Advances in Differential Equations Volume 14, Numbers 5-6 (2009), 401–432

# POSITIVE SOLUTIONS FOR EQUATIONS AND SYSTEMS WITH $p$-LAPLACE-LIKE OPERATORS

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(Submitted by: Jean Mawhin)

**Abstract.** We prove the existence of positive solutions to boundary-value problems of the form

$$(\phi(u'))' + f(t,u) = 0, \quad t \in (0,1)$$
$$\theta(u(0)) = \beta\theta(u'(0)), \quad \theta(u(1)) = -\delta\theta(u'(1)), \quad \beta,\delta \geq 0,$$

where $\phi$ and $\theta$ are odd increasing homeomorphisms of the real line. We also prove the existence of positive solutions to related systems. Our approach is via a priori estimates and Leray-Schauder degree.

## 1. Introduction

Let us consider the problem

$$\begin{aligned}(\phi(u'))' + f(t,u) = 0, \quad t \in (0,1)\\ \theta(u(0)) = \beta\theta(u'(0)), \quad \theta(u(1)) = -\delta\theta(u'(1)), \quad \beta,\delta \geq 0,\end{aligned} \tag{1.1}$$

where we assume that the function $\phi$ that generates the differential operator $(\phi(u'))'$ and the function $\theta$ in the boundary conditions are in general nonhomogeneous functions that satisfy

$(\phi_1)$ $\quad \phi, \theta$ are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$.

Accepted for publication: December 2008.
AMS Subject Classifications: 34B15, 34B18.
MG-H was supported by Fondecyt Grant 1070951, RM was supported by Fondap M.A. and Milenio grant-P05-004F.

The function $f$ satisfies

$$(f_1) \qquad \begin{cases} f : [0,1] \times [0,\infty) \to [0,\infty) \text{ is continuous,} \\ f(t,0) = 0, \;\; f(t,s) > 0 \quad \text{for all } s > 0 \text{ and } t \in [0,1]. \end{cases}$$

By a solution to (1.1) we mean a function $u \in C^1[0,1]$, such that $\phi(u') \in C^1[0,1]$, that satisfies the equation as well as the boundary conditions in (1.1). By a *positive solution* we mean a solution $u$ that satisfies $u(t) > 0$ for all $t \in (0,1)$.

The operator $(\phi(u'))'$ generalizes the well-known $p$-Laplace operator. In the last fifteen years nonlinear boundary-value problems containing the operator $(\phi(u'))'$ have been studied with increasing interest by a series of authors, see for example [8], [9], [10], [3] and [4]. Recently boundary-value problems including vector versions of this operator have been studied in [6],[7], [11], [13], and [15].

In this paper we will first show some existence of positive solutions results to (1.1) and then we will extend our analysis to some related systems cases that we will describe below.

We point out that in [1] and [2] boundary-value problems with an operator generated by homeomorphisms $\phi$ from $\mathbb{R}$ into a proper subset of $\mathbb{R}$, or from a proper subset of $\mathbb{R}$ onto $\mathbb{R}$, have been considered. It would be interesting to see to what extent the results of this paper can be extended to those situations.

Setting

$$f_0(t) := \lim_{s \to 0} \frac{f(t,s)}{\phi(s)}, \quad f_\infty(t) := \lim_{s \to \infty} \frac{f(t,s)}{\phi(s)},$$

we establish our first two results for the scalar case.

**Theorem 1.1.** *Let $\phi$ and $f$ satisfy*

$$f_0(t) = 0 \quad \textit{and} \quad f_\infty(t) = \infty \quad \textit{uniformly for} \quad t \in [0,1] \tag{1.2}$$

*and*

$$\liminf_{s \to 0} \frac{\phi(\tau s)}{\phi(s)} > 0 \quad \textit{for all } 0 < \tau < 1, \quad \limsup_{s \to \infty} \frac{\phi(\tau s)}{\phi(s)} < \infty \quad \textit{for all } \tau > 1. \tag{1.3}$$

*Furthermore, in the case that $\delta > 1$ we assume that every time $\{s_k\}$ and $\{r_k\}$ are sequences such that $s_k \to 0$ and $r_k \to 0$, as $k \to \infty$, it holds that*

$$\lim_{k \to \infty} \frac{\theta(r_k s_k)}{\theta(s_k)} = 0. \tag{1.4}$$

*Then problem* (1.1) *has at least one positive solution.*

The conditions in (1.2) can be read as $f$ being sublinear with respect to $\phi$ at zero and superlinear with respect to $\phi$ at infinity. On the other hand the second condition for $\phi$ in (1.3) turns out to be a sufficient condition, see Proposition 2.3 below, for a priori boundedness of positive solutions of problem (1.1). Both conditions in (1.3) are trivially satisfied if, for example, $\phi(s) = |s|^{p-2}s$, $p > 1$, the $p$-Laplace operator. Thus, in this case, and under the conditions of Proposition 2.3, positive solutions to problem (1.1) are a priori bounded.

Finally, condition (1.4) for the function $\theta$ arises as a technical condition when $\delta > 1$ in the boundary conditions. If $0 \leq \delta \leq 1$ then condition (1.4) is not needed in Theorem 1.1. Clearly (1.4) is automatically satisfied if for example $\theta(s) = |s|^{q-2}s$, $q > 1$.

Our next result is complementary to the previous theorem. More precisely, we consider the case in which $\phi, \theta$ and $f$ still satisfy respectively assumptions $(\phi_1)$, $(f_1)$, but now we assume that $f_0(t) = \infty$ ($f$ superlinear with respect to $\phi$ at zero) and $f_\infty(t) = 0$ uniformly for $t \in [0,1]$ ($f$ sublinear with respect to $\phi$ at infinity).

**Theorem 1.2.** *Let $\phi$ and $f$ satisfy*

$$f_0(t) = \infty \quad \text{and} \quad f_\infty(t) = 0 \quad \text{uniformly for} \quad t \in [0,1], \tag{1.5}$$

$$\limsup_{s\to 0} \frac{\phi(\tau s)}{\phi(s)} < \infty \ \text{ for all } \tau > 1 \ \text{ and } \ \liminf_{s\to\infty} \frac{\phi(\tau s)}{\phi(s)} > 0 \ \text{ for all } 0 < \tau < 1. \tag{1.6}$$

*In addition if $\delta > 1$ assume that every time $\{s_k\}$ and $\{\tau_k\}$ are sequences such that, $s_k \to \infty$ and $\tau_k \to 0$, as $k \to \infty$, it holds that*

$$\lim_{k\to\infty} \frac{\theta(\tau_k s_k)}{\theta(s_k)} = 0. \tag{1.7}$$

*Then problem* (1.1) *has a positive solution.*

**Remark 1.1.** In problem (1.1) we could have considered boundary conditions of the form

$$\theta(u(0)) = \beta\theta(u'(0)), \quad \gamma(u(1)) = -\delta\gamma(u'(1)), \quad \beta, \delta \geq 0,$$

where $\theta$ and $\gamma$ are two odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. In (1.1) we took $\theta = \gamma$ in order to simplify the presentation of our results.

A simple example illustrating the previous theorems is the following.

**Example 1.1.** Consider the problem

$$\begin{gathered}(|u'|^{p-2}u'\log(1+|u'|))' + f(t,u) = 0, \quad t \in (0,1)\\ u(0) = \beta u'(0), \quad u(1) = -\delta u'(1), \quad \beta, \delta \geq 0,\end{gathered} \tag{1.8}$$

where $p > 1$, and $f$ satisfies $(f_1)$. If in addition

$$f_0(t) = 0 \quad \text{and} \quad f_\infty(t) = \infty \quad \text{uniformly for} \quad t \in [0,1], \tag{1.9}$$

or

$$f_0(t) = \infty \quad \text{and} \quad f_\infty(t) = 0 \quad \text{uniformly for} \quad t \in [0,1], \tag{1.10}$$

then problem (1.8) has a positive solution.

Indeed it can be directly verified that for all $\tau > 0$

$$\lim_{s\to 0} \frac{\phi(\tau s)}{\phi(s)} = \tau^p \quad \text{and} \quad \lim_{s\to\infty} \frac{\phi(\tau s)}{\phi(s)} = \tau^{p-1},$$

and hence conditions (1.3) and (1.6) are verified. Also conditions (1.4) and (1.7) are trivially satisfied. Thus in case (1.9) holds the result follows from Theorem 1.1 and if (1.10) holds then the result follows from Theorem 1.2.

In [5] Erbe and Wang initiated the study of this type of problem by considering the case when $\phi(s) = \theta(s) = s$ in problem (1.1). More precisely they consider the problem

$$u'' + a(t)f(u) = 0, \quad \alpha u(0) = \beta u(1), \quad \gamma u(1) = -\delta u'(1).$$

Under sublinear, superlinear conditions (at zero or infinity ) for $f$ they proved existence of positive solutions. In [12], a generalization of the results of Erbe and Wang was obtained. Indeed, in Theorem 3.1 of [12] existence of positive solutions for problem (1.1) is proved, with $f(t,u) = a(t)f(u)$, $\theta(s) = s$, and under the following additional condition on the homeomorphism $\phi$:

$$\phi(x)\phi(y) = \phi(xy) \quad \text{for all } x,\ y \in \mathbb{R}.$$

As is well known, see for example [14], this condition implies that $\phi(s) = |s|^{p-2}s$, for some $p > 1$, in other words $\phi$ is a homogeneous function. As a consequence the results of Example 1.1 do not follow from [12].

In Section 6 we will consider systems of the form

$$\begin{gathered}(\phi(u'))' + g(t,v) = 0, \quad (\psi(v'))' + f(t,u) = 0, \quad t \in (0,1)\\ u(0) = \beta_1 u'(0), \quad u(1) = -\delta_1 u'(1), \quad v(0) = \beta_2 v'(0), \quad v(1) = -\delta_2 v'(1),\end{gathered} \tag{1.11}$$

with $\beta_i,\ \delta_i \geq 0,\ i = 1,2$, and where

$(H_1)$ $\begin{cases}\phi,\ \psi \text{ are odd increasing homeomorphisms from } \mathbb{R} \text{ onto } \mathbb{R}\\ \text{such that } \phi(0) = \psi(0) = 0,\end{cases}$

and

$(H_2)$

$$\begin{cases} f,\ g : [0,1] \times [0,\infty) \to [0,\infty) \text{ are continuous}, & f(t,0) = g(t,0) = 0, \\ f(t,s) > 0,\ g(t,s) > 0 \quad \text{for all } s > 0 \text{ and } t \in [0,1]. \end{cases}$$

By a solution to (1.11) we mean functions $u, v \in C^1[0,1]$ such that $\phi(u') \in C^1[0,1]$, $\phi(v') \in C^1[0,1]$, with $u, v$ satisfying the equations as well as the boundary conditions in (1.11). We will be interested in existence of positive solutions for this problem. By a positive solution to (1.11), we mean a solution to this system that satisfies $u(t) > 0$ and $v(t) > 0$ for all $t \in (0,1)$.

Let us set

$$f_0(t) := \lim_{s\to 0} \frac{f(t,s)}{\phi(s)}, \quad f_\infty(t) := \lim_{s\to\infty} \frac{f(t,s)}{\phi(s)},$$

$$g_0(t) := \lim_{s\to 0} \frac{g(t,s)}{\psi(s)}, \quad g_\infty(t) := \lim_{s\to\infty} \frac{g(t,s)}{\psi(s)}.$$

In Section 6, we will prove the following extension of Theorem 1.1 to systems.

**Theorem 1.3.** *Let $\phi$ and $f$ satisfy*

$$f_0(t) = g_0(t) = 0 \quad \textit{and} \quad f_\infty(t) = g_\infty(t) = \infty \quad \textit{uniformly for} \quad t \in [0,1] \tag{1.12}$$

*and*

$$\liminf_{s\to 0} \frac{\phi(\tau s)}{\phi(s)} > 0, \quad \liminf_{s\to 0} \frac{\psi(\tau s)}{\psi(s)} > 0 \textit{ for all } 0 < \tau < 1, \tag{1.13}$$

$$\limsup_{s\to\infty} \frac{\phi(\tau s)}{\phi(s)} < \infty, \quad \limsup_{s\to\infty} \frac{\psi(\tau s)}{\psi(s)} < \infty \quad \textit{for all } \tau > 1. \tag{1.14}$$

*Then problem* (1.11) *has at least one solution* $(u, v)$ *such that* $u(t) > 0$ *and* $v(t) > 0$ *for all* $t \in (0,1)$.

Our aim in Section 6 is to show by means of one nontrivial case how our results for the scalar problem can be extended to some systems. We nevertheless point out that using similar arguments the following theorem can be proved. The proof is left to the reader.

**Theorem 1.4.** *Let $\phi$ and $f$ satisfy*

$$f_0(t) = g_0(t) = \infty \quad \textit{and} \quad f_\infty(t) = g_\infty(t) = 0 \quad \textit{uniformly for} \quad t \in [0,1], \tag{1.15}$$

$$\limsup_{s\to 0} \frac{\phi(\tau s)}{\phi(s)} < \infty \quad \textit{and} \quad \limsup_{s\to 0} \frac{\psi(\tau s)}{\psi(s)} < \infty \quad \textit{for all } \tau > 1 \quad \textit{and} \tag{1.16}$$

$$\liminf_{s\to\infty}\frac{\phi(\tau s)}{\phi(s)}>0 \quad \textit{and} \quad \liminf_{s\to\infty}\frac{\psi(\tau s)}{\psi(s)}>0 \quad \textit{for all } \tau<1. \tag{1.17}$$

*Then problem* (1.11) *has a solution* $(u,v)$ *such that* $u(t)>0$ *and* $v(t)>0$ *for all* $t\in(0,1)$.

**Remark 1.2.** One can think of many other interesting ways of extending our results from the scalar case to systems of the form (1.11). We do not treat these cases in this paper because of space limitation. We notice though that in problem (1.11) the boundary conditions have been simplified in order that they do not contain nonhomogeneous functions. This simplifies our presentation and the proof of the results.

A piece of notation: $C[0,1]$, endowed with the norm $||u||=\sup_{t\in[0,1]}|u(t)|$ denotes the usual Banach space of real continuous functions defined in $[0,1]$. We will also use the product space $C[0,1]\times C[0,1]$, with the norm

$$||(u,v)||=\sup_{t\in[0,1]}|u(t)|+\sup_{t\in[0,1]}|v(t)|=||u||+||v||.$$

Our paper is organized as follows. In Section 2 we state and prove some key preliminary results. We will use Leray-Schauder topological degree arguments to prove our results and hence we will write (1.1) and problem (1.11) in equivalent abstract forms in the spaces $C[0,1]$ and $C[0,1]\times C[0,1]$, respectively. The case of problem (1.1) is done in Section 3, while the case of problem (1.11) is done in Section 6, using the results of Section 3.

The proofs of Theorems 1.1 and 1.2 for the scalar case are done in Sections 4 and 5, respectively. In each case the proof is obtained through a series of preliminary propositions that finally yield the result. Finally, in Section 6 we prove Theorem 1.3; again here the result is obtained through a number of preliminary propositions.

## 2. Some key Preliminary results

We begin with the following.

**Proposition 2.1.** (Generalized maximum principle) *Let* $u\in C^1[0,1]$ *with* $\phi(u')\in C^1[0,1]$ *satisfying*

$$\begin{gathered}-(\phi(u'))'\geq 0, \quad \textit{for all} \quad t\in(0,1),\\ \theta(u(0))=\beta\theta(u'(0)), \quad \theta(u(1))=-\delta\theta(u'(1)), \quad \beta,\delta\geq 0.\end{gathered} \tag{2.1}$$

*Then either* $u\equiv 0$ *or* $u(t)>0$ *for all* $t\in(0,1)$. *In this last case it holds that* $u'(0)>0$ *and* $u'(1)<0$. *Moreover, if* $\beta\neq 0$, *then* $u(0)>0$, *and, if* $\delta\neq 0$, *then* $u(1) > 0$. *In all the cases it holds that*

$$u(t) \geq \min\{u(0), u(1)\}. \tag{2.2}$$

**Proof.** Let $u$ be a solution to (2.1) and assume that $u \not\equiv 0$. From the inequality in (2.1) we find that $-\phi(u')$ is nondecreasing, and by the monotonicity of $\phi$, $u'$ is nonincreasing, implying that $u$ is concave.

We claim that $u'(0) > 0$. Indeed, assuming by contradiction that $u'(0) \leq 0$, then by the boundary condition at zero and the fact that $\theta$ is odd, it follows that $u(0) \leq 0$, which combined with the concavity of $u$ implies that $u(t) \leq 0$ for all $t \in [0, 1]$ and is nonincreasing. Since $u \not\equiv 0$ we cannot have $u(1) = 0$ and hence $u(1) < 0$. But, by the boundary condition at 1, this implies that $\delta > 0$ and $u'(1) > 0$, which is a contradiction to $u'$ nonincreasing. Thus $u'(0) > 0$ (and hence $u(0) \geq 0$). We note that if $u'(t) > 0$ for all $t \in (0, 1)$, then $u(1) > 0$, which in turn implies that $\delta > 0$ and $u'(1) < 0$, a contradiction. Hence there must exist a first $\bar{t} \in (0, 1)$ such that $u'(\bar{t}) = 0$ and $u'(t) > 0$, $t \in [0, \bar{t}]$. This implies that $u'(1) \leq 0$. If $u'(1) = 0$ (and hence $u(1) = 0$), then $u'(t) = 0$ for all $t \in [\bar{t}, 1]$, and hence $0 < u(\bar{t}) = u(1) = 0$, a contradiction, and $u'(1) < 0$.

Now the facts that $u'(0) > 0$ and $u'(1) < 0$ together with the boundary conditions imply that if $\beta > 0$ then $u(0) > 0$ and if $\delta > 0$ then $u(1) > 0$. Finally, the validity of (2.2) follows from the concavity of $u$. □

**Remark 2.1.** In the last proof the point $\bar{t}$ was chosen such that $u'(t) > 0$, for all $t \in (0, \bar{t})$. In case there exists another point $\bar{t} < t^* < 1$ such that $u'(t^*) = 0$ and $u'(t) < 0$ for all $t \in (t^*, 1]$ it is clear that $u'(t) = 0$ for all $t \in [\bar{t}, t^*]$ and this interval is the only set of points in $[0, 1]$ where $u'$ can vanish.

In our next proposition we consider a solution $u$ of (2.1) which is nontrivial. Then we know $u$ is positive in $(0, 1)$. The following proposition will be used often in the paper.

**Proposition 2.2.** *Let $u$ be a nontrivial solution to problem* (2.1) *for which we assume there is a unique $\bar{t} \in (0, 1)$ such that $u'(\bar{t}) = 0$. Then for any fixed $\rho \in (0, 1/2)$ the solution $u$ satisfies*

$$u(t) \geq u(\bar{t}) \frac{\rho/2}{1 - \rho/2} \quad \textit{for all } t \in [\rho, 1 - \rho]. \tag{2.3}$$

**Proof.** Let $\rho \in (0, 1/2)$ be given and note that then $\rho < 1 - \rho$. Suppose first that $\bar{t} < 1 - \rho$. Then by considering the straight line joining $(\bar{t}, u(\bar{t}))$ with

$(1-\rho/2, u(1-\rho/2))$, i.e.,

$$\ell(t) = u(\bar{t}) + \frac{u(1-\rho/2) - u(\bar{t})}{1-\rho/2-\bar{t}}(t-\bar{t}),$$

and since by the concavity of $u$, $u(t) \geq \ell(t)$ for all $t \in [\bar{t}, 1-\rho/2]$, we have that

$$u(t) \geq u(\bar{t})\Big(1 - \frac{t-\bar{t}}{1-\rho/2-\bar{t}}\Big) \geq u(\bar{t})\frac{1-\rho/2-t}{1-\rho/2-\bar{t}}$$

and thus

$$u(t) \geq u(\bar{t})\frac{\rho/2}{1-\rho/2} \quad \text{for all } t \in [\bar{t}, 1-\rho]. \tag{2.4}$$

Suppose next that $\bar{t} > \rho$. Then by considering the straight line joining $(\bar{t}, u(\bar{t}))$ with $(\rho/2, u(\rho/2))$ and arguing similarly as above, we conclude that

$$u(t) \geq u(\bar{t})\frac{\rho/2}{1-\rho/2} \quad \text{for all } t \in [\rho, \bar{t}\,]. \tag{2.5}$$

Thus if $\bar{t} \leq \rho$ then (2.3) follows from (2.4). If $\bar{t} \in (\rho, 1-\rho)$, then (2.4) and (2.5) imply (2.3). Finally, if $\bar{t} \geq 1-\rho$, then (2.3) follows from (2.5). □

We need later the following auxiliary result.

**Proposition 2.3.** *Consider the problem*

$$(I_1) \qquad \begin{gathered} -(\phi(u'))' \geq f(t, |u(t)|), \quad t \in (0,1) \\ \theta(u(0)) = \beta\theta(u'(0)), \quad \theta(u(1)) = -\delta\theta(u'(1)), \quad \beta, \delta \geq 0, \end{gathered}$$

*where $f$ satisfies condition $(f_1)$ and*

$$f_\infty(t) = \infty \quad \textit{uniformly for} \quad t \in [0,1]. \tag{2.6}$$

*Suppose that $(I_1)$ has a sequence of solutions $\{u_k\}$, with $||u_k|| \to \infty$. Then*

- *(i) for any $\rho \in (0, 1/2)$, it holds that $u_k(t) \to \infty$ uniformly for $t \in [\rho, 1-\rho]$, and*
- *(ii) there exists $\tau > 1$ such that*

$$\limsup_{s\to\infty} \frac{\phi(\tau s)}{\phi(s)} = \infty. \tag{2.7}$$

**Proof.** $(i)$ For each $k \in \mathbb{N}$, $u_k$ is a nontrivial solution of problem (2.1), thus by Proposition 2.1 it holds that $u_k(t) > 0$ for all $t \in (0,1)$ and that there is at least one $t_k \in (0,1)$ such that $u_k'(t_k) = 0$. We claim there is just one such point. Indeed, if there exists $s_k \in (0,1)$ such that $u_k'(s_k) = 0$, say $s_k > t_k$, then $u_k'(t) = 0$ for all $t \in [t_k, s_k]$. Since then $f(t, u_k(t)) = 0$ for all $t \in [t_k, s_k]$, and hence $u_k(t) = 0$ for all $t \in [t_k, s_k]$, we would contradict the positivity of

$u_k$ in $(0,1)$. Thus there exists a unique $t_k \in (0,1)$ such that $u_k'(t_k) = 0$ with $u_k(t_k) = ||u_k||$.

It then follows from Proposition 2.2 that, for fixed $\rho \in (0,1/2)$,

$$u_k(t) \geq ||u_k|| \frac{\rho/2}{1-\rho/2} \quad \text{for all } t \in [\rho, 1-\rho], \tag{2.8}$$

and hence $u_k(t) \to \infty$ as $k \to \infty$ uniformly for $t \in [\rho, 1-\rho]$.
$(ii)$ Let $N > 0$ be a given number which is arbitrarily large. From (2.6), there exists $T > 0$ such that $f(t,s) > N\phi(s)$ for all $t \in [0,1]$ and all $s > T$. For $k \in \mathbb{N}$, let $t_k$ be as in the proof of $(i)$ and assume first there is a subsequence of $\{t_k\}$, renamed the same, such that $t_k \leq 1/2$ for all $k \in \mathbb{N}$. By (2.8) let $k_0 \in \mathbb{N}$ be such that

$$u_k(t) \geq T \quad \text{for all} \quad k > k_0 \quad \text{and for all} \quad t \in [\rho, 1-\rho], \tag{2.9}$$

where $\rho \in (0,1/2)$ is a fixed number. Then, by integrating the inequality in problem $(I_1)$ from $1/2$ to $\tau > 1/2$, and considering that $u'(1/2) \leq 0$, we first obtain that

$$-\phi(u'(\tau)) \geq \int_{\frac{1}{2}}^{\tau} f(s, |u_k(s)|) ds.$$

Next, by integrating this expression from $t \in [1/2, 1-\rho]$ to $1-\rho$, we find that

$$u_k(t) \geq \int_t^{1-\rho} \Big(\phi^{-1}\Big[\int_{\frac{1}{2}}^{\tau} f(s, |u_k(s)|) ds\Big]\Big) d\tau, \quad t \in [1/2, 1-\rho],$$

thus

$$u_k(t) \geq \int_t^{1-\rho} \Big(\phi^{-1}\Big[\int_{\frac{1}{2}}^{t} f(s, |u_k(s)|) ds\Big]\Big) d\tau \geq (1-\rho-t)\phi^{-1}\Big(N\phi(u_k(t))(t-\tfrac{1}{2})\Big)$$

for all $t \in [1/2, 1-\rho]$.

Noticing that $t \in [\frac{1}{2}+\rho, 1-2\rho]$ implies that $1-\rho-t \geq \rho$ and $t - \frac{1}{2} \geq \rho$; from the last inequality, it follows that

$$u_k(t) \geq \rho\phi^{-1}(N\phi(u_k(t))\rho), \quad \text{for all} \quad t \in [\tfrac{1}{2}+\rho, 1-2\rho].$$

This implies that

$$\frac{\phi(\frac{1}{\rho}u_k(t))}{\phi(u_k(t))} \geq N\rho, \quad \text{for all} \quad t \in [\tfrac{1}{2}+\rho, 1-2\rho].$$

Setting $s_k = u_k(t_0)$, for any fixed $t_0 \in [\frac{1}{2}+\rho, 1-2\rho]$, the previous inequality yields

$$\frac{\phi(\frac{1}{\rho}s_k)}{\phi(s_k)} \geq N\rho,$$

where $\frac{1}{\rho} > 2$. By letting $k \to \infty$ we find that

$$\limsup_{s\to\infty} \frac{\phi(\frac{1}{\rho}s)}{\phi(s)} \geq N\rho, \tag{2.10}$$

where $N$ can be made arbitrarily large, implying (2.7).

Finally, the case when $t_k > 1/2$, for all $k \in \mathbb{N}$ except for a finite number of elements can be reduced to the previous one by the following argument. By making the change of variables

$$t = 1-s, \quad \tilde{f}(s,x) = f(1-s,x), \quad u_k(t) = u_k(1-s) = v_k(s),$$

one obtains that $v_k$ satisfies

$$\begin{gathered} -(\phi(v_k'))' \geq \tilde{f}(s, |v_k(s)|), \quad s \in (0,1) \\ \theta(v_k(0)) = \delta\theta(v_k'(0)), \quad \theta(v_k(1)) = -\beta\theta(v_k'(1)), \end{gathered}$$

where now $' = \frac{d}{ds}$. From the fact that $0 = u_k'(t_k) = -v_k'(s_k)$ with $s_k = 1-t_k$ and $t_k \in (\frac{1}{2}, 1)$ we find that $s_k \in (0, \frac{1}{2})$. Then since $\tilde{f}$ satisfies $(f_1)$ and $\tilde{f}_\infty(s) = \infty$, uniformly for $s \in [0,1]$, the previous argument yields again (2.10), ending the proof of the proposition. □

## 3. Abstract Formulation

Let $h \in L^1(0,1)$ and consider the problem

$$\begin{gathered} -(\phi(u'))' = |h(t)|, \quad t \in (0,1) \\ \theta(u(0)) = \beta\theta(u'(0)), \quad \theta(u(1)) = -\delta\theta(u'(1)), \quad \beta,\delta \geq 0. \end{gathered} \tag{3.1}$$

We next show that this problem is equivalent to an abstract equation. Thus let $u$ be a solution to problem (3.1). By integrating the equation in (3.1) over $(0,t)$, we obtain that

$$u'(t) = \phi^{-1}\Big(\phi(u'(0)) - \int_0^t |h(\tau)|d\tau\Big). \tag{3.2}$$

Integrating (3.2) over $(t,1)$, we find that

$$u(t) = u(1) - \int_t^1 \phi^{-1}\Big(\phi(u'(0)) - \int_0^s |h(\tau)|d\tau\Big)ds. \tag{3.3}$$

Let us set $r=\phi(u'(0))$ and $\gamma(s)=\theta^{-1}(\delta\theta(\phi^{-1}(s)))$. For $\delta>0$, the function $\gamma$ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$; clearly, if $\delta=0$, then $\gamma\equiv 0$.

Using the boundary condition $\theta(u(1))=-\delta\theta(u'(1))$ and (3.2) to compute $u'(1)$, we first obtain that

$$u(1)=-\gamma(r-\int_0^1|h(\tau)|d\tau).$$

Then, from (3.3),

$$u(t)=-\gamma\Big(r-\int_0^1|h(\tau)|d\tau\Big)-\int_t^1\phi^{-1}\Big(r-\int_0^s|h(\tau)|d\tau\Big)ds. \tag{3.4}$$

Next, evaluating this expression at $t=0$, we find that $r$ must satisfy $G(r,h)=0$, where $G:\mathbb{R}\times L^1(0,1)\to\mathbb{R}$ is defined by

$$G(r,h):=\mu(r)+\gamma\Big(r-\int_0^1|h(\tau)|d\tau\Big)+\int_0^1\phi^{-1}\Big(r-\int_0^s|h(\tau)|d\tau\Big)ds, \tag{3.5}$$

with $\mu(s)=(\theta^{-1}\beta\theta\phi^{-1})(s)$. For $\beta>0$, the function $\mu$ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$; if $\beta=0$ then $\mu\equiv 0$.

We have that, for fixed $h\in L^1(0,1)$, $G$ is a strictly increasing function of $r$ which tends to $-\infty$ as $r\to-\infty$ and to $+\infty$ as $r\to+\infty$. We conclude in this form that for any such given $h$ there is a unique $r=\chi(h)$ such that $G(r,h)=0$ and hence $u$ as in (3.4) gives a unique solution to (3.1) .

Notice also that $G(r,0)=0$ if and only if $r=0$. Furthermore one can observe directly from $G(r,h)=0$ that necessarily $r=\phi(u'(0))$ must be positive if $h\not\equiv 0$, in agreement with Proposition 2.1.

The functions $G,\chi$ satisfy additional properties. We have the following.

**Proposition 3.1.** *$G:\mathbb{R}\times L^1(0,1)\to\mathbb{R}$ and $\chi:L^1(0,1)\to\mathbb{R}$ are continuous.*

**Proof.** First we prove that $G$ is continuous. Indeed, let $r_n\to r$ in $\mathbb{R}$ and $h_n\to h$ in $L^1(0,1)$. Then $\mu(r_n)\to\mu(r)$, and since

$$\Big|\int_0^\tau|h_n(s)|ds-\int_0^\tau|h(s)|ds\Big|\le\int_0^1|h_n(s)-h(s)|ds\to 0$$

as $n\to\infty$, we have that

$$\int_0^\tau|h_n(s)|ds \quad \text{converges uniformly to} \quad \int_0^\tau|h(s)|ds \quad \text{for } \tau\in[0,1],$$

and

$$\phi^{-1}\Big(r_n-\int_0^s|h_n(\tau)|d\tau\Big)\to\phi^{-1}\Big(r-\int_0^s|h(\tau)|d\tau\Big)$$

uniformly in $[0,1]$, and

$$\gamma\Big(r_n - \int_0^1 |h_n(\tau)|d\tau\Big) \to \gamma\Big(r - \int_0^1 |h(\tau)|d\tau\Big),$$

implying that $G(r_n, h_n) \to G(r, h)$ in $\mathbb{R}$.

Assume now that $h_n \to h$ in $L^1(0,1)$ and set $r_n = \chi(h_n)$ and $r = \chi(h)$ . From $G(r_n, h_n) = 0$; i.e.,

$$G(r_n, h_n) := \mu(r_n) + \gamma\Big(r_n - \int_0^1 |h_n(\tau)|d\tau\Big) + \int_0^1 \phi^{-1}\Big(r_n - \int_0^s |h_n(\tau)|d\tau\Big)ds = 0,$$

it is immediate to see that $0 < r_n \leq ||h_n||_1$, implying that $\{r_n\}$ is a bounded sequence in $\mathbb{R}$. Let $\{r_{n_k}\}$ be any subsequence of $\{r_n\}$. Then this subsequence contains a subsequence $\{r_{n'_k}\}$ converging to some $\tilde{r} \in \mathbb{R}$. As $G(r_{n'_k}, h_{n'_k}) = 0$ and $G$ is continuous in $\mathbb{R} \times L^1(0,1)$, we find that $G(\tilde{r}, h) = 0$ and hence $\tilde{r} = \chi(h)$. It is then clear that $r_n \to \chi(h) = r$. □

Defining $\mathcal{T} : L^1(0,1) \mapsto C[0,1]$ by

$$\mathcal{T}(h)(t) = -\gamma\Big(\chi(h) - \int_0^1 |h(\tau)|d\tau\Big) - \int_t^1 \phi^{-1}\Big(\chi(h) - \int_0^s |h(\tau)|d\tau\Big)ds, \quad (3.6)$$

the unique solution to (3.1) satisfies

$$u = \mathcal{T}(h). \quad (3.7)$$

If now $u$ is given by (3.7) then it is straightforward to verify that $u$ is of class $C^1[0,1]$ and satisfies problem (3.1), with $u'(0) = \phi^{-1}(\chi(h))$.

In light of the last proposition it is standard to show the validity of the following.

**Proposition 3.2.** *The operator $\mathcal{T}$ given by* (3.6) *is continuous and maps bounded sets of $L^1(0,1)$ into relatively compact sets of $C[0,1]$, and hence it is a completely continuous operator.*

We now rewrite problem (1.1) in the following form:

$$\begin{gathered}(\phi(u'))' + f(t,|u|) = 0, \quad t \in (0,1)\\ \theta(u(0)) = \beta\theta(u'(0)), \quad \theta(u(1)) = -\delta\theta(u'(1)), \quad \beta, \delta \geq 0.\end{gathered} \quad (3.8)$$

By Proposition 2.1, if $u$ is a nontrivial solution to this problem, then $u(t) > 0$, for all $t \in (0,1)$ and hence it is a positive solution to problem (1.1).

Let us define $\mathcal{F} : C[0,1] \mapsto C[0,1]$ by $\mathcal{F}(u)(t) = f(t, |u(t)|)$, $t \in [0,1]$, then it is well known that $\mathcal{F}$ is continuous and maps bounded sets into bounded sets. By setting $T := \mathcal{T} \circ \mathcal{F}$, then $T : C[0,1] \mapsto C[0,1]$ and is given by

$$T(u) = -\gamma\Big(\chi(f(\cdot,|u(\cdot)|)) - \int_0^1 f(\tau,|u(\tau)|)d\tau\Big) \\ - \int_t^1 \phi^{-1}\Big(\chi(f(\cdot,|u(\cdot)|)) - \int_0^s f(\tau,|u(\tau)|)d\tau\Big)ds. \tag{3.9}$$

It is then straightforward to check that $T$ is a completely continuous operator and that finding solutions to problem (3.8) is equivalent to finding solutions to the fixed-point problem

$$u = T(u). \tag{3.10}$$

## 4. Proof of Theorem 1.1

The proof of Theorem 1.1 will follow from some propositions that we prove next.

**Proposition 4.1.** *Let $\phi$ and $f$ satisfy*

$$f_0(t) = 0 \quad \textit{uniformly for } t \in [0,1], \tag{4.1}$$

*and*

$$\liminf_{s\to 0} \frac{\phi(\tau s)}{\phi(s)} > 0 \quad \textit{for all } 0 < \tau < 1. \tag{4.2}$$

*In the case that $\delta > 1$ assume furthermore that every time $\{s_k\}$ and $\{r_k\}$ are sequences such that $s_k \to 0$ and $r_k \to 0$, as $k \to \infty$, it holds that*

$$\lim_{k\to\infty} \frac{\theta(r_k s_k)}{\theta(s_k)} = 0. \tag{4.3}$$

*Then there exists $\rho_0 > 0$, small, such that the Leray-Schauder degree $deg_{LS}(I - T, B(0,\rho), 0)$ is well defined for $0 < \rho < \rho_0$, and*

$$deg_{LS}(I - T, B(0,\rho), 0) = 1, \tag{4.4}$$

*and hence the index of $T$ with respect to 0,*

$$ind(T, 0, 0) = \lim_{\rho\to 0} deg_{LS}(I - T, B(0,\rho), 0) = 1. \tag{4.5}$$

**Proof.** We consider the family of problems

$$u = \lambda T(u), \quad \text{where} \quad \lambda \in [0,1], \tag{4.6}$$

and claim that there exists $\rho_0 > 0$ such that, for any $\lambda \in (0,1]$, the only solution to this problem in $B(0,\rho_0)$, the open ball in $C[0,1]$, center 0 and radius $\rho_0$ is $u = 0$. By contradiction, we assume that there exist sequences $\{u_k\}$ in $C[0,1]$, $\{\lambda_k\}$ in $[0,1]$ such that

$$u_k = \lambda_k T(u_k),$$

with $||u_k|| = \sup_{t\in[0,1]} u_k(t) = \rho_k$, $\rho_k > 0$ and $\rho_k \to 0$ as $k \to \infty$. Then from (3.9), and the fact that $\chi(f(\cdot, |u_k(\cdot)|)) > 0$, we find that

$$u_k(t) \le \gamma\Big(\int_0^1 f(\tau, |u_k(\tau)|)d\tau\Big) + \phi^{-1}\Big(\int_0^1 f(\tau, |u_k(\tau)|)d\tau\Big). \tag{4.7}$$

Now, let $\{\varepsilon_k\}$ be a sequence of real numbers such that $0 < \epsilon_k \to 0$. From (4.1) and the fact that $\{u_k\}$ tends to zero in $C[0,1]$, by redefining the sequence if necessary, we can assume that there exists $k_0 \in \mathbb{N}$ such that $f(t, |u_k(t)|) \le \epsilon_k \phi(u_k(t))$ for all $k \ge k_0$ and $t \in [0,1]$. Hence, $f(t, |u_k(t)|) \le \epsilon_k \phi(||u_k||)$, and from (4.7), we find that

$$1 \le \frac{\gamma(\epsilon_k \phi(||u_k||))}{||u_k||} + \frac{\phi^{-1}(\epsilon_k \phi(||u_k||))}{||u_k||}. \tag{4.8}$$

For $k \in \mathbb{N}$, set $\tau_k = \dfrac{\phi^{-1}(\epsilon_k \phi(||u_k||))}{||u_k||}$, then $\tau_k < 1$ for all $k \in \mathbb{N}$ large, and $\phi(\tau_k(||u_k||)) = \epsilon_k \phi(||u_k||)$. We have that (4.8) can be written as

$$1 \le \frac{\theta^{-1}(\delta\theta(\tau_k ||u_k||))}{||u_k||} + \tau_k. \tag{4.9}$$

Suppose $\{\tau_k\}$ has a subsequence, rename the same, such that $\tau_k \ge m_0$, where $m_0$ is a positive constant. Then $m_0 < 1$ and by the definition of $\tau_k$

$$m_0 \le \frac{\phi^{-1}(\epsilon_k \phi(||u_k||))}{||u_k||};$$

this implies

$$\frac{\phi(m_0 ||u_k||)}{\phi(||u_k||)} \le \epsilon_k,$$

contradicting (4.2). Thus it must be that the entire sequence $\{\tau_k\}$ tends to 0, as $k \to \infty$. Hence there is $0 < m_1 < 1$ such that $m_1 \le 1 - \tau_k$, for all large $k$. In this form (4.9) can be written as

$$\frac{1}{\delta} \le \frac{\theta(\tau_k ||u_k||)}{\theta(m_1 ||u_k||)},$$

which, by setting $s_k = m_1 ||u_k||$, becomes

$$\frac{1}{\delta} \le \frac{\theta(\frac{\tau_k}{m_1} s_k)}{\theta(s_k)} < 1 \quad \text{for all large}\, k. \tag{4.10}$$

If $\delta \le 1$, (4.10) gives immediately a contradiction to the existence of the sequences $\{u_k\}$ and $\{\lambda_k\}$, with $||u_k|| \to 0$. Let us reinforce that the above contradiction argument holds for $\beta \ge 0$ and $0 \le \delta \le 1$.

If now $\delta > 1$, then by letting $k \to \infty$ in this expression we get a contradiction from (4.3).

Thus there exists $\rho_0 > 0$, small, such that if $(u, \lambda)$, $u \in B(0, \rho_0)$ satisfies (4.6) then $u = 0$. This, in particular implies that for any $0 < \rho < \rho_0$ the Leray-Schauder degree $\deg_{LS}(I - \lambda T, B(0, \rho), 0)$ is well defined, and from properties of the degree

$$\deg_{LS}(I - \lambda T, B(0, \rho), 0) = constant, \quad \text{for all} \quad \lambda \in [0, 1],$$

implying immediately that (4.4) and (4.5) hold. □

Next let us consider the problem

$$(P_M) \qquad \begin{gathered} (\phi(u'))' + f(t, |u(t)|) + M = 0, \quad t \in (0, 1) \\ \theta(u(0)) = \beta\theta(u'(0)), \quad \theta(u(1)) = -\delta\theta(u'(1)), \quad \beta, \delta \ge 0, \end{gathered}$$

where $M > 0$. We have the following.

**Proposition 4.2.** *Assume that*

$$f_\infty(t) = \infty \quad \textit{uniformly for} \quad t \in [0, 1] \tag{4.11}$$

*and*

$$\limsup_{s \to \infty} \frac{\phi(\tau s)}{\phi(s)} < \infty \quad \textit{for all } \tau > 1. \tag{4.12}$$

*Then there exists $M_0 > 0$ such that problem $(P_M)$ does not have solutions for any $M \ge M_0$.*

**Proof.** We argue by contradiction and thus we assume there is a sequence $\{M_k\}$, $M_k \to \infty$ as $k \to \infty$, and corresponding solutions $\{u_k\}$ to $(P_M)$ with $M = M_k$, such that

$$(P_{M_k}) \qquad \begin{gathered} (\phi(u_k'))' + f(t, |u_k(t)|) + M_k = 0, \quad t \in (0, 1) \\ \theta(u_k(0)) = \beta\theta(u_k'(0)), \quad \theta(u_k(1)) = -\delta\theta(u_k'(1)). \end{gathered}$$

Then $u_k$ satisfies

$$\begin{gathered} -(\phi(u_k'))' \ge M_k, \quad t \in (0, 1) \\ \theta(u_k(0)) = \beta\theta(u_k'(0)), \quad \theta(u_k(1)) = -\delta\theta(u_k'(1)), \end{gathered} \tag{4.13}$$

and as before $u_k(t) > 0$ for all $t \in (0, 1)$ and there is a unique $t_k \in (0, 1)$ such that $u'(t_k) = 0$ with $u(t_k) = ||u_k||$. It then follows that $\{t_k\}$ has either a subsequence contained in $(0, 1/2]$ or a subsequence contained in $(1/2, 1)$. By renaming the subsequence as $\{t_k\}$, in the first case we have that $0 < t_k \le \frac{1}{2}$, for all $k \in \mathbb{N}$.

Let $\rho \in (0, \frac{1}{4})$ be a fixed number. By integrating the inequality in (4.13) from $t_k$ to $t \in (t_k, 1)$, we first find that

$$-u_k'(t) \ge \phi^{-1}\Big(M_k(t - t_k)\Big),$$

and then by integrating this expression from $t$ to $1-\rho/4$, with $t \in [\frac{1}{2}, 1-\rho/4]$, we obtain that

$$-u_k(1-\rho/4) + u_k(t) = \int_t^{1-\rho/4} \phi^{-1}\Big(M_k(\tau - t_k)\Big)d\tau.$$

Hence,

$$u_k(t) \ge \int_t^{1-\rho/4} \phi^{-1}\Big[M_k(\tau - \tfrac{1}{2})\Big]d\tau \ge \phi^{-1}\Big(M_k(t-\tfrac{1}{2})\Big)(1-\rho/4-t), \quad (4.14)$$

for all $t \in [\frac{1}{2}, 1-\rho/4]$. Then for $t \in [\frac{1}{2}+\rho/2, 1-\rho/2]$, we have that $t-\frac{1}{2} \ge \rho/2$ and $1-\rho/4-t \ge \rho/4$. Thus from (4.14) $u_k(t) \ge \frac{\rho}{4}\phi^{-1}(M_k\rho/2)$, implying that $||u_k|| \to \infty$ as $k \to \infty$. Hence, by the second part of Proposition 2.3, there is a $\tau_0 > 1$ such that

$$\limsup_{s\to\infty} \frac{\phi(\tau_0 s)}{\phi(s)} = \infty,$$

contradicting (4.12). Thus the original sequence $\{t_k\}$ cannot have a subsequence contained in $(0, 1/2]$.

We next show that $\{t_k\}$ cannot have a subsequence contained in $(\frac{1}{2}, 1)$ either. Indeed, by making the change of variables

$$t = 1-s, \quad u_k(t) = u_k(1-s) = v_k(s), \quad \tilde{f}(s, v_k(s)) = f(t, u_k(t)),$$

one obtains that $v_k$ satisfies

$$(\phi(v_k'))' + \tilde{f}(s, |v_k(s)|) + M_k = 0, \quad s \in (0,1)$$
$$\theta(v_k(0)) = \delta\theta(v_k'(0)), \quad \theta(v_k(1)) = -\beta\theta(v_k'(1)),$$

where now $' = \frac{d}{ds}$. From $0 = u_k'(t_k) = -v_k'(s_k)$ with $s_k = 1 - t_k$ and $t_k \in (\frac{1}{2}, 1)$ we find that $s_k \in (0, \frac{1}{2})$. Since $M_k \to \infty$ the previous argument tells us that the sequence $\{s_k\}$ cannot have a subsequence contained in $(0, \frac{1}{2})$, meaning that the sequence $\{t_k\}$ cannot have a subsequence contained in $(\frac{1}{2}, 1)$, contradicting the existence of the sequence $\{M_k\}$ with $M_k \to \infty$ as $k \to \infty$. Hence there exists $M_0 > 0$ such that problem $(P_M)$ does not have solutions for any $M \ge M_0$, ending the proof of the proposition. □

Let $M_0$ be as given by Proposition 4.2 and let us consider the family of problems

$$(P_\mu) \qquad \begin{array}{c} (\phi(u'))' + f(t, |u(t)|) + \mu M_0 = 0, \quad t \in (0,1) \\ \theta(u(0)) = \beta\theta(u'(0)), \quad \theta(u(1)) = -\delta\theta(u'(1)), \quad \beta, \delta \geq 0, \end{array}$$

where $\mu \in [0,1]$. This family of problems can be equivalently written as

$$u = S(\mu, u), \tag{4.15}$$

where $S : [0,1] \times C[0,1] \mapsto C[0,1]$, is given by

$$\begin{aligned} S(\mu, u)(t) = &-\gamma(\chi(f(\cdot, |u(\cdot)|) + \mu M_0)) - \int_0^1 (f(\tau, |u(\tau)|) + \mu M_0) d\tau \\ &- \int_t^1 \phi^{-1}\Big(\chi(f(\cdot, |u(\cdot)|) + \mu M_0) - \int_0^s (f(\tau, |u(\tau)|) + \mu M_0) d\tau\Big) ds. \end{aligned}$$

It is not difficult to see that $S$ is completely continuous and that $S(0, u) = T(u)$. We have the following.

**Proposition 4.3.** *Let $\phi$ and $f$ satisfy conditions* (4.11) *and* (4.12). *Then solutions to* (4.15) *are a priori bounded; i.e., there exists a constant $R_0 > 0$ such that, if $(u, \mu)$ satisfy* (4.15), *then*

$$||u|| < R_0, \tag{4.16}$$

*where $R_0$ is independent of $\mu$. Furthermore, we have*

$$deg_{LS}(I - T, B(0, R), 0) = 0, \tag{4.17}$$

*for any $R \geq R_0$.*

**Proof.** Assume by contradiction that there exists a sequence $\{(\mu_k, u_k)\}$, $\mu_k \in [0,1]$, $u_k \in C[0,1]$, of solutions to (4.15) with

$$||u_k|| = \sup_{t \in [0,1]} u_k(t) \to \infty \quad \text{as } k \to \infty. \tag{4.18}$$

Then $\mu_k$, $u_k$ satisfy

$$(I_\mu) \qquad \begin{array}{c} -(\phi(u_k'))' \geq f(t, |u_k(t)|), \quad t \in (0,1) \\ \theta(u_k(0)) = \beta\theta(u_k'(0)), \quad \theta(u_k(1)) = -\delta\theta(u_k'(1)), \end{array}$$

and hence we are in the situation of Proposition 2.3. But then (2.7) of that proposition and (4.12) are in contradiction and hence there exists a constant $R_0 > 0$ such that, if $(u, \mu)$ satisfy (4.15), then (4.16) holds.

We take $R_0 > M_0$, $M_0$ as in Proposition 4.2. Now by (4.16), for any $R \geq R_0$, the degree $\deg_{LS}(I - S(\mu, \cdot), B(0, R), 0)$ is well defined and satisfies

$$\deg_{LS}(I - S(\mu, \cdot), B(0, R), 0) = constant,$$

and hence

$$\deg_{LS}(I-T,B(0,R),0)=\deg_{LS}(I-S(0,\cdot),B(0,R),0)$$
$$=\deg_{LS}(I-S(1,\cdot),B(0,R),0)=0.$$

This last degree being zero, the result follows by Proposition 4.2. □

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Problem (3.8) can be equivalently written as the fixed-point problem

$$u=T(u). \tag{4.19}$$

By Proposition 4.3, there is $R>0$ such that

$$\deg_{LS}(I-T,B(0,R),0)=0,$$

and by Proposition 4.1, there is $0<\rho<R$, such that

$$\deg_{LS}(I-T,B(0,\rho),0)=1.$$

Thus, from classical theorems of Leray-Schauder degree, it follows that $\deg_{LS}(I-T,B(0,R)\setminus\overline{B(0,\rho)},0)$ is different from zero. This implies that (4.19) has a fixed point $u$, with $\rho<||u||<R$, which by the previous arguments is a solution to problem (1.1), and satisfies $u(t)>0$, for all $t\in(0,1)$. □

## 5. Proof of Theorem 1.2

To prove Theorem 1.2 we need to prove some previous propositions.

**Proposition 5.1.** *Assume $\phi$ and $f$ satisfy*

$$f_0(t)=\infty \text{ uniformly with respect to } t\in[0,1], \tag{5.1}$$

*and for all $\gamma>1$*

$$\limsup_{s\to0}\frac{\phi(\gamma s)}{\phi(s)}<\infty. \tag{5.2}$$

*Then there exists $\rho_0>0$ such that for any $M>0$ problem $(P_M)$ has no solutions in the closed ball $\overline{B(0,\rho_0)}\subset C[0,1]$.*

**Proof.** Suppose the proposition is false. Then there exists a sequence of positive numbers $\{M_k\}$ and a sequence $\{u_k\}\subset C[0,1]$ such that, for each $k$, $u_k$ is a solution of $(P_M)$ with $M=M_k$, and $u_k\to0$ as $k\to\infty$. For each $k$ we have that $u_k$ satisfies

$$-(\phi(u_k'(t)))'\ge M_k>0 \quad \text{for} \quad t\in(0,1)$$
$$\theta(u_k(0))=\beta\theta(u_k'(0)),\quad \theta(u_k(1))=-\delta\theta(u_k'(1)),\quad \beta,\delta\ge0.$$

It follows that $u_k(t) > 0$, concave down in $(0,1)$, and there is a unique $t_k \in (0,1)$ with $u(t_k) = \|u_k\|$ and $u_k'(t_k) = 0$.

Now $\|u_k\| \to 0$ and since $f_0(t) = \infty$, uniformly on $[0,1]$, for each $N > 0$ we can find $k_0 \geq 1$ such that whenever $k \geq k_0$ we have

$$f(t, |u_k(t)|) \geq N\phi(|u_k(t)|). \tag{5.3}$$

There are two cases, depending on whether $t_k \in (0, \frac{1}{2}]$ or $t_k \in [\frac{1}{2}, 1)$. The two cases are similar, so we give the argument only for the case $t_k \in (0, \frac{1}{2}]$. So, assuming $t_k \in (0, \frac{1}{2}]$, and integrating the equation in $(P_{M_k})$ over $[t_k, t]$ and using (5.3), we get for $t \in [t_k, 1]$

$$-u_k'(t) \geq \phi^{-1}\Big(N \int_{t_k}^{t} \phi(u_k(s))\, ds\Big).$$

Therefore for $t \in [1/2, 7/8]$ we find that

$$u_k(t) \geq \int_t^{7/8} \phi^{-1}\Big(N \int_{1/2}^{\tau} \phi(u_k(s))\, ds\Big) d\tau \geq \int_t^{7/8} \phi^{-1}\Big(N \int_{1/2}^{t} \phi(u_k(s))\, ds\Big) d\tau$$

and hence, using the fact that $\phi(u_k(s))$ decreases in $[t_k, 1]$,

$$u_k(t) \geq \phi^{-1}\Big(N \int_{1/2}^{t} \phi(u_k(s))\, ds\Big)(7/8 - t) \geq \phi^{-1}\Big(N(t - 1/2)\phi(u_k(t))\Big)(\tfrac{7}{8} - t).$$

Choosing now $t \in [5/8, 3/4]$ we obtain that

$$8u_k(t) \geq \phi^{-1}((N/8)\phi(u_k(t))),$$

implying that

$$\frac{\phi(8u_k(t))}{\phi(u_k(t))} \geq \frac{N}{8}.$$

This inequality is a contradiction to (5.2) since $0 < u_k(t) \leq \|u_k\| \to 0$ as $k \to \infty$ and $N$ is arbitrarily large. This contradiction proves the proposition. $\square$

**Proposition 5.2.** *Under the same assumptions of Proposition* 5.1, *there exists* $\rho_1 > 0$ *such that* (3.8) *has only the trivial solution in the closed ball* $\overline{B(0, \rho_1)} \subset C[0,1]$.

**Proof.** The proof is by contradiction, with arguments almost identical to those in the proof of Proposition 5.1. If the proposition is false, there is a sequence $\{u_n\}$ of solutions with $0 \neq \|u_n\| \to 0$ as $n \to \infty$. Because each $u_n$ satisfies

$$-(\phi(u_n'))' = f(t, |u_n(t)|) \quad t \in (0,1),$$
$$\theta(u_n(0)) = \beta\theta(u_n'(0)), \quad \theta(u_n(1)) = -\delta\theta(u_n'(1)), \quad \beta, \delta \geq 0,$$

we must have that each $u_n$ is positive and concave down in $(0,1)$. Now the proof proceeds as in Proposition 5.1. □

In the rest of this section $T$ denotes the operator defined in (3.9). We can now prove the following.

**Proposition 5.3.** *Under the same assumptions of Proposition* 5.1, *there exists* $\rho_2 > 0$ *such that, for all* $0 < \rho \leq \rho_2$,

$$deg_{LS}(I - T, B(0,\rho), 0) = 0,$$

*and hence*

$$ind(T,0,0) = \lim_{\rho \to 0} deg_{LS}(I - T, B(0,\rho), 0) = 0. \tag{5.4}$$

**Proof.** Let $\rho_0$ be as in Proposition 5.1, let $M_0 > 0$ be fixed, and let $\rho_1$ be as in Proposition 5.2. Choose $\rho_2 = \min\{\rho_0, \rho_1\}$. By the latter two propositions, there are no solutions on the boundary of $B(0,\rho)$, $0 < \rho \leq \rho_2$, to the family of equations

$$u = S(\mu, u)$$

for any $\mu \in [0,1]$, with $S(\mu,u)$ as defined in (4.15). It follows that the Leray-Schauder degree $deg_{LS}(I - S(\mu,\cdot), B(0,\rho), 0)$ is defined and independent of $\mu \in [0,1]$. Since there are no solutions in $B(0,\rho)$ for $\mu = 1$ and $T = S(0,.)$, it follows that

$$deg_{LS}(I - T, B(0,\rho), 0) = deg_{LS}(I - S(1,.), B(0,\rho), 0) = 0,$$

for any $0 < \rho \leq \rho_2$ and hence (5.4) holds. □

We now consider our problem on large balls.

**Proposition 5.4.** *Let* $\phi$ *and* $f$ *satisfy*

$$f_\infty(t) = 0 \quad \textit{uniformly with respect to } t \in [0,1] \tag{5.5}$$

*and for all* $0 < \tau < 1$

$$\liminf_{s \to \infty} \frac{\phi(\tau s)}{\phi(s)} > 0. \tag{5.6}$$

*Also in the case that* $\delta > 1$ *assume furthermore that every time* $\{s_k\}$ *and* $\{r_k\}$ *are sequences such that* $s_k \to \infty$ *and* $r_k \to 0$, *as* $k \to \infty$, *it holds that*

$$\lim_{k \to \infty} \frac{\theta(r_k s_k)}{\theta(s_k)} = 0. \tag{5.7}$$

*Then there exists* $R_0 > 0$ *such that, if* $u \in C[0,1]$ *is a solution to*

$$u = \lambda T(u)$$

*for some* $\lambda \in [0,1]$, *then* $\|u\| < R_0$.

**Proof.** Assume by way of contradiction that there are sequences $\{u_k\}$ in $C[0,1]$, $\{\lambda_k\}$ in $[0,1]$ such that

$$u_k = \lambda_k T(u_k) \quad \text{with} \quad \|u_k\| \to \infty. \tag{5.8}$$

By (5.5), if $\{\varepsilon_j\}$, $\varepsilon_j > 0$, $2\varepsilon_j < 1$, is a sequence such that $\varepsilon_j \to 0$, then for each $j \in \mathbb{N}$, we can find a positive constant $C_j$ such that

$$f(t,s) \leq \varepsilon_j \phi(s) + C_j \tag{5.9}$$

for all $t \in [0,1]$ and all $s \geq 0$. Now one can find a subsequence of $\{u_k\}$, that we rename as $\{u_j\}$, such that $\varepsilon_j \phi(||u_j||) \geq C_j$. Hence, for each $j \in \mathbb{N}$,

$$\int_0^1 f(\tau, |u_j(\tau)|)d\tau \leq \varepsilon_j \phi(||u_j||) + C_j \leq 2\varepsilon_j \phi(||u_j||). \tag{5.10}$$

Next, since as in (4.7)

$$u_j(t) \leq \gamma\Big(\int_0^1 f(\tau, |u_j(\tau)|)d\tau\Big) + \phi^{-1}\Big(\int_0^1 f(\tau, |u_j(\tau)|)d\tau\Big), \tag{5.11}$$

from (5.10), we find that

$$1 \leq \frac{\gamma(2\varepsilon_j \phi(||u_j||))}{||u_j||} + \frac{\phi^{-1}(2\varepsilon_j \phi(||u_j||))}{||u_j||}, \tag{5.12}$$

for all $j$. We notice here that if $\delta = 0$ then (5.12) gives immediately a contradiction. Hence we assume $\delta > 0$ for the rest of the argument.

Set $\tau_j = \dfrac{\phi^{-1}(2\varepsilon_j \phi(||u_j||))}{||u_j||}$, then $\tau_j < 1$, and $\phi(\tau_j ||u_j||) = 2\varepsilon_j \phi(||u_j||)$. We have that (5.12) can be written as

$$1 \leq \frac{\theta^{-1}(\delta\theta(\tau_j ||u_j||))}{||u_j||} + \tau_j. \tag{5.13}$$

Now, by repeating the argument that goes from formula (4.9) to (4.10), we find that the sequence $\{\tau_j\}$ tends to 0, as $j \to \infty$, and that there is $0 < m_1 < 1$ such that $m_1 \leq 1 - \tau_j$, for all large $j$ such that

$$\frac{1}{\delta} \leq \frac{\theta(\tau_j ||u_j||)}{\theta(m_1 ||u_j||)},$$

which as before, by setting $s_j = m_1 ||u_j||$, becomes

$$\frac{1}{\delta} \leq \frac{\theta(\frac{\tau_j}{m_1} s_j)}{\theta(s_j)} < 1. \tag{5.14}$$

If $\delta \leq 1$, (5.14) gives immediately a contradiction. If now $\delta > 1$, then by letting $j \to \infty$ in this expression we get a contradiction to (5.7), ending the proof of the proposition. □

**Proposition 5.5.** *Under the assumptions of Proposition* 5.4, *there exists* $R_0 > 0$ *such that, for all* $R \geq R_0$,

$$deg_{LS}(I - T, B(0,R), 0) = 1,$$

*where* $T$ *is defined in* (3.9).

**Proof.** By Proposition 5.4 there exists an $R_0 > 0$ such that for any $R \geq R_0$ the family of equations

$$u = \lambda T(u)$$

has no solutions on the boundary of $B(0,R) \subset C[0,1]$. Thus, $deg_{LS}(I - \lambda T, B(0,R), 0)$ is, for all $R \geq R_0$, defined and independent of $\lambda \in [0,1]$. Hence,

$$deg_{LS}(I - T, B(0,R), 0) = deg_{LS}(I, B(0,R), 0) = 1. \qquad \square$$

We can now prove Theorem 1.2.

**Proof of Theorem 1.2.** By Propositions 5.3 and 5.5 there are numbers $0 < \rho_1 < R_1$ such that

$$deg_{LS}(I - T, B(0,\rho_1), 0) = 0 \quad \text{and} \quad deg_{LS}(I - T, B(0,R_1), 0) = 1.$$

By the excision property of the Leray-Schauder degree,

$$deg_{LS}(I - T, B(0,R_1)/\overline{B(0,\rho_1)}, 0) \neq 0.$$

This proves there is a nontrivial solution to problem (3.8). But any nontrivial solution of (3.8) must be positive on $[0,1]$, and hence is a positive solution of problem (1.1). □

## 6. Proof of Theorem 1.3

In this section we will prove Theorem 1.3; to this end we will consider the problem that follows

$$\begin{gathered}(\phi(u'))' + g(t,|v|) = 0, \quad (\psi(v'))' + f(t,|u|) = 0, \quad t \in (0,1) \\ u(0) = \beta_1 u'(0),\ u(1) = -\delta_1 u'(1). \quad v(0) = \beta_2 v'(0),\ v(1) = -\delta_2 v'(1).\end{gathered} \tag{6.1}$$

By Proposition 2.1, nontrivial solutions of this problem give positive solutions of problem (1.11). We recall that we are assuming $\beta_i, \delta_i \geq 0,\ i = 1,2$ and the functions $\phi, \psi$ satisfy condition $(H_1)$ and $f, g$ condition $(H_2)$ of the Introduction.

Let $(u, v)$ denote a solution to (6.1). We first note that if $(u, v)$ is a solution to (6.1) then $u \equiv 0$ if and only if $v \equiv 0$. Thus if $u \not\equiv 0$ (and hence $v \not\equiv 0$) by Proposition 2.1 it follows that $u(t) > 0$ and $v(t) > 0$ for all $t \in (0,1)$. Moreover, in this case both $u$ and $v$ are strictly concave down and $u'(0) > 0$, $u'(1) < 0$, $v'(0) > 0$, $v'(1) < 0$.

As in (3.6), define $\mathcal{T}_1, \mathcal{T}_2 : L^1(0,1) \mapsto C[0,1]$ by

$$\begin{aligned}\mathcal{T}_1(h)(t) = &-\delta_1\phi^{-1}\Big(\phi(\chi_1(h)) - \int_0^1 |h(\tau)|d\tau\Big)\\ &- \int_t^1 \phi^{-1}\Big(\phi(\chi_1(h)) - \int_0^s |h(\tau)|d\tau\Big)ds,\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}\mathcal{T}_2(h)(t) = &-\delta_2\psi^{-1}\Big(\psi(\chi_2(h)) - \int_0^1 |h(\tau)|d\tau\Big)\\ &- \int_t^1 \psi^{-1}\Big(\psi(\chi_2(h)) - \int_0^s |h(\tau)|d\tau\Big)ds,\end{aligned} \tag{6.3}$$

where $\chi_1(h)$ and $\chi_2(h)$ are the unique solutions to (3.5), with $\beta$ and $\delta$ replaced by $\beta_1$ and $\delta_1$ and $\beta_2$ and $\delta_2$, respectively. By Proposition 3.2 both $\mathcal{T}_1$ and $\mathcal{T}_2$ are completely continuous operators.

Let next $\mathcal{F}, \mathcal{G} : C[0,1] \mapsto C[0,1]$ be defined by $\mathcal{F}(u)(t) = f(t, |u(t)|)$ and $\mathcal{G}(v)(t) = g(t, |v(t)|)$, $t \in [0,1]$. Then $\mathcal{F}$ and $\mathcal{G}$ are continuous and map bounded sets into bounded sets. Defining

$$T(u, v) := (\mathcal{T}_1 \circ \mathcal{G}(v), \mathcal{T}_2 \circ \mathcal{F}(u)), \tag{6.4}$$

then $T : (C[0,1])^2 \mapsto (C[0,1])^2$ is continuous and maps bounded subsets of $(C[0,1])^2$ into relatively compact subsets of $(C[0,1])^2$, and hence it is a completely continuous operator.

It is easy to check that finding solutions to problem (6.1) is equivalent to finding solutions to the fixed-point problem

$$(u, v) = T(u, v), \tag{6.5}$$

or equivalently to finding solutions to the system

$$u = \mathcal{T}_1 \circ \mathcal{G}(v) \quad v = \mathcal{T}_2 \circ \mathcal{F}(u). \tag{6.6}$$

As in the scalar case the proof of this theorem is a consequence of some propositions which we establish and prove next.

**Proposition 6.1.** *Let $\phi, \psi$ and $f, g$ satisfy*

$$f_0(t) = g_0(t) = 0 \quad \textit{uniformly for } t \in [0,1]. \tag{6.7}$$

*Assume furthermore that*

$$\liminf_{s\to 0}\frac{\phi(\tau s)}{\phi(s)} > 0 \quad \textit{and} \quad \liminf_{s\to 0}\frac{\psi(\tau s)}{\psi(s)} > 0 \textit{ for all } 0<\tau<1. \tag{6.8}$$

*Then there exists* $\rho_0 > 0$*, small, such that* $(u, v) \equiv (0, 0)$ *is the unique solution to* (6.5) *in the ball* $\mathcal{B}(0, \rho_0)$ *and for any fixed* $0 < \rho < \rho_0$

$$deg_{LS}(I - T, \mathcal{B}(0, \rho), 0) = 1. \tag{6.9}$$

Here $\mathcal{B}(0,\rho) = B(0,\rho)\times B(0,\rho) \subset (C[0,1])^2$ where $B(0,\rho)$ is the ball with center at 0 and radius $\rho$ in $C[0,1]$.

**Proof.** In order to prove these assertions we consider the family of problems

$$(u, v) = \lambda T(u, v), \quad \text{where} \quad \lambda \in [0, 1], \tag{6.10}$$

and assume by way of contradiction that there exist sequences $\{(u_k, v_k)\}$ in $(C[0,1])^2$, $\{\lambda_k\}$ in $[0,1]$ such that $(u_k, v_k) = \lambda_k T(u_k, v_k)$, with $||(u_k, v_k)|| = ||u_k|| + ||v_k|| = \rho_k, \quad \rho_k > 0$ and $\rho_k \to 0$ as $k \to \infty$; then from (6.2), (6.3), (6.10), and the fact that $\chi_1(g(\cdot, |v_k(\cdot)|)) > 0$ and $\chi_2(f(\cdot, |u_k(\cdot)|)) > 0$, we find that

$$\begin{aligned} u_k(t) &\le \delta_1\phi^{-1}\Big(\int_0^1 g(\tau, |v_k(\tau)|)d\tau\Big) + \phi^{-1}\Big(\int_0^1 g(\tau, |v_k(\tau)|)d\tau\Big)(1-t)\\ &\le \phi^{-1}\Big(\int_0^1 g(\tau, |v_k(\tau)|)d\tau\Big)(\delta_1 + 1 - t), \end{aligned}$$

and

$$\begin{aligned} v_k(t) &\le \delta_2\psi^{-1}\Big(\int_0^1 f(\tau, |u_k(\tau)|)d\tau\Big) + \psi^{-1}\Big(\int_0^1 f(\tau, |u_k(\tau)|)d\tau\Big)(1-t)\\ &\le \psi^{-1}\Big(\int_0^1 f(\tau, |u_k(\tau)|)d\tau\Big)(\delta_2 + 1 - t). \end{aligned}$$

Hence, for all $t \in (0, 1)$,

$$\frac{u_k(t)}{1+\delta_1} \le \phi^{-1}\Big(\int_0^1 g(\tau, |v_k(\tau)|)d\tau\Big) \tag{6.11}$$

$$\frac{v_k(t)}{1+\delta_2} \le \psi^{-1}\Big(\int_0^1 f(\tau, |u_k(\tau)|)d\tau\Big). \tag{6.12}$$

Now, from (6.7) and the fact that $\{(u_k, v_k)\}$ tends to zero in $(C[0,1])^2$, and by redefining the sequence if necessary, we can assume that for given $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$g(t, |v_k(t)|) \le \epsilon\psi(u_k(t)) \quad \text{and} \quad f(t, |v_k(t)|) \le \epsilon\phi(v_k(t))$$

for all $k \geq k_0$ and $t \in [0,1]$. Hence,

$$g(t, |v_k(t)|) \leq \epsilon\psi(||v_k||) \quad \text{and} \quad f(t, |u_k(t)|) \leq \epsilon\phi(||u_k||).$$

Replacing these expressions into (6.11) and (6.12) respectively, we obtain that

$$\frac{||u_k||}{1+\delta_1} \leq \phi^{-1}\Big(\epsilon\psi(||v_k||)\Big) \quad \text{and} \quad \frac{||v_k||}{1+\delta_2} \leq \psi^{-1}\Big(\epsilon\phi(||u_k||)\Big), \tag{6.13}$$

and thus

$$\frac{\phi\big(\frac{||u_k||}{1+\delta_1}\big)}{\phi(||u_k||)} \frac{\psi\big(\frac{||v_k||}{1+\delta_2}\big)}{\psi(||v_k||)} \leq \epsilon^2.$$

The fact that at this point $\epsilon$ can be made arbitrarily small implies that either

$$\liminf_{s\to 0} \frac{\phi(\frac{1}{1+\delta_1}s)}{\phi(s)} = 0 \quad \text{or} \quad \liminf_{s\to 0} \frac{\psi(\frac{1}{1+\delta_2}s)}{\psi(s)} = 0,$$

which is a contradiction to (6.8). Thus, there exists $\rho_0 > 0$ small, such that $((u,v),\lambda)$ satisfies (6.10) implies that $(u,v) = (0,0)$ is the unique solution to (6.5) in the ball $\mathcal{B}(0,\rho_0)$. From here for any fixed $0 < \rho < \rho_0$ the Leray-Schauder degree $\deg_{LS}(I-\lambda T, \mathcal{B}(0,\rho), 0)$ is well defined, and from properties of this degree

$$\deg_{LS}(I - \lambda T, \mathcal{B}(0,\rho), 0) = \textit{constant}, \quad \text{for all} \quad \lambda \in [0,1],$$

implying immediately that (6.9) holds. □

Our following proposition generalizes Proposition 2.3 to systems.

**Proposition 6.2.** *Consider the problem*

$(I_2)$

$$\begin{cases} -(\phi(u'))' \geq g(t, |v(t)|), \quad -(\psi(v'))' \geq f(t, |u(t)|), \ t \in (0,1) \\ u(0) = \beta_1 u'(0), \quad u(1) = -\delta_1 u'(1), \quad v(0) = \beta_2 v'(0), \quad v(1) = -\delta_2 v'(1), \end{cases}$$

*where as before* $\beta_i, \delta_i \geq 0,\ i = 1,2$, *the functions* $\phi, \psi$ *satisfy condition* $(H_1)$, $f, g$ *condition* $(H_2)$ *of the Introduction, and*

$$f_\infty(t) = g_\infty(t) = \infty \quad \textit{uniformly for} \quad t \in [0,1]. \tag{6.14}$$

*Suppose that* $(I_2)$ *has a sequence of solutions* $\{(u_k, v_k)\}$, *with*

$$||(u_k, v_k)|| = ||u_k|| + ||v_k|| \to \infty. \tag{6.15}$$

*Then*

(i) *there exists a subsequence of* $\{(u_k, v_k)\}$ *(which we rename the same) such that, for any* $0 < \rho < \frac{1}{2}$, *it holds that* $u_k(t) \to \infty$ *and* $v_k(t) \to \infty$ *uniformly for* $t \in [\rho, 1-\rho]$;

(ii) *there exists* $\tau > 1$ *such that either*

$$\limsup_{s\to\infty} \frac{\phi(\tau s)}{\phi(s)} = \infty \quad or \quad \limsup_{s\to\infty} \frac{\psi(\tau s)}{\psi(s)} = \infty. \tag{6.16}$$

**Proof.** Since $u_k$, $v_k$ satisfy

$$-(\phi(u_k'))' \geq 0, \quad -(\psi(v_k'))' \geq 0, \quad t \in (0,1)$$

$$u_k(0) = \beta_1 u_k'(0), \quad u_k(1) = -\delta_1 u_k'(1), \quad v_k(0) = \beta_2 v_k'(0), \quad v_k(1) = -\delta_2 v_k'(1),$$

it follows that $u_k(t) > 0$, $v_k(t) > 0$ are concave down in $(0,1)$, and there are unique $t_k$, $r_k \in (0,1)$ with $u_k(t_k) = \|u_k\|$ and $u_k'(t_k) = 0$, $v_k(r_k) = \|v_k\|$ and $v_k'(r_k) = 0$.

We observe next that from (6.15) at least one of the sequences $\{u_k\}$ or $\{v_k\}$ must contain a subsequence (renamed the same) such that either $||u_k|| \to \infty$ or $||v_k|| \to \infty$. Assume that $||u_k|| \to \infty$ ( if $||v_k|| \to \infty$ the proof is similar), then, as in the proof of Proposition 2.3, we conclude that $u_k(t) \to \infty$ uniformly in $[\rho, 1-\rho]$ for any fixed $\rho \in (0, 1/2)$.

Now, from (6.14), for given $N > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$f(s, |u_k(s)|) \geq N\phi(u_k(s))$$

for all $s \in [\rho, 1-\rho]$ and all $k \geq k_0$. Let us assume first that $\{t_k\}$ contains a subsequence (still denoted the same) contained in $(0, 1/2]$. Then, $f(s, |u_k(s)|) \geq N\phi(u_k(s))$ for all $s \in [t_k, 1-\rho]$ and all $k \geq k_0$ ( if $t_k < \rho$, then use the fact that $u_k$ is convex and attains its maximum at $t_k$). Hence, by integrating the second inequality in $(I_2)$ over $[t_k, t]$, $t \leq 1-\rho$, we obtain that

$$-v_k'(t) \geq \psi^{-1}\Big(\int_{t_k}^{t} N\phi(u_k(s))ds\Big),$$

and by setting $\alpha_k = \max(t_k, \rho) \leq 1/2$, we have that

$$-v_k'(t) \geq \psi^{-1}\Big(\int_{\alpha_k}^{t} N\phi(u_k(s))ds\Big),$$

for all $\alpha_k \leq t \leq 1-\rho$. Then,

$$-v_k'(t) \geq \psi^{-1}\Big(Nm_k(t-\alpha_k)\Big), \tag{6.17}$$

where $m_k = \min_{s\in[\alpha_k, 1-\rho]} \phi(u_k(s))$. Clearly, $m_k \to \infty$ as $k \to \infty$. Let next $t \in [\frac{3-2\rho}{4}, 1-\rho)$, and notice that $1/2 < \frac{3-2\rho}{4} < 1-\rho$. By integrating (6.17) over $[t, 1-\rho]$, we find that

$$v_k(t) \geq \int_t^{1-\rho} \psi^{-1}\Big(Nm_k(s-\alpha_k)\Big)ds \geq \psi^{-1}\big(Nm_k(t-1/2)\big)(1-\rho-t).$$

Hence,

$$\begin{aligned} v_k(t_k) \geq v_k(\frac{3-2\rho}{4}) &\geq \psi^{-1}\Big(Nm_k(\frac{3-2\rho}{4}-1/2\Big)(1-\rho-\frac{3-2\rho}{4}) \\ &\geq \psi^{-1}\Big(Nm_k(\frac{1-2\rho}{4})\Big)(\frac{1-2\rho}{4}), \end{aligned}$$

implying that $v_k(t_k) \to \infty$ as $k \to \infty$. By Proposition 2.2, it follows that $v_k(t) \to \infty$ uniformly in $[\rho, 1-\rho]$, for any $\rho \in (0, 1/2)$.

If the sequence $\{t_k\}$ has a subsequence (renamed the same) contained this time in $(1/2, 1]$, then a slight modification of the argument at the end of Proposition 2.3 that uses a suitable change of variables will yield again that $v_k(t) \to \infty$ uniformly in $[\rho, 1-\rho]$, for any $\rho \in (0, 1/2)$, ending the proof of (i).

We will prove next (ii); i.e., we will prove that (6.16) holds. To this end let $N > 0$ be given and denote by $\{t_k\}$ and $\{r_k\}$ the unique points such that $||u_k|| = u_k(t_k)$ and $||v_k|| = v_k(r_k)$ respectively. We may assume, by passing to subsequences if necessary, that $t_k \to t_0$ and $r_k \to r_0$ as $k \to \infty$, and without loss of generality that $t_0 \leq r_0$.

Assume first that $t_0 \in (0, 1]$. Then, for $0 < \rho \leq t_0/4$, there exists $k_0 \in \mathbb{N}$ such that $t_k \geq t_0 - \rho$ and $r_k \geq t_0 - \rho$ for all $k \geq k_0$. Also, from $(i)$ we may assume that for $k \geq k_0$ it holds that

$$f(t, |u_k(t)|) \geq N\phi(u_k(t)) \quad \text{and} \quad g(t, |v_k(t)|) \geq N\psi(v_k(t)) \tag{6.18}$$

for all $t \in [\rho, 1-\rho]$. Now, for $s \in [\rho, t_0 - \rho]$, by integrating the inequalities in $(I_2)$ over $[s, t_k]$ and $[s, r_k]$ respectively, and by using (6.18), we obtain that

$$u_k'(s) \geq \phi^{-1}\Big(\int_s^{t_k} g(\tau, |v_k(\tau)|)d\tau\Big) \geq \phi^{-1}\Big(N\int_s^{t_0-\rho} \psi(v_k(\tau))d\tau\Big) \tag{6.19}$$

$$v_k'(s) \geq \psi^{-1}\Big(\int_s^{r_k} f(\tau, |u_k(\tau)|)d\tau\Big) \geq \psi^{-1}\Big(N\int_s^{t_0-\rho} \phi(u_k(\tau))d\tau\Big). \tag{6.20}$$

Thus, for $t \in [\rho, t_0 - \rho]$, by integrating (6.19) and (6.20) over $[\rho, t]$, it follows that

$$u_k(t) \geq \int_\rho^t \phi^{-1}\Big(N\int_s^{t_0-\rho} \psi(v_k(\tau))d\tau\Big)ds \tag{6.21}$$

$$v_k(t) \geq \int_\rho^t \psi^{-1}\Big(N\int_s^{t_0-\rho} \phi(u_k(\tau))d\tau\Big)ds, \tag{6.22}$$

and hence, for $t \in [2\rho, t_0 - 2\rho]$, by using the fact that both $u_k$ and $v_k$ are increasing over $[\rho, t_0 - \rho]$, we find that

$$u_k(t) \geq \int_\rho^t \phi^{-1}\Big(N\int_t^{t_0-\rho} \psi(v_k(t))d\tau\Big)ds \geq \rho\phi^{-1}\Big(N\rho\psi(v_k(t))\Big) \tag{6.23}$$

$$v_k(t) \geq \int_\rho^t \psi^{-1}\Big(N\int_t^{t_0-\rho}\phi(u_k(t))d\tau\Big)ds \geq \rho\psi^{-1}\Big(N\rho\phi(u_k(t))\Big). \quad (6.24)$$

Thus, for $t \in [2\rho, t_0 - 2\rho]$, we have that

$$\phi(\frac{1}{\rho}u_k(t)) \geq N\rho\psi(v_k(t)) \quad \text{and} \quad \psi(\frac{1}{\rho}v_k(t)) \geq N\rho\phi(u_k(t)),$$

and therefore

$$\frac{\phi(\frac{1}{\rho}u_k(t))}{\phi(u_k(t))}\frac{\psi(\frac{1}{\rho}v_k(t))}{\psi(v_k(t))} \geq (N\rho)^2.$$

Since this implies that

$$\Big(\limsup_{s\to\infty}\frac{\phi(\frac{1}{\rho}s)}{\phi(s)}\Big)\Big(\limsup_{s\to\infty}\frac{\psi(\frac{1}{\rho}s)}{\psi(s)}\Big) \geq (N\rho)^2,$$

the fact that $N$ can be made arbitrarily large implies that (6.16) holds.

The case when $t_0 = 0$ and $r_0 < 1$ can be handled similarly by an obvious modification of the previous argument, arguing this time over an interval of the form $[r_0 + \rho, 1 - \rho]$, where both $u_k$ and $v_k$ are decreasing. This case can also be reduced to the previous one by a suitable change of variables.

In this form we are left with the last case when $t_0 = 0$ and $r_0 = 1$, where we proceed as follows. Let $0 < \rho < 1/4$ and choose $k_0 \in \mathbb{N}$ such that $t_k \leq \rho/2$ and $r_k \geq 1 - \rho/2$ for all $k \geq k_0$. Then, from (2.3) we have that

$$u_k(t) \geq u_k(t_k)\frac{\rho/2}{1-\rho/2} \quad \text{for all } t \in [\rho, 1-\rho], \quad (6.25)$$

and

$$v_k(t) \geq v_k(r_k)\frac{\rho/2}{1-\rho/2} \quad \text{for all } t \in [\rho, 1-\rho]. \quad (6.26)$$

For $s \in [\rho, 1-\rho]$, by integrating the inequalities in $(I_2)$ over $[t_k, s]$ and $[s, r_k]$ respectively and by using (6.18), we obtain that

$$-u_k'(s) \geq \phi^{-1}\Big(\int_{t_k}^s g(\tau, |v_k(\tau)|)d\tau\Big) \geq \phi^{-1}\Big(N\int_\rho^s \psi(v_k(\tau))d\tau\Big) \quad (6.27)$$

$$v_k'(s) \geq \psi^{-1}\Big(\int_s^{r_k} f(\tau, |u_k(\tau)|)d\tau\Big) \geq \psi^{-1}\Big(N\int_s^{1-\rho}\phi(u_k(\tau))d\tau\Big). \quad (6.28)$$

Thus, for $t \in [\rho, 1-\rho]$, by integrating (6.27) and (6.28) over $[t, 1-\rho]$ and $[\rho, t]$ respectively, we find

$$u_k(t) \geq \int_t^{1-\rho}\phi^{-1}\Big(N\int_\rho^s \psi(v_k(\tau))d\tau\Big)ds \geq \int_t^{1-\rho}\phi^{-1}\Big(N\int_\rho^t \psi(v_k(\tau))d\tau\Big)ds$$

$$v_k(t) \geq \int_\rho^t \psi^{-1}\Big(N\int_s^{1-\rho}\phi(u_k(\tau))d\tau\Big)ds \geq \int_\rho^t \psi^{-1}\Big(N\int_t^{1-\rho}\phi(u_k(\tau))d\tau\Big)ds.$$

Hence, for $t \in [\rho, 1-\rho]$, we get

$$u_k(t) \geq (1-\rho-t)\phi^{-1}\Big(N(t-\rho)\psi(v_k(\rho))\Big) \tag{6.29}$$

$$v_k(t) \geq (t-\rho)\psi^{-1}\Big(N(1-\rho-t)\phi(u_k(1-\rho))\Big), \tag{6.30}$$

and thus, for $t \in [2\rho, 1-2\rho]$, we obtain that

$$u_k(t) \geq \rho\phi^{-1}\Big(N\rho\psi(v_k(\rho))\Big) \quad \text{and} \quad v_k(t) \geq \rho\psi^{-1}\Big(N\rho\phi(u_k(1-\rho))\Big). \tag{6.31}$$

But from (6.25) and (6.26) we have that

$$u_k(1-\rho) \geq \frac{\rho/2}{1-\rho/2}||u_k|| \quad \text{and} \quad v_k(\rho) \geq \frac{\rho/2}{1-\rho/2}||v_k||,$$

which, combined with (6.31), yield

$$||u_k|| \geq u_k(t) \quad \geq \quad \rho\phi^{-1}\Big(N\rho\psi(\frac{\rho/2}{1-\rho/2}||v_k||)\Big)$$

and

$$||v_k|| \geq v_k(t) \quad \geq \quad \rho\psi^{-1}\Big(N\rho\phi(\frac{\rho/2}{1-\rho/2}||u_k||)\Big).$$

Then

$$\phi(\frac{1}{\rho}||u_k||) \quad \geq \quad N\rho\psi(\frac{\rho/2}{1-\rho/2}||v_k||)$$

and

$$\psi(\frac{1}{\rho}||v_k||) \quad \geq \quad N\rho\phi(\frac{\rho/2}{1-\rho/2}||u_k||),$$

which imply that

$$\frac{\phi(\frac{1}{\rho}||u_k||)}{\phi(\frac{\rho/2}{1-\rho/2}||u_k||)}\frac{\psi(\frac{1}{\rho}||v_k||)}{\psi(\frac{\rho/2}{1-\rho/2}||v_k||)} \geq (N\rho)^2.$$

Setting $\mu_k = \frac{\rho/2}{1-\rho/2}||u_k||$, $\nu_k = \frac{\rho/2}{1-\rho/2}||v_k||$, the last expression can be written as

$$\frac{\phi(\frac{2}{\rho^2}\mu_k)}{\phi(\mu_k)}\frac{\psi(\frac{2}{\rho^2}\nu_k)}{\psi(\nu_k)} \geq (N\rho)^2.$$

Finally, since $N$ can be made arbitrarily large, this implies that either

$$\limsup_{s\to\infty} \frac{\phi(\frac{2}{\rho^2}s)}{\phi(s)} = \infty \quad \text{or} \quad \limsup_{s\to\infty} \frac{\psi(\frac{2}{\rho^2}s)}{\psi(s)} = \infty$$

and thus (6.16) holds. □

Next let us consider the problem

$$(\phi(u'))' + g(t, |v(t)|) + M = 0, \quad (\psi(v'))' + f(t, |u(t)|) = 0, \quad t \in (0,1) \tag{6.32}$$

$$u(0) = \beta_1 u'(0), \quad u(1) = -\delta_1 u'(1), \quad v(0) = \beta_2 v'(0), \quad v(1) = -\delta_2 v'(1),$$

where $M > 0$. We have the following.

**Proposition 6.3.** *Assume that, in problem* (6.32), $\beta_i, \delta_i \geq 0,\ i = 1, 2,$ *the functions* $\phi, \psi$ *satisfy condition* $(H_1)$, $f, g$ *condition* $(H_2)$ *of the Introduction, and* (6.14). *If furthermore* $\phi, \psi$ *satisfy*

$$\limsup_{s\to\infty} \frac{\phi(\tau s)}{\phi(s)} < \infty, \quad \limsup_{s\to\infty} \frac{\psi(\tau s)}{\psi(s)} < \infty \text{ for all } \tau > 1, \tag{6.33}$$

*then there exists* $M_0 > 0$ *such that for any* $M \geq M_0$ *problem* (6.32) *does not have solutions.*

**Proof.** If the conclusion of the proposition is not true, then there is a sequence $\{M_k\}$, $M_k \to \infty$ as $k \to \infty$, and corresponding solutions $\{(u_k, v_k)\}$ to (6.32) with $M = M_k$. As in the proof of Proposition 4.2 we deduce that $||(u_k, v_k)|| \to \infty$ as $k \to \infty$ and thus we are in the situation of Proposition 6.2. Consequently, (6.16) holds, contradicting (6.33). □

Next let us consider the family of problems

$$(\phi(u'))' + g(t, |v(t)|) + \mu M = 0, \quad (\psi(v'))' + f(t, |u(t)|) = 0, \quad t \in (0,1)$$
$$u(0) = \beta_1 u'(0), \quad u(1) = -\delta_1 u'(1), \quad v(0) = \beta_2 v'(0), \quad v(1) = -\delta_2 v'(1), \tag{6.34}$$

where $\mu \in [0, 1]$ and $M \geq M_0$ is a fixed constant.

**Proposition 6.4.** *Under the hypotheses of the last proposition solutions to* (6.34) *are a priori bounded; i.e., there exists a positive constant* $R_0$ *such that, if* $(\mu, u, v)$ *satisfy* (6.34), *then*

$$||(u, v)|| = ||u|| + ||v|| < R_0. \tag{6.35}$$

*Furthermore, we have*

$$deg_{LS}(I - T, \mathcal{B}(0, R), 0) = 0, \tag{6.36}$$

*for any* $R \geq R_0$ *and where, as before,* $\mathcal{B}(0,R) = B(0,R) \times B(0,R) \subset (C[0,1])^2$.

**Proof.** Assume by contradiction that there exists a sequence $\{(\mu_k, u_k, v_k)\}$, $\mu_k \in [0,1]$, $u_k \in C[0,1]$, $v_k \in C[0,1]$, of solutions to (6.34) with

$$||(u_k, v_k)|| = ||u_k|| + ||v_k|| \to \infty \quad \text{as } k \to \infty. \tag{6.37}$$

Since $(\mu_k, u_k, v_k)$ also satisfy

$$(\phi(u_k'))' + g(t, |v_k(t)|) \geq 0, \quad (\psi(v_k'))' + f(t, |u_k(t)|) = 0, \quad t \in (0,1)$$
$$u_k(0) = \beta_1 u_k'(0),\ u_k(1) = -\delta_1 u_k'(1),\ v_k(0) = \beta_2 v_k'(0),\ v_k(1) = -\delta_2 v_k'(1),$$

we are in the situation of Proposition 6.2. By that proposition, and (6.37), we have that (6.16) holds, contradicting (6.33), and hence there is $R_0 > 0$ such that (6.35) holds.

Define $S : [0,1] \times C[0,1] \times C[0,1] \mapsto C[0,1] \times C[0,1]$ by

$$S(\mu, u, v) := (\mathcal{T}_1 \circ \tilde{\mathcal{G}}(v), \mathcal{T}_2 \circ \mathcal{F}(u)), \tag{6.38}$$

where as before $\mathcal{F}$ is defined by $\mathcal{F}(u)(t) = f(t, |u(t)|)$ and $\tilde{\mathcal{G}}(\mu, v)(t) = g(t, |v(t)|) + \mu M_0$, $t \in [0,1]$, $\mu \in [0,1]$. Then it is a routine matter to check that $S$ is a completely continuous operator. Notice that $S(0, \cdot, \cdot) = T$.

Let us take $R > \max\{M, R_0\}$; then by (6.35) the following is well defined and satisfies

$$\deg_{LS}(I - S(\mu, \cdot, \cdot), \mathcal{B}(0,R), 0) = \textit{constant}.$$

Hence,

$$\begin{aligned}\deg_{LS}(I - T, \mathcal{B}(0,R), 0) &= \deg_{LS}(I - S(0, \cdot, \cdot), \mathcal{B}(0,R), 0) \\ &= \deg_{LS}(I - S(1, \cdot, \cdot), \mathcal{B}(0,R), 0) = 0,\end{aligned}$$

this last degree being zero by Proposition 6.3. □

We are now in a position to prove Theorem 1.3.

**Proof of Theorem 1.3.** Problem (6.1) can be equivalently written as the fixed-point problem

$$(u, v) = T(u, v). \tag{6.39}$$

By Proposition 6.4, we have that

$$\deg_{LS}(I - T, \mathcal{B}(0,R), 0) = 0,$$

and by (6.9) of Proposition 6.1

$$\deg_{LS}(I - T, \mathcal{B}(0,\rho), 0) = 1,$$

where $R > \rho$. Thus, from classical theorems of Leray-Schauder degree, it follows that $\deg_{LS}(I - T, \mathcal{B}(0,R) \setminus \mathcal{B}(0,\rho), 0)$ is different from zero. This implies that (6.39) has a fixed point $(u,v)$, which by the previous arguments is a solution to problem (6.1) and hence a solution to problem (1.11) that satisfies $u(t) > 0$ and $v(t) > 0$ for all $t \in (0,1)$. □

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