 17.) In light of this, we need only determine the leading order behavior of $\lambda_*(\xi)$. We proceed, then, by Taylor expansion of $E(\lambda, \xi)$:

$$\begin{aligned} E(\lambda, \xi) &= E(0, 0) + \sum_{k=1}^{\infty} \frac{1}{k!} (\lambda \partial_{\bar{\lambda}} + \xi \partial_{\bar{\xi}})^k E(\bar{\lambda}, \bar{\xi}) \Big|_{(\bar{\lambda}, \bar{\xi})=(0,0)} \\ &= \frac{1}{2} E_{\lambda\lambda}(0, 0) \lambda^2 + \frac{1}{2} E_{\lambda\xi\xi}(0, 0) \lambda \xi^2 + \frac{1}{24} E_{\xi\xi\xi\xi}(0, 0) \xi^4 + \dots, \end{aligned} \quad (2.16)$$

where we have included only those terms relevant to the calculation. Expanding $\lambda_*(\xi)$ as a power series in ξ , we see immediately from the relation $E_{\xi\xi}(0,0) = 0$ that there is no term of order ξ . We take, then,

$$\lambda_*(\xi) = a_2\xi^2 + \mathbf{O}(\xi^3).$$

Upon setting $E(\lambda_*(\xi), \xi) = 0$, and matching coefficients of powers of ξ , we find that the first order is ξ^4 , and that we have

$$\frac{1}{2}E_{\lambda\lambda}(0,0)a_2^2 + \frac{1}{2}E_{\lambda\xi\xi}(0,0)a_2 + X^4 = 0,$$

from which we find

$$a_2 = \frac{-\frac{1}{2}E_{\lambda\xi\xi}(0,0) \pm \sqrt{\frac{1}{4}E_{\lambda\xi\xi}(0,0)^2 - 2E_{\lambda\lambda}(0,0)X^4}}{E_{\lambda\lambda}(0,0)}. \quad (2.17)$$

We observe that, in the event that $E_{\lambda\lambda}(0,0) < 0$, then regardless of the sign of $E_{\lambda\xi\xi}(0,0)$ we will have a branch of eigenvalues corresponding with $a_2 > 0$, and consequently we will have an interval of positive real eigenvalues. Indeed, this case appears to correspond with instability with respect to perturbations of period X . In the special case that $E_{\lambda\lambda}(0,0) = 0$, we have

$$a_2 = -\frac{2X^4}{E_{\lambda\xi\xi}(0,0)},$$

so that instability is implied by the condition $E_{\lambda\xi\xi}(0,0) < 0$. Finally, in the event that $E_{\lambda\lambda}(0,0) > 0$, there exists a solution to (2.17) with positive real part if and only if $E_{\lambda\xi\xi}(0,0) < 0$. \square

Though the sign of $E_{\lambda\xi\xi}(0,0)$ can be verified numerically for a given wave $\bar{u}(x)$, in the generality of Lemma 2.2 it is difficult to verify analytically. In the next section, we consider a symmetric setting in which direct evaluation is possible.

2.1. The Symmetric Case. In this section, we consider the case in which F —following the transformation to standard form—is an even double well function. For each such F , the binodal values are $u_1 = -u_2$ and u_2 , and for each $u_* \in (0, u_2)$ there exists a unique periodic solution with minimum $-u_*$ and maximum u_* . (These are by no means the only periodic solutions for such F ; see Theorem 1.4.) We refer to this as the symmetric case.

We can observe from (2.1) that we have considerable symmetry in this case. In particular, $\bar{u}(x)$ solves

$$\bar{u}_x(x) = \pm \sqrt{\frac{2}{\kappa}(F(\bar{u}(x)) - F(u_*))}; \quad \bar{u}(0) = 0. \quad (2.18)$$

(In the case that F has the polynomial form $F(u) = au^4 - bu^2$, this equation can be solved exactly for the symmetric $\bar{u}(x)$ in terms of appropriate Jacobi elliptic functions.) Letting X denote the period of $\bar{u}(x)$, we can describe $\bar{u}(x)$ as follows: restricted to the interval $[0, X/2]$, $\bar{u}(x)$ is even about the midpoint $X/4$, while on $[0, X]$ $\bar{u}(x)$ is odd about the midpoint $X/2$. The following lemma, which can be proven by direct calculation from the general expressions, follows from these symmetries.

Lemma 2.3. *Suppose the assumptions of Theorem 1.5 hold, and that $\bar{u}(x)$ is defined as there, with additionally $\bar{u}(0) = 0$. In this case, the basis functions $\phi_1(x; 0)$ and $\phi_3(x; 0)$ are both periodic with period X , and the following relations hold:*

$$\bar{u}_{xx}(0) = 0; \quad b'(0) = F'''(\bar{u}(0))\bar{u}_x(0) = 0; \quad \int_0^X \bar{u}(x)dx = 0; \quad \int_0^X \psi(x)dx = 0,$$

with additionally

$$\int_0^X \phi_3(x; 0)dx = 4 \int_0^{X/4} \frac{\bar{u}(X/4)^2 - \bar{u}(x)^2}{\bar{u}_x^2} dx - \frac{4K\bar{u}(X/4)^2}{\bar{u}_{xx}(X/4)}; \quad \psi(X) = \frac{4K\bar{u}_x(0)}{\bar{u}_{xx}(X/4)},$$

where

$$K := \lim_{x \rightarrow \frac{X}{4}^-} \left\{ \bar{u}_{xx}(X/4) \int_0^x \frac{dy}{\bar{u}_y^2} + \frac{1}{\bar{u}_x(x)} \right\}.$$

Applying Lemma 2.3 to $E_{\lambda\lambda}(0, 0)$ and $A'_2(0)$, we find

$$\frac{1}{2}E_{\lambda\lambda}(0, 0) = -\frac{4KX}{\kappa^2\bar{u}_{xx}(X/4)} \int_0^X \bar{u}(x)^2 dx \int_0^X \phi_3(x; 0)dx, \quad (2.19)$$

and additionally

$$\frac{1}{2}E_{\lambda\xi\xi}(0, 0) = A'_2(0) = \frac{4KX^2}{\kappa\bar{u}_{xx}(X/4)} \int_0^X \bar{u}(x)^2 dx - \frac{X^3}{\kappa} \int_0^X \phi_3(x; 0)dx. \quad (2.20)$$

Our first consideration will be the stability index associated with perturbations whose period is the same as that of the wave $\bar{u}(x)$. Recalling that $E_\lambda(0, 0) = 0$, and following [35], we define

$$\Gamma := \operatorname{sgn} [E_{\lambda\lambda}(0, 0)] \times \operatorname{sgn} \left[\lim_{\mathbb{R} \ni \lambda \rightarrow \infty} E(\lambda; 0) \right]. \quad (2.21)$$

In the event that $\Gamma = -1$, there must exist a positive real value of λ so that $E(\lambda, 0) = 0$; that is, $\Gamma = -1$ corresponds with spectral instability with respect to perturbations that are periodic with the same period as the wave $\bar{u}(x)$. On the other hand, if $\Gamma = +1$ we know only that $E(\lambda; 0)$ does not cross

0 an odd number of times, and consequently we cannot generally conclude anything about spectral stability.

Lemma 2.4. *For equation (1.1), under conditions (H0)–(H2), and for identity mobility $M(u) \equiv 1$, we have*

$$\operatorname{sgn} \left[\lim_{\mathbb{R} \ni \lambda \rightarrow \infty} E(\lambda; 0) \right] = +1,$$

where $E(\lambda, \xi)$ is as defined in (1.11).

Proof of Lemma 2.4. The proof of this lemma is straightforward, based on standard ODE asymptotics, and so we only briefly sketch the argument. We mention at the outset that there is a more elegant method outlined in [35], but the method is less standard and more details would be required for completeness.

We proceed by the rescaling argument of [19]. More precisely, we change variables in (1.4) to the stretched coordinate $y = |\lambda|^{\frac{1}{4}}x$, for which (1.4) becomes

$$\begin{aligned} -\kappa \varphi''''(y) &= \frac{\lambda}{|\lambda|} \varphi + |\lambda|^{-1} a'(y/|\lambda|^{1/4}) \varphi + |\lambda|^{-\frac{3}{4}} (-b'(y/|\lambda|^{1/4}) \\ &\quad + a(y/|\lambda|^{1/4})) \varphi' - |\lambda|^{-\frac{1}{2}} b(y/|\lambda|^{1/4}) \varphi''(y), \end{aligned}$$

where $\varphi(y) := \phi(y)$. We write this equation as a first-order system

$$W' = A(\bar{\lambda})W + \mathbf{O}(|\lambda|^{-\frac{1}{2}})W, \quad (2.22)$$

where $\bar{\lambda} := \lambda/|\lambda|$,

$$A(\bar{\lambda}) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\bar{\lambda}/\kappa & 0 & 0 & 0 \end{pmatrix},$$

and we remark that it is important to the argument that M is constant so that the order term is of size $|\lambda|^{-1/2}$. The eigenvalues of $A(\bar{\lambda})$ are the four roots of $\mu^4 = -\bar{\lambda}/\kappa$, and we order them as $\mu_1 = (-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) \sqrt[4]{(\bar{\lambda}/\kappa)}$, $\mu_2 = (-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \sqrt[4]{(\bar{\lambda}/\kappa)}$, $\mu_3 = (\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) \sqrt[4]{(\bar{\lambda}/\kappa)}$, and $\mu_4 = (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \sqrt[4]{(\bar{\lambda}/\kappa)}$. Looking for solutions to (2.22) of the form $W = e^{\mu_3 x} Z$, we find that Z satisfies the integral equation

$$Z(y) = e^{(A(\bar{\lambda}) - \mu_3(\bar{\lambda}))y} Z(0) + \int_0^y e^{(A(\bar{\lambda}) - \mu_3(\bar{\lambda}))(y-z)} \mathbf{O}(|\lambda|^{-1/2}) Z(z) dz. \quad (2.23)$$

We observe now that we only require Z at the value $y = \sqrt[4]{|\lambda|}X$ (so that solutions to (1.4) can be evaluated at $x = X$), and we can carry out the integral over this bounded region by taking a supremum of the integrand. In this way, a standard contraction mapping can be constructed for $|\lambda|$ sufficiently large. In order to isolate a solution associated with μ_3 , we take $Z(0)$ to be the eigenvector of A associated with μ_3 , namely $Z(0) = V_3 = (1, \mu_3, \mu_3^2, \mu_3^3)^{\text{tr}}$. In this way, we have

$$Z(\sqrt[4]{|\lambda|}X) = V_3 + \mathbf{O}(|\lambda|^{-\frac{1}{4}}).$$

Upon returning to our original coordinates (and proceeding similarly in the additional cases), we find

$$\phi_k(X; \lambda) = e^{\mu_k(\lambda)X} (V_k + \mathbf{O}(|\lambda|^{-\frac{1}{4}})), \quad (2.24)$$

where $V_k = (1, \mu_k, \mu_k^2, \mu_k^3)^{\text{tr}}$ is the eigenvector associated with μ_k . If we set $\Phi(x; \lambda)$ to be a standard fundamental solution constructed from the ϕ_k , then we can write the monodromy matrix as $M(\lambda) = \Phi(X; \lambda)\Phi(0; \lambda)^{-1}$. In particular, we now have

$$\begin{aligned} E(\lambda; 0) &= \det(\Phi(X; \lambda)\Phi(0; \lambda)^{-1} - \Phi(0; \lambda)\Phi(0; \lambda)^{-1}) \\ &= \det((\Phi(X; \lambda) - \Phi(0; \lambda))\Phi(0; \lambda)^{-1}) \\ &= \frac{\det(\Phi(X; \lambda) - \Phi(0; \lambda))}{\det \Phi(0; \lambda)} = \prod_{k=1}^4 (e^{\mu_k(\lambda)X} - 1) + \mathbf{O}(|\lambda|^{-\frac{1}{4}}). \end{aligned}$$

The claimed result is an immediate consequence of this formula. \square

It is clear from (2.21) and Lemma 2.4 that the stability index associated with $\xi = 0$ is simply

$$\Gamma = \text{sgn}[E_{\lambda\lambda}(0, 0)].$$

According to (2.19), this can only be negative if

$$\frac{4KX}{\kappa^2 \bar{u}_{xx}(X/4)} \int_0^X \bar{u}(x)^2 dx \int_0^X \phi_3(x; 0) dx > 0,$$

which since $\bar{u}_{xx}(X/4) < 0$ requires $K \int_0^X \phi_3(x; 0) dx < 0$.

Lemma 2.5. *Under the assumptions of Lemma 2.3, and for the wave $\bar{u}(x)$ described there, positivity of the stability index Γ is implied by positivity of K .*

Proof We need only combine (2.19) with Lemma 2.3. In particular, if K is positive, then so is $\int_0^X \phi_3(x; 0) dx$. \square

In general, the sign of K is easily checked for a given function F . In addition to this, we can prove the following result on the behavior of K for periodic waves whose maximum and minimum values are near the binodal values.

Lemma 2.6. *Under the assumptions of Lemma 2.3, there exists $\epsilon > 0$ so that, for $u_2 - u_* < \epsilon$, we have $K > 0$.*

Proof. We begin by observing that K can be written in the alternative form

$$K = \frac{1}{\bar{u}_x(0)} + \int_0^{\frac{X}{4}} \frac{\bar{u}_{xx}(X/4) - \bar{u}_{xx}(x)}{\bar{u}_x(x)^2} dx. \quad (2.25)$$

According to (2.1) and (2.18) we can re-write this as

$$K = \frac{1}{\sqrt{-(2/\kappa)F(u_*)}} + \frac{1}{2} \int_0^{\frac{X}{4}} \frac{(F'(u_*) - F'(\bar{u}(x)))}{(F(\bar{u}(x)) - F(u_*))} dx.$$

Finally, we make the change of variables $y = \bar{u}(x)$ to write

$$K(u_*) = \sqrt{\frac{\kappa}{2}} \left[\frac{1}{\sqrt{-F(u_*)}} - \frac{1}{2} \int_0^{u_*} \frac{F'(y) - F'(u_*)}{(F(y) - F(u_*))^{3/2}} dy \right].$$

Letting now u_s denote the positive spinodal value (i.e., $F''(u_s) = 0$), we observe that, for $u_* > u_s$, there exists some value $u_{**} < u_s$ so that $F'(u_{**}) = F'(u_*)$ and, for $y \in [u_{**}, u_*]$, there holds $F'(y) - F'(u_*) \leq 0$, in which case we have

$$\int_0^{u_*} \frac{F'(y) - F'(u_*)}{(F(y) - F(u_*))^{3/2}} dx \leq \int_0^{u_{**}} \frac{F'(y) - F'(u_*)}{(F(y) - F(u_*))^{3/2}} dx.$$

For $y \in [0, u_{**}]$, $(F(y) - F(u_*))$ is bounded below by $F(u_{**}) - F(u_*)$. We have, then, the estimate

$$\begin{aligned} K(u_*) &\geq \sqrt{\frac{\kappa}{2}} \left[\frac{1}{\sqrt{-F(u_*)}} - \frac{1}{2} \frac{1}{(F(u_{**}) - F(u_*))^{3/2}} \int_0^{u_{**}} F'(y) - F'(u_*) dy \right] \\ &= \sqrt{\frac{\kappa}{2}} \left[\frac{1}{\sqrt{-F(u_*)}} - \frac{1}{2} \frac{F(u_{**}) - u_{**}F'(u_*)}{(F(u_{**}) - F(u_*))^{3/2}} \right]. \end{aligned}$$

In the limit as $u_* \rightarrow u_2$, this approaches the limit

$$\frac{1}{\sqrt{-F(u_2)}} > 0. \quad \square$$

We conclude from this that, at least for periodic waves whose maximum and minimum values are near the binodal values, we cannot rule out the

possibility of spectral stability with respect to perturbations that are periodic with the same period as the wave. Indeed, numerical calculations seem to indicate that such waves are stable with respect to such perturbations.

We are now prepared to state a general lemma on the instability of symmetric periodic solutions.

Lemma 2.7. *Under the assumptions of Lemma 2.3, and for the wave $\bar{u}(x)$ described there, instability of $\bar{u}(x)$ is implied by positivity of $\int_0^X \phi_3(x; 0)dx$.*

Proof. Proceeding as in the proof of Lemma 2.2, we need only show that positivity of $\int_0^X \phi_3(x; 0)dx$ implies that $E_{\lambda\xi\xi}(0, 0) < 0$. We begin by observing from (2.19) that, with $\int_0^X \phi_3(x; 0)dx > 0$, the only way we can have $E_{\lambda\lambda}(0, 0) > 0$ is if $K > 0$. In this case, we can see additionally from (2.20) that $E_{\lambda\xi\xi}(0, 0) < 0$. Moreover, we can compute

$$\begin{aligned} & \frac{1}{4}E_{\lambda\xi\xi}(0, 0)^2 - 2E_{\lambda\lambda}(0, 0)X^4 = \\ & \left(\frac{X^2b(X)w(X)\psi'(X)}{\kappa^2\bar{u}_x(0)} \int_0^X \bar{u}(x)^2 dx - \frac{X^3}{\kappa} \int_0^X \phi_3(x; 0)dx \right)^2 \\ & + \frac{4b(X)\phi'(X)\omega(X)X^5}{\kappa^3\bar{u}_x(0)} \int_0^X \phi_3(x; 0)dx \int_0^X \bar{u}(x)^2 dx \\ & = \left(\frac{X^2b(X)w(X)\psi'(X)}{\kappa^2\bar{u}_x(0)} \int_0^X \bar{u}(x)^2 dx + \frac{X^3}{\kappa} \int_0^X \phi_3(x; 0)dx \right)^2, \end{aligned}$$

which is strictly positive for K and $\int_0^X \phi_3(x; 0)dx$ both positive. The significance of this last observation is that it verifies that we will have a real eigenvalue. \square

Theorem 1.5 can now be established by the following lemma on the positivity of the integral $\int_0^X \phi_3(x; 0)dx$.

Lemma 2.8. *Let the assumptions of Lemma 2.3 hold, and suppose additionally that for $\bar{u}(x)$ as described there we have that $F'''(u) \geq 0$ for all $u \in (0, u_*)$. Then*

$$\int_0^X \phi_3(x; 0)dx > 0.$$

Proof. According to Lemma 2.3, we need only show positivity of the expression

$$\int_0^{\frac{X}{4}} \frac{\bar{u}(X/4)^2 - \bar{u}(x)^2}{\bar{u}_x^2} dx - \frac{K\bar{u}(\frac{X}{4})^2}{\bar{u}_{xx}(\frac{X}{4})}.$$

Upon substitution of the relation (2.25) for K , we find that this last expression is equivalent to

$$-\frac{1}{\bar{u}_{xx}(X/4)} \left[\int_0^{\frac{X}{4}} \frac{\bar{u}_{xx}(X/4)\bar{u}(x)^2 - \bar{u}(X/4)^2\bar{u}_{xx}(x)}{\bar{u}_x(x)^2} dx + \frac{\bar{u}(X/4)^2}{\bar{u}_x(0)} \right]. \quad (2.26)$$

Observing that $-1/\bar{u}_{xx}(X/4) > 0$, we need only establish positivity of the expression in square brackets. Proceeding similarly as in the proof of Lemma 2.6, we set $y = \bar{u}(x)$, and we find that this expression (in square brackets in (2.26)) is equivalent to

$$\sqrt{\frac{\kappa}{2}} \left[\frac{1}{2} \int_0^{u_*} \frac{y^2 F'(u_*) - u_*^2 F'(y)}{(F(y) - F(u_*))^{3/2}} dy + \frac{u_*^2}{\sqrt{-F(u_*)}} \right]. \quad (2.27)$$

Finally, we observe that for functions F so that $F'''(y) \geq 0$ for $y \in [0, u_*]$, there holds

$$H(y) := y^2 F'(u_*) - u_*^2 F'(y) \geq 0,$$

from which positivity of (2.27) follows. In order to understand this final inequality, we observe that $F''(0) < 0$ for double-well F in standard form, and consequently $H(y)$ is positive for sufficiently small values of y . Moreover, for $F'''(y) \geq 0$, $H(y)$ is concave down for all $y \in (0, u_*)$, while additionally $H(u_*) = 0$. This eliminates the possibility of a zero of H between 0 and u_* , and thus it must be positive on the entire interval. \square

As discussed in the introduction, this condition on $F'''(u)$ is not a necessary condition for instability. Indeed, it is clear from our proof that significantly less is required of F for (2.27) to be positive. Our choice of specifying Theorem 1.5 in this form is motivated by the observation that this condition holds for the forms of F that make sense physically (see [1, 3]).

3. TRANSITION AND REVERSAL WAVES

In this section, we prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. We begin by observing that, for the eigenvalue problem (1.4), with $\bar{u}(x)$ a transition wave, if we are away from the essential spectrum the eigenfunctions $\phi(x; \lambda)$ must decay along with the derivatives at an exponential rate. Taking, then, $\lambda \neq 0$ away from the essential spectrum, we can integrate (1.4) to obtain the relation

$$\int_{-\infty}^{+\infty} \phi(x; \lambda) dx = 0, \quad (3.1)$$

for any L^2 eigenfunction $\phi(x; \lambda)$. This justifies defining the exponentially decaying integrated variable

$$w(x; \lambda) := \int_{-\infty}^x \phi(y; \lambda) dy,$$

which solves the eigenvalue problem

$$M(\bar{u}(x))(F''(\bar{u}(x))w_x - \kappa w_{xxx})_x = \lambda w. \quad (3.2)$$

In particular, we observe that away from the essential spectrum (also away from the eigenvalue $\lambda = 0$ embedded in the essential spectrum), $\phi(x; \lambda)$ is an L^2 eigenfunction of (1.4) if and only if $w(x; \lambda)$ is an L^2 eigenfunction of (3.2). In this way, for $\lambda \neq 0$, the point spectrum of (1.4) corresponds precisely with the point spectrum of (3.2).

For (3.2), we divide by $M(\bar{u}(x))$ and observe that the resulting equation,

$$(F''(\bar{u}(x))w_x - \kappa w_{xxx})_x = \lambda \frac{1}{M(\bar{u}(x))} w, \quad (3.3)$$

has a self-adjoint linear operator on the left-hand side. Consequently, the eigenvalues of both (1.4) and (3.2) are all real. Upon multiplication of (3.3) by w , integration over \mathbb{R} , and an application of integration by parts, we obtain the relation

$$- \int_{-\infty}^{+\infty} w_x H w_x dx = \lambda \int_{-\infty}^{+\infty} \frac{1}{M(\bar{u}(x))} w^2, \quad (3.4)$$

where H is the Schrödinger type operator

$$H := -\kappa \partial_{xx}^2 + F''(\bar{u}(x)).$$

The operator H has an eigenvalue at $\lambda = 0$ associated with an eigenfunction $\bar{u}_x(x)$ that is never 0, and consequently $\lambda = 0$ is the lowest eigenvalue of H . It follows from the spectral theorem that H is a positive operator, and we conclude from (3.4) that $\lambda \leq 0$. (A more detailed proof of the positivity of H is given in [1]; see also the relevant discussion on pages 1246–1247 of [40].)

For the eigenvalue at $\lambda = 0$, there remains to show that $\bar{D}''(0) \neq 0$. According to Lemma 2.5 of [20] (restated in the current notation), we have

$$\bar{D}''(0) = -\frac{2(u_+ - u_-)}{\kappa M(\bar{u}(0))} \det \begin{pmatrix} \phi_1^-(x; 0) & \bar{u}_x(x) & \phi_2^+(x; 0) \\ \phi_1^{-\prime}(x; 0) & \bar{u}_{xx}(x) & \phi_2^{+\prime}(x; 0) \\ \phi_1^{-\prime\prime}(x; 0) & \bar{u}_{xxx}(x) & \phi_2^{+\prime\prime}(x; 0) \end{pmatrix} \Big|_{x=0}. \quad (3.5)$$

We must show that the functions $\phi_1^-(x; 0)$, $\bar{u}_x(x)$, and $\phi_2^+(x; 0)$ are linearly independent. If not, then we can write $\phi_1^-(x; 0)$ as a linear combination of

\bar{u}_x and ϕ_2^+ for $x > 0$ and conclude from this that $\phi_1^-(x; 0)$ is bounded at both $\pm\infty$. We will proceed by showing that in fact neither $\phi_1^-(x; 0)$ nor $\phi_2^+(x; 0)$ can be bounded at both $\pm\infty$.

According to Lemma 2.1 of [20],

$$\begin{aligned}\partial_x^k \phi_1^-(x; \lambda) &= e^{\mu_3^-(\lambda)x} (\mu_3^-(\lambda))^k + \mathbf{O}(e^{-\alpha|x|}) \\ \partial_x^k \phi_2^+(x; \lambda) &= e^{\mu_2^+(\lambda)x} (\mu_2^+(\lambda))^k + \mathbf{O}(e^{-\alpha|x|}),\end{aligned}\tag{3.6}$$

for $k = 0, 1, 2, 3$, where

$$\mu_3^-(\lambda) = \sqrt{\frac{b_- - \sqrt{b_-^2 - 4\lambda c_-}}{2c_-}}; \quad \mu_2^+(\lambda) = -\sqrt{\frac{b_+ - \sqrt{b_+^2 - 4\lambda c_+}}{2c_+}},\tag{3.7}$$

and with $b_\pm := M(u_\pm)F''(u_\pm)$ and $c_\pm := \kappa M(u_\pm)$. In this way, we see that for each of the solutions $\phi_1^-(x; 0)$, $\bar{u}_x(x)$, and $\phi_2^+(x; 0)$, we can integrate (1.4) to obtain

$$(F''(\bar{u})\phi - \kappa\phi_{xx})_x = 0,\tag{3.8}$$

where $M(\bar{u})$ has been divided out. We must show, then, that any three solutions of (3.8) with the asymptotic behavior of $\phi_1^-(x; 0)$, $\bar{u}_x(x)$, and $\phi_2^+(x; 0)$ must necessarily be linearly independent. Observing that \bar{u}_x is one solution of $-\kappa\phi'' + F''(\bar{u})\phi$, and that by reduction of order a second solution is

$$\phi_A(x) := \bar{u}_x(x) \int_0^x \frac{dy}{\bar{u}_y(y)^2},$$

we can construct a third, linearly independent solution of (3.8) as the variation of parameters solution of

$$F''(\bar{u})\phi - \kappa\phi_{xx} = 1.$$

That is, we have a third solution

$$\phi_B(x) = \bar{u}_x(x) \int_0^x \frac{\bar{u}(y)}{\bar{u}_y(y)^2} dy.$$

By construction, $\bar{u}_x(x)$ decays at an exponential rate at both $\pm\infty$, and $\phi_A(x)$ grows at an exponential rate at both $\pm\infty$. Likewise, $\phi_B(x)$ grows at an exponential rate at ∞ unless $u_+ = 0$, in which case it grows at an exponential rate at $-\infty$. If we now look for a solution $\phi_0(x)$ that does not grow at either $\pm\infty$, we have

$$\phi_0(x) = \alpha_1 \bar{u}_x(x) + \alpha_2 \phi_A(x) + \alpha_3 \phi_B(x).$$

In order to keep $\phi_0(x)$ from growing at $-\infty$, we require $\alpha_2 + u_- \alpha_3 = 0$, while in order to keep $\phi_0(x)$ from growing at $+\infty$ we require $\alpha_2 + u_+ \alpha_3 = 0$. Since $u_- \neq u_+$, we conclude that $\alpha_2 = \alpha_3 = 0$, and consequently $\phi_0(x)$ is a constant multiple of $\bar{u}_x(x)$. \square

Proof of Theorem 1.3. For reversal waves, we proceed by defining and computing an appropriate stability index (see, for example, [6, 19]), though we note at the outset that Theorem 1.3 can also be obtained by applying min-max methods to an appropriate integrated equation.

As in the case of transition fronts, the Evans function for reversals is given by

$$\bar{D}(\rho) = W(\phi_1^-, \phi_2^-, \phi_1^+, \phi_2^+) \Big|_{x=0}, \quad (3.9)$$

where the ϕ_k^\pm are the asymptotically decaying solutions of the eigenvalue problem (1.4) (as described in the paragraph following (1.4)), and following [20], we take the convention

$$\phi_1^+(x; 0) = \bar{u}_x(x) = \phi_2^-(x; 0). \quad (3.10)$$

The remaining solutions $\phi_1^-(x; 0)$ and $\phi_2^+(x; 0)$ both solve the third-order equation

$$(F''(\bar{u}(x)) - \kappa \phi_{xx})_x = 0, \quad (3.11)$$

which has three linearly independent solutions: \bar{u}_x ,

$$\phi_A(x) = \bar{u}_x(x) \begin{cases} \int_{-\bar{x}}^x \frac{dy}{\bar{u}'(y)^2} & x < 0 \\ \int_{\bar{x}}^x \frac{dy}{\bar{u}'(y)^2} & x > 0, \end{cases}$$

where \bar{x} is the unique positive value so that $\bar{u}_{xx}(\bar{x}) = 0$, and where $\phi_A(0)$ can be defined to make $\phi_A(x)$ continuous, and

$$\phi_B(x) = \bar{u}_x(x) \begin{cases} \int_{-\bar{x}}^x \frac{\bar{u}(y)}{\bar{u}'(y)^2} dy & x < 0 \\ \int_{\bar{x}}^x \frac{\bar{u}(y)}{\bar{u}'(y)^2} dy & x > 0. \end{cases}$$

If we now expand $\phi_1^-(x; 0)$ and $\phi_2^+(x; 0)$ as linear combinations of $\bar{u}_x(x)$, $\phi_A(x)$, and $\phi_B(x)$, and insist on scalings such that

$$\lim_{x \rightarrow -\infty} \phi_1^-(x; 0) = \lim_{x \rightarrow +\infty} \phi_2^+(x; 0) = 1,$$

we find $\phi_1^-(x; 0) = \phi_2^+(x; 0)$ and

$$\phi_1^-(x; 0) = -\frac{F''(u_\infty)}{\kappa} \bar{u}_x(x) \begin{cases} \int_{-\bar{x}}^x \frac{\bar{u}(y) - u_\infty}{\bar{u}'(y)^2} dy & x < 0 \\ \int_{\bar{x}}^x \frac{\bar{u}(y) - u_\infty}{\bar{u}'(y)^2} dy & x > 0, \end{cases} \quad (3.12)$$

where our notation here is

$$\bar{u}(0) = u_0; \quad \lim_{x \rightarrow \pm\infty} \bar{u}(x) = u_\infty;$$

i.e., u_∞ is either u_3 or u_4 (q.v. the statement of Theorem 1.3) depending upon whether the reversal is up or down.

The identities $\phi_1^-(x; 0) = \phi_2^+(x; 0)$ and $\phi_2^-(x; 0) = \phi_1^+(x; 0)$ immediately give $\bar{D}(0) = 0$ and $\bar{D}'(0) = 0$. Moreover, by combining these identities with the analyticity of $\phi_1^+(x; \rho)$ and $\phi_2^-(x; \rho^2)$ in ρ^2 we can conclude $\bar{D}''(0) = 0$, a degeneracy at $\rho = 0$. In this way, the direction of $\bar{D}(\rho)$ as ρ moves away from 0 is determined by the first non-zero derivative

$$\begin{aligned} \bar{D}'''(0) &= 3W(\partial_\rho \phi_1^-, \partial_{\rho\rho}^2(\phi_2^- - \phi_1^+), \bar{u}_x, \phi_2^+) \Big|_{x=0} \\ &\quad + 3W(\phi_1^-, \partial_{\rho\rho}^2(\phi_2^- - \phi_1^+), \bar{u}_x, \partial_\rho \phi_2^+) \Big|_{x=0}. \end{aligned} \quad (3.13)$$

In order to evaluate these determinants, we will employ a technical lemma that can be established by direct calculation using techniques from the proof of Lemma 3.2 of [21] and Lemma 3.3 of [22].

Lemma 3.1. *Let T denote the linear operator*

$$T := -c(x)\partial_{xxx}^3 + b(x)\partial_x - a(x),$$

where

$$\begin{aligned} c(x) &= \kappa M(\bar{u}(x)), \quad b(x) = M(\bar{u}(x))F''(\bar{u}(x)) \\ a(x) &= -M(\bar{u}(x))F'''(\bar{u}(x))\bar{u}_x(x). \end{aligned}$$

Under the assumptions of Theorem 1.3, the solutions $\phi_k^\pm(x; 0)$ of the eigenvalue problem (1.4) satisfy the following relations:

- (i) $T\phi_k^\pm(x; 0) \equiv 0, \quad k = 1, 2$
- (ii) $T\frac{\partial\phi_1^-}{\partial\rho}(x; 0) = b_\infty^{3/2}$
- (iii) $T\frac{\partial\phi_2^+}{\partial\rho}(x; 0) = -b_\infty^{3/2}$
- (iv) $T\frac{\partial^2\phi_2^-}{\partial\rho\rho}(x; 0) = 2b_\infty^2(\bar{u}(x) - u_\infty)$
- (v) $T\frac{\partial^2\phi_1^+}{\partial\rho\rho}(x; 0) = 2b_\infty^2(\bar{u}(x) - u_\infty)$
- (vi) $W(\phi_1^-, \bar{u}_x)(x; 0) = W(\phi_2^+, \bar{u}_x)(x; 0) = \frac{F''(u_\infty)}{\kappa}(\bar{u}(x) - u_\infty).$

Employing Lemma 3.1, we find that for the first summand in (3.13) we have

$$\begin{aligned}
& W(\partial_\rho \phi_1^-, \partial_{\rho\rho}^2(\phi_2^- - \phi_1^+), \bar{u}_x, \phi_2^+)(x; 0) = \\
& \det \begin{pmatrix} \partial_\rho \phi_1^- & \partial_{\rho\rho}^2(\phi_2^- - \phi_1^+) & \bar{u}' & \phi_2^+ \\ (\partial_\rho \phi_1^-)' & \partial_{\rho\rho}^2(\phi_2^- - \phi_1^+)' & \bar{u}'' & \phi_2^{+'} \\ (\partial_\rho \phi_1^-)'' & \partial_{\rho\rho}^2(\phi_2^- - \phi_1^+)'' & \bar{u}''' & \phi_2^{+''} \\ -\frac{b_\infty^{3/2}}{c(0)} & 0 & 0 & 0 \end{pmatrix} \\
& = \frac{b_\infty^{3/2}}{c(x)} W(\partial_{\rho\rho}^2(\phi_2^- - \phi_1^+), \bar{u}', \phi_2^+) \\
& = \frac{b_\infty^{3/2}}{c(x)} W(\partial_{\rho\rho}^2 \phi_2^-, \bar{u}', \phi_2^+) - \frac{b_\infty^{3/2}}{c(x)} W(\partial_{\rho\rho}^2 \phi_1^+, \bar{u}', \phi_2^+).
\end{aligned} \tag{3.14}$$

Observing the limit

$$\lim_{x \rightarrow \infty} W(\partial_{\rho\rho}^2 \phi_1^+, \bar{u}', \phi_2^+)(x; 0) = 0,$$

we can write

$$W(\partial_{\rho\rho}^2 \phi_1^+, \bar{u}', \phi_2^+)(x; 0) = - \int_x^\infty \frac{d}{dy} W(\partial_{\rho\rho}^2 \phi_1^+, \bar{u}', \phi_2^+)(y; 0) dy.$$

Here,

$$\begin{aligned}
\frac{d}{dx} W(\partial_{\rho\rho}^2 \phi_1^+, \bar{u}', \phi_2^+)(x; 0) &= \det \begin{pmatrix} \partial_{\rho\rho}^2 \phi_1^+ & \bar{u}' & \phi_2^+ \\ (\partial_{\rho\rho}^2 \phi_1^+)' & \bar{u}'' & \phi_2^{+'} \\ -\frac{2b_\infty^2(\bar{u}(x) - u_\infty)}{c(x)} & 0 & 0 \end{pmatrix} \\
&= -\frac{2b_\infty^2(\bar{u}(x) - u_\infty)}{c(x)} W(\bar{u}, \phi_2^+) = \frac{2b_\infty^2 F''(u_\infty)}{\kappa c(x)} (\bar{u}(x) - u_\infty)^2,
\end{aligned}$$

where in this calculation we have used (v) and (vi) from Lemma 3.1. We conclude

$$W(\partial_{\rho\rho}^2 \phi_1^+, \bar{u}', \phi_2^+)(x; 0) \Big|_{x=0} = -\frac{2M(u_\infty)^2 F''(u_\infty)^3}{\kappa^2} \int_0^\infty \frac{(\bar{u}(x) - u_\infty)^2}{M(\bar{u}(x))} dx. \tag{3.15}$$

Proceeding similarly for the first Wronskian in the last line of (3.14), we find

$$W(\partial_{\rho\rho}^2 \phi_2^-, \bar{u}', \phi_2^+)(x; 0) \Big|_{x=0} = \frac{2M(u_\infty)^2 F''(u_\infty)^3}{\kappa^2} \int_{-\infty}^0 \frac{(\bar{u}(x) - u_\infty)^2}{M(\bar{u}(x))} dx,$$

so that

$$\begin{aligned} W(\partial_\rho \phi_1^-, \partial_{\rho\rho}^2(\phi_2^- - \phi_1^+), \bar{u}_x, \phi_2^+)(x; 0) \Big|_{x=0} = \\ \frac{2M(u_\infty)^{7/2} F''(u_\infty)^{9/2}}{\kappa^3 M(u_0)} \int_{-\infty}^{+\infty} \frac{(\bar{u}(x) - u_\infty)^2}{M(\bar{u}(x))} dx. \end{aligned} \quad (3.16)$$

An equivalent contribution can similarly be shown to arise from the second summand in the expression for $\bar{D}'''(0)$, and we conclude

$$\bar{D}'''(0) = \frac{12M(u_\infty)^{7/2} F''(u_\infty)^{9/2}}{\kappa^3 M(u_0)} \int_{-\infty}^{+\infty} \frac{(\bar{u}(x) - u_\infty)^2}{M(\bar{u}(x))} dx.$$

We note in particular that $\bar{D}'''(0) > 0$. We are now in a position to define an appropriate stability index for this problem.

Definition 3.1. *The stability index for a reversal wave $\bar{u}(x)$ is defined as*

$$\Gamma = \text{sgn} \bar{D}'''(0) \cdot \lim_{\lambda \rightarrow \infty} \text{sgn} D(\lambda).$$

Clearly, the condition $\Gamma = -1$ guarantees the existence of a positive real eigenvalue, and since we have $\bar{D}'''(0) > 0$, we need only understand the limit.

For λ real and sufficiently large, the behavior of solutions of the eigenvalue problem (1.4) are governed by the fourth-order term, and we can proceed as in [6], in which the authors analyze thin-film equations of the general form

$$u_t + f(u)_x = (b(u)u_x)_x - (c(u)u_{xxx})_x$$

(see also [23, 24]). In order to state the salient result from [6], we first observe that the eigenvalue problem (1.4) can be written as a first-order system of four equations, with components $W_1 = \phi$, $W_2 = \phi'$, $W_3 = \phi''$, and $W_4 = \phi'''$, and that this system can be written as $W' = \mathbb{A}(x; \lambda)W$, and also in the asymptotic form

$$W' = \mathbb{A}_\infty(\lambda)W + E(x; \lambda)W, \quad (3.17)$$

where

$$\mathbb{A}_\infty(\lambda) = \lim_{x \rightarrow \infty} \mathbb{A}(x; \lambda), \quad (3.18)$$

and $E(x; \lambda)$ decays at an exponential rate in $|x|$ as $|x| \rightarrow \infty$. We are now in a position to adapt Proposition 2.6 from [6] to the current setting. (For ease of reference, we adopt the notation of [6].)

Lemma 3.2. *Let \bar{S}^+ denote the subspace of solutions of $W' = \mathbb{A}_\infty(\lambda)W$ that decay at $+\infty$, and let \bar{U}^- denote the subspace of solutions of $W' = \mathbb{A}_\infty(\lambda)W$ that decay at $-\infty$. In addition, let V_j^+ , $j = 1, 2$, denote a choice of basis*

vectors for $\bar{\mathcal{S}}^+$, and let V_j^- , $j = 3, 4$, denote a choice of basis vectors for $\bar{\mathcal{U}}^-$. Then for $\lambda > 0$ real and sufficiently large

$$\operatorname{sgn} D(\lambda) = \operatorname{sgn} \left\{ \det \begin{pmatrix} V_{11}^+ & V_{21}^+ \\ V_{12}^+ & V_{22}^+ \end{pmatrix} \det \begin{pmatrix} V_{31}^- & V_{41}^- \\ V_{32}^- & V_{42}^- \end{pmatrix} \right\} \Big|_{\lambda=0}.$$

Here, the V_{jk}^\pm are components of the vector V_j^\pm .

According to Lemma 3.2, the sign of $D(\lambda)$ for large values of λ is determined by the bases of $\bar{\mathcal{S}}^+$ and $\bar{\mathcal{U}}^-$ at $\lambda = 0$. By our choice of scaling convention $\phi_1^-(x; 0)$ and $\phi_2^+(x; 0)$ approach 1 as x approaches $-\infty$ and $+\infty$. For the case $u_\infty > 0$ $\bar{u}_x(x)$ is positive for $x < 0$ and negative for $x > 0$, while $\bar{u}_{xx}(x)$ is positive for all $|x|$ sufficiently large; for the case $u_\infty < 0$ $\bar{u}_x(x)$ is negative for $x < 0$ and positive for $x > 0$, while $\bar{u}_{xx}(x)$ is negative for all $|x|$ sufficiently large. We can conclude, for $u_\infty > 0$,

$$\operatorname{sgn} \det \begin{pmatrix} V_{11}^+ & V_{21}^+ \\ V_{12}^+ & V_{22}^+ \end{pmatrix} = \lim_{x \rightarrow \infty} \operatorname{sgn} \det \begin{pmatrix} \bar{u}_x(x) & \phi_2^+(x; 0) \\ \bar{u}_{xx}(x) & \phi_2^{+\prime}(x; 0) \end{pmatrix} = -1,$$

and similarly

$$\operatorname{sgn} \det \begin{pmatrix} V_{31}^- & V_{41}^- \\ V_{32}^- & V_{42}^- \end{pmatrix} = \lim_{x \rightarrow -\infty} \operatorname{sgn} \det \begin{pmatrix} \phi_1^-(x; 0) & \bar{u}_x(x) \\ \phi_1^{-\prime}(x; 0) & \bar{u}_{xx}(x) \end{pmatrix} = +1,$$

and each sign is opposite for $u_\infty < 0$. We conclude that for λ real and sufficiently large

$$D(\lambda) < 0,$$

and according to the remark immediately following Definition 3.1 this guarantees that there is a positive real eigenvalue associated with the reversal wave. \square

Acknowledgements. This research was partially supported by the National Science Foundation under Grant No. DMS-0500988.

REFERENCES

- [1] V. Alexiades and E. C. Aifantis, *On the thermodynamic theory of fluid interfaces: infinite intervals, equilibrium solutions, and minimizers*, J. Colloid and Interface Science, **111** (1986), 119–132.
- [2] J. Alexander, R. Gardner, and C. K. R. T. Jones, *A topological invariant arising in the analysis of traveling waves*, J. Reine Angew. Math., **410** (1990), 167–212.
- [3] E. C. Aifantis and J. B. Serrin, *Equilibrium solutions in the mechanical theory of fluid microstructures*, J. Colloid and Interface Science **96** (1983), 530–547.
- [4] A. Bertozzi, S. Esedoglu, and A. Gillette, *Analysis of a two-scale Cahn–Hilliard model for image inpainting*, Preprint 2006.

- [5] F. Bai, C. M. Elliot, A. Gardiner, A. Spence, and A. M. Stuart, *The viscous Cahn–Hilliard equation. Part I: computations*, *Nonlinearity* **8** (1995), 131–160.
- [6] A. L. Bertozzi, A. Münch, M. Shearer, and K. Zumbrun, *Stability of compressive and undercompressive thin film travelling waves: the dynamics of thin film flows*, *European J. Appl. Math.* **12** (2001), 253–291.
- [7] J. W. Cahn, *Phase separation by spinodal decomposition in isotropic systems*, *J. Chemical Physics* **42** (1965), 93–99.
- [8] J. W. Cahn, *Spinodal Decomposition*, *Trans. Metallurgical Soc. AIME* **242** (1968), 166–180.
- [9] J. W. Cahn and J. E. Hilliard, *Free energy of a uniform system I: Interfacial free energy*, *J. Chem. Phys.* **28** (1958), 258–267.
- [10] D. S. Cohen and J. D. Murray, *A generalized diffusion model for growth and dispersal of a population*, *J. Math. Biology* **12** (1981), 237–249.
- [11] J. W. Evans, *Nerve axon equations I–IV*, *Indiana U. Math. J.* **21** (1972), 877–885; **22** (1972), 75–90; **22** (1972), 577–594; **24** (1975), 1169–1190.
- [12] C. M. Elliot and D. A. French, *Numerical studies for the Cahn–Hilliard equation for phase separation*, *IMA J. Appl. Math.* **38** (1987), 97–128.
- [13] D. J. Eyre, *Systems of Cahn–Hilliard equations*, *SIAM J. Appl. Math.* **53** (1993), 1686–1712.
- [14] C. P. Grant, *The dynamics of pattern selection for the Cahn–Hilliard equation*, Ph.D. Thesis, The University of Utah, 1991, Advisor: Paul Fife.
- [15] C. P. Grant, *Spinodal decomposition for the Cahn–Hilliard equation*, *Comm. Partial Differential Equations* **18** (1993), 453–490.
- [16] R. A. Gardner, *On the structure of the spectra of periodic travelling waves*, *J. Math. Pures Appl.* **72** (1993), 415–439.
- [17] R. A. Gardner, *Instability of oscillatory shock profile solutions of the generalized Burgers–KdV equation*, *Physica D* **90** (1996), 366–386.
- [18] R. A. Gardner, *Spectral analysis of long wavelength periodic waves and applications*, *J. Reine Angew. Math.* **491** (1997), 149–181.
- [19] R. Gardner and K. Zumbrun, *The gap lemma and geometric criteria for instability of viscous shock profiles*, *Comm. Pure Appl. Math.* **51** (1998), 789–847.
- [20] P. Howard, *Asymptotic behavior near transition fronts for equations of generalized Cahn–Hilliard form*, *Commun. Math. Phys.* **269** (2007), 765–808.
- [21] P. Howard, *Asymptotic behavior near planar transition fronts for the Cahn–Hilliard equation*, *Physica D* **229** (2007), 123–165.
- [22] P. Howard, *Spectral analysis of planar transition fronts for the Cahn–Hilliard equation*, *J. Differential Equations* **245** (2008), 594–615.
- [23] P. Howard and C. Hu, *Pointwise Green’s function estimates toward stability for multidimensional fourth order viscous shock fronts*, *J. Differential Equations* **218** (2005), 325–389.
- [24] P. Howard and C. Hu, *Nonlinear stability for multidimensional fourth order shock fronts*, *Arch. Rational Mech. Anal.* **181** (2006), 201–260.
- [25] J. M. Hyde, M. K. Miller, M. G. Hetherington, A. Cerezo, G. D. W. Smith, and C. M. Elliot, *Spinodal decomposition in Fe–Cr alloys: experimental study at the atomic level and comparison with computer models—II. development of domain size and composition amplitude*, *Acta Metall. Mater.* **43** (1995), 3403–3413.

- [26] J. M. Hyde, M. K. Miller, M. G. Hetherington, A. Cerezo, G. D. W. Smith, and C. M. Elliot, *Spinodal decomposition in Fe–Cr alloys: experimental study at the atomic level and comparison with computer models—I. introduction and methodology*, Acta Metall. Mater. **43** (1995), 3415–3426.
- [27] S. A. Hill and B. Ralph, *Continuous phase separation in a Nickel–Aluminum alloy*, Acta Metall. **30** (1982), 2219–2225.
- [28] B. A. Huberman, *Striations in chemical reactions*, J. Chem. Phys. **65** (1976), 2013–2019.
- [29] C. K. R. T. Jones, *Stability of the traveling wave solution of the FitzHugh–Nagumo system*, Trans. Amer. Math. Soc. **286** (1984), 431–469.
- [30] J. S. Langer, *Theory of spinodal decomposition in alloys*, Annals of Physics **65** (1971), 53–86.
- [31] M. K. Miller, J. M. Hyde, M. G. Hetherington, A. Cerezo, G. D. W. Smith, and C. M. Elliot, *Spinodal decomposition in Fe–Cr alloys: experimental study at the atomic level and comparison with computer models—I. introduction and methodology*, Acta Metall. Mater. **43** (1995), 3385–3401.
- [32] W. McKinney, Thesis, Department of Mathematics, University of Tennessee, Knoxville, TN 1989. Partially reprinted in: N. Alikakos, P. W. Bates, and G. Fusco, *Slow motion for the Cahn–Hilliard equation in one space dimension*, J. Differential Equations **90** (1991), 81–135.
- [33] E. V. L. de Mello and O. T. dS. Filho, *Numerical study of the Cahn–Hilliard equation in one, two, and three dimensions*, Physica A **347** (2005), 429–443.
- [34] A. A. Nepomniashchii, *On stability of secondary flows of a viscous fluid in unbounded space*, J. Appl. Math. Mech. **40** (1976), 836–841.
- [35] M. Oh and K. Zumbrun, *Stability of periodic solutions of conservation laws with viscosity: analysis of the Evans function*, Arch. Rational Mech. Anal. **166** (2003), 99–166.
- [36] M. Oh and K. Zumbrun, *Stability of periodic solutions of conservation laws with viscosity: pointwise bounds on the Green’s function*, Arch. Rational Mech. Anal. **166** (2003), 167–196.
- [37] V. G. Papanicolaou, *The spectral theory of the vibrating periodic beam*, Commun. Math. Phys. **170** (1995), 359–373.
- [38] G. V. Rozenblum, M. A. Shubin, and M. Z. Solomyak, *Spectral theory of differential operators*, in Encyclopedia of Mathematical Sciences Vol. 64, Partial Differential Equations VII, Ed. M. A. Shubin.
- [39] V. G. Papanicolaou, *The periodic Euler–Bernoulli equation*, Trans. AMS **355** (2003), 3727–3759.
- [40] M. Schatzman, *On the stability of the saddle solution of Allen–Cahn’s equation*, Proceedings of the Royal Society of Edinburgh **125A** (1995), 1241–1275.
- [41] G. Schneider, *Diffusive stability of spatial periodic solutions of the Swift–Hohenberg equation*, Commun. Math. Phys. **178** (1996), 679–702.
- [42] G. Schneider, *Nonlinear diffusive stability of spatially periodic solutions—abstract theorem and higher space dimensions*, in: Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997), , 159–167, Tohoku Math. Publ., 8, Tohoku Univ., Sendai, 1998.

- [43] G. Schneider, *Nonlinear stability of Taylor vortices in infinite cylinders*, Arch. Rational Mech. Anal. **144** (1998), 121–200.
- [44] G. Schneider, *Cahn–Hilliard description of secondary flows of a viscous incompressible fluid in an unbounded domain*, Z. Angew. Math. Mech. **79** (1999), 615–626.
- [45] J. D. van der Waals, “The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density,” (in Dutch), Verhandel. Konink. Akad. Wet. Amsterdam (Sect. 1), **1** (1893), . Published in English translation: J. Statistical Physics **20** (1979), 200–244, J. S. Rowlinson translator.