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LINEAR NON-AUTONOMOUS CAUCHY PROBLEMS AND EVOLUTION SEMIGROUPS

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Abstract. The paper is devoted to the problem of existence of propagators for an abstract linear non-autonomous evolution Cauchy problem of hyperbolic type in separable Banach spaces. The problem is solved using the so-called evolution semigroup approach which reduces the existence problem for propagators to a perturbation problem of semigroup generators. The results are specified to abstract linear non-autonomous evolution equations in Hilbert spaces where the assumption is made that the domains of the quadratic forms associated with the generators are independent of time. Finally, these results are applied to time-dependent Schrödinger operators with moving point interactions in 1D.

1. Introduction and setup of the Problem

The aim of the present paper is to develop an approach to Cauchy problems for linear non-autonomous evolution equations of the type

$$\frac{\partial}{\partial t}u(t) + A(t)u(t) = 0, \quad u(s) = u_s \in X, \quad t, s \in \mathcal{I}, \tag{1.1}$$

where $\mathcal{I}$ is a bounded open interval of $\mathbb{R}$ and $\{A(t)\}_{t\in\mathcal{I}}$ is a family of closed linear operators in the separable Banach space X. Evolution equations of that type are called forward evolution equations if $s \leq t$, backward if $s \geq t$ and bidirectional evolution equations if s and t are arbitrary. The main question concerning the Cauchy problem (1.1) is to find a so-called "solution operator" or propagator $U(t,s)$ such that $u(t) := U(t,s)u_s$ is in some sense a solution of (1.1) satisfying the initial condition $u(s) = u_s$.

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Usually it is assumed that either $\{A(t)\}_{t\in\mathcal{I}}$ or $\{-A(t)\}_{t\in\mathcal{I}}$ are families of generators of C_0-semigroups in X. In order to distinguish both cases we call an operator A a *generator* if it generates a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$. We call A an *anti-generator* if $-A$ generates a C_0-semigroup $\{e^{-tA}\}_{t\geq 0}$; i.e., the operator $-A$ is the *generator* of a semigroup. If simultaneously A is an anti-generator and a generator, then A is called a group generator.

Very often the Cauchy problem (1.1) is attacked for a suitable dense subset of initial data u_s by solving it directly in the same manner as an ordinary differential equation, which immediately implies the existence of the propagator, see e.g. [45]. For this purpose one assumes that $\{A(t)\}_{t\in\mathcal{I}}$ is a family of anti-generators of C_0-semigroups such that they uniformly belong to the class of quasi-bounded semigroups $\mathcal{G}(M,\beta)$, cf. [23, Chapter IX]. If $\{A(t)\}_{t\in\mathcal{I}}$ is a family of anti-generators of class $\mathcal{G}(M,\beta)$ which are simultaneously anti-generators of *holomorphic* C_0-semigroups, then the evolution equation is called of "parabolic" type. If it is not holomorphic, then it is called of "hyperbolic" type. In this paper we are only interested in the "hyperbolic" case.

There is a rich literature on "hyperbolic" evolution equations problems. The first author who discussed these problems was Phillips [39]. A more general case was considered by Kato in [21, 22] and by Mizohata in [31]. These results were generalized in the sixties in [12, 17, 26, 53, 54, 29, 16, 14]. Kato has improved these results in two important papers [24, 25], where for the first time he introduced the assumptions of *stability* and *invariance*. In the seventies and eighties Kato's result were generalized in [11, 20, 27, 51, 52]. For related results see also [10, 15, 28]. Recently several new results were obtained in [3, 38, 37, 46, 47, 48]. In the following we refer to these results as the "standard approach" or the "standard methods." Their common feature is that the propagator is constructed by using certain approximations of the family $\{A(t)\}_{t\in\mathcal{I}}$ for which the corresponding Cauchy problem can be easily solved. After that one has only to verify that the obtained sequence of propagators converges to the propagator of the original problem. Widely used approximations are the so-called *Yosida approximations* introduced in [54], *piecewise constant* approximations proposed by Kato, cf. [24, 25], as well as a combination of both, see [26].

In contrast to the *standard methods* another approach was developed in [13, 19, 33, 34, 35, 36]. It does not rely on any approximation, since it is based on the fact that the existence problem for the propagator in question is *equivalent* to an operator extension problem for a suitably defined operator in a vector-valued Banach space $L^p(\mathcal{I},X)$ for some $p\in[1,\infty)$. More precisely, it turns out that any *forward propagator* $\{U(t,s)\}_{(t,s)\in\Delta_\mathcal{I}}$, $\Delta_\mathcal{I} := \{(t,s) \in \mathcal{I}\times\mathcal{I} : s \le t\}$, (see Definition 2.1) defines a C_0-semigroup in $L^p(\mathcal{I},X)$ by

$$(\mathcal{U}(\sigma)f)(t) := U(t,t-\sigma)\chi_\mathcal{I}(t-\sigma)f(t-\sigma), \quad f \in L^p(\mathcal{I},X), \quad \sigma \ge 0, \tag{1.2}$$

where $\chi_\mathcal{I}(\cdot)$ is the characteristic function of the open interval $\mathcal{I}$. C_0-semigroups in $L^p(\mathcal{I},X)$ admitting a forward propagator representation (1.2) are called *forward evolution semigroups*. The anti-generator K of the semigroup $\{\mathcal{U}(\sigma)\}_{\sigma\in\mathbb{R}_+}$, i.e. $\mathcal{U}(\sigma) = e^{-\sigma K}$, $\sigma \in \mathbb{R}_+$, is called the *forward generator*. Our approach is based on the important fact that the set of forward generators can be described explicitly, and that there is a *one-to-one correspondence* between forward propagators and forward generators, see [34].

Now, let us assume that the forward propagator $\{U(t,s)\}_{(t,s)\in\Delta_\mathcal{I}}$ is found by the standard approach and that it solves the forward evolution equation (1.1) in some sense. Then it turns out that the forward generator $K_\mathcal{I}$ defined by (1.2) is an extension of the so-called *evolution operator* $\widetilde{K}_\mathcal{I}$ given by

$$(\widetilde{K}_\mathcal{I}f)(t) = D_\mathcal{I}f + Af, \quad f \in \mathrm{dom}(\widetilde{K}_\mathcal{I}) = \mathrm{dom}(D_\mathcal{I}) \cap \mathrm{dom}(A), \tag{1.3}$$

in $L^p(\mathcal{I},X)$ for some $p \in [1,\infty)$, where $D_\mathcal{I}$ is the anti-generator of the *right-shift semigroup* in $L^p(\mathcal{I},X)$ and A is the *multiplication* operator in $L^p(\mathcal{I},X)$ induced by the family $\{A(t)\}_{t\in\mathcal{I}}$, see Section 2.

This remark leads to the main idea of our approach: to solve the evolution equation (1.1) by extending the evolution operator $\widetilde{K}_\mathcal{I}$ to an anti-generator of an (forward) evolution semigroup. Notice that in contrast to the standard approach now the focus has moved from the problem of constructing a propagator to the problem of finding a certain operator extension. This so-called "extension approach" or "extension method" has a lot of advantages, since it works in a very general setting, and it is quite flexible and transparent. The approach becomes very simple, if the closure of the evolution operator $\widetilde{K}_\mathcal{I}$ is already an anti-generator, in other words, if $\widetilde{K}_\mathcal{I}$ is an *essential* anti-generator. In this case one gets the forward generator by closing $\widetilde{K}_\mathcal{I}$, see Theorem 2.4, which immediately implies the existence of a unique forward propagator for the non-autonomous Cauchy problem (1.1). Some recent results related to the extension method can be find in e.g. [30, 32, 37, 41, 42].

Below we exploit this approach extensively and we show how this method can be applied to evolution equations of type (1.1). We prove that, under the stability and invariance assumptions of Kato [24, 25] and the Radon-Nikodym property of certain Banach spaces involved, the evolution operator $\widetilde{K}_\mathcal{I}$ is already an *essential* anti-generator, which means that its closure $K_\mathcal{I}$ is a forward generator which extends the results of [4, 5]. Furthermore, in addition to [4, 5] we demonstrate that this fact immediately yields the existence of the propagator. Since we are not interested in the problem in which sense the obtained propagator satisfies the evolution equation (1.1) measurability assumptions are sufficient in the following.

We apply also the extension method to *bidirectional* evolution equations of the type

$$i\frac{\partial}{\partial t}u(t) = H(t)u(t), u(s) = u_s, \quad s,t \in \mathbb{R}, \tag{1.4}$$

on $\mathbb{R}$ in Hilbert spaces, where $\{H(t)\}_{t\in\mathbb{R}}$ is a family of non-negative self-adjoint operators. Using the extension method we restore and obtain some generalizations of the Kisyński result [26]. Moreover, we show that Kisyński's propagator is in fact the propagator of an auxiliary evolution equation problem closely related to (1.4). The solution of the auxiliary problem implies a solution for (1.4). The uniqueness of the auxiliary solution does not imply, however, uniqueness of the original problem (1.4), in general.

The paper is organized as follows. In Section 2 we recall some basic facts of the theory of evolution semigroups. Section 3 is devoted to a perturbation theorem for generators of these semigroups, which is used then in Section 4 to show that the closure $K_{\mathcal{I}}$ of the evolution operator (1.3) is an anti-generator. The results of Section 4 are specified in Section 5 to families $\{A(t)\}_{t\in\mathbb{R}}$ of the form $A(t) = iH(t)$ where $H(t)$ are semi-bounded self-adjoint operators with *time-independent* form domains in a Hilbert space. In Section 6, we apply these results of Section 5 to Schrödinger operators with time-dependent point interactions of the form

$$H(t) := -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx} + V(x) + \sum_{j=1}^{N}\kappa_j(t)\delta(x-x_j) \quad x\in\mathbb{R}, \quad t\in\mathbb{R},$$

as well as to the case of moving point interactions of the form:

$$H(t) := -\frac{d^2}{dx^2} + \kappa_1(t)\delta(x - x_1(t)) + \kappa_2(t)\delta(t)(x - x_2(t))$$

where the coupling constants $\kappa_j(\cdot)$ are non-negative Lipschitz continuous functions in $t \in \mathbb{R}$ and $x_j(t)$ are C^2-trajectories in $\mathbb{R}$. These kind of problems were the subject of publications [6, 8, 40, 44, 43, 50].

2. Evolution generators

In the following we are interested not only in the *forward* evolution equations but also in the *backward* ones as well as in the *bidirectional* evolution equations. The interest in these evolution equations arises from time reversible problems in quantum mechanics, which we consider in the conclusion of this paper as applications. For this purpose we show in Section 2.2 how one has to modify the extension approach for backward evolution equations. Moreover, in applications to quantum mechanics we are concerned with *infinite* time intervals, in particular, with $\mathcal{I} = \mathbb{R}$. In order to apply our approach to this situation it is useful to *localize* it in time; this means that instead of considering the Cauchy problem on $\mathbb{R}$ we consider it on arbitrary finite subintervals of $\mathbb{R}$. In this case, however, one has to ensure that propagators for different time intervals are *compatible*.

2.1. **Forward generators.** We start with the definition of a forward propagator in a separable Banach space.

Definition 2.1. *Let X be a separable Banach space and let $\mathcal{I}$ be a (arbitrary) bounded open interval of $\mathbb{R}$. A strongly continuous operator-valued function $U(\cdot,\cdot) : \Delta_{\mathcal{I}} \longrightarrow \mathcal{B}(X)$ is called a forward propagator on $\Delta_{\mathcal{I}} = \{(t,s) \in \mathcal{I}\times\mathcal{I} : s \le t\}$, if*

(i) $U(t,t) = I_X$ *for* $t \in \mathcal{I}$,
(ii) $U(t,r)U(r,s) = U(t,s)$ *for* $(t,r,s) \in \mathcal{I}^3$, $s \le r \le t$,
(iii) $\|U\|_{\mathcal{B}(X)} := \sup_{(t,s)\in\Delta_{\mathcal{I}}} \|U(t,s)\|_{\mathcal{B}(X)} < \infty$.

We call a strongly continuous operator-valued function $U(\cdot,\cdot)$ defined on $\Delta_{\mathbb{R}} := \{(t,s) \in \mathbb{R}\times\mathbb{R} : s \le t\}$ a forward propagator, if for any bounded interval $\mathcal{I}$, the restriction of $U(\cdot,\cdot)$ to $\Delta_{\mathcal{I}}$ is a forward propagator.

Another important notion is the so-called *evolution operator*. To explain this notion we introduce the Banach space $L^p(\mathcal{I},X)$, $p \in [1,\infty)$, where X is a separable Banach space. In $L^p(\mathcal{I},X)$ we define the *multiplication* operator

$$(M(\phi)f)(t) := \phi(t)f(t), \quad \mathrm{dom}(M(\phi)) = L^p(\mathcal{I},X), \quad \phi \in L^\infty(\mathcal{I}). \tag{2.1}$$

Definition 2.2. *A linear operator K in $L^p(\mathcal{I},X)$, $p \in [1,\infty)$, is called an evolution operator, if it satisfies the following conditions:*

(i)
$$\mathrm{dom}(K) \subseteq C(\overline{\mathcal{I}},X), \tag{2.2}$$
$$M(\phi)\mathrm{dom}(K) \subseteq \mathrm{dom}(K), \quad \phi \in H^{1,\infty}(\mathcal{I}), \tag{2.3}$$

and

$$KM(\phi)f - M(\phi)Kf = M(\dot\phi)f, \quad f \in \mathrm{dom}(K), \quad \phi \in H^{1,\infty}(\mathcal{I}), \tag{2.4}$$

where $\dot\phi := d\phi/dt$;

(ii) *its domain* dom(K) *has a dense cross-section in* X*; this means that*

$$[\mathrm{dom}(K)]_t := \{x \in X : \exists f \in \mathrm{dom}(K) \text{ such that } f(t) = x\}$$

is dense in X *for each* $t \in \mathcal{I}$*.*

If in addition K *is an anti-generator or a generator in* $L^p(\mathcal{I}, X)$*, then* K *is called a forward or backward generator, respectively.*

The density of the cross-section is not a trivial condition. However, one has to mention that it is important to ensure the *continuity* of the propagator. Notice that, if K is an evolution operator, then its domain dom(K) is already dense in $L^p(\mathcal{I}, X)$, $1 \leq p < \infty$.

A crucial role is played by Theorem 4.12 of [34] in the following. It establishes a *one-to-one correspondence* between the set of forward propagators and the set of forward generators such that (1.2) holds. That means, for every forward generator K in $L^p(\mathcal{I}, X)$, cf. Definition 2.2, there is a forward propagator $U(\cdot,\cdot)$ given on $\Delta_\mathcal{I}$ such that the representation (1.2) is valid. Conversely, for any forward propagator $U(\cdot,\cdot)$ defined on $\Delta_\mathcal{I}$ the relation (1.2) defines a C_0-semigroup in $L^p(\mathcal{I}, X)$ such that its anti-generator K satisfies (2.2)-(2.4) and the domain dom(K) has a dense cross-section. The correspondence is unique.

Let $S_r(\sigma)$ be the right-shift semigroup in $L^p(\mathcal{I}, X)$, $1 \leq p < +\infty$, given by

$$(S_r(\sigma)f)(t) := f(t-\sigma)\chi_\mathcal{I}(t-\sigma), \quad f \in L^p(\mathcal{I}, X). \tag{2.5}$$

This is a C_0-semigroup of class $\mathcal{G}(1,0)$. Its generator is given by $-D_\mathcal{I}$, where

$$(D_\mathcal{I} f)(t) = \frac{\partial}{\partial t} f(t), \quad f \in \mathrm{dom}(D_\mathcal{I}) := H^{1,p}_a(\mathcal{I}, X), \quad \mathcal{I} = (a,b).$$

According to our convention the operator $D_\mathcal{I}$ is an anti-generator. Here

$$H^{1,p}_a(\mathcal{I}, X) := \{f \in H^{1,p}(\mathcal{I}, X) : f(a) = 0\},$$

and $H^{1,p}_a(\mathcal{I}, X)$ is the Sobolev space of X-valued absolutely continuous functions on $\mathcal{I}$ with p-summable derivative.

Notice that a family $\{A(t)\}_{t\in\mathcal{I}}$ of closed and densely defined linear operators is called *measurable*, if there is a $z \in \mathbb{C}$ such that z belongs to the *resolvent set* $\varrho(A(t))$ of $A(t)$ for almost every (a.e.) $t \in \mathcal{I}$ and for each $x \in X$ the function

$$f(t) := (A(t) - z)^{-1}x, \quad t \in \mathcal{I},$$

is *strongly* measurable. If the family $\{A(t)\}_{t\in\mathcal{I}}$ is measurable, then one can show that the multiplication operator A,

$$(Af)(t) := A(t)f(t), \quad f \in \mathrm{dom}(A), \tag{2.6}$$

$$\operatorname{dom}(A) := \left\{ f \in L^p(\mathcal{I}, X) : \begin{array}{l} f(t) \in \operatorname{dom}(A(t)) \text{ for a.e. } t \in \mathcal{I}, \\ A(t)f(t) \in L^p(\mathcal{I}, X) \end{array} \right\} \tag{2.7}$$

is densely defined and closed in $L^p(\mathcal{I}, X)$.

Instead of solving the Cauchy problem (1.1) for a suitable set of initial data u_s we consider the operator

$$\widetilde{K}_{\mathcal{I}} f := D_{\mathcal{I}} f + Af, \quad f \in \operatorname{dom}(\widetilde{K}_{\mathcal{I}}) := \operatorname{dom}(D_{\mathcal{I}}) \cap \operatorname{dom}(A), \tag{2.8}$$

in $L^p(\mathcal{I}, X)$, $p \in [1, \infty)$. If the domain $\operatorname{dom}(\widetilde{K}_{\mathcal{I}})$ has a dense cross-section, then by the definition above $\widetilde{K}_{\mathcal{I}}$ is an evolution operator. This leads naturally to the following definitions:

Definition 2.3. *Let $\{A(t)\}_{t \in \mathcal{I}}$ be a measurable family of closed and densely defined linear operators in the separable Banach space X.*

(i) *The forward evolution equation* (1.1) *is well posed on $\mathcal{I}$ for some $p \in [1, \infty)$ if $\widetilde{K}_{\mathcal{I}}$ is an evolution operator.*
(ii) *A forward propagator $\{U(t,s)\}_{(t,s) \in \Delta_{\mathcal{I}}}$ is called a solution of the well-posed forward evolution equation* (1.1) *on $\mathcal{I}$ if the corresponding forward generator $K_{\mathcal{I}}$, cf.* (1.2)*, is an extension of $\widetilde{K}_{\mathcal{I}}$.*
(iii) *The well-posed forward evolution equation* (1.1) *on $\mathcal{I}$ has a unique solution if $\widetilde{K}_{\mathcal{I}}$ admits only one extension which is a forward generator.*

It is quite possible that the forward evolution equation (1.1) has several solutions, which means that the evolution operator $\widetilde{K}_{\mathcal{I}}$ admits *several* extensions, and each of them is a forward generator. The dense cross-section property of the evolution operator is not sufficient to show that the evolution equation admits a unique solution.

In the following the next statement will be important for our reasoning.

Theorem 2.4. *Let $\{A(t)\}_{t \in \mathcal{I}}$ be a measurable family of closed and densely defined linear operators in the separable Banach space X. Assume that the forward evolution equation* (1.1) *is well posed on $\mathcal{I}$ for some $p \in [1, \infty)$. If the evolution operator $\widetilde{K}_{\mathcal{I}}$ is closable in $L^p(\mathcal{I}, X)$ and its closure $K_{\mathcal{I}}$ is an anti-generator, then the forward evolution equation* (1.1) *on $\mathcal{I}$ has a unique solution.*

Proof. Since the evolution equation is well posed, the domain $\operatorname{dom}(\widetilde{K}_{\mathcal{I}})$ is densely defined in $L^p(\mathcal{I}, X)$. By assumptions the closure $K_{\mathcal{I}}$ is an anti-generator. Hence, it remains to show that the closure $K_{\mathcal{I}}$ satisfies the conditions (2.2)-(2.4). It is easy to verify that the closure $K_{\mathcal{I}}$ satisfies the conditions (2.3) and (2.4). To show (2.2) let us assume that $K_{\mathcal{I}}$ belongs to $\mathcal{G}(M,\beta)$. By Lemma 2.16 of [35] the closure $K_{\mathcal{I}}$ admits the estimate

$$\|f(t)\|_X \leq \frac{M}{(\xi-\beta)^{(p-1)/p}}\|(K_{\mathcal{I}}+\xi)f\|_{L^p(\mathcal{I},X)}, \quad f \in \mathrm{dom}(K_{\mathcal{I}}), \quad p \in [1,\infty),$$

for almost every $t \in \overline{\mathcal{I}}$ and $\xi > \beta$. In particular, we have

$$\|f\|_{C(\overline{\mathcal{I}},X)} \leq \frac{M}{(\xi-\beta)^{(p-1)/p}}\|(\widetilde{K}_{\mathcal{I}}+\xi)f\|_{L^p(\mathcal{I},X)}, \quad f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}}). \tag{2.9}$$

Since $\widetilde{K}_{\mathcal{I}}$ has a closure $K_{\mathcal{I}}$, there is a sequence of elements $\{f_n\}_{n\in\mathbb{N}}$ for any $f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}})$ such that $f_n \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}})$, $f_n \longrightarrow f$ and $\widetilde{K}_{\mathcal{I}}f_n \longrightarrow K_{\mathcal{I}}f$ in the $L^p(\mathcal{I},X)$ sense when $n \longrightarrow \infty$. By (2.9) one gets that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(\overline{\mathcal{I}},X)$. Hence $f \in C(\overline{\mathcal{I}},X)$; that proves (2.2). Since $\mathrm{dom}(\widetilde{K}_{\mathcal{I}})$ has a dense cross-section for each $t \in \mathcal{I}$, one gets that its closure $K_{\mathcal{I}}$ has a dense cross-section for each $t \in \mathcal{I}$. Hence, $K_{\mathcal{I}}$ is a forward generator.

Let $K_{\mathcal{I}}$ and $K'_{\mathcal{I}}$ be two different extensions of $\widetilde{K}_{\mathcal{I}}$, which are both forward generators. Since $K_{\mathcal{I}}$ is the closure of $\widetilde{K}_{\mathcal{I}}$ one has $K_{\mathcal{I}} \subseteq K'_{\mathcal{I}}$. Since $K_{\mathcal{I}}$ and $K'_{\mathcal{I}}$ are generators of a C_0-semigroup, one gets $K_{\mathcal{I}} = K'_{\mathcal{I}}$. Hence the evolution equation (1.1) is uniquely solvable. □

2.2. **Backward generators.** In the following we are also interested in the so-called *backward* evolution equation (1.1), $t \leq s$, $t,s \in \mathcal{I}$. Equations of that type require the introduction of the notion of *backward propagator*:

Definition 2.5. *A strongly continuous operator-valued function $V(\cdot,\cdot)$: $\nabla_{\mathcal{I}} \to \mathcal{B}(X)$ is called a backward propagator on $\nabla_{\mathcal{I}} := \{(t,s) \in \mathcal{I}\times\mathcal{I} : t \leq s\}$, if*

- (i) *$V(t,t) = I_X$ for $t \in \mathcal{I}$,*
- (ii) *$V(t,r)V(r,s) = V(t,s)$ for $(t,r,s) \in \mathcal{I}^3$, $t \leq r \leq s$,*
- (iii) $\sup_{(t,s)\in\nabla_{\mathcal{I}}} \|V(t,s)\|_{\mathcal{B}(X)} < \infty$.

We call a strongly continuous operator-valued function $V(\cdot,\cdot)$ defined on $\nabla_{\mathbb{R}} := \{(t,s) \in \mathbb{R}\times\mathbb{R} : s \leq t\}$ a backward propagator if for any bounded interval $\mathcal{I}$ the restriction of $V(\cdot,\cdot)$ to $\nabla_{\mathcal{I}}$ is a backward propagator.

Similar to forward propagators there is a one-to-one correspondence between backward propagators and backward generators given by

$$(e^{\sigma K}f)(t) = V(t,t+\sigma)\chi_{\mathcal{I}}(t+\sigma)f(t+\sigma), \quad f \in L^p(\mathcal{I},X), \quad \sigma \geq 0, \tag{2.10}$$

$p \in [1,\infty)$. With the backward evolution equation we associated the operator $\widetilde{K}^{\mathcal{I}}$

$$\widetilde{K}^{\mathcal{I}}f = D^{\mathcal{I}}f + Af, \quad f \in \mathrm{dom}(\widetilde{K}^{\mathcal{I}}) := \mathrm{dom}(D^{\mathcal{I}}) \cap \mathrm{dom}(A), \tag{2.11}$$

where

$$(D^{\mathcal{I}}f)(t) = \frac{\partial}{\partial t}f(t), \quad f \in \mathrm{dom}(D^{\mathcal{I}}) := \{f \in H^{1,p}_b(\mathcal{I}, X) : f(b) = 0\}$$

is the generator of the *left-shift* semigroup $S_l(\sigma) = e^{\sigma D^{\mathcal{I}}}$ on $L^p(\mathcal{I}, X)$; that is,

$$(S_l(\sigma)f)(t) = f(t+\sigma)\chi_{\mathcal{I}}(t+\sigma), \quad t \in \mathcal{I}, \quad f \in L^p(\mathcal{I}, X), \quad \sigma \geq 0.$$

Definition 2.6. *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of closed and densely defined linear operators in the separable Banach space X.*

(i) *The backward evolution equation* (1.1) *is well posed on $\mathcal{I}$ for some $p \in [1, \infty)$, if $\widetilde{K}^{\mathcal{I}}$ is an evolution operator.*
(ii) *A backward propagator $\{V(t,s)\}_{(t,s)\in\nabla_{\mathcal{I}}}$ is called a solution of the well-posed backward evolution equation* (1.1) *on $\mathcal{I}$ if the corresponding backward generator $K^{\mathcal{I}}$, cf.* (2.10)*, is an extension of $\widetilde{K}^{\mathcal{I}}$.*
(iii) *The well-posed backward evolution equation* (1.1) *on $\mathcal{I}$ has a solution if $\widetilde{K}_{\mathcal{I}}$ admits only one extension which is a backward generator.*

Now, following the same line of reasoning as in Theorem 2.4 we obtain a similar statement concerning the backward evolution equation (1.1):

Theorem 2.7. *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of closed and densely defined linear operators in the separable Banach space X. Assume that the backward evolution equation* (1.1) *is well posed on $\mathcal{I}$ for some $p \in [1, \infty)$. If the evolution operator $\widetilde{K}^{\mathcal{I}}$ is closable in $L^p(\mathcal{I}, X)$ and its closure $K^{\mathcal{I}}$ is a generator, then the backward evolution equation* (1.1) *on $\mathcal{I}$ has a unique solution.*

2.3. **Bidirectional problems.** Crucial for studying *bidirectional* evolution equations on bounded intervals is the following proposition.

Proposition 2.8. *Let $\{U(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}}}$ and $\{V(t,s)\}_{(t,s)\in\nabla_{\mathcal{I}}}$ be for- and backward propagators which correspond to the for- and backward generators $K_{\mathcal{I}}$ and $K^{\mathcal{I}}$, respectively. The relation*

$$V(s,t)U(t,s) = U(t,s)V(s,t) = I_X, \quad (t,s) \in \Delta_{\mathcal{I}}, \tag{2.12}$$

holds if and only if for each $\phi \in H^{1,\infty}_a(\mathcal{I}) \cap H^{1,\infty}_b(\mathcal{I})$ the conditions

$$M(\phi)\mathrm{dom}(K_{\mathcal{I}}) \subseteq \mathrm{dom}(K^{\mathcal{I}}) \quad \text{and} \quad M(\phi)\mathrm{dom}(K^{\mathcal{I}}) \subseteq \mathrm{dom}(K_{\mathcal{I}}) \tag{2.13}$$

and

$$K^{\mathcal{I}}M(\phi)f = K_{\mathcal{I}}M(\phi)f, \quad f \in \mathrm{dom}(K_{\mathcal{I}}) \quad \text{or} \quad f \in \mathrm{dom}(K^{\mathcal{I}}) \tag{2.14}$$

are satisfied.

Proof. We set

$$g(\sigma) := e^{\sigma K^{\mathcal{I}}} M(\phi) e^{-\sigma K_{\mathcal{I}}} f, \quad f \in L^p(\mathcal{I}, X), \quad \phi \in H^{1,\infty}(\mathcal{I}).$$

Taking into account (1.2) and (2.10) we find

$$(g(\sigma))(t) = V(t, t+\sigma)\phi(t+\sigma)U(t+\sigma, t)\chi_{\mathcal{I}}(t+\sigma)\chi_{\mathcal{I}}(t)f(t), \quad t \in \mathcal{I}.$$

Using (2.12) we obtain

$$(g(\sigma))(t) = \phi(t+\sigma)\chi_{(a,b-\sigma)}(t)f(t), \quad t \in \mathcal{I}, \quad 0 \le \sigma < b-a. \tag{2.15}$$

Since

$$(g(\sigma) - M(\phi))f = (e^{\sigma K^{\mathcal{I}}} - I)M(\phi)f + e^{\sigma K^{\mathcal{I}}} M(\phi)(e^{-\sigma K_{\mathcal{I}}} - I)f, \tag{2.16}$$

by (2.15) we get that

$$\lim_{\sigma \to +0} \frac{1}{\sigma}(g(\sigma) - M(\phi))f = M(\dot{\phi})f, \quad f \in L^p(\mathcal{I}, X).$$

Assuming $f \in \mathrm{dom}(K_{\mathcal{I}})$ we immediately find from (2.16) that $M(\phi)f \in \mathrm{dom}(K^{\mathcal{I}})$ and (2.13). Interchanging $K^{\mathcal{I}}$ and $K_{\mathcal{I}}$ we prove $M(\phi)f \in \mathrm{dom}(K_{\mathcal{I}})$ and (2.14).

Conversely, assuming (2.13) and (2.14) we get that the function $g(\sigma)$ is differentiable and

$$\frac{d}{d\sigma} g(\sigma) = e^{\sigma K^{\mathcal{I}}} \left(K^{\mathcal{I}} M(\phi) - M(\phi)K_{\mathcal{I}}\right) e^{-\sigma K_{\mathcal{I}}} f, \quad \sigma \ge 0.$$

By virtue of (2.4) we find

$$\frac{d}{d\sigma} g(\sigma) = e^{\sigma K^{\mathcal{I}}} M(\dot{\phi}) e^{-\sigma K_{\mathcal{I}}} f, \quad \sigma \ge 0$$

which yields

$$e^{\sigma K^{\mathcal{I}}} M(\phi) e^{-\sigma K} f = M(\phi)f + \int_0^\sigma d\tau \; e^{\tau K^{\mathcal{I}}} M(\dot{\phi}) e^{-\tau K} f, \quad \sigma \ge 0.$$

Therefore, using representations (1.2) and (2.10) we obtain

$$\begin{aligned} &V(t, t+\sigma)U(t+\sigma, t)\phi(t+\sigma)\chi_{\mathcal{I}}(t+\sigma)f(t) \\ &\quad = \phi(t)f(t) + \int_0^\sigma d\tau \; V(t, t+\tau)U(t+\tau, t)\dot{\phi}(t+\tau)\chi_{\mathcal{I}}(t+\tau)f(t) \end{aligned}$$

for $t \in \mathcal{I}$ and $\sigma \ge 0$. Put $s := t + \sigma$. Then we get

$$V(t,s)U(s,t)\phi(s)\chi_{\mathcal{I}}(s)f(t) = \phi(t)f(t) + \int_t^s dr \; V(t,r)U(r,t)\dot{\phi}(r)\chi_{\mathcal{I}}(r)f(t)$$

for $(s,t) \in \Delta_{\mathcal{I}}$. Let $\overline{\mathcal{I}}_0 \subset \mathcal{I}$ be a closed subinterval such that the restriction $\phi \restriction \overline{\mathcal{I}}_0 = 1$. If $s,t \in \overline{\mathcal{I}}_0$, then

$$V(t,s)U(s,t)f(t) = f(t)$$

for $t \in \overline{\mathcal{I}}$. Since $[\mathrm{dom}(K_{\mathcal{I}})]_t$ is dense in X for each $t \in \mathcal{I}$, we prove the first part of the equality (2.12). To prove the second part one has to interchange the generators $K_{\mathcal{I}}$ and $K^{\mathcal{I}}$. □

Corollary 2.9. *Let $\widetilde{K}_{\mathcal{I}}$ and $\widetilde{K}^{\mathcal{I}}$, $p \in [1,\infty)$, be evolution operators in $L^p(\mathcal{I},X)$. Assume that for each $\phi \in H^{1,\infty}_a(\mathcal{I}) \cap H^{1,\infty}_b(\mathcal{I})$ one has*

$$M(\phi)\mathrm{dom}(\widetilde{K}_{\mathcal{I}}) \subseteq \mathrm{dom}(\widetilde{K}^{\mathcal{I}}) \quad \textit{and} \quad M(\phi)\mathrm{dom}(\widetilde{K}^{\mathcal{I}}) \subseteq \mathrm{dom}(\widetilde{K}_{\mathcal{I}}) \tag{2.17}$$

and

$$\widetilde{K}^{\mathcal{I}}M(\phi)f = \widetilde{K}_{\mathcal{I}}M(\phi)f, \quad f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}}) \quad \textit{or} \quad f \in \mathrm{dom}(\widetilde{K}^{\mathcal{I}}). \tag{2.18}$$

If the closures $K_{\mathcal{I}}$ and $K^{\mathcal{I}}$ of the evolution operators $\widetilde{K}_{\mathcal{I}}$ and $\widetilde{K}^{\mathcal{I}}$ exist and are (*respectively*) *for- and backward generators, then the corresponding for- and backward propagators $\{U(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}}}$ and $\{V(t,s)\}_{(t,s)\in\nabla_{\mathcal{I}}}$ satisfy the relation* (2.12).

Proof. Let $f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}})$. Then from (2.18) and (2.4) we get

$$\widetilde{K}^{\mathcal{I}}M(\phi)f = M(\phi)\widetilde{K}_{\mathcal{I}}f - M(\dot{\phi})f.$$

Since $\widetilde{K}_{\mathcal{I}}$ is closable, for each $f \in \mathrm{dom}(K_{\mathcal{I}})$ there is a sequence $\{f_n\}_{n\in\mathbb{N}}$, $f_n \in \mathrm{dim}(\widetilde{K}_{\mathcal{I}})$, such that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} \widetilde{K}_{\mathcal{I}}f_n = K_{\mathcal{I}}f$. Since

$$\widetilde{K}^{\mathcal{I}}M(\phi)f_n = M(\phi)\widetilde{K}_{\mathcal{I}}f_n - M(\dot{\phi})f_n, \quad n \in \mathbb{N},$$

we get $M(\phi)f \in \mathrm{dom}(K^{\mathcal{I}})$ and

$$K^{\mathcal{I}}M(\phi)f = M(\phi)K_{\mathcal{I}}f - M(\dot{\phi})f$$

for $f \in \mathrm{dom}(K_{\mathcal{I}})$. Using (2.4) we prove $M(\phi)\mathrm{dom}(K_{\mathcal{I}}) \subseteq \mathrm{dom}(K^{\mathcal{I}})$ and (2.14). Similarly, we prove also $M(\phi)\mathrm{dom}(K^{\mathcal{I}}) \subseteq \mathrm{dom}(K_{\mathcal{I}})$ and (2.14). Then an application of Proposition 2.8 completes the proof. □

Now it makes sense to introduce the following definition.

Definition 2.10. *A strongly continuous operator-valued function $G(\cdot,\cdot)$: $\mathcal{I}\times\mathcal{I} \longrightarrow \mathcal{B}(X)$ is called a bidirectional propagator on $\mathcal{I}\times\mathcal{I}$ if*

(i) $G(t,t) = I_X$ *for* $t \in \mathcal{I}$,
(ii) $G(t,r)G(r,s) = G(t,s)$ *for* $(t,r,s) \in \mathcal{I}^3$,
(iii) $\sup_{(t,s)\in\mathcal{I}\times\mathcal{I}} \|G(t,s)\|_{\mathcal{B}(X)} < \infty$.

A strongly continuous operator-valued function $G(\cdot,\cdot)$ *defined on* $\mathbb{R}\times\mathbb{R}$ *is called a bidirectional propagator on* $\mathbb{R}\times\mathbb{R}$*, if for any bounded interval* $\mathcal{I}$ *the restriction of* $G(\cdot,\cdot)$ *to* $\mathcal{I}\times\mathcal{I}$ *is a bidirectional propagator.*

One can easily verify that, if $G(\cdot,\cdot)$ is a bidirectional propagator, then $U(\cdot,\cdot) := G(\cdot,\cdot)\restriction\Delta_{\mathcal{I}}$ and $V(\cdot,\cdot) := G(\cdot,\cdot)\restriction\nabla_{\mathcal{I}}$ are, respectively, for- and backward propagators related by (2.12). Conversely, if $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ are, respectively, for- and backward propagators, which are related by (2.12), then

$$G(t,s) := \begin{cases} U(t,s), & (t,s)\in\Delta_{\mathcal{I}} \\ V(t,s), & (t,s)\in\nabla_{\mathcal{I}} \end{cases} \tag{2.19}$$

defines a bidirectional propagator.

Definition 2.11. *Let* $\{A(t)\}_{t\in\mathcal{I}}$ *be a measurable family of closed and densely defined linear operators in the separable Banach space* X*.*

(i) *The evolution equation* (1.1) *is well posed on* $\mathcal{I}$ *if the for- and backward evolution equations* (1.1) *are well posed on* $\mathcal{I}$ *for some* $p\in[1,\infty)$*.*

(ii) *The bidirectional propagator* $\{G(t,s)\}_{(t,s)\in\mathcal{I}\times\mathcal{I}}$ *is called a solution of the bidirectional evolution equation* (1.1) *on* $\mathcal{I}$ *if the for- and backward propagators* $\{U(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}}}$*,* $U(\cdot,\cdot) := G(\cdot,\cdot)\restriction\Delta_{\mathcal{I}}$*, and* $\{V(t,s)\}_{(t,s)\in\nabla_{\mathcal{I}}}$*,* $V(\cdot,\cdot) := G(\cdot,\cdot)\restriction\nabla_{\mathcal{I}}$*, are solutions of the for- and backward equations* (1.1) *on* $\mathcal{I}$*.*

(iii) *The well-posed evolution equation* (1.1) *has a unique solution if the for- and backward evolution equation* (1.1) *has unique solutions.*

Theorem 2.12. *Let* $\{A(t)\}_{t\in\mathcal{I}}$ *be a measurable family of closed and densely defined linear operators in the separable Banach space* X*. Assume that the bidirectional evolution equation* (1.1) *is well posed on* $\mathcal{I}$ *for some* $p\in[1,\infty)$*. If the closures* $K_{\mathcal{I}}$ *and* $K^{\mathcal{I}}$ *of the evolution operators* $\widetilde{K}_{\mathcal{I}}$ *and* $\widetilde{K}^{\mathcal{I}}$ *exist in* $L^p(\mathcal{I},X)$ *and are anti-generators and generators, respectively, then the bidirectional evolution equation* (1.1) *has a unique solution on* $\mathcal{I}$*.*

Proof. One easily verifies that the operators $\widetilde{K}_{\mathcal{I}}$ and $\widetilde{K}^{\mathcal{I}}$ defined by (2.8) and (2.11) satisfy the conditions (2.17), (2.18). Then application of Corollary 2.9 completes the proof. □

2.4. **Problems on** $\mathbb{R}$**.** Let us consider the forward evolution equation (1.1) on $\mathbb{R}$. A natural way to study this problem is to consider the equation (1.1) on bounded open intervals $\mathcal{I}\subset\mathbb{R}$. In this case one gets a solution $\{U_{\mathcal{I}}(t,s)\}_{(t,s)\in\mathbb{R}}$ for any bounded interval $\mathcal{I}$. Then we have to guarantee that two solutions $\{U_{\mathcal{I}_1}(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}_1}}$ and $\{U_{\mathcal{I}_2}(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}_2}}$, which correspond to different bounded open intervals $\mathcal{I}_1$ and $\mathcal{I}_2$, are *compatible*; i.e., one has

$$U_{\mathcal{I}_1}(t,s) = U_{\mathcal{I}_2}(t,s), \quad (t,s) \in \Delta_1 \subseteq \Delta_2, \tag{2.20}$$

for $\mathcal{I}_1 \subseteq \mathcal{I}_2$. Below we clarify this compatibility of propagators in terms of evolution generators.

If $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $L^p(\mathcal{I}_1, X)$ is a subspace of $L^p(\mathcal{I}_2, X)$. Let $Q_{\mathcal{I}_1}$ denote the projection from $L^p(\mathcal{I}_2, X)$ onto the subspace $L^p(\mathcal{I}_1, X)$ given by

$$(Q_{\mathcal{I}_1} f)(t) := \chi_{\mathcal{I}_1}(t) f(t), \quad f \in L^p(\mathcal{I}_2, X).$$

Let intervals $\mathcal{I}_1 = (a_1, b_1)$ and $\mathcal{I}_2 = (a_2, b_2)$ be related by $a_2 \le a_1 < b_1 \le b_2$. We set $\mathcal{I}' = (a_1, b_2)$.

Proposition 2.13. *Let $\mathcal{I}_1 = (a_1, b_1)$ and $\mathcal{I}_2 = (a_2, b_2)$ be two bounded intervals such that $\mathcal{I}_1 \subseteq \mathcal{I}_2$. Further, let $K_{\mathcal{I}_1}$ and $K_{\mathcal{I}_2}$ be forward generators in $L^p(\mathcal{I}_1, X)$ and $L^p(\mathcal{I}_2, X)$, respectively. The corresponding propagators $\{U_{\mathcal{I}_1}(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}_1}}$ and $\{U_{\mathcal{I}_2}(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}_2}}$ are compatible if and only if for any $f \in L^p(\mathcal{I}_2, X)$ obeying $Q_{\mathcal{I}'} f \in \mathrm{dom}(K_{\mathcal{I}_2})$ one has $Q_{\mathcal{I}_1} f \in \mathrm{dom}(K_{\mathcal{I}_1})$ and the relation*

$$K_{\mathcal{I}_1} Q_{\mathcal{I}_1} f = Q_{\mathcal{I}_1} K_{\mathcal{I}_2} Q_{\mathcal{I}'} f. \tag{2.21}$$

Proof. We put $K_j := K_{\mathcal{I}_j}$, $U_j(t,s) := U_{\mathcal{I}_j}$, $Q_j := Q_{\mathcal{I}_j}$, $j = 1,2$, and $Q' := Q_{\mathcal{I}'}$. Assume that the propagators $U_1(t,s)$ and $U_2(t,s)$ are compatible. In this case one easily verifies that

$$e^{-\sigma K_1} Q_1 f = Q_1 e^{-\sigma K_2} Q' f, \quad \sigma \ge 0, \quad f \in L^p(\mathcal{I}_2, X).$$

Moreover, by the fact that

$$\frac{1}{\sigma}(I - e^{-\sigma K_1}) Q_1 f = \frac{1}{\sigma} Q_1 (I - e^{-\sigma K_2}) Q' f, \quad \sigma > 0,$$

one gets that $Q' f \in \mathrm{dom}(K_2)$ yields $Q_1 f \in \mathrm{dom}(K_1)$ as well as (2.21).

To prove the converse we set

$$W(\sigma) f := e^{-(\tau-\sigma)K_1} Q_1 e^{-\sigma K_2} Q' f, \quad 0 \le \sigma \le \tau.$$

If $g := Q' f \in \mathrm{dom}(K_2)$, then $g(\sigma) := e^{-\sigma K_2} Q' f \in \mathrm{dom}(K_2)$ for $\sigma \ge 0$. Since

$$Q' e^{-\sigma K_2} Q' f = e^{-\sigma K_2} Q' f, \quad f \in L^p(\mathcal{I}_2, X), \quad \sigma \ge 0, \tag{2.22}$$

we obtain $Q' g(\sigma) = Q' e^{-\sigma K_1} Q' f \in \mathrm{dom}(K_2)$, which yields $Q_1 e^{-\sigma K_2} Q' f \in \mathrm{dom}(K_1)$. Hence,

$$\frac{d}{d\sigma} W(\sigma) f = e^{-(\tau-\sigma)K_1} \left(K_1 Q_1 - Q_1 K_2\right) e^{-\sigma K_2} Q' f, \quad 0 \le \sigma \le \tau.$$

Applying to this equation the relation (2.21), we obtain $\partial_\sigma W(\sigma)f = 0$, which yields

$$W(\tau)f = W(0)f, \quad \tau \geq 0,$$

or

$$Q_1 e^{-\sigma K_2} Q' f = e^{-\sigma K_1} Q_1 f, \quad \sigma \geq 0, \tag{2.23}$$

for all those $f \in L^p(\mathcal{I}_2, X)$ such that $Q'f \in \mathrm{dom}(K_2)$.

Now we notice that the set $\mathcal{D}'$,

$$\mathcal{D}' := \{f \in L^p(\mathcal{I}_2, X) : Q'f \in \mathrm{dom}(K_2)\},$$

is dense in $L^p(\mathcal{I}', X)$. Indeed, let $\phi \in H^{1,\infty}(\mathcal{I}_2)$ such that $\mathrm{supp}(\phi) \subseteq \overline{\mathcal{I}'}$. By (2.3) we have $M(\phi)f \in \mathrm{dom}(K_2)$ for $f \in \mathrm{dom}(K_2)$. By virtue of the fact that $Q'M(\phi)f = M(\phi)f$, we get $M(\phi)\mathrm{dom}(K_2) \subseteq \mathcal{D}'$. Since this holds for any $\phi \in H^{1,\infty}(\mathcal{I}_2)$ obeying $\mathrm{supp}(\phi) \subseteq \overline{\mathcal{I}'}$, we immediately find that $\mathcal{D}'$ is dense in $L^p(\mathcal{I}', X)$.

Hence, the relation (2.21) holds for any $f \in L^p(\mathcal{I}_2, X)$. This implies the compatibility of the propagators $U_1(t,s)$ and $U_2(t,s)$. □

Corollary 2.14. *Let $\widetilde{K}_{\mathcal{I}_1}$ and $\widetilde{K}_{\mathcal{I}_2}$, $\mathcal{I}_1 \subseteq \mathcal{I}_2$, be evolution operators such that $Q_{\mathcal{I}'} f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}_2})$ yields $Q_{\mathcal{I}_1} f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}_1})$ and the relation*

$$\widetilde{K}_{\mathcal{I}_1} Q_{\mathcal{I}_1} f = Q_{\mathcal{I}_1} \widetilde{K}_{\mathcal{I}_2} Q_{\mathcal{I}'} f \tag{2.24}$$

holds. If the closures $K_{\mathcal{I}_1}$ and $K_{\mathcal{I}_2}$ of the evolution operators $\widetilde{K}_{\mathcal{I}_1}$ and $\widetilde{K}_{\mathcal{I}_2}$ exist in $L^p(\mathcal{I}, X)$, $p \in [1, \infty)$, and they are forward generators, then the corresponding forward propagators are compatible.

Proof. Let $\mathcal{I}_1 = (a_1, b_1)$ and $\mathcal{I}_2 = (a_2, b_2)$, $\mathcal{I}_1 \subseteq \mathcal{I}_2$. As above we set $K_j := K_{\mathcal{I}_j}$, $U_j(t,s) := U_{\mathcal{I}_j}$, $Q_j := Q_{\mathcal{I}_j}$, $j = 1, 2$, and $Q' := Q_{\mathcal{I}'}$, where $\mathcal{I}' = (a_1, b_2)$, see above. Further, let $\widetilde{K}_j := \widetilde{K}_{\mathcal{I}_j}$, $j = 1, 2$. Let $g := Q'f \in \mathrm{dom}(K_2)$. Then there is a sequence $\{g_n\}_{n \in \mathbb{N}}$, $g_n \in \mathrm{dom}(\widetilde{K}_2)$, such that $g_n \longrightarrow g$ and $\widetilde{K}_2 g_n \longrightarrow K_2 g$ as $n \to \infty$. Let $\phi \in H^{1,\infty}(\mathcal{I}_2)$ such that $\mathrm{supp}(\phi) \subseteq \overline{\mathcal{I}}'$. By (2.3) we have $M(\phi)g_n \in \mathrm{dom}(\widetilde{K}_2)$ and $Q'M(\phi)g_n = M(\phi)g_n$, $n \in \mathbb{N}$. Then taking into account (2.24) we obtain

$$\widetilde{K}_1 Q_1 M(\phi) g_n = Q_1 \widetilde{K}_2 Q' M(\phi) g_n, \quad n \in \mathbb{N}.$$

Using (2.4) we find

$$Q_1 \widetilde{K}_2 Q' M(\phi) g_n = Q_1 \widetilde{K}_2 M(\phi) g_n = M(\phi) Q_1 \widetilde{K}_2 g_n + M(\dot{\phi}) Q_1 g_n, \quad n \in \mathbb{N},$$

which yields

$$Q_1 \widetilde{K}_2 Q' M(\phi) g_n \longrightarrow Q_1 K_2 M(\phi) Q' f \quad \text{as} \quad n \to \infty.$$

Hence, we obtain

$$\widetilde{K}_1 Q_1 M(\phi) g_n \longrightarrow Q_1 K_2 M(\phi) Q' f \quad \text{as} \quad n \to \infty,$$

which proves

$$\widetilde{K}_1 Q_1 M(\phi) g_n \longrightarrow K_1 Q_1 M(\phi) f \quad \text{as} \quad n \to \infty$$

and

$$K_1 Q_1 M(\phi) f = Q_1 K_2 M(\phi) Q' f.$$

Using (2.4) we also get

$$K_1 Q_1 M(\phi) f = M(\phi) Q_1 K_2 Q' f + M(\dot{\phi}) Q_1 f \tag{2.25}$$

for $\phi \in H^{1,\infty}(\mathcal{I}_2)$ obeying $\mathrm{supp}(\phi) \subseteq \overline{\mathcal{I}}'$.

Let us put

$$\phi_\delta(t) := \begin{cases} 0 & t \in (a_2, a_1] \\ (t - a_1)/\delta & t \in (a_1, a_1 + \delta) \\ 1 & t \in [a_1 + \delta, b_2), \end{cases}$$

where $\delta > 0$. Then by (2.25) we obtain

$$K_1 Q_1 M(\phi_\delta) f = M(\phi_\delta) Q_1 K_2 Q' f + \frac{1}{\delta} M(\chi_{(a_1, a_1+\delta)}) Q_1 f \tag{2.26}$$

for any $\delta > 0$. Since $g = Q' f \in \mathrm{dom}(K_2)$ and the function is continuous at $t = a_1$ we get $g(a_1) = 0$. Furthermore, from (2.26) we find that

$$(K_1 + \xi) Q_1 M(\phi_\delta) f = M(\phi_\delta) Q_1 (K_2 + \xi) Q' f + \frac{1}{\delta} M(\chi_{(a_1, a_1+\delta)}) Q_1 Q' f, \quad \xi > 0,$$

which yields

$$Q_1 M(\phi_\delta) f = \tag{2.27}$$
$$(K_1 + \xi)^{-1} M(\phi_\delta) Q_1 (K_2 + \xi) Q' f + \frac{1}{\delta} (K_1 + \xi)^{-1} M(\chi_{(a_1, a_1+\delta)}) Q_1 g$$

for $\xi > 0$. From Theorem 4.10 of [34] we get the representation

$$\frac{1}{\delta} \left((K_1 + \xi)^{-1} M(\chi_{(a_1, a_1+\delta)}) Q_1 g \right)(t)$$
$$= \frac{1}{\delta} \int_{a_1}^{t} e^{-\xi(t-s)} U_1(t,s) \chi_{(a_1, a_1+\delta)}(s) g(s) ds, \qquad t \in \mathcal{I}_1,$$

$t \in \mathcal{I}_1$, where the propagator $U_1(\cdot,\cdot)$ corresponds to the forward generator K_1. We have

$$\left\| \frac{1}{\delta} \int_{a_1}^{t} e^{-\xi(t-s)} U_1(t,s) \chi_{(a_1, a_1+\delta)}(s) g(s) ds \right\| \le \|U_1\|_{\mathcal{B}(X)} \frac{1}{\delta} \int_{a_1}^{a_1+\delta} \|g(s)\| ds$$

for $t \in \mathcal{I}_1$. Since g is continuous and $g(a_1) = 0$ one gets

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_{a_1}^{a_1+\delta} \|g(s)\| ds = 0$$

which yields

$$\lim_{\delta\to 0} \frac{1}{\delta}(K_1 + \xi)^{-1} M(\chi_{(a_1,a_1+\delta)}) Q_1 g = 0$$

in $L^p(\mathcal{I}_1, X)$. We obtain from (2.27) that

$$Q_1 f = (K_1 + \xi)^{-1} Q_1 (K_2 + \xi) Q' f, \quad \xi > 0,$$

which shows $Q_1 f \in \mathrm{dom}(K_1)$ and $K_1 Q_1 f = Q_1 K_2 Q' f$. Applying Proposition 2.13 one completes the proof. □

Definition 2.15. *Let $\{A(t)\}_{t\in\mathbb{R}}$ be a measurable family of closed and densely defined linear operators in the separable Banach space X.*

(i) *The forward evolution equation* (1.1) *is well posed on $\mathbb{R}$ for some $p \in [1, \infty)$ if for any bounded open interval $\mathcal{I}$ of $\mathbb{R}$ the operator $\widetilde{K}_\mathcal{I}$ is an evolution operator.*
(ii) *A forward propagator $\{U(t, s)\}_{(t,s)\in\Delta_\mathbb{R}}$ is called a solution of the well-posed forward evolution equation* (1.1) *on $\mathbb{R}$ if $\{U_\mathcal{I}(t, s)\}_{(t,s)\in\Delta_\mathcal{I}}$, $U_\mathcal{I}(\cdot, \cdot) := U(\cdot, \cdot) \restriction \Delta_\mathcal{I}$, is a solution of the forward evolution equation* (1.1) *for any bounded interval $\mathcal{I}$ of $\mathbb{R}$.*
(iii) *The well-posed forward evolution equation* (1.1) *on $\mathbb{R}$ has a unique solution if for any bounded interval $\mathcal{I} \subseteq \mathbb{R}$ the forward evolution equation* (1.1) *admits a unique solution.*

This definition can be extended (*mutatis mutandis*) to backward and to bidirectional evolution equations on $\mathbb{R}$.

Theorem 2.16. *Let $\{A(t)\}_{t\in\mathbb{R}}$ be a measurable family of closed and densely defined linear operators in the separable Banach space X. Assume that the forward evolution equation* (1.1) *is well posed on $\mathbb{R}$ for some $p \in [1, \infty)$. If for any bounded open interval $\mathcal{I}$ of $\mathbb{R}$ the closure $K_\mathcal{I}$ of the evolution operator $\widetilde{K}_\mathcal{I}$ exists in $L^p(\mathcal{I}, X)$, $p \in [1, \infty)$, and it is an anti-generator, then the forward evolution equation* (1.1) *has a unique solution on $\mathbb{R}$.*

Proof. Let $\mathcal{I}_1 \subseteq \mathcal{I}_2$. One can easily verify that the evolution operators $\widetilde{K}_{\mathcal{I}_1}$ and $\widetilde{K}_{\mathcal{I}_2}$, which are given by (2.8), satisfy the condition (2.24). Since the operators $\widetilde{K}_{\mathcal{I}_1}$ and $\widetilde{K}_{\mathcal{I}_2}$ are closable and their closures are already forward evolution generators, one gets from Corollary 2.14 that the corresponding forward propagators (they exist and are unique by Theorem 2.4) are compatible. □

Proposition 2.13, Corollary 2.15 and Theorem 2.16 can be generalized (*mutatis mutandis*) to backward and bidirectional evolution equations.

3. Semigroup perturbations

Theorem 2.4 shows that the problem of the unique solution of the forward or backward evolution equation (1.1) can be transformed to the question whether the evolution operators $\widetilde{K}_{\mathcal{I}}$ or $\widetilde{K}^{\mathcal{I}}$ are closable and their closures $K_{\mathcal{I}}$ or $K^{\mathcal{I}}$ are anti-generators or generators in $L^p(\mathcal{I}, X)$ for some $p \in [1, \infty)$. In applications $\{A(t)\}_{t\in\mathcal{I}}$ is often a measurable family of anti-generators or generators belonging uniformly to the class $\mathcal{G}(M, \beta)$, for some constants M and β. One can easily verify that in this case the induced multiplication operator A is an anti-generator or generator in $L^p(\mathcal{I}, X)$.

This reduces the problem to the following one: Let T and A be anti-generators or generators in some Banach space space $\mathfrak{X}$; is it possible to find conditions ensuring that their operator sum $\widetilde{K}$:

$$\widetilde{K}f = Tf + Af, \quad \mathrm{dom}(T) \cap \mathrm{dom}(A), \tag{3.1}$$

is closable in $\mathfrak{X}$ and its closure K is an anti-generator or generator? To prove this kind of result we rely on the following theorem.

Theorem 3.1. *Let the operators T and A be generators in $\mathfrak{X}$ both belonging to the class $\mathcal{G}(1, 0)$. If $\mathrm{dom}(T) \cap \mathrm{dom}(A)$ is dense in $\mathfrak{X}$ and $\mathrm{ran}(T + A + \xi)$ is dense in $\mathfrak{X}$ for $\xi < 0$, then $\widetilde{K}$ is closable and its closure K is a generator from the class $\mathcal{G}(1, 0)$.*

This theorem was originally proved by Kato, see [23, Theorem IX.2.11], however, under the additional assumption that $\widetilde{K}$ is closable. This condition was dropped by Da Prato in [4, Proposition 1.1], see also Da Prato and Grisvard in [5, Theorem 5.6].

In general, the assumption $T, A \in \mathcal{G}(1, 0)$ is too restrictive for our purposes. So, we modify this assumption. It is known that in general it is possible to find for the Banach space $\mathfrak{X}$ a new norm such that one of the operators, T or A, becomes a generator of a contraction semigroup on $\mathfrak{X}$. Indeed, since T is the generator of a C_0 semigroup; i.e., $T \in \mathcal{G}(M, \beta_T)$, one has

$$\|e^{\sigma T} f\| \le M_T e^{\beta_T \sigma}. \tag{3.2}$$

Setting

$$|||f||| := \sup_{\sigma>0} e^{-\beta_T\,\sigma}\|e^{\sigma\,T}f\|$$

one immediately gets that

$$|||e^{\tau T}f||| = e^{\beta_T\,\tau}\sup_{\sigma>0} e^{-\beta_T\,\{\sigma+\tau\}}\|e^{\tau T}f\|.$$

This observation shows that in the Banach space $\mathfrak{X}$ endowed with the norm $|||\cdot|||$ the semigroup $\{e^{\sigma T}\}_\sigma$ belongs to the class $\mathcal{G}(1,\beta_T)$ of *quasi-contractive* semigroups. Since

$$\|f\| \leq |||f||| \leq M_T\|f\|,$$

the norm $|||\cdot|||$ is equivalent to $\|\cdot\|$. The same reasoning can be applied to the semigroup $\{e^{\sigma A}\}_{\sigma\geq 0}$, but in general it is *impossible* to find an equivalent norm such that *both* semigroups become quasi-contractive.

Definition 3.2. *Let T and A be generators of C_0-semigroups $e^{\sigma T}$ and $e^{\sigma A}$ in $\mathfrak{X}$. The pair $\{T,A\}$ is called renormalizable with constants β_A and β_T if for any sequences $\{\tau_k\}_{k=1}^N$, $\tau_k \geq 0$, and $\{\sigma_k\}_{k=1}^N$, $\sigma_k \geq 0$, $n \in \mathbb{N}$, one has*

$$\sup_{\substack{\tau_1\geq 0,\ldots,\tau_n\geq 0\\ \sigma_1\geq 0,\ldots,\sigma_n\geq 0\\ n\in\mathbb{N}}} e^{-\beta_T\sum\tau_k}e^{-\beta_A\sum\sigma_k}\|e^{\tau_1 T}e^{\sigma_1 A}\cdots e^{\tau_n T}e^{\sigma_n A}f\| < \infty \tag{3.3}$$

for each $f \in \mathfrak{X}$. In an obvious manner the definition carries over to pairs $\{T,A\}$ of anti-generators.

Remark 3.3. In the following we formulate the statements in terms of pairs of generators. However, it is easy to see that these statements remain true for pairs of anti-generators.

Lemma 3.4 (Lemma 5.1, [36]). *Let T and A be generators of C_0-semigroups in $\mathfrak{X}$. There is an equivalent norm $|||\cdot|||$ on $\mathfrak{X}$ such that $T \in \mathcal{G}(1,\beta_T)$ and $A \in \mathcal{G}(1,\beta_A)$ if and only if the pair $\{T,A\}$ is renormalizable with constants β_T and β_A.*

Proof. Let the pair $\{T,A\}$ be renormalizable with constants β_T and β_A. On the space $\mathfrak{X}$ we define a norm by

$$|||f||| := \sup_{\substack{\tau_1\geq 0,\ldots,\tau_n\geq 0\\ \sigma_1\geq 0,\ldots,\sigma_n\geq 0\\ n\in\mathbb{N}}} e^{-\beta_T\sum\tau_k}e^{-\beta_A\sum\sigma_k}\|e^{\tau_1 T}e^{\sigma_1 A}\cdots e^{\tau_n T}e^{\sigma_n A}f\|$$

Obviously, we have $\|f\| \leq |||f|||$, $f \in \mathfrak{X}$. On the other hand, by the *uniform boundedness principle*, see e.g. [23, Theorem I.1.29], we find that the value of

$$M := \sup_{\substack{\tau_1 \geq 0, \ldots, \tau_n \geq 0 \\ \sigma_1 \geq 0, \ldots, \sigma_n \geq 0 \\ n \in \mathbb{N}, \|f\| \leq 1}} e^{-\beta_T \sum \tau_k} e^{-\beta_A \sum \sigma_k} \|e^{\tau_1 T} e^{\sigma_1 A} \cdots e^{\tau_n T} e^{\sigma_n A} f\|$$

is finite, which yields $|||f||| \leq M\|f\|$, $f \in \mathfrak{X}$. Hence, the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent. Moreover, it turns out that $T \in \mathcal{G}(M, \beta_T)$ and $A \in \mathcal{G}(M, \beta_A)$. Furthermore, a straightforward computation shows that

$$|||e^{\tau T} f||| \leq e^{\beta_T \tau} |||f|||, \quad f \in \mathfrak{X},$$

$$|||e^{\sigma A} f||| \leq e^{\beta_A \sigma} |||f|||, \quad f \in \mathfrak{X}.$$

Therefore, in the Banach space $\{\mathfrak{X}, |||\cdot|||\}$ the generators T and A belong to $\mathcal{G}(1, \beta_T)$ and $\mathcal{G}(1, \beta_A)$, respectively.

Conversely, if there is an equivalent norm $|||\cdot|||$ in the Banach space $\mathfrak{X}$ such that $T \in \mathcal{G}(1, \beta_T)$ and $A \in \mathcal{G}(1, \beta_A)$, then a straightforward computation yields (3.3); i.e., the pair $\{T, A\}$ is renormalizable with constants β_T and β_A. □

Definition 3.5. *Let $\mathfrak{Y}$ be a Banach space which is densely and continuously embedded into the Banach space $\mathfrak{X}$; i.e. $\mathfrak{Y} \hookrightarrow \mathfrak{X}$, and let the operator T be the generator of a C_0-semigroup in $\mathfrak{X}$. The Banach space $\mathfrak{Y}$ is called admissible with respect to T, if the space $\mathfrak{Y}$ is invariant with respect to the semigroup $e^{\sigma T}$; i.e.,*

$$e^{\sigma T} \mathfrak{Y} \subseteq \mathfrak{Y}, \quad \sigma \geq 0,$$

and the restriction $e^{\sigma \widehat{T}} := e^{\sigma T} \upharpoonright \mathfrak{Y}$, $\sigma \geq 0$, is a C_0-semigroup on $\mathfrak{Y}$.

If $J : \mathfrak{Y} \longrightarrow \mathfrak{X}$ is the *embedding operator* of $\mathfrak{Y}$ into $\mathfrak{X}$, then we get

$$e^{\sigma T} J f = J e^{\sigma \widehat{T}} f, \quad f \in \mathfrak{Y},$$

which yields

$$T J f = J \widehat{T} f, \quad f \in \mathrm{dom}(\widehat{T}).$$

Lemma 3.6. *Let $\widehat{T}$ and $\widehat{A}$ be generators of C_0-semigroups of class $\mathcal{G}(1, 0)$ in the Banach space $\mathfrak{Y}$. If either $\mathrm{dom}(\widehat{T}^*)$ or $\mathrm{dom}(\widehat{A}^*)$ are dense in $\mathfrak{Y}^*$, then for any $\xi < 0$ one gets the inequality*

$$|\xi| \|g\|_{\mathfrak{Y}^*} \leq \|\widehat{T}^* g + \widehat{A}^* g + \xi g\|_{\mathfrak{Y}^*}, \quad g \in \mathrm{dom}(\widehat{T}^*) \cap \mathrm{dom}(\widehat{A}^*). \tag{3.4}$$

Proof. Let $\mathrm{dom}(\widehat{A}^*)$ be dense in $\mathfrak{Y}^*$. We define

$$\widehat{A}_\alpha := \widehat{A}(I + \alpha\widehat{A})^{-1}, \quad \alpha < 0.$$

Since $\widehat{A} \in \mathcal{G}(1,0)$ we have $\widehat{A}_\alpha \in \mathcal{G}(1,0)$ for $\alpha < 0$. Further, we set

$$\widehat{K}_\alpha f := \widehat{T} f + \widehat{A}_\alpha f, \quad f \in \mathrm{dom}(\widehat{K}) := \mathrm{dom}(\widehat{T}), \quad \alpha < 0.$$

Since $\widehat{T} \in \mathcal{G}(1,0)$ and $\widehat{A}_\alpha \in \mathcal{G}(1,0)$ we find that $\widehat{K}_\alpha \in \mathcal{G}(1,0)$, $\alpha < 0$. This yields the estimate

$$\|(\widehat{K}_\alpha + \xi)^{-1} f\|_{\mathfrak{Y}} \leq \frac{1}{|\xi|}\|f\|_{\mathfrak{Y}}, \quad f \in \mathfrak{Y}, \quad \alpha < 0, \quad \xi < 0.$$

Hence, we obtain

$$\|(\widehat{K}^*_\alpha + \xi)^{-1} g\|_{\mathfrak{Y}^*} \leq \frac{1}{|\xi|}\|g\|_{\mathfrak{Y}^*}, \quad g \in \mathfrak{Y}^*, \quad \alpha < 0, \quad \xi < 0,$$

or

$$|\xi|\|g\|_{\mathfrak{Y}^*} \leq \|(\widehat{K}^*_\alpha + \xi) g\|_{\mathfrak{Y}^*}, \; g \in \mathrm{dom}(\widehat{K}^*_\alpha) = \mathrm{dom}(\widehat{T}^*), \; \alpha < 0, \; \xi < 0. \quad (3.5)$$

Note that

$$\widehat{K}^*_\alpha g = \widehat{T}^* g + \widehat{A}^*_\alpha g, \quad g \in \mathrm{dom}(\widehat{T}^*), \; \alpha < 0, \; \xi < 0.$$

Now, since $\mathrm{dom}(\widehat{A}^*)$ is dense in $\mathfrak{Y}^*$, we get

$$s - \lim_{\alpha \to 0} (I + \alpha\widehat{A}^*)^{-1} = I, \quad \alpha < 0,$$

which yields

$$\lim_{\alpha \to 0} \widehat{K}^*_\alpha g = \widehat{T}^* g + \widehat{A}^* g, \quad \alpha < 0,$$

for $g \in \mathrm{dom}(\widehat{T}^*) \cap \mathrm{dom}(\widehat{A}^*)$. Then for $\alpha \to 0$ the inequality (3.5) yields (3.4).

The proof is similar, if one supposes that $\mathrm{dom}(\widehat{T}^*)$ is dense in $\mathfrak{Y}^*$. □

Corollary 3.7. *Let T and A be generators of C_0-semigroups of class $\mathcal{G}(1,0)$ on $\mathfrak{X}$. Further, let $\mathfrak{Y} \hookrightarrow \mathfrak{X}$ be admissible with respect to T, A and let the operator A be such that*

$$\mathfrak{Y} \subseteq \mathrm{dom}(A). \quad (3.6)$$

Assume that the induced generators $\widehat{T}$ and $\widehat{A}$ are of the class $\mathcal{G}(1,0)$. If $\mathrm{dom}(A^*)$ *is dense in $\mathfrak{X}^*$, then*

$$|\xi|\|g\|_{\mathfrak{Y}^*} \leq \|\widehat{T}^* g + \widehat{A}^* g + \xi g\|_{\mathfrak{Y}^*}, \quad g \in \mathrm{dom}(\widehat{T}^*) \cap J^*\mathfrak{X}^*, \quad (3.7)$$

for $\xi < 0$ where $J : \mathfrak{Y} \longrightarrow \mathfrak{X}$ is the embedding operator.

Proof. By condition (3.6) we get that $\mathrm{dom}(\widehat{A}^*) \supseteq J^*\mathfrak{X}^*$. Let $g \in \mathrm{dom}(\widehat{T}^*) \cap J^*\mathfrak{X}^*$. Then there is $h \in \mathfrak{X}^*$ such that $g = J^*h$. Hence,

$$\widehat{K}^*_\alpha J^*h = \widehat{T}^* J^*h + \widehat{A}^* J^*(I + \alpha A^*)^{-1}h, \quad \alpha < 0.$$

By condition (3.6) the operator $B := AJ : \mathfrak{Y} \longrightarrow \mathfrak{X}$ is bounded. This yields the representation

$$\widehat{K}^*_\alpha J^*h = \widehat{T} J^*h + B^*(I + \alpha A^*)^{-1}h.$$

Since $\mathrm{dom}(A^*)$ is dense in $\mathfrak{X}^*$ we have $s - \lim_{\alpha \to 0}(I + \alpha A^*)^{-1} = I$. Hence

$$\lim_{\alpha \to 0} \widehat{K}^*_\alpha J^*g = \widehat{T} J^*h + B^*h = \widehat{T}^*g + \widehat{A}^*g.$$

Using (3.5) we get (3.7). $\square$

Theorem 3.8 (Theorem 5.5, [36])**.** *Let $\{T, A\}$ be a renormalizable pair of generators of C_0-semigroups on $\mathfrak{X}$. Further, let the Banach space $\mathfrak{Y} \hookrightarrow \mathfrak{X}$ be admissible with respect to the operators T and A. Assume that A satisfies condition* (3.6) *and that the pair $\{\widehat{T}, \widehat{A}\}$ is renormalizable. If either one of the domains* $\mathrm{dom}(\widehat{T}^*)$, $\mathrm{dom}(\widehat{A}^*)$ *is dense in $\mathfrak{Y}^*$, or* $\mathrm{dom}(A^*)$ *is dense in $\mathfrak{X}^*$, then the closure K of $\widetilde{K}$,*

$$\widetilde{K}f := Tf + Af, \quad \mathrm{dom}(\widetilde{K}) = \mathrm{dom}(T) \cap \mathrm{dom}(A),$$

exists and K is the generator of a C_0-semigroup.

Proof. Since the pairs $\{T, A\}$ and $\{\widehat{T}, \widehat{A}\}$ are renormalizable we can assume without lost of generality that $T, A \in \mathcal{G}(1, 0)$ as well as $\widehat{T}, \widehat{A} \in \mathcal{G}(1, 0)$. It is obvious that

$$TJf = J\widehat{T}f, \quad f \in \mathrm{dom}(\widehat{T}),$$

and

$$AJf = J\widehat{A}f, \quad f \in \mathrm{dom}(\widehat{A}).$$

By condition (3.6) we get that $J^*\mathfrak{X}^* \subseteq \mathrm{dom}(\widehat{A}^*)$. Since $\mathrm{dom}(\widehat{T})$ is dense in $\mathfrak{Y}$ and $\mathfrak{Y}$ is densely embedded in $\mathfrak{X}$, we get that the operator $\widetilde{K}$ is densely defined. In particular, we have

$$\mathfrak{D} := J \,\mathrm{dom}(\widehat{T}) \subseteq \mathrm{dom}(\widetilde{K}).$$

We set $\widetilde{K}_{\mathfrak{D}} := \widetilde{K} \restriction \mathfrak{D} \subseteq \widetilde{K}$. Let $g \in \mathrm{dom}(\widetilde{K}^*_{\mathfrak{D}}) \subseteq \mathfrak{X}^*$. Then we have

$$\left\langle \widetilde{K}_{\mathfrak{D}} Jf, g \right\rangle = \langle TJf, g\rangle + \langle Bf, g\rangle = \left\langle J\widehat{T}f, g \right\rangle + \langle f, B^*g\rangle$$

for $f \in \mathrm{dom}(\widehat{T})$ where the operator $B : \mathfrak{Y} \longrightarrow \mathfrak{X}$ is defined by $Bf := AJf$, $f \in \mathfrak{Y}$. By assumption (3.6) the operator B is well defined and bounded. Hence,

$$\left\langle \widehat{T} f, J^* g \right\rangle = \langle f, B^* g\rangle - \left\langle f, J^* \widetilde{K}^*_{\mathfrak{D}} g \right\rangle, \quad f \in \mathrm{dom}(\widehat{T}),$$

which yields $J^*\mathrm{dom}(\widetilde{K}^*_{\mathfrak{D}}) \subseteq \mathrm{dom}(\widehat{T}^*)$. Since $J^*\mathfrak{X}^* \subseteq \mathrm{dom}(\widehat{A}^*)$ we obtain

$$J^*\mathrm{dom}(\widetilde{K}^*_{\mathfrak{D}}) \subseteq \mathrm{dom}(\widehat{T}^*) \cap \mathrm{dom}(\widehat{A}^*). \tag{3.8}$$

Now, assume that $\mathrm{ran}(\widetilde{K}_{\mathfrak{D}} + \xi)$ is not dense in $\mathfrak{X}$ for some $\xi < 0$. In this case there is a $g \in \mathfrak{X}^*$ such that

$$\left\langle (\widetilde{K}_{\mathfrak{D}} + \xi) f, g \right\rangle = 0, \quad f \in \mathrm{dom}(\widetilde{K}_{\mathfrak{D}}) = \mathfrak{D}.$$

Hence $g \in \mathrm{dom}(\widetilde{K}^*_{\mathfrak{D}})$ and $(\widetilde{K}^*_{\mathfrak{D}} + \xi)g = 0$. By (3.8) we obtain

$$J^* g \in \mathrm{dom}(\widehat{T}^*) \cap \mathrm{dom}(\widehat{A}^*).$$

If either $\mathrm{dom}(\widehat{T}^*)$ or $\mathrm{dom}(\widehat{A}^*)$ is dense in $\mathfrak{Y}^*$, then by Lemma 3.6 we get $J^*g = 0$, which yields $g = 0$. If $\mathrm{dom}(A^*)$ is dense in $\mathfrak{X}^*$, then we apply Corollary 3.7 and find also $J^*g = 0$, which yields $g = 0$. Hence the range $\widetilde{K}_{\mathfrak{D}} + \xi$, $\xi < 0$, is dense in $\mathfrak{X}$. Since $\widetilde{K}_{\mathfrak{D}} \subseteq \widetilde{K}$ the range $\widetilde{K} + \xi$, $\xi < 0$, is also dense in $\mathfrak{X}$. Applying Theorem 3.1 we find that $\widetilde{K}$ is closable and its closure K belongs to $\mathcal{G}(0, 1)$. □

Remark 3.9. Under the assumptions of Theorem 3.8 it follows that the set $\mathfrak{D} := J\mathrm{dom}(\widehat{T}) \subseteq \mathfrak{X}$ is a core of K; i.e., the closure $K_{\mathfrak{D}}$ of $\widetilde{K}_{\mathfrak{D}} = K \restriction \mathfrak{D}$ coincides with K. Indeed, using the fact that the range of $\widetilde{K}_{\mathfrak{D}} + \xi$, $\xi < 0$, is dense in $\mathfrak{X}$ and applying again Theorem 3.1 we get that $\widetilde{K}_{\mathfrak{D}}$ is closable and its closure $K_{\mathfrak{D}}$ belongs to $\mathcal{G}(0, 1)$. Since $K_{\mathfrak{D}} \subseteq K$ and both operators belong to $\mathcal{G}(0, 1)$ we obtain $K_{\mathfrak{D}} = K$. Hence $\mathfrak{D}$ is a core of K.

Taking into account Theorem 3.8 and [49] one immediately obtains the following corollary.

Corollary 3.10. *Let the assumptions of Theorem 3.8 be satisfied. If either one of the domains* $\mathrm{dom}(\widehat{T}^*)$, $\mathrm{dom}(\widehat{A}^*)$ *is dense in* $\mathfrak{Y}^*$, *or* $\mathrm{dom}(A^*)$ *is dense in* $\mathfrak{X}^*$, *then the Trotter product formula*

$$s - \lim_{n\to\infty} \left(e^{\sigma\, T/n} e^{\sigma\, A/n} \right)^n = e^{\sigma\, K}$$

holds uniformly in $\sigma \in [0, \sigma_0]$, *for any* $\sigma_0 > 0$.

4. Solutions of evolution equations

4.1. **Solutions of forward evolution equations.** Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of anti-generators of class $\mathcal{G}(M,\beta)$, in the separable Banach space X. By A we denote the multiplication operator induced by (2.6) and (2.7) in the Banach space $\mathfrak{X} = L^p(\mathcal{I},X)$, $1 \le p < \infty$. Notice that A is an anti-generator of a C_0-semigroup on $\mathfrak{X} = L^p(\mathcal{I},X)$ of class $\mathcal{G}(M,\beta)$.

Definition 4.1 ([24, 25])**.** *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of anti-generators of C_0-semigroups in the separable Banach space X. The family is called forward stable, if there are constants $M > 0$ and $\beta \ge 0$ such that the estimate*

$$\|e^{-\sigma_1 A(t_1)}e^{-\sigma_2 A(t_2)}\cdots e^{-\sigma_n A(t_n)}\|_{\mathcal{B}(X)} \le Me^{\beta\sum_{k=1}^n \sigma_k}$$

holds for each of the sequences $\{\sigma_k\}_{k=1}^n$, $\sigma_k \ge 0$, and almost every $(t_1,t_2,\ldots,t_n) \in \Delta_n := \{(t_1,t_2,\ldots,t_n) \in \mathbb{R}^n : a < t_n \le t_{n-1} \le \cdots \le t_1 < b\}$ with respect to the $\mathbb{R}^n$-Lebesgue measure.

It is clear that, if $\{A(t)\}_{t\in\mathcal{I}}$ is forward stable, then the anti-generators $A(t)$ belong to $\mathcal{G}(M,\beta)$ for almost every $t \in \mathcal{I}$.

Lemma 4.2 (Lemma 5.9, [36])**.** *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of anti-generators of C_0-semigroups in the separable Banach space X. The pair of anti-generators $\{D_{\mathcal{I}},A\}$ is renormalizable on $\mathfrak{X} = L^p(\mathcal{I},X)$, $1 \le p < \infty$, if and only if the family of anti-generators $\{A(t)\}_{t\in\mathcal{I}}$ is forward stable.*

Definition 4.3. *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of anti-generators (generators) of class $\mathcal{G}(M,\beta)$ in the separable Banach space X. Further, let Y be a separable Banach space which is densely and continuously embedded into X. The Banach space Y is called admissible with respect to the family $\{A(t)\}_{t\in\mathcal{I}}$ if*

- (i) *for almost every $t \in \mathcal{I}$, the Banach space Y is admissible with respect to $A(t)$,*
- (ii) *there are constants $\widehat{M}$ and $\widehat{\beta}$ such that the anti-generators (generators) $\{\widehat{A}(t)\}_{t\in\mathcal{I}}$ of the induced semigroups belong to $\mathcal{G}(\widehat{M},\widehat{\beta})$ for almost every $t \in \mathcal{I}$,*
- (iii) *the family $\{\widehat{A}(t)\}_{t\in\mathcal{I}}$ is measurable in Y.*

We note that the condition (iii) in Definition 4.3 is redundant if X^* is densely embedded into the Banach space Y^*.

Lemma 4.4 (Lemma 5.11, [36])**.** *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of anti-generators in the separable Banach space X belonging to $\mathcal{G}(M,\beta)$ and* *let the separable Banach space Y be densely and continuously embedded into X. The Banach space $\mathfrak{Y} = L^p(\mathcal{I}, Y)$, $1 \le p < \infty$, is admissible with respect to the anti-generator A if and only if the family $\{A(t)\}_{t\in\mathcal{I}}$ is admissible with respect to Y.*

Summing up all those properties it is useful for further purposes to introduce the following definition:

Definition 4.5. *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of anti-generators in the separable Banach space X. Further, let Y be a separable Banach space which is densely and continuously embedded into X. We say the family $\{A(t)\}_{t\in\mathcal{I}}$ satisfies the forward Kato condition if*

(i) *$\{A(t)\}_{t\in\mathcal{I}}$ is forward stable in X,*
(ii) *the Banach space Y is admissible with respect to the family $\{A(t)\}_{t\in\mathcal{I}}$,*
(iii) *the induced family $\{\widehat{A}(t)\}_{t\in\mathcal{I}}$ is forward stable in Y,*
(iv) $Y \subseteq \mathrm{dom}(A(t))$ *holds for almost every $t \in \mathcal{I}$,*
(v) $A(\cdot)\restriction Y$ *is strongly measurable and* $\operatorname{ess\,sup}_{t\in\mathcal{I}} \|A(t)\restriction Y\|_{B(Y,X)} < \infty$.

In the following we use the so-called *Radon-Nikodym property* of certain Banach spaces, see e.g. [9].

We recall that a scalar-valued measure $\mu(\cdot)$ defined on the Borel sets of $\mathbb{R}$ satisfies the Radon-Nikodym property if, for instance, its continuity with respect to the Lebesgue measure implies the existence of a locally summable function $f(\cdot)$ such that $\mu(\delta) = \int_\delta f(x)dx$ for any bounded Borel set $\delta \subset \mathbb{R}$. In general, this property *does not* extend to measures taking their values in Banach spaces. However, there are classes of Banach spaces where this Radon-Nikodym property still holds. For example, *dual* spaces of *separable* Banach spaces admit this property if and only if they are themselves separable. This, in particular, yields that the dual Banach space $L^p(\mathcal{I}, Y)^*$, $1 < p < \infty$, is isometric to $L^q(\mathcal{I}, Y^*)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4.6. *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of anti-generators in a separable Banach space X. Further, let Y be a separable Banach space which is densely and continuously embedded into X. If $\{A(t)\}_{t\in\mathcal{I}}$ obeys the forward Kato condition and if, in addition, one of the following conditions:*

(A_1) *Y^* satisfies the Radon-Nikodym property,*
(A_2) $\mathrm{dom}(\widehat{A}^*(t))$ *is dense in Y^* for almost every $t \in \mathcal{I}$,*
(A_3) $\mathrm{dom}(A(t)^*)$ *is dense in X^* for almost every $t \in \mathcal{I}$ holds,*
then the forward evolution equation (1.1) *is well posed on $\mathcal{I}$ for some $p \in (1, \infty)$ and has a unique solution.*

Proof. By Lemma 4.2 the pair $\{D_{\mathcal{I}}, A\}$ of anti-generators is renormalizable. Further, let us consider the Banach space $\mathfrak{Y} = L^p(\mathcal{I}, Y)$, $1 < p < \infty$. Since Y is densely and continuously embedded into X the Banach space $\mathfrak{Y}$ is densely and continuously embedded in $\mathfrak{X} = L^p(\mathcal{I}, X)$. Since the family $\{A(t)\}_{t\in\mathcal{I}}$ is admissible with respect to Y, the operator A is admissible with $\mathfrak{Y}$, cf. Lemma 4.4. Then from conditions (iv) and (v) of Definition 4.5 we find that $\mathfrak{Y} \subseteq \mathrm{dom}(A)$.

Let (see (A_1)) Y^* satisfy the Radon-Nikodym property. Then $\mathfrak{Y}^* = L^p(\mathcal{I}, Y)^* = L^q(\mathcal{I}, Y^*)$, $1/p + 1/q = 1$, which yields that $\mathrm{dom}(\widehat{D}^*_{\mathcal{I}})$ is dense in $\mathfrak{Y}^*$. Applying Theorem 3.8 we immediately get that $\widetilde{K}_{\mathcal{I}}$ is closable and its closure K generates a C_0-semigroup. Taking into account Theorem 2.4 and Theorem 3.8 we complete the proof of the theorem under condition (A_1).

If Y does not satisfy the Radon-Nikodym property, then the dual space $\mathfrak{Y}^*$ can be identified with a space $L^q_w(\mathcal{I}, Y^*)$, cf. [2]. The space $L^q_w(\mathcal{I}, Y^*)$ consists of equivalence classes $[g]$ of w^*-measurable functions $g(\cdot) : \mathcal{I} \longrightarrow Y^*$ such that $\int_0^T \|g(t)\|^q_{Y^*}\, dt < \infty$. Two functions $g_1(\cdot) : \mathcal{I} \longrightarrow Y^*$ and $g_2(\cdot) : \mathcal{I} \longrightarrow Y^*$ are called *equivalent*, if $< x, g_1(t) >=< x, g_2(t) >$ holds for almost every $t \in \mathcal{I}$ for each $x \in Y$. Recall that a function $g(\cdot) : \mathcal{I} \longrightarrow Y^*$ is *w^*-measurable*, if $< x, g(\cdot) >$ is measurable for each $x \in Y$. By a straightforward computation we obtain that $(\alpha\widehat{A}^* + \xi)^{-1}$, $\xi > \beta$, $\alpha > 0$, admits the representation

$$\left((\alpha\widehat{A}^* + \xi)^{-1} g\right)(t) = (\alpha\widehat{A}(t)^* + \xi)^{-1} g(t), \quad g \in L^q_w(\mathcal{I}, Y^*). \tag{4.1}$$

Hence,

$$\left\|\left(\xi(\alpha\widehat{A}^* + \xi)^{-1} g - g\right)\right\|^q_{\mathfrak{Y}^*} = \int_a^b \left\|\xi(\alpha\widehat{A}(t)^* + \xi)^{-1} g(t) - g(t)\right\|^q_{Y^*}\, dt.$$

Note that for almost every $t \in \mathcal{I}$ we have the estimate

$$\left\|\xi(\alpha\widehat{A}(t)^* + \xi)^{-1} g(t)\right\|_{Y^*} \le \frac{\widehat{M}\xi}{\xi - \alpha\widehat{\beta}}\, \|g(t)\|_{Y^*},$$

which yields

$$\left\|\xi(\alpha\widehat{A}(t)^* + \xi)^{-1} g(t) - g(t)\right\|_{Y^*} \le \left\{1 + \frac{\widehat{M}\xi}{\xi - \alpha\widehat{\beta}}\right\} \|g(t)\|_{Y^*}$$

for almost every $t \in \mathcal{I}$. Since the domain $\mathrm{dom}(\widehat{A}(t)^*)$ is dense in Y^* for almost every $t \in \mathcal{I}$, by *assumption* (A_2) we get

$$\lim_{\alpha\to 0} \left\|\xi(\alpha\widehat{A}(t)^* + \xi)^{-1} g(t) - g(t)\right\|_{Y^*} = 0$$

for almost every $t \in \mathcal{I}$. Hence, by the Lebesgue dominated convergence theorem we obtain

$$\lim_{\alpha \to 0} \left\| \left(\xi(\alpha \widehat{A}^* + \xi)^{-1} g - g \right) \right\| = 0,$$

which shows that $\mathrm{dom}(\widehat{A}^*)$ is dense in $\mathfrak{Y}^*$. Taking into account Theorem 2.4 and Theorem 3.8 we again conclude that the forward evolution equation (1.1) is well posed and uniquely solvable.

Finally, by the same reasoning we obtain that under the assumption (A_3) the domain $\mathrm{dom}(A^*)$ is dense in $\mathfrak{X}^*$. Applying again Theorem 2.4 and Theorem 3.8 we deduce that the evolution equation is well posed and uniquely solvable. □

Theorem 4.6 improves the result of Kato in [24] because the existence of the propagator is proven under weaker assumptions. On the other hand, the result is weaker since nothing is said in which sense the propagator solves the evolution equation (1.1).

Notice that using (2.5) we get the following representation:

$$\begin{aligned}&\left(\left((e^{-\sigma\, D_{\mathcal{I}}/n} e^{-\sigma\, A/n})\right)^n f\right)(t)\\ &\qquad = e^{-\sigma\, A(t-\sigma/n)/n} e^{-\sigma\, A(t-2\sigma/n)/n} \cdots e^{-\sigma\, A(t-\sigma)/n} \chi_{\mathcal{I}}(t-\sigma) f(t-\sigma)\end{aligned}$$

for almost every $t \in \mathcal{I}$ and $\sigma \geq 0$.

Corollary 4.7. *If the assumptions of Theorem 4.6 are satisfied, then the propagator can be approximated as follows:*

$$\lim_{n\to\infty} \int_a^{b-\sigma} \left\| e^{-\frac{\sigma}{n}\, A(s+\frac{n-1}{n}\sigma)} e^{-\frac{\sigma}{n}\, A(s+\frac{n-2}{n}\sigma)} \cdots e^{-\frac{\sigma}{n}\, A(s)} x - U(s+\sigma, s)x \right\|^p ds = 0$$

for each $x \in X$ and $0 \leq \sigma \leq b-a$, $1 < p < \infty$.

4.2. **Backward and bidirectional evolution equations.** To solve the backward evolution equation (1.1) we assume that $\{A(t)\}_{t\in\mathcal{I}}$ is a measurable family of generators of C_0-semigroups of the class $\mathcal{G}(M,\beta)$. We note that the multiplication operator defined by (2.6) and (2.7) generates a C_0-semigroup of class $\mathcal{G}(M,\beta)$.

Definition 4.8. *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of generators of class $\mathcal{G}(M,\beta)$ in a separable Banach space X. The family $\{A(t)\}_{t\in\mathcal{I}}$ is called backward stable if*

$$\|e^{\sigma_1 A(t_1)} e^{\sigma_2 A(t_2)} \cdots e^{\sigma_n A(t_n)}\|_{\mathcal{B}(X)} \leq M e^{\beta \sum_{k=1}^n \sigma_k}$$

is valid for each sequence $\{\sigma_k\}_{k=1}^n$, $\sigma_k \geq 0$ *and almost every* $t \in \nabla_n := \{(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n : a < t_1 \leq t_2 \leq \ldots \leq t_n < b\}$.

Then Lemma 4.2 admits the following analogue.

Lemma 4.9. *Let* $\{A(t)\}_{t\in\mathcal{I}}$ *be a measurable family of semigroup generators in the separable Banach space* X*, which is supposed to belong to* $\mathcal{G}(M,\beta)$*. The pair* $\{D^{\mathcal{I}}, A\}$ *is renormalizable on* $\mathfrak{X} = L^p(\mathcal{I}, X)$, $1 \leq p < \infty$*, if and only if the family of generators* $\{A(t)\}_{t\in\mathcal{I}}$ *is backward stable.*

Proof. Let $\mathcal{I} = (a, b)$. We introduce the isometry $\mho : L^p(\mathcal{I}, X) \longrightarrow L^p(\mathcal{I}, X)$, defined by

$$(\mho f)(t) = f(a + b - t), \quad t \in \mathcal{I}, \quad f \in \mathrm{dom}(\mho) := L^p(\mathcal{I}, X). \tag{4.2}$$

Notice that $\mho^2 = I$ which yields $\mho^{-1} = \mho$. A straightforward computation shows that $\mho^{-1}D^{\mathcal{I}}\mho = \mho D^{\mathcal{I}}\mho = -D_{\mathcal{I}}$. Introducing the family

$$A'(t) := -A(a + b - t), \quad t \in \mathcal{I},$$

and the multiplication operator A' in $L^p(\mathcal{I}, X)$ we get that $\mho^{-1}A\mho = \mho A\mho = A'$. Hence, $\mho^{-1}\{D^{\mathcal{I}}, A\}\mho = \mho\{D^{\mathcal{I}}, A\}\mho = \{-D_{\mathcal{I}}, -A'\}$. Thus, the generator pair $\{D^{\mathcal{I}}, A\}$ is renormalizable if and only if the corresponding anti-generator pair $\{D_{\mathcal{I}}, A'\}$ is renormalizable. From Lemma 4.2 we obtain that $\{D_{\mathcal{I}}, A'\}$ is renormalizable if and only if the family $\{A'(t)\}_{t\in\mathcal{I}}$ is forward stable. On the other hand, $\{A'(t)\}_{t\in\mathcal{I}}$ is forward stable if and only if $\{A(t)\}_{t\in\mathcal{I}}$ is backward stable; that finishes the proof. □

Definition 4.10. *Let* $\{A(t)\}_{t\in\mathcal{I}}$ *be a measurable family of generators on the separable Banach space* X *and let* Y *be a separable Banach space which is densely and continuously embedded into* X*. We say the family* $\{A(t)\}_{t\in\mathcal{I}}$ *satisfies the backward Kato condition if*

(i) $\{A(t)\}_{t\in\mathcal{I}}$ *is backward stable in* X*,*
(ii) *the Banach space* Y *is admissible with respect to the family* $\{A(t)\}_{t\in\mathcal{I}}$*,*
(iii) *the induced family* $\{\widehat{A}(t)\}_{t\in\mathcal{I}}$ *(see Definition* 4.3) *is backward stable in* Y*,*
and, in addition, we assume that conditions (*iv*) *and* (*v*) *of Definition* 4.5 *are valid.*

Then, applying Theorem 3.8 we immediately obtain the following statement:

Theorem 4.11. *Let* $\{A(t)\}_{t\in\mathcal{I}}$ *be a measurable family of generators in the separable Banach space* X*. Further, let* Y *be a separable Banach space* *which is densely and continuously embedded into X. If $\{A(t)\}_{t\in\mathcal{I}}$ obeys the backward Kato condition and if in addition one of the conditions (A_1)-(A_3) (Theorem 4.6) holds, then the backward evolution equation (1.1) is well posed on $\mathcal{I}$ for some $p \in (1,\infty)$ and has a unique solution.*

Corollary 4.12. *If the assumptions of Theorem 4.11 are satisfied, then we obtain an approximation of the propagator in the form*

$$\lim_{n\to\infty}\int_{a+\sigma}^{b}\left\|e^{\frac{\sigma}{n}A(s-\frac{n-1}{n}\sigma)}e^{\frac{\sigma}{n}A(s-\frac{n-2}{n}\sigma)}\cdots e^{\frac{\sigma}{n}A(s)}x - U(s-\sigma,s)x\right\|^p ds = 0$$

for each $x \in X$ and $0 \le \sigma \le b-a$, $1 < p < \infty$.

The proofs of Theorem 4.11 and Corollary 4.12 follow directly from Theorem 4.6 and Corollary 4.7 by using transformation (4.2).

Theorem 4.13. *Let $\{A(t)\}_{t\in\mathcal{I}}$ be a measurable family of group generators in the separable Banach space X and let Y be a separable Banach space, which is densely and continuously embedded into X. If the family $\{A(t)\}_{t\in\mathcal{I}}$ obeys the forward and backward Kato conditions and if one of the conditions (A_1)-(A_3) (Theorem 4.6) holds, then the bidirectional evolution equation (1.1) is well posed on $\mathcal{I}$ for some $p \in (1,\infty)$ and has a unique solution.*

The proof follows directly from Theorem 2.12, Theorem 4.6 and Theorem 4.11. Finally, let us consider bidirectional evolution equations (1.1) on $\mathbb{R}$.

Theorem 4.14. *Let $\{A(t)\}_{t\in\mathbb{R}}$ be a measurable family of group generators in the separable Banach space X. Further, let Y be a separable Banach space which is densely and continuously embedded into X. If for any bounded open interval of $\mathbb{R}$ the family $\{A(t)\}_{t\in\mathcal{I}}$ obeys the forward and backward Kato conditions and if one of the conditions (A_1)-(A_3) (Theorem 4.6) holds, then the bidirectional equation (1.1) is well posed on $\mathbb{R}$ for some $p \in (1,\infty)$ and admits a unique solution.*

The proof follows from a bidirectional modification of Theorem 2.16 and from Theorem 4.13.

5. Evolution equations in Hilbert spaces

Our next aim is to apply the above results to evolution equations for families of semi-bounded self-adjoint operators $\{H(t)\}_{t\in\mathbb{R}}$ with *time independent form-domains.*

This case was studied by Kisyński in [26]. The main Theorem 8.1 of [26] states that if, for all elements of the form-domain, the corresponding closed quadratic form is continuously differentiable for $t \in \mathbb{R}$, then one can associate with the bidirectional evolution equation

$$\frac{1}{i}\frac{\partial}{\partial t}u(t) + H(t)u(t) = 0, \quad u(s) = u_s, \quad s,t \in \mathbb{R}, \tag{5.1}$$

a unique propagator which is called the solution of (5.1). In the present section we elucidate and improve this result.

5.1. **Preliminaries.** Let $\{H(t)\}_{t\in\mathbb{R}}$ be a family of non-negative self-adjoint operators in a separable Hilbert space $\mathfrak{H}$. In the following we consider the non-autonomous Cauchy problem (1.4). As above we assume that the family of operators $\{H(t)\}_{t\in\mathbb{R}}$ is measurable. As in [26] we assume also that

$$\mathfrak{D}^+ = \mathrm{dom}(H(t)^{1/2}) \subseteq \mathfrak{H}, \quad t \in \mathbb{R},$$

which means that the domain $\mathrm{dom}(H(t)^{1/2})$ is independent of $t \in \mathbb{R}$. Introducing the scalar products

$$(f,g)^+_t := (\sqrt{H(t)}f, \sqrt{H(t)}g) + (f,g), \quad t \in \mathbb{R}, \quad f,g \in \mathfrak{D},$$

one defines a family of Hilbert spaces $\{\mathfrak{H}^+_t\}_{t\in\mathbb{R}}$, which is densely and continuously embedded, $\mathfrak{H}^+_t \hookrightarrow \mathfrak{H}$, into $\mathfrak{H}$. The corresponding vector norm is denoted by $\|\cdot\|^+_t$. The natural embedding operator of $\mathfrak{H}^+_t$ into $\mathfrak{H}$ is denoted by $J^+_t : \mathfrak{H}^+_t \longrightarrow \mathfrak{H}$.

By the *closed graph principle* it follows that for each $t, s \in \mathbb{R}$ the constants

$$c(t,s) := \left\|(H(t)+I)^{1/2}(H(s)+I)^{-1/2}\right\|_{\mathcal{B}(\mathfrak{H})}$$

are finite. Obviously, we have

$$\|f\|^+_t \le c(t,s)\|f\|^+_s, \quad f \in \mathfrak{D}, \quad t,s \in \mathbb{R},$$

which yields the estimates

$$\frac{1}{c(t,s)}\|f\|^+_t \le \|f\|^+_s \le c(s,t)\|f\|^+_t, \quad f \in \mathfrak{D}, \quad t,s \in \mathbb{R}. \tag{5.2}$$

This means that the norms $\|\cdot\|^+_t$ are *mutually equivalent*.

We note that for each $t \in \mathbb{R}$ the Hilbert space $\mathfrak{H}^+_t$ is admissible with respect to $-iH(t)$. The corresponding *induced group* (see Definition 3.5) is denoted by $U^+_t(\sigma)$ and is unitary. Its generator is denoted by $-iH^+(t)$; i.e., $U^+_t(\sigma) = e^{-i\sigma H^+(t)}$. Using the embedding operator J^+_t one gets that

$$U_t(\sigma)J^+_t f = J^+_t U^+_t(\sigma)f, \quad f \in \mathfrak{H}^+_t, \quad \sigma \in \mathbb{R}. \quad t \in \mathbb{R}. \tag{5.3}$$

Notice that

$$H^+(t)f = H(t)f, \quad f \in \mathrm{dom}(H^+(t)) := \{f \in \mathrm{dom}(H(t)) : H(t)f \in \mathfrak{H}_t^+\},$$

which gives

$$\mathrm{dom}(H^+(t)) = \mathrm{dom}(H(t)^{3/2}).$$

The dual space with respect to the scalar product $(\cdot,\cdot)$ is denoted by $\mathfrak{H}_t^-$. We note that

$$\mathfrak{H}_t^+ \hookrightarrow \mathfrak{H} \hookrightarrow \mathfrak{H}_t^-, \quad t \in \mathbb{R}.$$

The dual space can be obtained as the completion of the Hilbert space $\mathfrak{H}$ with respect to the norm

$$\|f\|_t^- := \|(H(t)+I)^{-1/2}f\|, \quad f \in \mathfrak{H}.$$

Then from (5.2) we get

$$\frac{1}{c(s,t)}\|f\|_t^- \le \|f\|_s^- \le c(t,s)\|f\|_t^-, \quad f \in \mathfrak{H}, \quad t,s \in \mathbb{R},$$

which shows that the set $\mathfrak{D}^- := \mathfrak{H}_t^-$ is independent of t and the norms $\|\cdot\|_t^-$, $t \in \mathbb{R}$, are mutually equivalent. The natural embedding operator of $\mathfrak{H}$ into $\mathfrak{H}_t^-$ is denoted by $J_t^- : \mathfrak{H} \longrightarrow \mathfrak{H}_t^-$. Obviously, we have

$$J_t^- = (J_t^+)^* \quad \text{and} \quad J_t^+ = (J_t^-)^*, \quad t \in \mathbb{R}. \tag{5.4}$$

The group $U_t(\sigma)$, $\sigma \in \mathbb{R}$, $t \in \mathbb{R}$, admits a unitary extension to the Hilbert space $\mathfrak{H}_t^-$, which we denote by $U_t^-(\sigma)$, $\sigma \in \mathbb{R}$, $t \in \mathbb{R}$. The generator of this group is $-iH_t^-$; i.e. $U_t^-(\sigma) = e^{-i\sigma H_t^-}$, $\sigma \in \mathbb{R}$, $t \in \mathbb{R}$, and its domain is given by

$$\mathrm{dom}(H^-(t)) = \mathrm{dom}(H(t)^{1/2}) = \mathfrak{D}^+. \tag{5.5}$$

One can verify that the Hilbert space $\mathfrak{H}$ is admissible with respect to $-iH_t^-$, $t \in \mathbb{R}$. The corresponding unitary group coincides with $U_t(\sigma)$. One also has

$$U_t^-(\sigma)J_t^- f = J_t^- U_t(\sigma)f, \quad f \in \mathfrak{H}, \quad \sigma \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{5.6}$$

and

$$\mathrm{dom}(H(t)) = \{f \in \mathrm{dom}(H^-(t)) : H^-(t)f \in \mathfrak{H}\}.$$

Since $\mathfrak{H}_t^+$ is admissible with respect to $-iH(t)$, one gets that $\mathfrak{H}_t^+$ is admissible with respect to $-iH_t^-$. The natural embedding operator is given by $J_t := J_t^- J_t^+ : \mathfrak{H}_t^+ \longrightarrow \mathfrak{H}_t^-$, thus we obtain

$$U_t^-(\sigma)J_t f = J_t U_t^+(\sigma)f, \quad f \in \mathfrak{H}_t^+, \quad \sigma \in \mathbb{R}, \quad t \in \mathbb{R},$$

which shows that

$$\mathrm{dom}(H^+(t)) = \{f \in \mathrm{dom}(H^-(t)) : H^-(t)f \in \mathfrak{H}_t^+)\}.$$

Moreover, regarding the operator $H(t)$ as an operator acting from $\mathfrak{H}^+_t$ into $\mathfrak{H}^-_t$, one finds that $H(t)$ can be extended to a *contraction* $B(t)$ acting from $\mathfrak{H}^+_t$ into $\mathfrak{H}^-_t$. Indeed, this follows from the estimate

$$\|B(t)f\|^-_t = \tag{5.7}$$
$$\|(H(t)+I)^{-1/2}H(t)f\|_t \leq \|H(t)^{1/2}f\|_t \leq \|f\|^+_t, \quad f \in \mathrm{dom}(H(t)).$$

Finally, taking into account (5.3)-(5.6) we get the relations

$$U^+_t(\sigma)^* = U^-_t(-\sigma) \quad \text{and} \quad U^-_t(\sigma)^* = U^+_t(-\sigma), \quad \sigma \in \mathbb{R}, \quad t \in \mathbb{R}.$$

5.2. **Auxiliary evolution equation.** We consider the Hilbert space

$$X := \mathfrak{H}^-_{t=0} \quad \text{with} \quad \|\cdot\|_X := \|\cdot\|^-_{t=0}$$

and the auxiliary bidirectional evolution equation

$$\frac{\partial}{\partial t}u(t) + iH^-(t)u(t) = 0 \tag{5.8}$$

on $\mathbb{R}$. To apply results from Section 4 we set $A(t) = iH^-(t)$, $t \in \mathbb{R}$. Obviously, $\{A(t)\}_{t\in\mathbb{R}}$ is a family of group generators in X. Further, we set

$$Y := \mathfrak{H}^+_{t=0} \quad \text{with} \quad \|\cdot\|_Y := \|\cdot\|^+_{t=0}. \tag{5.9}$$

It turns out that the Hilbert space $Y = \mathfrak{H}^+_0$ is densely and continuous embedded into X and admissible with respect to $\{A(t)\}_{t\in\mathbb{R}}$.

Lemma 5.1. *Let $\{H(t)\}_{t\in\mathbb{R}}$ be a measurable family of non-negative self-adjoint operators defined in a separable Hilbert space $\mathfrak{H}$ such that $\mathrm{dom}(H(t)^{1/2})$ is independent of $t \in \mathbb{R}$. If $\mathcal{I}$ is a bounded open interval and*

$$c_{\mathcal{I}} := \sup_{(t,s)\in\mathcal{I}\times\mathcal{I}} c(t,s) < \infty,$$

then there are constants $M_{\mathcal{I}}$ and $\beta_{\mathcal{I}}$ such that $\{A(t)\}_{t\in\mathcal{I}}$ is a measurable family of group generators belonging to $\mathcal{G}(M_{\mathcal{I}}, \beta_{\mathcal{I}})$.

If the Hilbert space Y is given by (5.9) *and there is a constant $\gamma_{\mathcal{I}} > 0$ such that*

$$c(t,s) \leq e^{\gamma_{\mathcal{I}}|t-s|}, \quad t,s \in \mathcal{I}, \tag{5.10}$$

holds, then the families $\{A(t)\}_{t\in\mathcal{I}}$ obey the forward and backward Kato conditions, respectively.

Proof. The measurability of the family $\{A(t)\}_{t\in\mathcal{I}}$ follows from the equivalence of weak and strong measurability, see e.g. [18]. Next, we have

$$\|e^{\sigma A(t)}x\|_X = \|e^{i\sigma H^-(t)}x\|^-_0 \leq c(0,t)\|e^{i\sigma H^-(t)}x\|^-_t$$

$$\leq c(0,t)\|x\|_t^- \leq c(0,t)c(t,0)\|x\|_0^- = c(0,t)c(t,0)\|x\|_X,$$

$\sigma \in \mathbb{R}$. Hence,

$$\|e^{\sigma A(t)}x\|_X \leq M_{\mathcal{I}}\|x\|_X, \quad x \in X, \quad \sigma \in \mathbb{R}, \quad t \in \mathcal{I},$$

where $M_{\mathcal{I}} := c_{\mathcal{I}}^2$, which yields the fact that $A(t)$ generates a group of the class $\mathcal{G}(M_{\mathcal{I}}, 0)$.

If condition (5.10) is satisfied, then the forward and backward stability of $\{A(t)\}_{t\in\mathcal{I}}$ follows from [45, Theorem 4.3.2].

To prove the measurability of $\{\widehat{A}(t)\}_{t\in\mathcal{I}}$ we note that Y is admissible for almost every $t \in \mathcal{I}$. Using (5.2) we obtain that the generator $\widehat{A}(t)$ of the induced group (Definition 3.5) belongs to $\mathcal{G}(M_{\mathcal{I}}, 0)$, too. The measurability of the induced family $\{\widehat{A}(t)\}_{t\in\mathcal{I}}$ follows from the equivalence of strong and weak measurability.

The forward and backward stability of $\{\widehat{A}(t)\}_{t\in\mathcal{I}}$ follows again from condition (5.10) and [45, Theorem 4.3.2].

The condition $Y \subseteq \mathrm{dom}(A(t))$ for almost every $t \in \mathcal{I}$ is obtained from (5.5). Condition (v) of Definition 4.5 follows from (5.7). □

Theorem 5.2. *Let $\{H(t)\}_{t\in\mathbb{R}}$ be a measurable family of non-negative self-adjoint operators defined in a separable Hilbert space $\mathfrak{H}$ such that the domain* $\mathrm{dom}(H(t)^{1/2})$ *is independent of $t \in \mathbb{R}$. If for any bounded open interval $\mathcal{I}$ the condition* (5.10) *is satisfied, then the auxiliary bidirectional evolution problem* (5.8) *is well posed on $\mathbb{R}$ for $p \in (1,\infty)$ and has a unique solution $\{G^-(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ obeying the estimate*

$$\|G^-(t,s)x\|_t^- \leq e^{\gamma_{\mathcal{I}}(t-s)}\|x\|_s^-, \quad x \in \mathfrak{H}_s^-, \tag{5.11}$$

for all $(t,s) \in \mathcal{I}\times\mathcal{I}$.

Proof. Since $Y = \mathfrak{H}_0^+$ is a Hilbert space, all conditions (A_1)-(A_3) are satisfied. Using Lemma 5.1 and Theorem (4.14) one gets that the bidirectional evolution equation (5.8) has a unique solution $\{G^-(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ on $\mathbb{R}$.

By Corollary 4.7 there is a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that one has

$$U_{\mathcal{I}}^-(s+\sigma,s)x = s-\lim_{k\to\infty} e^{-i\frac{\sigma}{n}H^-(s+\frac{n_k-1}{n_k}\sigma)}e^{-i\frac{\sigma}{n}H^-(s+\frac{n_k-2}{n_k}\sigma)}\cdots e^{-i\frac{\sigma}{n}H^-(s)}x$$

for each $x \in \mathfrak{H}_s^-$ and almost every $s \in (a, b-\sigma)$, $0 \leq \sigma \leq b-a$, where $U_{\mathcal{I}}^-(\cdot,\cdot) := G^-(\cdot,\cdot) \restriction \Delta_{\mathcal{I}}$. This yields the estimate

$$\|U_{\mathcal{I}}^-(s+\sigma,t)x\|_{s+\sigma}^- \leq e^{\gamma_{\mathcal{I}}\sigma}\|x\|_s^-, \quad x \in \mathfrak{H}_s^-,$$

for almost every $s \in (a, b-\sigma)$, $0 \le \sigma \le b-a$. Since $U^-_{\mathcal{I}}(\cdot,\cdot)$ is strongly continuous, this holds for any $s \in (a, b-\sigma)$. Setting $t := s + \sigma$ we obtain

$$\|U^-_{\mathcal{I}}(t,s)x\|^-_t \le e^{\gamma_{\mathcal{I}}|t-s|}\|x\|^-_s, \quad x \in \mathfrak{H}^-_t, \quad (t,s) \in \Delta_{\mathcal{I}}. \tag{5.12}$$

Similarly, using Corollary 4.12 we obtain

$$\|V^-_{\mathcal{I}}(s-\sigma,t)x\|^-_{s-\sigma} \le e^{\gamma\sigma}\|x\|^-_s, \quad x \in \mathfrak{H}^-_s,$$

for $s \in (a+\sigma, b)$, $0 \ge \sigma \ge b-a$, where $V^-_{\mathcal{I}}(\cdot,\cdot) := G^-(\cdot,\cdot) \restriction \nabla_{\mathcal{I}}$. Hence one gets the inequality

$$\|V^-_{\mathcal{I}}(t,s)x\|^-_t \le e^{\gamma_{\mathcal{I}}|t-s|}\|x\|^-_s, \quad x \in \mathfrak{H}^-_t, \quad (t,s) \in \nabla_{\mathcal{I}}. \tag{5.13}$$

Using (5.12) and (5.13) we immediately obtain (5.11). □

5.3. **Back to the original problem.** Our Theorem 5.2 gives no information about *solvability* of the bidirectional evolution equation (1.4) on $\mathbb{R}$. This goes back to the fact that in general the evolution equation might be *not* well posed. In fact, it may happen that the cross-sections of the sets

$$\mathrm{dom}(\widetilde{K}_{\mathcal{I}}) := \mathrm{dom}(D_{\mathcal{I}}) \cap \mathrm{dom}(H_{\mathcal{I}}) = H^{1,p}_a(\mathcal{I},\mathfrak{H}) \cap \mathrm{dom}(H_{\mathcal{I}})$$

and

$$\mathrm{dom}(\widetilde{K}^{\mathcal{I}}) = \mathrm{dom}(D^{\mathcal{I}}) \cap \mathrm{dom}(H_{\mathcal{I}}) = H^{1,p}_b(\mathcal{I},\mathfrak{H}) \cap \mathrm{dom}(H_{\mathcal{I}}),$$

$p \in (1,\infty)$, are *not dense* in $\mathfrak{H}$ for intervals $\mathcal{I} = (a,b) \subseteq \mathbb{R}$. Recall that $H_{\mathcal{I}}$ is defined as the multiplication operator induced by the family $\{H(t)\}_{t\in\mathcal{I}}$ in $L^p(\mathcal{I},\mathfrak{H})$.

To avoid this situation we assume in the following that the bidirectional evolution problem (5.1) is well posed on $\mathbb{R}$. Naturally, then we face up to the question whether under this condition the evolution equation (5.1) admits a solution on $\mathbb{R}$.

Lemma 5.3. *Let $\{H(t)\}_{t\in\mathbb{R}}$ be a measurable family of non-negative self-adjoint operators defined in the separable Hilbert space $\mathfrak{H}$ such that $\mathrm{dom}(H(t)^{1/2})$ is independent of $t \in \mathbb{R}$. If for any bounded open interval $\mathcal{I}$ the condition* (5.10) *is satisfied, then there is a unitary bidirectional propagator $\{G(t,s)\}_{(t,s)\in\mathbb{R}^2}$ on $\mathfrak{H}$, such that*

$$J^-_0 G(t,s) = G^-(t,s)J^-_0, \quad (t,s) \in \mathbb{R}^2. \tag{5.14}$$

Moreover, there is a bidirectional propagator $\{G^+(t,s)\}_{(t,s)\in\mathbb{R}^2}$ on $\mathfrak{H}^+_0$, such that

$$J_0 G^+(t,s) = G^-(t,s)J_0, \quad (t,s) \in \mathbb{R}^2 \tag{5.15}$$

and

$$J_0^+ G^+(t,s) = G(t,s) J_0^+, \quad (t,s) \in \mathbb{R}^2. \tag{5.16}$$

Proof. Let $J^+ := J_0^+$, $J^- := J_0^-$ and $J := J_0$. We consider the forward case. Let $\mathcal{I} = (a,b)$ be a bounded open interval of $\mathbb{R}$ and let $0 \le \sigma \le b-a$. By Corollary 4.7 we get that

$$\begin{aligned} &U^-(\cdot+\sigma,\cdot)J^- x_0 \\ &\quad = s \overset{L^p(\mathcal{I}_\sigma,X)}{\longrightarrow} \lim_{n\to\infty} e^{-i\frac{\sigma}{n}H^-(\cdot+\frac{n-1}{n}\sigma)} e^{-i\frac{\sigma}{n}H^-(\cdot+\frac{n-2}{n}\sigma)} \cdots e^{-i\frac{\sigma}{n}H^-(\cdot)} J^- x_0, \end{aligned}$$

$\mathcal{I}_\sigma := (a, b-\sigma)$, for each $x_0 \in \mathfrak{H}$. Since

$$\begin{aligned} &e^{-i\frac{\sigma}{n}H^-(s+\frac{n-1}{n}\sigma)} e^{-i\frac{\sigma}{n}H^-(s+\frac{n-2}{n}\sigma)} \cdots e^{-i\frac{\sigma}{n}H^-(s)} J^- x_0 \\ &\quad = J^- e^{-i\frac{\sigma}{n}H(s+\frac{n-1}{n}\sigma)} e^{-i\frac{\sigma}{n}H(s+\frac{n-2}{n}\sigma)} \cdots e^{-i\frac{\sigma}{n}H(s)} x_0 \end{aligned}$$

for almost every $s \in \mathcal{I}_\sigma$ and since

$$\{e^{-i\frac{\sigma}{n}H(\cdot+\frac{n-1}{n}\sigma)} e^{-i\frac{\sigma}{n}H(\cdot+\frac{n-2}{n}\sigma)} \cdots e^{-i\frac{\sigma}{n}H(\cdot)}\}_{n\in\mathbb{N}}$$

is bounded in $L^p(\mathcal{I}_\sigma, \mathfrak{H})$, we obtain that the weak limit

$$U(\cdot+\sigma,\cdot)x_0 := w \overset{L^p(\mathcal{I}_\sigma,\mathfrak{H})}{\longrightarrow} \lim_{n\to\infty} e^{-i\frac{\sigma}{n}H(\cdot+\frac{n-1}{n}\sigma)} e^{-i\frac{\sigma}{n}H(\cdot+\frac{n-2}{n}\sigma)} \cdots e^{-i\frac{\sigma}{n}H(\cdot)} x_0$$

exists for each $x_0 \in \mathfrak{H}$ and for each $\sigma \in (0, b-a)$. Hence, we get

$$J^- U(s+\sigma,s)x_0 = U^-(s+\sigma,s)J^- x_0$$

for almost every $s \in \mathcal{I}_\sigma$, $\sigma \in (0,b-a)$ and any $x_0 \in \mathfrak{H}$. We note that

$$\|U(s+\sigma,s)x_0\|_{\mathfrak{H}} \le \|x_0\|_{\mathfrak{H}}$$

for almost every $s \in \mathcal{I}_\sigma$ and $\sigma \in (0,b-a)$, $x_0 \in \mathfrak{H}$. Taking into account the fact that the propagator $\{U^-(t,s)\}_{(t,s)\in\Delta_\mathcal{I}}$ is strongly continuous, one gets that $\{U(t,s)_{(t,s)\in\Delta_\mathcal{I}}$ is a weakly continuous family of contractions obeying

$$J^- U(t,s)x_0 = U^-(t,s)J^- x_0 \tag{5.17}$$

for any $(t,s) \in \Delta_\mathcal{I}$ and for each $x_0 \in \mathfrak{H}$. Similarly one proves that there is a weakly continuous family of contractions $\{V(t,s)\}_{(t,s)\in\nabla_\mathcal{I}}$ such that

$$J^- V(t,s)x_0 = V^-(t,s)J^- x_0 \tag{5.18}$$

holds for $(t,s) \in \nabla_\mathcal{I}$ and $x_0 \in \mathfrak{H}$. Setting $G(t,s) := U(t,s)$, $(t,s) \in \Delta_\mathcal{I}$, and $G(t,s) := V(t,s)$, $(t,s) \in \nabla_\mathcal{I}$, and taking into account the fact that $\mathcal{I}$ is arbitrary, we obtain a weakly continuous family $\{G(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ of contractions obeying

$$G(t,s) = G(s,t)^{-1}, \quad (t,s) \in \mathbb{R}\times\mathbb{R}.$$

Since for $(t,s) \in \mathbb{R}\times\mathbb{R}$ and any $x_0 \in \mathfrak{H}$ one has

$$\|x_0\|_{\mathfrak{H}} = \|G(s,t)G(t,s)x_0\|_{\mathfrak{H}} \le \|G(t,s)x_0\|_{\mathfrak{H}} \le \|x_0\|_{\mathfrak{H}},$$

$\|G(t,s)x_0\|_{\mathfrak{H}} = \|x_0\|_{\mathfrak{H}}$, which shows that $\{G(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ is a weakly continuous family of unitary operators. However, this immediately yields that $\{G(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ is in fact a strongly continuous family of unitary operators obeying

$$J^- G(t,s) = G^-(t,s)J^-, \quad (t,s)\in\mathbb{R}\times\mathbb{R}, \tag{5.19}$$

which yields the fact that $\{G(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ is a unitary propagator.

Now we put

$$V^+(s,t) := U^-(t,s)^*, \ \ (t,s)\in\Delta_{\mathbb{R}}, \quad \text{and} \quad U^+(s,t) := V^-(t,s)^*,$$

$(t,s)\in\nabla_{\mathbb{R}}$, as well as

$$G^+(s,t) := G^-(t,s)^*, \quad (t,s)\in\mathbb{R}^2.$$

Then one can easily verify that $\{G^+(t,s)\}_{(t,s)\in\mathcal{I}\times\mathcal{I}}$ is a weakly continuous propagator for any bounded interval $\mathcal{I}$. Taking into account (5.11) and (5.12) we obtain

$$\|V^+(s,t)y\|_s^+ \le e^{\gamma(t-s)}\|y\|_t^+, \quad y\in\mathfrak{H}_t^+,$$

and

$$\|U^+(t,s)y\|_t^+ \le e^{\gamma(t-s)}\|y\|_s^+, \ \ y\in\mathfrak{H}_s^+,$$

for $s\le t$. Using the scalar product $(f,g)_s^+ := (\sqrt{H(s)+I}f, \sqrt{H(s)+I}g)$, $f,g\in\mathfrak{D}^+$, we get

$$(\|U^+(t,s)y-y\|_s^+)^2 = (\|U^+(t,s)y\|_s^+)^2 + (\|y\|_s^+)^2 - 2\Re\mathrm{e}(U^+(t,s)y,y)_s^+.$$

Now, using (5.11) we find

$$\|U^+(t,s)y\|_s^+ \le e^{\gamma(t-s)}\|U^+(t,s)y\|_t^+ \le e^{2\gamma(t-s)}\|y\|_s^+,$$

which implies

$$(\|U^+(t,s)y-y\|_s^+)^2 \le e^{4\gamma(t-s)}(\|y\|_s^+)^2 + (\|y\|_s^+)^2 - 2\Re\mathrm{e}(U^+(t,s)y,y)_s^+.$$

By the weak continuity of the forward propagator $\{U^+(t,s)\}_{(t,s)\in\Delta_{\mathbb{R}}}$ we obtain $\lim_{t\to s+0}U^+(t,s) = I$. Hence, $\lim_{t\to s+0}\|U^+(t,s)y-y\|_s^+ = 0$ for each $y\in\mathfrak{H}_s^+$. Since the norms $\|\cdot\|_t^+$ and $\|\cdot\|_0^+$ are equivalent, we find $\lim_{t\to s+0}\|U^+(t,s)y-y\|_0^+ = 0$ for each $y\in Y=\mathfrak{H}_0^+$. Similarly we prove $\lim_{t\to s-0}\|V^+(t,s)y-y\|_0^+ = 0$ for each $y\in Y=\mathfrak{H}_0^+$. Using the representation

$$G^+(t,s) = G^+(t,0)G^+(0,s),$$

where

$$G^+(t,0) = \begin{cases} U^+(t,0), & t \geq 0, \\ V^+(t,0), & t \leq 0, \end{cases} \quad \text{and} \quad G^+(0,s) = \begin{cases} V^+(0,s), & s \geq 0, \\ U^+(0,s), & s \leq 0, \end{cases}$$

one proves the strong continuity of the families $\{G^+(t,0)\}_{(t\in\mathbb{R}}$ and $\{G^+(0,s)\}_{s\in\mathbb{R}}$, which yields the strong continuity of $\{G^+(t,s)\}_{(t,s)\in\mathbb{R}^2}$.

Finally, by $(J^-)^* = J^+$ and $J = J^-J^+$ we find the equation

$$J^+G^+(s,t) = G(s,t)J^+, \quad (s,t) \in \mathbb{R}\times\mathbb{R},$$

which by virtue of (5.19) proves (5.16). Hence we get that

$$\begin{aligned} JG^+(s,t) &= J^-J^+G^+(s,t) \\ &= J^-G(s,t)J^+ = G^-(s,t)J^-J^+ = G^-(s,t)J, \qquad (s,t) \in \mathbb{R}\times\mathbb{R}, \end{aligned}$$

which proves (5.15). □

Now it is useful to introduce the following definition.

Definition 5.4. *Let $\{G(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ be a bidirectional propagator in a separable Banach space X and let Y be a separable Banach space, which is densely and continuously embedded into X. The Banach space Y is called admissible with respect to the family $\{G(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ if there is a bidirectional propagator $\{\widehat{G}(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ in Y such that*

$$G(t,s)J = J\widehat{G}(t,s), \quad (t,s) \in \mathbb{R}\times\mathbb{R}, \tag{5.20}$$

holds where J is the embedding operator of Y into X.

The following theorem generalizes Theorem 8.1 of [26]. Our proof is quite independent from that in [26].

Theorem 5.5. *Let $\{H(t)\}_{t\in\mathbb{R}}$ be a measurable family of non-negative self-adjoint operators defined in the separable Hilbert space $\mathfrak{H}$ such that $\mathrm{dom}(H(t)^{1/2})$ is independent of $t \in \mathbb{R}$. If the bidirectional evolution equation (5.1) is well posed on $\mathbb{R}$ for some $p \in (1,\infty)$ and the condition (5.10) is satisfied for any bounded open interval, then the bidirectional evolution equation (5.1) admits on $\mathbb{R}$ a unitary solution $\{G(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ for which the Hilbert space $\mathfrak{H}_0^+$ is admissible. Moreover, if for any bounded open interval $\mathcal{I} = (a,b)$ the sets*

$$H^{1,p}_a(\mathcal{I},\mathfrak{H}_0^+) \cap \mathrm{dom}(H_\mathcal{I}) \quad \textit{and} \quad H^{1,p}_b(\mathcal{I},\mathfrak{H}_0^+) \cap \mathrm{dom}(H_\mathcal{I}), \quad p \in (1,\infty), \tag{5.21}$$

are dense in $H^{1,p}_a(\mathcal{I},\mathfrak{H}_0^+)$ and $H^{1,p}_b(\mathcal{I},\mathfrak{H}_0^+)$, respectively, then there is only one unitary solution for which the Hilbert space $\mathfrak{H}_0^+$ is admissible.

Proof. We have to show that the evolution operator $\widetilde{K}_{\mathcal{I}}$,

$$\widetilde{K}_{\mathcal{I}} f = D_{\mathcal{I}} + iH_{\mathcal{I}} f, \quad f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}}) = \mathrm{dom}(D_{\mathcal{I}}) \cap \mathrm{dom}(H_{\mathcal{I}}),$$

which is associated with the forward evolution equation (5.1), can be extended to a forward generator. Let $\widetilde{K}_{\mathcal{I}}^{-}$ be the evolution operator

$$\widetilde{K}_{\mathcal{I}}^{-} g = D_{\mathcal{I}}^{-} g + iH_{\mathcal{I}}^{-} g, \quad g \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}}^{-}) = \mathrm{dom}(D_{\mathcal{I}}^{-}) \cap \mathrm{dom}(H_{\mathcal{I}}^{-}),$$

associated with (5.8), where $D_{\mathcal{I}}^{-}$ is the anti-generator of the right-shift semigroup in $L^p(\mathcal{I}, \mathfrak{H}_0^-)$, and let $H_{\mathcal{I}}^{-}$ be the multiplication operator induced by $\{H^-(t)\}_{t\in\mathcal{I}}$. By $\mathcal{J}^-$ we denote the embedding operator of $L^p(\mathcal{I}, \mathfrak{H})$, $p \in (1, \infty)$, into $L^p(\mathcal{I}, \mathfrak{H}_0^-)$, defined as

$$(\mathcal{J}^- f)(t) = J^- f(t), \quad f \in L^p(\mathcal{I}, \mathfrak{H}),$$

where $J^- := J_0^-$. One can easily verify that $\mathcal{J}^- \mathrm{dom}(\widetilde{K}_{\mathcal{I}}) \subseteq \mathrm{dom}(\widetilde{K}_{\mathcal{I}}^{-})$ and

$$\widetilde{K}_{\mathcal{I}}^{-} \mathcal{J}^- f = \mathcal{J}^- \widetilde{K}_{\mathcal{I}} f, \quad f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}}).$$

By Theorem 5.2 the forward evolution equation (5.8) is uniquely solvable. This means that the operator $\widetilde{K}_{\mathcal{I}}^{-}$ admits only one extension $K_{\mathcal{I}}^{-}$, which is a forward generator. In fact, it has been already proven that the closure of $\widetilde{K}_{\mathcal{I}}^{-}$ coincides with $K_{\mathcal{I}}^{-}$.

By Lemma 5.3 there is a forward generator

$$\{U_{\mathcal{I}}(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}}}, \qquad U_{\mathcal{I}}(t,s) := G(t,s) \restriction \Delta_{\mathcal{I}}$$

obeying (5.14). By the relation

$$(e^{-\sigma K_{\mathcal{I}}} f)(t) = U_{\mathcal{I}}(t, t-\sigma)\chi_{\mathcal{I}}(t-\sigma) f(t-\sigma), \quad f \in L^p(\mathcal{I}, \mathfrak{H}),$$

one defines a forward generator $K_{\mathcal{I}}$ in $L^p(\mathcal{I}, \mathfrak{H})$. Obviously, we have

$$e^{-\sigma K_{\mathcal{I}}^{-}} \mathcal{J}^- f = \mathcal{J}^- e^{-\sigma K_{\mathcal{I}}} f, \quad f \in L^p(\mathcal{I}, \mathfrak{H}).$$

Hence, $\mathcal{J}^- \mathrm{dom}(K_{\mathcal{I}}) \subseteq \mathrm{dom}(K_{\mathcal{I}}^{-})$ and

$$K_{\mathcal{I}}^{-} \mathcal{J}^- f = \mathcal{J}^- K_{\mathcal{I}} f, \qquad f \in \mathrm{dom}(K_{\mathcal{I}}).$$

Notice that

$$e^{-\sigma K_{\mathcal{I}}^{-}} g = g - \int_0^\sigma d\tau\, e^{-\tau K_{\mathcal{I}}^{-}} K_{\mathcal{I}}^{-} g, \quad g \in \mathrm{dom}(K_{\mathcal{I}}^{-}).$$

Then choosing $g = \mathcal{J}^- f$, $f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}}^{-})$, we obtain

$$\mathcal{J}^- e^{-\sigma K_{\mathcal{I}}} f = \mathcal{J}^- f - \mathcal{J}^- \int_0^\sigma d\tau\, e^{-\tau K_{\mathcal{I}}} \widetilde{K}_{\mathcal{I}} f$$

which yields

$$e^{-\sigma K_{\mathcal{I}}} f = f - \int_0^\sigma d\tau\, e^{-\tau K_{\mathcal{I}}} \widetilde{K}_{\mathcal{I}} f, \quad f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}}).$$

Therefore, $\widetilde{K}_{\mathcal{I}} \subseteq K_{\mathcal{I}}$, which shows that $\{U_{\mathcal{I}}(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}}}$ is a solution of the forward evolution equation (5.1) on $\mathcal{I}$. The same procedure can be applied to the backward evolution equation (5.1) on $\mathcal{I}$. Hence, the unitary bidirectional propagator $\{G(t,s)\}_{(t,s)\in\mathbb{T}\times\mathbb{R}}$ defined by (5.14) is, in fact, a solution of the bidirectional evolution equation (5.1) on $\mathbb{R}$.

Assume now that $\{Z(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ is another unitary solution of the bidirectional evolution equation (5.1) such that the Hilbert space $\mathfrak{H}_0^+$ is admissible with respect to $\{Z(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$. Then from

$$J^+ \widehat{Z}(t,s) = Z(t,s) J^+, \quad (t,s) \in \mathbb{R}\times\mathbb{R},$$

we obtain

$$\widehat{Z}(t,s)^* J^- = J^- Z(t,s)^*, \quad (t,s) \in \mathbb{R}\times\mathbb{R},$$

where we used the fact that $J^- = (J^+)^*$. We set $Z^-(t,s) := \widehat{Z}(s,t)^*$, $(t,s) \in \mathbb{R}\times\mathbb{R}$. Since $\{Z(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ is unitary, we have $Z(t,s) = Z(s,t)^*$. By this we find

$$Z^-(t,s) J^- = J^- Z(t,s), \quad (t,s) \in \mathbb{R}\times\mathbb{R}.$$

Since $\{Z(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ and $\{Z^+(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ are bidirectional propagators in $\mathfrak{H}$ and $\mathfrak{H}_0^+$, respectively, one easily gets that $\{Z^-(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ is a bidirectional propagator in $\mathfrak{H}_0^-$. For any bounded interval $\mathcal{I}$ in $\mathbb{R}$ a forward generator $L^-_{\mathcal{I}}$ corresponds to the forward propagator $\{Z^-_{\mathcal{I}}(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}}}$, $Z^-_{\mathcal{I}}(\cdot,\cdot) := Z^-(\cdot,\cdot) \restriction \mathcal{I}\times\mathcal{I}$ by the relation

$$(e^{-\sigma L^-_{\mathcal{I}}} f)(t) := Z^-_{\mathcal{I}}(t, t-\sigma)\chi_{\mathcal{I}}(t-\sigma) f(t-\sigma), \quad t \in \mathcal{I}, \quad f \in L^P(\mathcal{I}, \mathfrak{H}_0^-).$$

It is obvious that

$$e^{-\sigma L^-_{\mathcal{I}}} \mathcal{J}^- = \mathcal{J}^- e^{-\sigma L_{\mathcal{I}}}, \quad \sigma \ge 0,$$

where $L_{\mathcal{I}}$ denotes the forward generator, which corresponds to $\{Z_{\mathcal{I}}(t,s)\}_{(t,s)\in\Delta_{\mathcal{I}}}$, $Z_{\mathcal{I}}(t\cdot,\cdot) := Z(\cdot,\cdot) \restriction \Delta_{\mathcal{I}}$. Hence, $\mathcal{J}^-\mathrm{dom}(L_{\mathcal{I}}) \subseteq \mathrm{dom}(L^-_{\mathcal{I}})$ and

$$L^-_{\mathcal{I}} \mathcal{J}^- f = \mathcal{J}^- L_{\mathcal{I}} f, \quad f \in \mathrm{dom}(K_{\mathcal{I}}).$$

Since $L_{\mathcal{I}}$ is an extension of $\widetilde{K}_{\mathcal{I}}$, we obtain

$$L^-_{\mathcal{I}} \mathcal{J}^- f = \mathcal{J}^- \widetilde{K}_{\mathcal{I}} f, \quad f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}}),$$

which shows that $L^-_{\mathcal{I}}$ is an extension of $\widetilde{L}^-_{\mathcal{I}} := L^-_{\mathcal{I}} \upharpoonright \mathcal{J}^- \mathrm{dom}(\widetilde{K}_{\mathcal{I}})$. Since

$$K^-_{\mathcal{I}} \mathcal{J}^- f = \mathcal{J}^- \widetilde{K}_{\mathcal{I}} f, \quad f \in \mathrm{dom}(\widetilde{K}_{\mathcal{I}}),$$

holds one gets that $K^-_{\mathcal{I}}$ is also an extension of $\widetilde{L}^-_{\mathcal{I}}$. Since the intersection $H^{1,p}_a(\mathcal{I}, \mathfrak{H}^+_0) \cap \mathrm{dom}(H_{\mathcal{I}})$, cf. (5.21), is dense in $H^{1,p}_a(\mathcal{I}, \mathfrak{H}^+_0)$, the domain $\mathrm{dom}(\bar{L}^-_{\mathcal{I}})$ of the closure $\bar{L}^-_{\mathcal{I}}$ of $\widetilde{L}^-_{\mathcal{I}}$ contains $H^{1,p}_a(\mathcal{I}, \mathfrak{H}^+)$. By Remark 3.9 the set $H^{1,p}_a(\mathcal{I}, \mathfrak{H}^+_0)$ is a *core* of $K^-_{\mathcal{I}}$, which shows that $K^-_{\mathcal{I}} = \bar{L}^-_{\mathcal{I}}$. Hence $L^-_{\mathcal{I}} = K^-_{\mathcal{I}}$, which yields $Z^-_{\mathcal{I}}(t,s) = U^-_{\mathcal{I}}(t,s)$, $(t,s) \in \Delta_{\mathcal{I}}$, for any bounded interval $\mathcal{I}$ of $\mathbb{R}$. The same can be proven for the backward evolution equation, which ensures that the bidirectional evolution (5.1) admits only *one* solution for which the Hilbert space $\mathfrak{H}^+_0$ is admissible. □

6. EXAMPLES

6.1. **Point interactions with varying coupling constant.** We consider a family $\{H(t)\}_{t\in\mathbb{R}}$ of self-adjoint operators associated in the Hilbert space $\mathfrak{H} = L^2(\mathbb{R})$ with the sesquilinear forms

$$\mathfrak{h}_t[f,g] := \int_{\mathbb{R}} \left\{ \frac{1}{2m(x)} f'(x)\overline{(g'(x)} \right\} + V(x) f(x) \overline{g(x)} + \sum_{j=1}^{N} \kappa_j(t) f(x_j) \overline{g(x_j)}, \tag{6.1}$$

where $f,g \in \mathrm{dom}(\mathfrak{h}_t) := H^{1,2}(\mathbb{R})$, $1 \le N \le \infty$. We assume that

$$m(x) > 0, \quad \frac{1}{m} + m \in L^\infty(\mathbb{R}), \quad \text{and} \quad V \in L^\infty(\mathbb{R})$$

$x_j \in \mathbb{R}$, $j = 1,2,\dots,N$, and that the coupling constants $\kappa_j(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}_+$ are measurable functions. The family $\{H(t)\}_{t\in\mathbb{R}}$ is uniformly semibounded from below. Indeed, we have

$$H(t) \ge -\|V\|_{L^\infty(\mathbb{R})}, \quad t \in \mathbb{R}.$$

Therefore, without loss of generality we assume that $V(x) \ge 0$ for almost every $x \in \mathbb{R}$, which yields the fact that $\{H(t)\}_{t\in\mathbb{R}}$ is a family of non-negative self-adjoint operators. Moreover, one can easily verify that $\{H(t)\}_{t\in\mathbb{R}}$ is a measurable family of self-adjoint operators. For finite N the domain $\mathrm{dom}(H(t))$ admits an explicit description. Indeed, in this case the operators $H(t)$ are given by the sum of operators in the form-sense (6.1):

$$H(t) = -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx} \dotplus V(x) \dotplus \sum_{j=1}^{N} \kappa_j(t)\delta(x - x_j)$$

with domain defined by

$$\operatorname{dom}(H(t)) := \tag{6.2}$$

$$\left\{ f \in H^{1,2}(\mathbb{R}) : \begin{array}{l} \frac{1}{m}f' \in H^{1,2}(\mathbb{R} \setminus \bigcup_{j=1}^{N}\{x_j\}), \\ \left(\frac{1}{2m}f'\right)(x_j - 0) - \left(\frac{1}{2m}f'\right)(x_j + 0) = \kappa_j(t)f(x_j), \\ j = 1, 2, \ldots, N < \infty \end{array} \right\}$$

for $t \in \mathcal{I}$, cf. [1, Chapter II.1]. In the following we assume (*convergence* condition) that

$$\sup_{t\in\mathcal{I}} \sum_{j=1}^{N} \kappa_j(t) < \infty, \quad 1 \leq N \leq \infty, \tag{6.3}$$

for each bounded subinterval $\mathcal{I} \subset \mathbb{R}$. Furthermore, we assume (*continuity* condition) that for each bounded subinterval $\mathcal{I} \subset \mathbb{R}$ there is a constant $C_{\mathcal{I}} > 0$ such that

$$\sum_{j=1}^{N} |\kappa_j(t) - \kappa_j(s)| \leq C_{\mathcal{I}}|t - s|, \quad t, s \in \mathcal{I}. \tag{6.4}$$

Since $\mathfrak{D}^+ := \operatorname{dom}(H(t)^{1/2}) = \operatorname{dom}(\mathfrak{h}_t) = H^{1,2}(\mathbb{R})$ is independent of $t \in \mathbb{R}$, Theorem 5.2 is applicable in this case: the *auxiliary* bidirectional evolution equation (5.8) admits a unique solution, if the estimate (5.10) is satisfied for each bounded subinterval $\mathcal{I} \subset \mathbb{R}$.

To show this it is sufficient to verify that the estimate

$$\|\sqrt{H(t) + I}f\| \leq e^{\gamma_{\mathcal{I}}|t-s|}\|\sqrt{H(s) + I}f\|, \quad f, g \in \mathfrak{D}^+ \tag{6.5}$$

holds for any $t, s \in \mathcal{I}$. Indeed, one obviously has

$$|f(x_j)|^2 = 2\Re\mathrm{e}\left\{\int_{-\infty}^{x_j} f'(x)\overline{f(x)}dx\right\}, \quad f \in H^{1,2}(\mathbb{R}), \quad j \in 1, 2, \ldots, N,$$

which yields

$$|f(x_j)|^2 \leq \int_{\mathbb{R}} \left\{|f'(x)|^2 + |f(x)|^2\right\} dx, \ j = 1, 2, \ldots, N. \tag{6.6}$$

Hence,

$$|f(x_j)|^2 \leq \max\{1, 2\|m\|_{L^\infty}\}\|\sqrt{H(s) + I}f\|^2, \quad j = 1, 2, \ldots, N. \tag{6.7}$$

Therefore, we have

$$\left|\|\sqrt{H(t) + I}f\|^2 - \|\sqrt{H(s) + I}f\|^2\right| \leq \sum_{j=1}^{N} |\kappa_j(t) - \kappa_j(s)|\, |f(x_j)|^2,$$

and consequently, by (6.7) we obtain

$$\Big|\|\sqrt{H(t)+I}f\|^2 - \|\sqrt{H(s)+I}f\|^2\Big| \le \max\{1, 2\|m\|_{L^\infty}\}\|\sqrt{H(s)+I}f\|^2 \sum_{j=1}^{N} |\kappa_j(t) - \kappa_j(s)|.$$

Using (6.4) we get

$$\Big|\|\sqrt{H(t)+I}f\|^2 - \|\sqrt{H(s)+I}f\|^2\Big| \le 2\gamma_{\mathcal{I}}\ |t-s|\ \|\sqrt{H(s)+I}f\|^2 \tag{6.8}$$

for $t, s \in \mathcal{I}$, where

$$\gamma_{\mathcal{I}} := \frac{1}{2} C_{\mathcal{I}} \max\{1, 2\|m\|_{L^\infty}\}.$$

From (6.8) it follows that

$$\|\sqrt{H(t)+I}f\| \le \sqrt{1 + 2\gamma_{\mathcal{I}}\ |t-s|}\ \|\sqrt{H(s)+I}f\|,$$

which yields

$$\|\sqrt{H(t)+I}f\| \le (1 + \gamma_{\mathcal{I}}\ |t-s|)\ \|\sqrt{H(s)+I}f\|,$$

for $t, s \in \mathcal{I}$. Since $1 + \gamma_{\mathcal{I}}\ |t-s| \le e^{\gamma_{\mathcal{I}}|t-s|}$, for any $t, s \in \mathcal{I}$, we obtain (6.5).

Then by Theorem 5.5 the *original* bidirectional evolution equation (5.1) admits a solution for which the Hilbert space $H^{1,2}(\mathbb{R})$ is admissible. It is more complicated to solve the problem whether this solution of the *original* problem is *unique*. To this end one has to verify the additional condition (5.21) of Theorem 5.5. This condition is satisfied if the sets $(I + H_{\mathcal{I}})^{-1} H^{1,2}_a(\mathcal{I}, \mathfrak{H})$ and $(I + H_{\mathcal{I}})^{-1} H^{1,2}_b(\mathcal{I}, \mathfrak{H})$ are dense in $H^{1,2}_a(\mathcal{I}, \mathfrak{H}^+_0)$ and $H^{1,2}_b(\mathcal{I}, \mathfrak{H}^+_0)$, for any bounded interval $\mathcal{I} = (a, b)$, respectively.

To prove this we introduce linear operators $C_j : L^2(\mathbb{R}) \longrightarrow \mathbb{C}$ defined by

$$C_j f := ((I + H(0))^{-1/2} f)(x_j), \qquad f \in L^2(\mathbb{R}), \qquad j = 1, 2, \ldots, N.$$

Using the estimate (6.7) we find $|C_j f| \le C\|f\|_{L^2(\mathbb{R})}$, where C is given by $C := \max\{1, 2\|m\|_{L^\infty}\}$. Setting $B_j := C_j^* C_j$ we obtain the representation

$$(I + H(t))^{-1} = (I + H(0))^{-1/2} R(t) (I + H(0))^{-1/2}, \qquad t \in \mathbb{R},$$

where

$$R(t) := \Big(I + \sum_{j=1}^{N} \kappa_j(t) B_j\Big)^{-1}, \qquad t \in \mathbb{R}.$$

Since the coupling constants are locally Lipschitz continuous, see (6.4), we get that $R(t)x \in H^{1,2}_a(\mathcal{I}, \mathfrak{H})$, $x \in \mathfrak{H}$, for any bounded open interval $\mathcal{I} \subseteq \mathbb{R}$.

Hence, $R(t)f(t) \in H^{1,2}_a(\mathcal{I},\mathfrak{H})$ for $f \in H^{1,2}_a(\mathcal{I},\mathfrak{H})$ and any bounded open interval $\mathcal{I} \subseteq \mathbb{R}$. Hence we get $(I+H(t))^{-1}f(t) \in H^{1,2}_a(\mathcal{I},\mathfrak{H}^+_0)$ for $f \in H^{1,2}_a(\mathcal{I},\mathfrak{H})$ and $\mathcal{I} \subseteq \mathbb{R}$. Now we show that the set of elements $(I+H(t))^{-1}f(t)$, $f \in H^{1,2}_a(\mathcal{I},\mathfrak{H})$, is *dense* in $H^{1,2}_a(\mathcal{I},\mathfrak{H}^+_0)$. Note that the standard norm of $H^{1,2}_a(\mathcal{I},\mathfrak{H}^+_0)$ is equivalent to the norm

$$\|f\|_{H^{1,2}_a(\mathcal{I},\mathfrak{H}^+_0)} = \left(\int_{\mathcal{I}} \|\sqrt{I+H(0)}f'(t)\|^2_{\mathfrak{H}}\, dt\right)^{1/2}.$$

If the elements $(I+H(t))^{-1}f(t)$, $f \in H^{1,2}_a(\mathcal{I},\mathfrak{H})$, are not dense in $H^{1,2}_a(\mathcal{I},\mathfrak{H}^+_0)$, then there is an element $g \in H^{1,2}_a(\mathcal{I},\mathfrak{H}^+_0)$ such that

$$\int_{\mathcal{I}} \left(\sqrt{I+H(0)}\,(R_0(t)f(t))', \sqrt{I+H(0)}g'(t)\right) dt = 0$$

for any $f \in H^{1,2}_a(\mathcal{I},\mathfrak{H})$ where

$$R_0(t) := (I+H(0))^{-1/2}R(t)(I+H(0))^{-1/2}.$$

Hence, we obtain

$$\int_{\mathcal{I}}(R'(t)(I+H(0))^{-1/2}f(t)+R(t)(I+H(0))^{-1/2}f'(t), \sqrt{I+H(0)}g'(t))dt = 0.$$

Setting $h(t) := (I+H(0))^{-1/2}f(t) \in H^{1,2}_a(\mathcal{I},\mathfrak{H}^+_0)$ and $k(t) := \sqrt{I+H(0)}g'(t) \in L^2(\mathcal{I},\mathfrak{H})$ we find that

$$\int_{\mathcal{I}}(R'(t)h(t)+R(t)h'(t),k(t))dt = 0 \tag{6.9}$$

for any $h \in H^{1,2}_a(\mathcal{I},\mathfrak{H}^+_0)$. Since $H^{1,2}_a(\mathcal{I},\mathfrak{H}^+_0)$ is dense in $H^{1,2}_a(\mathcal{I},\mathfrak{H})$ one gets that (6.9) holds for any $h \in H^{1,2}_a(\mathcal{I},\mathfrak{H})$. From (6.9) we obtain

$$\int_{\mathcal{I}}(h'(t),R(t)k(t))dt = -\int_{\mathcal{I}}(h(t),R'(t)k(t))dt$$

for any $h \in H^{1,2}_a(\mathcal{I},\mathfrak{H})$, which yields $z(t) := R(t)k(t) \in H^{1,2}_b(\mathcal{I},\mathfrak{H})$ and

$$\frac{d}{dt}R(t)k(t) - R'(t)k(t) = 0 \tag{6.10}$$

for almost every $t \in \mathcal{I}$. From the representation

$$k(t) = \Big(I + \sum_{j=1}^{N}\kappa_j(t)B_j\Big)z(t)$$

and condition (6.4) we obtain the fact that $k(t) \in H_b^{1,2}(\mathcal{I}, \mathfrak{H})$. Taking into account this last observation we get from (6.10) that $R(t)k'(t) = 0$ for almost every $t \in \mathcal{I}$. Since $\ker(R(t)) = \{0\}$ for $t \in \mathcal{I}$, we find that $k'(t) = 0$, which implies $k(t) = constant$. But since $k(b) = 0$, we get $k(t) = 0$ for $t \in \mathcal{I}$. Hence $g'(t) = 0$ for $t \in \mathcal{I}$, which yields $g(t) = 0$ for $t \in \mathcal{I}$. Consequently, the set $(I + H_{\mathcal{I}})^{-1} H_a^{1,2}(\mathcal{I}, \mathfrak{H})$ is dense in $H_a^{1,2}(\mathcal{I}, \mathfrak{H}_0^+)$ for any bounded open interval $\mathcal{I} = (a, b)$.

Similarly, one proves that the set $(I + H_{\mathcal{I}})^{-1} H_b^{1,2}(\mathcal{I}, \mathfrak{H})$ is dense in $H_b^{1,2}(\mathcal{I}, \mathfrak{H}_0^+)$ for any bounded open interval $\mathcal{I} = (a, b)$.

Taking into account the second part of Theorem 5.5 one finds that there is a *unique* solution of the original problem (5.1) such that $\mathfrak{H}_0^+$ is admissible.

Therefore, summing up this line of reasoning we obtain the proof of the following theorem:

Theorem 6.1. *Let* $0 \le V \in L^\infty(\mathbb{R})$, $m > 0$ *and* $1/m + m \in L^\infty(\mathbb{R})$. *Further, let* $\{x_j\}_{j\in\mathbb{N}}$ *be a (infinite) sequence of real numbers which are mutually different and let* $\kappa_j(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}_+$ *be non-negative locally Lipschitz continuous functions. Moreover, let* $\{H(t)\}_{t\in\mathbb{R}}$ *be a family of non-negative self-adjoint Schrödinger operators associated with the sesquilinear forms* (6.1). *If the conditions* (6.3) *and* (6.4) *are satisfied, then the bidirectional evolution equation* (5.1) *is well posed on* $\mathbb{R}$ *for* $p = 2$ *and possesses a unique solution* $\{G(t, s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ *such that* $H^{1,2}(\mathbb{R})$ *is admissible.*

A similar problem was treated in three dimensions by [43] for the case of finite point interactions and $m(x) = const$. In contrast to Theorem 6.1 their results concern the case of coupling constants $\kappa_j(t)$ which are *twice continuously differentiable*, cf. [43, Theorem 1]. In this case the bidirectional evolution equation is verified in the strong sense. Moreover, only the existence of a bidirectional propagator was established under the weaker assumption that the coupling constants $\kappa_j(t)$ are locally L^∞-function, cf. [43, Theorem 2]. The first results were improved in [7], where the smoothness of the coupling constants was reduced to a certain Hölder continuity. However, it seems to be difficult to extend the technique used in [7, 43] to the case of an *infinite* number of point interactions and to a *non-smooth* position dependent effective mass m.

In conclusion we would like to remark that Theorem 6.1 covers rather *bizarre* situations. For instance, let $\{x_j\}_{j\in\mathbb{N}}$ be an enumeration of the rational numbers $\mathbb{Q}$ and let $\{\kappa_j(t)\}$ be a sequence of coupling constants such that conditions (6.3) and (6.4) are satisfied. Moreover, let us assume that for any $t \in \mathbb{R}$ the values $\kappa_j(t)$ are pairwise different. In this case one has

$\bigcap_{t\in\mathcal{I}} \mathrm{dom}(H(t)) = \{0\}$ for any bounded open interval $\mathcal{I} \subseteq \mathbb{R}$. Nevertheless, the sets $(I + H_{\mathcal{I}})^{-1}H^{1,2}_a(\mathcal{I}, \mathfrak{H})$ and $(I + H_{\mathcal{I}})^{-1}H^{1,2}_b(\mathcal{I}, \mathfrak{H})$ are dense in $H^{1,2}_a(\mathcal{I}, \mathfrak{H}^+_0)$ and $H^{1,2}_b(\mathcal{I}, \mathfrak{H}^+_0)$, respectively!

6.2. **Moving potentials.** In this section we consider an example which is more involved than what we studied above. Here we consider the Hamiltonian of two *moving point particles*:

$$H(t) = -\frac{1}{2}\frac{d^2}{dx^2} \dot{+} \kappa_1(t)\delta(x - x_1(t)) \dot{+} \kappa_2(t)\delta(x - x_2(t)), \tag{6.11}$$

whose domain is described by

$$\mathrm{dom}(H(t)) := \tag{6.12}$$
$$\left\{ f \in H^{1,2}(\mathbb{R}) : \begin{array}{c} f' \in H^{1,2}(\mathbb{R} \setminus \{x_1(t), x_2(t)\}), \\ (f'/2)\,(x_1(t) - 0) - (f'/2)\,(x_1(t) + 0) = \kappa_1(t)f(x_1(t)), \\ (f'/2)\,(x_2(t) - 0) - (f'/2)\,(x_2(t) + 0) = \kappa_2(t)f(x_2(t)), \end{array} \right\}$$

in the Hilbert space $L^2(\mathbb{R})$, cf. [1, Chapter II.1]. In the following we assume that $\kappa_j(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}_+$ are *continuous differentiable* functions. Moreover, we suppose that

$$x_1(t) < x_2(t) \tag{6.13}$$

for $t \in \mathbb{R}$. The sesquilinear form associated with $H(t)$ is given by

$$\mathfrak{h}_t[f, g] = \frac{1}{2}\int_{\mathbb{R}} f'(x)\overline{g'(x)}dx + \kappa_1(t)f(x_1(t))\overline{g(x_1(t))} + \kappa_2(t)f(x_2(t))\overline{g(x_2(t))},$$

$f, g \in \mathrm{dom}(\mathfrak{h}_t) := H^{1,2}(\mathbb{R})$. Notice that the sesquilinear form $\mathfrak{h}_t$ is nonnegative.

To handle this case we start with some formal manipulations. Using the momentum operator P,

$$Pf = \frac{1}{i}\,\frac{\partial}{\partial x}f(x), \quad f \in \mathrm{dom}(P) := H^{1,2}(\mathbb{R}),$$

we get the representation

$$\mathfrak{h}_t[f, g] = \tfrac{1}{2}(Pf, Pg) + \kappa_1(t)f(x_1(t))\overline{g(x_1(t))} + \kappa_2(t)f(x_2(t))\overline{g(x_2(t))},$$

$f, g \in H^{1,2}(\mathbb{R})$. The momentum operator generates the right-shift group $S(\tau) := e^{-i\tau P}$, $\tau \in \mathbb{R}$, acting as

$$(S(\tau)f)(x) = f(x - \tau), \quad f \in L^2(\mathbb{R}), \quad \tau \in \mathbb{R}.$$

Obviously, one has that

$$S(\tau)^{-1}H(t)S(\tau) = \tfrac{1}{2}P^2 + \kappa_1(t)\delta(x - x_1(t) + \tau) + \kappa_2(t)\delta(x - x_2(t) + \tau).$$

In particular, for $y(t) := \frac{1}{2}(x_1(t) + x_2(t))$ we obtain

$$\begin{aligned} H^S(t) &:= S(y(t))^{-1}H(t)S(y(t)) = \\ &e^{iy(t)P}H(t)e^{-iy(t)P} = \frac{1}{2}P^2 + \kappa_1(t)\delta(x + x(t)) + \kappa_2(t)\delta(x - x(t)), \end{aligned}$$

where the relative coordinate obeys

$$x(t) := \frac{x_2(t) - x_1(t)}{2} > 0, \quad t \in \mathbb{R},$$

by (6.13). Further, we define the unitary transformations $W(\theta) : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}))$, $\theta > 0$,

$$W(\theta)f)(x) := \sqrt{\theta}f(\theta x), \quad f \in L^2(\mathbb{R}).$$

Let X be the multiplication operator $(Xf) := xf(x)$ in $L^2(\mathbb{R})$. Then

$$L = \frac{1}{2}(XP + PX)$$

is a so-called *dilation operator*, which is self-adjoint in $L^2(\mathbb{R})$. The operator iL generates the *dilation group* given by

$$(e^{isL}f)(x) = e^{s/2}f(e^s x), \quad f \in L^2(\mathbb{R}), \quad s \in \mathbb{R}.$$

Then we obviously get $W(\theta) = e^{i\ln(\theta)L}$, $\theta > 0$, and

$$\begin{aligned} W(\theta)^{-1}H^S(t)W(\theta) &= \\ &-\frac{\theta^2}{2}\frac{d^2}{dx^2} + \kappa_1(t)\theta\delta(x + \theta x(t)) + \kappa_2(t)\theta\delta(x - \theta x(t)). \end{aligned}$$

If we set $\theta = 1/x(t)$, then

$$\begin{aligned} H^{SW}(t) &:= W(1/x(t))^{-1}H^S(t)W(1/x(t)) = \\ &e^{i\ln(x(t))L}H^S(t)e^{-i\ln(x(t))L} = \frac{1}{2x(t)^2}P^2 + \varkappa_1(t)\delta(x+1) + \varkappa_2(t)\delta(x-1), \end{aligned}$$

where

$$\varkappa_1(t) := \frac{\kappa_1(t)}{x(t)} \quad \text{and} \quad \varkappa_2(t) := \frac{\kappa_2(t)}{x(t)}.$$

The relation between this Hamiltonian and (6.11) has the form

$$H(t) = e^{-iy(t)P}e^{-i\ln(x(t))L}H^{SW}(t)e^{i\ln(x(t))L}e^{iy(t)P}.$$

Now, in the Hilbert space $L^2(\mathbb{R}, \mathfrak{H})$, $\mathfrak{H} := L^2(\mathbb{R})$, we introduce the operator

$$(Df)(t,x) = \left(\frac{1}{i}\frac{\partial}{\partial t}f\right)(t,x), \qquad \operatorname{dom}(D) := H^{1,2}(\mathbb{R}, \mathfrak{H}).$$

The multiplication operator $S := M(S(y(t)))$, $y(t) = \frac{1}{2}(x_1(t) + x_2(t))$ (i.e., $(Sf)(t,x) := (S(y(t))f)(t,x) = f(t, x - y(t))$, see (2.1)), defines a unitary operator on $L^2(\mathbb{R}, \mathfrak{H})$, and we have

$$D^S := S^{-1} D\, S = D - \dot{y}(t)P.$$

Similarly, the multiplication operator $W := M(W(1/x(t)))$, $x(t) = \frac{1}{2}(x_2(t) - x_1(t))$, induces a unitary operator on $L^2(\mathbb{R}, \mathfrak{H})$. We set

$$D^{SW} := W^{-1} D^S W.$$

Since the multiplication operator $W = M(e^{-i\ln(x(t))L})$, by the commutation relation $LP - PL = iP$ one gets that

$$D^{SW} = D - i\frac{\dot{x}(t)}{x(t)}L - i\frac{\dot{y}(t)}{x(t)}P.$$

Now we set

$$H^{SW} := W^{-1} S^{-1} H\, S\, W \quad \text{and} \quad \widetilde{K}^{SW} := D^{SW} + H^{SW}$$

with domain $\mathrm{dom}(\widetilde{K}^{SW}) := \mathrm{dom}(D^{SW}) \cap \mathrm{dom}(H^{SW})$. Then a straightforward computation shows that this operator is equal to

$$\widetilde{K}^{SW} := D + L_0$$

with domain $\mathrm{dom}(\widetilde{K}^{SW}) = \mathrm{dom}(D) \cap \mathrm{dom}(L_0)$, where

$$\begin{aligned} L_0(t) \;\; := \;\; & \frac{1}{2x(t)^2}(P - x(t)(\dot{x}(t)X + \dot{y}(t))^2 - \frac{1}{2}(\dot{x}(t)X + \dot{y}(t))^2 \\ & + \varkappa_1(t)\delta(x+1) + \varkappa_2(t)\delta(x-1). \end{aligned}$$

Finally, let us introduce the gauge transformation

$$(\Gamma(t)f)(x) := e^{i\int_0^t\left((\dot{x}(s)x + \dot{y}(s))^2 + x^2\right)ds/2} f(x), \quad f \in L^2(\mathbb{R}),$$

which induces the multiplication operator $\Gamma := M(\Gamma(t))$ on $L^2(\mathcal{I}, \mathfrak{H})$. Then we find

$$\widetilde{\mathbf{K}} := \widetilde{K}^{SW\Gamma} := \Gamma^{-1} K^{SW} \Gamma = D + L,$$

where the operator

$$\begin{aligned} & L(t) := \\ & \qquad \frac{1}{2x(t)^2}(P + \beta_1(t)X + \beta_0(t))^2 + \frac{1}{2}X^2 + \varkappa_1(t)\delta(x+1) + \varkappa_2(t)\delta(x-1) \end{aligned}$$

with

$$\beta_1(t) := \int_0^t (\dot{x}(s)^2 + 1)ds - x(t)\dot{x}(t)$$

and

$$\beta_0(t) := \int_0^t \dot{y}(s)\dot{x}(s)ds - x(t)\dot{y}(t).$$

As above the family $\{L(t)\}_{t\in\mathbb{R}}$, is measurable and defines a densely defined self-adjoint multiplication operator $L := M(L(t))$ on $L^2(\mathcal{I},\mathfrak{H})$. Then the operators $\widetilde{K} := D + H$ and $\widetilde{\mathbf{K}} = D + L$ are related by

$$\widetilde{K} = S\,W\,\Gamma\,\widetilde{\mathbf{K}}\,\Gamma^{-1}\,W^{-1}\,S^{-1}. \tag{6.14}$$

Instead to solve the bidirectional evolution equation (5.1) we consider the modified bidirectional evolution equation

$$\frac{1}{i}\,\frac{\partial}{\partial t}u(t) + L(t)u(t) = 0. \tag{6.15}$$

Following Section 5 we introduce the family of quadratic forms $\mathfrak{l}_t[\cdot,\cdot]$

$$\begin{aligned}\mathfrak{l}_t[f,g] \;:=\; & \frac{1}{2x(t)^2}(Pf + \beta_1(t)Xf + \beta_0(t)f, Pf + \beta_1(t)Xg + \beta_0(t)g) + \\ & \frac{1}{2}(Xf,Xg) + \varkappa_1(t)f(-1)\overline{g(-1)} + \varkappa_2(t)f(1)\overline{g(1)} + (f,g),\end{aligned}$$

$f,g \in \mathrm{dom}(\mathfrak{l}_t) := \mathrm{dom}(P)\cap\mathrm{dom}(X)$ corresponding to operators $L(t)$, and define the norm

$$\|f\|_t^+ := \|\sqrt{L(t)+I}f\| = \sqrt{\mathfrak{l}_t[f,f] + \|f\|^2},$$

$f \in \mathrm{dom}(\sqrt{L(t)+I}) = \mathrm{dom}(\mathfrak{l}_t) = \mathrm{dom}(P)\cap\mathrm{dom}(X)$. It is easy to check that the domain $\mathrm{dom}(\mathfrak{l}_t)$ is independent of $t\in\mathbb{R}$. By $\mathfrak{L}_t^+$ we denote the Hilbert space, which arises when we endow the domain $\mathrm{dom}(\mathfrak{l}_t)$ with the scalar product $(f,g)_t^+ := \mathfrak{l}_t[f,g] + (f,g)$. Note that the norm $\|\cdot\|_t^+$ is equivalent to the norm $\|f\|_{PX} = \sqrt{\|Pf\|^2 + \|Xf\|^2}$, $f \in \mathrm{dom}(P)\cap\mathrm{dom}(X)$.

Now we proceed as in the previous section. First we find

$$\begin{aligned}\frac{d}{dt}(\|f\|_t^+)^2 = & \frac{\dot{x}(t)}{x(t)^3}\|Pf + \beta_1(t)Xf + \beta_0(t)f\|^2 \\ & + \frac{2}{x(t)^2}\Re\mathrm{e}(Pf + \beta_1(t)Xf + \beta_0(t)f, \dot{\beta}_1(t)Xf + \dot{\beta}_0(t)f) \\ & + \dot{\varkappa}_1(t)|f(-1)|^2 + \dot{\varkappa}_2(t)|f(1)|^2.\end{aligned}$$

A straightforward computation shows that for any bounded interval $\mathcal{I}$ there is a constant $\gamma_{\mathcal{I}}$ such that

$$\left|\frac{d}{dt}(\|f\|_t^+)^2\right| \le 2\gamma_{\mathcal{I}}(\|f\|_t^+)^2$$

for $t \in \mathcal{I}$ which yields

$$-\gamma_{\mathcal{I}} \leq \frac{d}{dt}\ln(\|f\|_t^+) \leq \gamma_{\mathcal{I}}.$$

Hence we obtain the estimate

$$-\gamma_{\mathcal{I}}(t-s) \leq \ln(\|f\|_t^+) - \ln(\|f\|_s^+) \leq \gamma_{\mathcal{I}}(t-s)$$

for $t, s \in \mathcal{I}$ and $s \leq t$, which yields

$$\|f\|_t^+ \leq e^{\gamma_{\mathcal{I}}(t-s)}\|f\|_s^+, \quad t, s \in \mathcal{I}, \quad s \leq t.$$

The last relation implies (5.10). By virtue of Theorem 5.2 we get that the *auxiliary* bidirectional evolution equation

$$\frac{\partial}{\partial t}u(t) + iL^-(t)u(t) = 0$$

admits a unique solution $\{\Lambda^-(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ on $\mathbb{R}$. By Theorem 5.5 the *original* bidirectional evolution equation (6.15) admits a solution for which the Hilbert space $\mathfrak{L}_0^+$ is admissible. By the same line of reasoning as for non-moving point interactions one can prove that there is a unique unitary solution $\{\Lambda(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ of the bidirectional evolution equation (6.15) for which $\mathfrak{L}_0^+$ is admissible.

These results allow us to prove that the original forward evolution equation (5.8) on $\mathbb{R}$ admits a solution. To this end one has to verify that for any bounded interval $\mathcal{I}$ the extension of the forward generator $\mathbf{K}_{\mathcal{I}}$ of $\widetilde{\mathbf{K}}_{\mathcal{I}}$ defines an extension of the forward generator $K_{\mathcal{I}}$ of $\widetilde{K}_{\mathcal{I}}$ defined by $K_{\mathcal{I}} := S\ W\ \Gamma\ \mathbf{K}_{\mathcal{I}}\ \Gamma^{-1}\ W^{-1}\ S^{-1}$. However, this is evident since it follows from the representation (6.14). Similarly, one proves that for any bounded interval $\mathcal{I}$ the backward generator extension $\mathbf{K}^{\mathcal{I}}$ of $\widetilde{\mathbf{K}}^{\mathcal{I}}$ defines a backward generator extension $K^{\mathcal{I}}$ of $\widetilde{K}^{\mathcal{I}}$ by $K^{\mathcal{I}} := S\ W\ \Gamma\ \mathbf{K}^{\mathcal{I}}\ \Gamma^{-1}\ W^{-1}\ S^{-1}$. From these facts we immediately obtain the fact that the bidirectional propagator $\{G(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$ defined by

$$G(t,s) := e^{-iy(t)P}e^{-i\ln(x(t))L}\Gamma(t)\Lambda(t,s)\Gamma(s)^{-1}e^{i\ln(x(s))L}e^{iy(s)P}, \qquad (6.16)$$

for any $(t,s) \in \mathbb{R}\times\mathbb{R}$, is a solution of the bidirectional evolution equation (5.1).

It remains only to identify the subspace which is admissible with respect to $\{G(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$. We recall that $\mathfrak{L}_0^+$ is the subspace which is admissible with respect to $\{\Lambda(t,s)\}_{(t,s)\in\mathbb{R}\times\mathbb{R}}$. If we set

$$H^\Gamma(t) := \Gamma(t)L(t)\Gamma(t)^{-1}, \quad t \in \mathbb{R},$$

then a straightforward computation shows that

$$\begin{aligned} H^{\Gamma}(t) \ &:= \ \frac{1}{2x(t)^2}(P - x(t)\dot{x}(t)X - x(t)\dot{y}(t))^2 + \\ & \quad \frac{1}{2}X^2 + \varkappa_1(t)(t)\delta(x+1) + \varkappa_2(t)\delta(x-1). \end{aligned}$$

Further, setting

$$H^{\Gamma W}(t) := e^{-i\ln(x(t))L}\Gamma(t)L(t)\Gamma(t)^{-1}e^{i\ln(x(t))L}$$

we find that

$$H^{\Gamma W}(t) = \frac{1}{2}(P - \frac{\dot{x}(t)}{x(t)}X - \dot{y}(t))^2 + \frac{1}{2x(t)^2}X^2 + \kappa_1(t)\delta(x + x(t)) + \kappa_2(t)\delta(x - x(t)).$$

Finally, we introduce the family:

$$H^{\Gamma WS}(t) := e^{-iy(t)P}e^{-i\ln(x(t))L}\Gamma(t)L(t))\Gamma(t)^{-1}e^{i\ln(x(t))L}e^{iy(t)P}$$

which implies

$$\begin{aligned} H^{\Gamma WS}(t) &= \frac{1}{2}\left(P - \frac{\dot{x}(t)}{x(t)}(X - y(t))\right)^2 + \\ & \frac{1}{2x(t)^2}(X - y(t))^2 + \kappa_1(t)\delta(x - x_1(t)) + \kappa_2(t)\delta(x - x_2(t)). \end{aligned}$$

For a shorthand let $Z(t) := H^{\Gamma WS}(t)$. Then the quadratic form associated with $Z(t)$ we denote by $\mathfrak{z}_t[\cdot,\cdot]$. One can easily verify that the domain $\mathrm{dom}(\mathfrak{z}_t)$ is independent of $t \in \mathbb{R}$. The Hilbert space which is associated with $\mathfrak{z}_t$ is denoted by $\mathfrak{Z}_t^+$. A straightforward computation shows that for any $t \in \mathbb{R}$ the Hilbert space $\mathfrak{Z}_t^+$ can be identified with $\mathfrak{H}_{PX} := \{\mathrm{dom}(P) \cap \mathrm{dom}(X), \|\cdot\|_{PX}\}$. It is obvious, that the Hilbert space $\mathfrak{H}_{PX}$ is admissible for the bidirectional propagator $\{G(t,s)\}_{(t,s)\in\mathbb{R}}$ defined by (6.16). Summing up one gets the following theorem:

Theorem 6.2. *Let $\kappa_j(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}_+$ and $x_j(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ be continuously differentiable functions. Further, let $\{H(t)\}_{t\in\mathbb{R}}$ be the family of non-negative self-adjoint operators given by* (6.11) *and* (6.12)*. If the condition* (6.13) *is satisfied for any $t \in \mathbb{R}$, then the bidirectional evolution equation* (5.1) *is well posed on $\mathbb{R}$ for $p = 2$ and possesses a unique unitary solution for which the Hilbert space $\mathfrak{H}_{PX}$ is admissible.*

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