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MULTIPLICITY OF POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR PROBLEMS

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Abstract. In this paper we prove the existence and multiplicity of nontrivial weak solutions for quasilinear elliptic equations of the form $-L_pu+V(x)|u|^{p-2}u = h(u)$ in $\mathbb{R}^N$, where $L_pu \doteq \epsilon^p\Delta_pu+\epsilon^p\Delta_p(u^2)u$ and V is a positive continuous potential bounded away from zero satisfying some conditions and the nonlinear term $h(u)$ has a subcritical growth type. Here, we use a variational method to get the multiplicity of positive solutions involving the Lusternick-Schnirelman category of the set where V achieves its minimum value.

1. Introduction

This paper is concerned with the existence of multiple positive solutions for a quasilinear problem of the form

$$-L_pu + V(x)|u|^{p-2}u = h(u), \quad u \in W^{1,p}(\mathbb{R}^N), \qquad (P_\varepsilon)$$

where

$$L_pu \doteq \epsilon^p\Delta_pu + \epsilon^p\Delta_p(u^2)u,$$

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$\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $2 \leq p < N$, $\varepsilon > 0$ is a small parameter and $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

(V_1) $0 < V_0 < \liminf_{|x|\to\infty} V(x) =: V_\infty$ where $V_0 = \inf_{x\in\mathbb{R}^N} V(x)$,

and either $V_\infty < \infty$ or $V_\infty = \infty$ must hold. Regarding the nonlinearity h, we assume that $h : \mathbb{R} \to \mathbb{R}$ is of class C^1 and satisfies the following conditions:

(H_0) $h(s) = 0$ for $s < 0$ and $h'(s) = o(|s|^{p-2})$ at the origin;
(H_1) $\lim_{|s|\to\infty} h'(s)|s|^{-q-1} = 0$ for some $q \in (2p-1, 2p^*-1)$ where $p^* = Np/(N-p)$;
(H_2) there exists $\theta > 2p$ such that $0 < \theta H(s) \leq sh(s)$ for all $s > 0$;
(H_3) the function $s \to h(s)/s^{2p-1}$ is increasing for $s > 0$.

In all of this paper, a function $u : \mathbb{R}^N \to \mathbb{R}$ is called a weak solution of (P_ε) if $u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ and

$$\begin{aligned}\epsilon^p \int_{\mathbb{R}^N} &[(1 + 2^{p-1}|u|^p)|\nabla u|^{p-2}\nabla u\nabla\varphi + 2^{p-1}|\nabla u|^p|u|^{p-2}u\varphi]\mathrm{d}x \\ = \int_{\mathbb{R}^N} &[h(u) - V(x)|u|^{p-2}u]\varphi \,\mathrm{d}x \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).\end{aligned} \tag{1.1}$$

Equations of the kind (P_ε) arise in various branches of mathematical physics. For example, solutions of (P_ε), in the case $p = 2$ and $\epsilon = 1$, are related to the existence of solitary wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t\psi = -\Delta\psi + V(x)\psi - \widetilde{h}(|\psi|^2)\psi - \kappa\Delta[\rho(|\psi|^2)]\rho'(|\psi|^2)\psi \tag{1.2}$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $V = V(x)$, $x \in \mathbb{R}^N$, is a given potential, κ is a real constant and $\rho, \widetilde{h}$ are real functions. The semilinear case corresponding to $\kappa = 0$ has been studied extensively in recent years, see for example [5], [16], [19] and references therein. Quasilinear equations of the form (1.2) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of ρ. The case $\rho(s) = s$ was used for the superfluid film equation in plasma physics by Kurihura in [22](cf. [23]). In the case $\rho(s) = (1+s)^{1/2}$, equation (1.2) models the self-channeling of a high-power ultra short laser in matter, see [7], [8], [9], [32] and references in [12]. Equation (1.2) also appears in plasma physics and fluid mechanics [4], [21], [31], [35], in mechanics [18] and in condensed matter theory [28]. Considering the case $\rho(s) = s$, $\kappa > 0$ and putting

$$\psi(t, x) = \exp(-iFt)u(x), \quad F \in \mathbb{R},$$

we obtain a corresponding equation

$$-\Delta u - \Delta(u^2)u + V(x)u = h(u) \quad \text{in} \quad \mathbb{R}^N \tag{1.3}$$

where we have renamed $V(x) - F$ to be $V(x)$, $h(u) = \widetilde{h}(u^2)u$ and we assume, without loss of generality, that $\kappa = 1$.

The quasilinear Schrödinger equation (1.3) has received special attention in the past several years. This problem was studied in [11], [15], [25], [26], [27], [30] and references therein. Many important results on the existence of nontrivial solutions of (1.3) were obtained in these papers and give us very good insight into this quasilinear Schrödinger equation. The existence of a positive ground state solution has been proved in [30] by using a constrained minimization argument, which gives a solution of (1.3) with an unknown Lagrange multiplier λ in front of the nonlinear term. In [26], by a change of variables the quasilinear problem was transformed into a semilinear one and an Orlicz space framework was used as the working space, and they were able to prove the existence of positive solutions of (1.3) by the mountain-pass theorem. The same method of change of variables was used recently also in [11], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space and they studied a different class of nonlinearities. In [15], for $N = 2$ the authors treated the case where the nonlinearity $h : \mathbb{R} \to \mathbb{R}$ has critical exponential growth; that is, h behaves like $exp(4\pi s^4) - 1$ as $|s| \to \infty$. They establish an existence result for the problem by combining the Ambrosetti-Rabinowitz mountain-pass theorem with a version of the Trudinger-Moser inequality in $\mathbb{R}^2$. In [27], the existence of both one-sign and nodal ground states of soliton type solutions was established by the Nehari method.

Motivated by the works just described and by results found in [2] and [10], the present paper establishes the existence of multiple positive solutions to (P_ϵ) for ϵ small enough, by using the Lusternick-Schnirelman category. More precisely, setting the sets $M = \{x \in \mathbb{R}^N : V(x) = V_0\}$ and

$$M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \leq \delta\}, \quad \text{for} \quad \delta > 0,$$

we have proved that (P_ϵ) has at least $cat_{M_\delta}(M)$ for small enough ϵ. Note that the condition (V_1) implies that $M \neq \emptyset$ and M is bounded. Our main result completes the studies made in [10] and [2], in the sense that those papers have established the existence of multiple solutions by using Lusternick-Schnirelman category, the set M and the operators Δu (Laplacian) and $\Delta_p u$ (p-Laplacian), for $p \geq 2$, respectively, which do not have the quasilinear term $-\Delta_p(u^2)u$. The presence of the term $-\Delta_p(u^2)u$ in the problem implies that several estimates used in [10] and [2] cannot be repeated for the functional energy associated to (P_ϵ). As observed in [33] and [34], there are some technical difficulties to applying directly variational methods to the functional associated to (P_ε) given by

$$J(u) = \frac{\epsilon^p}{p}\int_{\mathbb{R}^N}(1+2^{p-1}|u|^p)|\nabla u|^p \,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(u)\,\mathrm{d}x,$$

where $H(s) = \int_0^s h(t)dt$. The main difficulty is related to the fact that J is not well defined in all of $W^{1,p}(\mathbb{R}^N)$ for $N > p > 1$. For example, if $u \in C_0^1(\mathbb{R}^N\backslash\{0\})$ is defined by

$$u(x) = |x|^{(p-N)/2p} \quad \text{for} \quad x \in B_1\backslash\{0\},$$

we have $u \in W^{1,p}(\mathbb{R}^N)$, while the function $|u|^p|\nabla u|^p$ does not belong to $L^1(\mathbb{R}^N)$. To overcome this difficulty, we use arguments developed in [33] and [34], which generalize some arguments found in Liu, Wang and Wang [26] and Colin-Jeanjean [11] for the case $p = 2$. More precisely, we make the change of variables $v = f^{-1}(u)$, where f is defined by

$$\begin{aligned} f'(t) &= \frac{1}{(1+2^{p-1}|f(t)|^p)^{1/p}} && \text{on} \quad [0,+\infty), \\ f(t) &= -f(-t) && \text{on} \quad (-\infty,0]. \end{aligned} \tag{1.4}$$

Therefore, after the change of variables, the functional $J(u)$ can be rewritten in the following way:

$$\widetilde{J}(v) \doteq J(f(v)) = \frac{\epsilon^p}{p}\int_{\mathbb{R}^N}|\nabla v|^p \,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|f(v)|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(v))\,\mathrm{d}x \tag{1.5}$$

which is well defined on the Banach space X defined by

$$X = \left\{ v \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|f(v)|^p dx < \infty \right\}$$

endowed with the norm

$$\|v\| = |\nabla v|_p + \inf_{\xi>0}\left[\frac{1}{\xi} + \int_{\mathbb{R}^N} V(x)|f(\xi v)|^p dx\right].$$

The reader can find in [33] and [34] more details about the function f and the proof that $\|\ \|$ is a norm in X.

Under the conditions (V_1) and $(H_0)-(H_2)$, a straightforward computation shows that the functional $\widetilde{J} : X \to \mathbb{R}$ is of class C^1 with

$$\langle \widetilde{J}'(v), w\rangle = \epsilon^p \int_{\mathbb{R}^N}[|\nabla v|^{p-2}\nabla v \nabla w \ + V(x)|f(v)|^{p-2}f(v)f'(v)w]dx$$

$$- \int_{\mathbb{R}^N} h(f(v))f'(v)wdx$$

for $v, w \in X$. Thus, the critical points of $\widetilde{J}$ correspond exactly to the weak solutions of the quasilinear problem

$$\begin{cases} -\epsilon^p \Delta_p v + V(x)|f(v)|^{p-2} f(v)f'(v) = h(f(v))f'(v) \ \text{ in } \ \mathbb{R}^N \\ v \in W^{1,p}(\mathbb{R}^N) \ \text{ with } \ 2 \le p < N, \\ v(x) > 0, \forall\ x \in \mathbb{R}^N. \end{cases} \tag{D_ϵ}$$

The problem above has a strong relation with problem (P_ϵ), because if $v \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ is a critical point of the functional $\widetilde{J}$, then $u = f(v)$ is a weak solution of (P_ϵ). In all of this paper, we will work to find nontrivial solutions for (D_ϵ) with this property.

The main result of this paper is the following.

Theorem 1.1. *Assume that* (V_1) *and* $(H_0) - (H_3)$ *hold. Then, given* $\delta > 0$, *there exists* $\overline{\varepsilon} = \overline{\varepsilon}(\delta) > 0$ *such that the equation* (P_ε) *has at least* $cat_{M_\delta}(M)$ *positive weak solutions. Moreover, each solution decays to zero at infinity and if* u_ϵ *denotes one of these positive solutions and* $\eta_\epsilon \in \mathbb{R}^N$ *its global maximum, then*

$$\lim_{\epsilon \to 0} V(\eta_\epsilon) = V_0.$$

We would like to emphasize that Theorem 1.1 can be seen as a complement of the study made in some papers mentioned in this introduction, because in the papers where the operator $L_p u$ has been studied, the existence of multiple positive solutions was not considered by employing the Lusternick-Schnirelman category. Moreover, we work only with the (PS) sequence related to $\widetilde{J}$, while a lot of papers have worked with Cerami sequences associated to $\widetilde{J}$. Here, we have proved some properties involving the functions h, f and $\| \ \|$, which allows us to work with the (PS) sequence, see technical results in Sections 2 and 3. These properties allow us to prove that $\widetilde{J}$ satisfies the well-known Palais-Smale condition in a specific interval, which is a crucial point to applying Lusternick-Schnirelman category.

Notation. In this paper we make use of the following notation:

- C, C_0, C_1, C_2, ... denote positive (possibly different) constants.
- B_R denotes the open ball centered at the origin and radius $R > 0$.
- $C_0^\infty(\mathbb{R}^N)$ denotes the functions infinitely differentiable with compact support in $\mathbb{R}^N$.
- For $1 \le p \le \infty$, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space endowed with the usual norms.

- By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between X and its dual X^*.
- We denote the weak convergence in X by " $\rightharpoonup$ " and the strong convergence by " $\to$ ".
- We omit the symbol dx in integrals over $\mathbb{R}^N$ when no confusion can arise.

2. Preliminary results

We begin with some preliminary results. Let us collect some properties of the change of variables $f : \mathbb{R} \to \mathbb{R}$ defined in (1.4), which will be usual in the sequel of the paper. The proof of the lemma below can be found in [33] and [34].

Lemma 2.1. *The function $f(t)$ and its derivative enjoy the following properties:*

(1) *f is uniquely defined, C^2 and invertible;*
(2) $|f'(t)| \le 1$ *for all* $t \in \mathbb{R}$;
(3) $|f(t)| \le |t|$ *for all* $t \in \mathbb{R}$;
(4) $f(t)/t \to 1$ *as* $t \to 0$;
(5) $|f(t)| \le 2^{1/2p}|t|^{1/2}$ *for all* $t \in \mathbb{R}$;
(6) $f(t)/2 < tf'(t) < f(t)$ *for all* $t > 0$;
(7) $f(t)/\sqrt{t} \to a > 0$ *as* $t \to +\infty$;
(8) *there exists a positive constant C such that*

$$|f(t)| \ge \begin{cases} C|t|, & |t| \le 1 \\ C|t|^{1/2}, & |t| \ge 1; \end{cases}$$

(9) $|f(t)f'(t)| \le 1/2^{(p-1)/p}$ *for all* $t \in \mathbb{R}$.

Corollary 2.2. *The following properties involving the functions f and h hold:*

(i) *the function* $(f(t))^{p-1}f'(t)t^{1-p}$ *is decreasing for* $t > 0$;
(ii) *the function* $(f(t))^{2p-1}f'(t)t^{1-p}$ *is increasing for* $t > 0$;
(iii) *the function* $h(f(t))f'(t)t^{1-p}$ *is increasing for* $t > 0$.

Proof. By using Lemma 2.1(6), it is easy to see that $f(t)/t$ is nonincreasing for $t > 0$. Thus, we get

$$\frac{d}{dt}\Big(\frac{(f(t))^{p-1}f'(t)}{t^{p-1}}\Big) = (p-1)\Big(\frac{f(t)}{t}\Big)^{p-2}\frac{d}{dt}\Big(\frac{f(t)}{t}\Big)f'(t) + \frac{(f(t))^{p-1}}{t^{p-1}}f''(t) < 0$$

for all $t > 0$, which shows (i).

To prove (ii), we observe that

$$\frac{d}{dt}\left(\frac{(f(t))^{2p-1}f'(t)}{t^{p-1}}\right) = \frac{1}{t^{2(p-1)}}\Big\{(2p-1)(f(t))^{2p-2}(f'(t))^2t^{p-1} - 2^{p-1}(f(t))^{3p-2}(f'(t))^{p+2}t^{p-1} - (p-1)(f(t))^{2p-1}f'(t)t^{p-2}\Big\}.$$

Hence,

$$\frac{d}{dt}\Big(\frac{(f(t))^{2p-1}f'(t)}{t^{p-1}}\Big) \geq f'(t)(f(t))^{2p-2}t^{p-2}\frac{(2p-1)f'(t)t - f'(t)t - (p-1)f(t)}{t^{2(p-1)}}$$

from which it follows that

$$\frac{d}{dt}\Big(\frac{(f(t))^{2p-1}f'(t)}{t^{p-1}}\Big) \geq f'(t)(f(t))^{2p-2}t^{p-2}(p-1)\frac{2f'(t)t - f(t)}{t^{2(p-1)}} > 0$$

for all $t > 0$, where we have used (6) and (9) in Lemma 2.1. The last inequality proves (ii). The proof of (iii) follows by using (H_3), (ii) and the equality

$$\frac{h(f(t))f'(t)}{t^{p-1}} = \Big[\frac{h(f(t))}{(f(t))^{2p-1}}\Big]\Big[\frac{(f(t))^{2p-1}f'(t)}{t^{p-1}}\Big] \text{ for } t > 0. \qquad \square$$

The next result will be used in the proof of some results later on.

Proposition 2.3. *Let $B : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative measurable function and (v_n) be a sequence in $W^{1,p}(\mathbb{R}^N)$ satisfying*

$$\int_{\mathbb{R}^N} B(x)|f(v_n)|^p \to 0 \quad \text{as } n \to \infty.$$

Then

$$\inf_{\xi>0}\left\{\frac{1}{\xi} + \int_{\mathbb{R}^N} B(x)|f(\xi v_n)|^p\right\} \to 0 \;\; \text{as } \; n \to \infty.$$

Proof. Since f is odd and $\frac{f(t)}{t}$ is nonincreasing for $t > 0$, for each $\xi > 1$, we have

$$\frac{1}{\xi} + \int_{\mathbb{R}^N} B(x)|f(\xi v_n)|^p \leq \frac{1}{\xi} + \xi^p\int_{\mathbb{R}^N} B(x)|f(v_n)|^p.$$

Hence, for any $\delta > 0$, fixing ξ_* sufficiently large such that $\frac{1}{\xi_*} < \frac{\delta}{2}$, we get

$$\inf_{\xi>0}\left\{\frac{1}{\xi} + \int_{\mathbb{R}^N} B(x)|f(\xi v_n)|^p\right\} \leq \frac{\delta}{2} + \xi_*^p\int_{\mathbb{R}^N} B(x)|f(v_n)|^p.$$

Thus,

$$\limsup_{n\to\infty}\left(\inf_{\xi>0}\left\{\frac{1}{\xi}+\int_{\mathbb{R}^N}B(x)|f(\xi v_n)|^p\right\}\right)\leq\frac{\delta}{2}\quad\forall\delta>0$$

which proves the proposition. □

3. The Palais-Smale condition

In this section, we will show that the functional $\widetilde{J}$ satisfies the Palais-Smale condition at a specific interval.

3.1. **Autonomous problem.** We start this subsection studying the existence of a positive ground-state solution for the following quasilinear problem:

$$\begin{cases} -\Delta_p v+\mu|f(v)|^{p-2}f(v)f'(v)=h(f(v))f'(v) \text{ in } \mathbb{R}^N\\ v\in W^{1,p}(\mathbb{R}^N),\\ v(x)>0,\forall\, x\in\mathbb{R}^N, \end{cases}\tag{A_μ}$$

where $\mu>0$ and $2\leq p<N$. Here, we consider the Banach space $W_\mu\equiv W^{1,p}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|=|\nabla u|_p+\inf_{\xi>0}\left[\frac{1}{\xi}+\int_{\mathbb{R}^N}\mu|f(\xi u)|^p\right],$$

which is equivalent to the usual norm in $W^{1,p}(\mathbb{R}^N)$.

The energy functional associated to (A_μ) is given by

$$E_\mu(v)=\frac{1}{p}\int_{\mathbb{R}^N}|\nabla v|^p+\frac{1}{p}\int_{\mathbb{R}^N}\mu|f(v)|^p-\int_{\mathbb{R}^N}H(f(v)).$$

By a direct calculation, it follows that $E_\mu\in C^1(W_\mu,\mathbb{R})$ with

$$\begin{aligned}\left\langle E'_\mu(v),w\right\rangle=&\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla w+\int_{\mathbb{R}^N}\mu|f(v)|^{p-2}f(v)f'(v)w\\ &-\int_{\mathbb{R}^N}h(f(v))f'(v)w\end{aligned}$$

for $v,w\in W_\mu$. Hereafter, let us denote by $\mathcal{M}_\mu$ the Nehari manifold associated to (A_μ); that is,

$$\mathcal{M}_\mu=\{u\in W_\mu: u\not\equiv 0 \text{ and } \left\langle E'_\mu(u),u\right\rangle=0\}.$$

Notice that, if $v\in\mathcal{M}_\mu$, then $v_+=\max\{v,0\}\not\equiv 0$. Indeed, suppose that there is $v\in\mathcal{M}_\mu$ with $v_+\equiv 0$. Thereby, $v\equiv -v_-$ where $v_-=\max\{-v,0\}$. From (H_0) together with the fact that f is odd, we reach

$$\int_{\mathbb{R}^N}|\nabla v_-|^p+\mu\int_{\mathbb{R}^N}|f(v_-)|^{p-2}f(v_-)f'(v_-)v_-=0.$$

The last equality combined with Lemma 2.1 yields

$$\int_{\mathbb{R}^N} |\nabla v_-|^p + \int_{\mathbb{R}^N} \mu |f(v_-)|^p = 0.$$

This together with Proposition 2.3 implies that $\|v_-\| = 0$, and consequently $v \equiv 0$, which is impossible, because $v \in \mathcal{M}_\mu$.

3.1.1. *Mountain pass geometry.* In this subsection, we will prove that E_μ exhibits the mountain pass geometry. For that matter, for $\rho > 0$ let us consider the set

$$\mathcal{S}(\rho) := \{v \in W_\mu : \mathcal{Q}(v) = \rho^p\},$$

where $\mathcal{Q} : W_\mu \to \mathbb{R}$ is given by

$$\mathcal{Q}(v) = \int_{\mathbb{R}^N} [|\nabla v|^p + \mu |f(v)|^p\ ].$$

Since $\mathcal{Q}(v)$ is continuous, $\mathcal{S}(\rho)$ is a closed subset of W_μ and it disconnects this space.

Lemma 3.1. *Under the hypotheses* (V_1) *and* $(H_0)-(H_2)$*, there exist* $\rho_0,\ \delta_0 > 0$ *such that*

$$E_\mu(v) \geq \delta_0 \quad \textit{for all} \quad v \in \mathcal{S}(\rho_0).$$

Proof. For $v \in \mathcal{S}(\rho)$ and $\epsilon > 0$, from $(H_0) - (H_1)$ we obtain

$$\int_{\mathbb{R}^N} H(f(v)) \ \leq \frac{\epsilon}{p} \int_{\mathbb{R}^N} |f(v)|^p \ + C \int_{\mathbb{R}^N} |f(v)|^{2p^*}$$

for some positive constant C. Now, by Lemma 2.1(5), we get the inequality

$$\int_{\mathbb{R}^N} H(f(v)) \ \leq \frac{\epsilon}{p} \int_{\mathbb{R}^N} |f(v)|^p \ + C \int_{\mathbb{R}^N} |v|^{p^*},$$

which implies

$$\int_{\mathbb{R}^N} H(f(v)) \ \leq \frac{\epsilon}{p} \int_{\mathbb{R}^N} |f(v)|^p \ + C_1 \Big(\int_{\mathbb{R}^N} |\nabla v|^p \ \Big)^{p^*/p}.$$

Thereby, for all $v \in \mathcal{S}(\rho)$, we have

$$E_\mu(v) \geq \frac{1}{p} |\nabla v|_p^p + \frac{(\mu - \epsilon)}{p} \int_{\mathbb{R}^N} |f(v)|^p \, \mathrm{d}x - C_1 |\nabla v|_p^{p*}.$$

On the other hand, since

$$\mathcal{Q}(v) = \int_{\mathbb{R}^N} [|\nabla v|^p + \mu |f(v)|^p\ ] \, \mathrm{d}x = \rho^p$$

we derive that

$$E_\mu(v) \geq C_2 \rho^p - C_1 \rho^{p^*}$$

for some positive constants C_1 and C_2. Therefore, for $\rho = \rho_0$ sufficiently small, we have $E_\mu(v) \geq \delta_0 > 0$, for all $v \in S(\rho_0)$. □

Lemma 3.2. *For each $v \in W_\mu \backslash \{0\}$, we have the following limits:*

(i) *if $v_+ \neq 0$, then*

$$E_\mu(tv) \to -\infty \quad as \quad t \to +\infty; \tag{3.1}$$

(ii) *if $v_+ = 0$, then*

$$E_\mu(tv) \to +\infty \quad as \quad t \to +\infty. \tag{3.2}$$

In particular, by (i), there exists $v \in W_\mu$ such that $\mathcal{Q}(v) > \rho_0^p$ and $E_\mu(v) < 0$.

Proof. First, we will prove (i). By (H_2), there exist positive constants C_1 and C_2 such that $H(s) \geq C_1 s^\theta - C_2$ for all $s \geq 0$. If $v_+ \neq 0$, there exists a bounded set Ω with positive measure, that is, $|\Omega| > 0$, such that $v_+ > 0$ in Ω. Thereby,

$$E_\mu(tv) \leq t^p\Big(\frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p + \frac{\mu}{p}\int_{\mathbb{R}^N} \frac{|f(tv)|^p}{t^p} - C_1\int_\Omega \frac{|f(tv_+)|^\theta}{t^p} + \frac{C_2|\Omega|}{t^p}\Big),$$

which implies

$$E_\mu(tv) \leq t^p\Big(\frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p + \frac{\mu}{p}\int_{\mathbb{R}^N} |v|^p - C_1\int_\Omega \frac{|f(tv_+)|^\theta}{t^p} + \frac{C_2|\Omega|}{t^p}\Big).$$

In the last inequality, we have used Lemma 2.1(3). On the other hand,

$$\int_\Omega \frac{f(tv_+)^\theta}{t^p} = \int_\Omega \left(\frac{f(tv_+)}{\sqrt{tv_+}}\right)^{2p} f(tv_+)^{\theta-2p} v_+^p$$

and since

$$\Big(\frac{f(tv_+(x))}{\sqrt{tv_+(x)}}\Big)^{2p} f(tv_+(x))^{\theta-2p} v_+^p(x) \to +\infty \quad \text{as} \quad t \to +\infty,$$

from Fatou's lemma, it follows that

$$\int_\Omega \frac{f(tv_+)^\theta}{t^p} \to +\infty \quad \text{as} \quad t \to +\infty$$

and therefore $E_\mu(tv) \to -\infty \quad \text{as} \quad t \to +\infty$, and the proof of (i) is complete. Now, if $v_+ = 0$, we see that

$$E_\mu(tv) \geq \frac{t^p}{p}\int_{\mathbb{R}^N} |\nabla v|^p,$$

implying that $E_\mu(tv) \to +\infty$ as $t \to \infty$, and the proof of (ii) is over. □

3.1.2. *Estimate of the mountain pass level.*

Lemma 3.3. *For every $v \in W_\mu \setminus \{0\}$ with $v_+ \neq 0$, there exists a unique $t_v > 0$ such that $t_v v \in \mathcal{M}_\mu$. Moreover, the maximum of $E_\mu(tv)$ for $t \geq 0$ is achieved at $t = t_v$.*

Proof. For each $v \in W_\mu \setminus \{0\}$ with $v_+ \neq 0$ let us consider the function $g(t) \doteq E_\mu(tv)$ for $t \geq 0$. Notice that $g'(t) = \langle E'_\mu(tv), v\rangle = 0$ if, and only if, $tv \in \mathcal{M}_\mu$. Furthermore, $g'(t) = 0$ is equivalent to

$$\int_{\mathbb{R}^N} |\nabla v|^p = \int_{\mathbb{R}^N} \frac{h(f(tv))f'(tv)v}{t^{p-1}} - \int_{\mathbb{R}^N} \frac{\mu|f(tv)|^{p-2}f(tv)f'(tv)v}{t^{p-1}}$$
$$= \int_{\{x;v(x)>0\}} \frac{h(f(tv))f'(tv)}{(tv)^{p-1}}v^p + \int_{\{x;v(x)\neq 0\}} \left[-\frac{\mu f(t|v|)^{p-1}f'(t|v|)}{(t|v|)^{p-1}}|v|^p\right].$$

From Corollary 2.2, the right-hand side in the equality above is an increasing function of t, because for each $x \in \mathbb{R}^N$ such that $v(x) > 0$, the function $c : (0,\infty) \to \mathbb{R}$ given by

$$c(t) = \frac{h(f(tv(x)))}{(f(tv(x)))^{2p-1}} \frac{(f(tv(x)))^{2p-1}f'(tv(x))}{(tv(x))^{p-1}}$$

is increasing and for each $x \in \mathbb{R}^N$ such that $v(x) \neq 0$ the function $d : (0,\infty) \to \mathbb{R}$ given by $d(t) = -(f(t|v(x)|))^{p-1}f'(t|v(x)|)/(t|v(x)|)^{p-1}$ is increasing. Next, choose $\delta > 0$ satisfying

$$\int_{\mathbb{R}^N} |\nabla v|^p \mathrm{d}x - \frac{\delta p}{\theta}\int_{\mathbb{R}^N} |v|^p x > 0.$$

From $(H_0) - (H_1)$, there exists $C > 0$ such that

$$\frac{h(f(tv))f(tv)}{t^p} \leq \delta|v|^p + Ct^{(q+1-2p)/2}|v|^{(q+1)/2},$$

and thus,

$$\lim_{t\to 0^+} \int_{\mathbb{R}^N} \frac{h(f(tv))f(tv)}{t^p} \leq \delta \int_{\mathbb{R}^N} |v|^p \mathrm{d}x.$$

Since

$$g(t) \geq t^p \left(\frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p - \frac{1}{\theta}\int_{\mathbb{R}^N} \frac{h(f(tv))f(tv)}{t^p}\right),$$

it follows that $g(t) > 0$ for $t > 0$ small. Moreover, $g(0) = 0$ and $g(t) = E_\mu(tv) < 0$ for t large. The previous analysis shows that there exists a unique $t_v > 0$ such that $g'(t_v) = 0$; that is, $t_v v \in \mathcal{N}$ and $g(t_v) = \max_{t\geq 0} g(t)$. □

From Lemma 3.3, it is possible to prove that the mountain pass level c_μ, given by

$$c_\mu = \inf_{\gamma\in\Gamma_\mu}\max_{t\in[0,1]} E_\mu(\gamma(t)),$$

where

$$\Gamma_\mu = \{\gamma \in C([0,1], W_\mu) : \gamma(0) = 0,\ E_\mu(\gamma(1)) < 0\},$$

satisfies the following equalities:

$$c_\mu = \inf_{\mathcal{M}_\mu} E_\mu = \inf_{v\in W_\mu\setminus\{0\}}\max_{t\geq 0} E_\mu(tv). \tag{3.3}$$

The next lemma establishes some properties related to $(PS)_c$ sequences of E_μ. We recall that $(v_n) \subset W_\mu$ is a $(PS)_c$ sequences of E_μ if

$$E_\mu(v_n) \to c \text{ and } E'_\mu(v_n) \to 0 \text{ as } n \to \infty.$$

Lemma 3.4. *Let (v_n) be a $(PS)_c$ sequences of E_μ. Then,*

(i) *(v_n) is bounded in W_μ and for some subsequence, still denoted by itself, $v_n \rightharpoonup v$ for some $v \in W_\mu$;*
(ii) *$E'_\mu(v) = 0$;*
(iii) *$v_n \geq 0$ for $n \in \mathbb{N}$.*

Proof. i) From (H_2) and Lemma 2.1(6), we have

$$C_1 + o_n(1)\|v_n\| \geq (\frac{1}{p} - \frac{1}{\theta})\int_{\mathbb{R}^N} |\nabla v_n|^p + \mu(\frac{1}{p} - \frac{1}{2\theta})\int_{\mathbb{R}^N} |f(v_n)|^p, \tag{3.4}$$

where $o_n(1) \to 0$ as $n \to \infty$. Recalling that $|\nabla v_n|_p \leq 1 + |\nabla v_n|_p^p$, we obtain

$$C_1 + o_n(1)\|v_n\| \geq C_2\Big(|\nabla v_n|_p - 1 + \mu\int_{\mathbb{R}^N} |f(v_n)|^p\Big) \tag{3.5}$$

from which follows the inequality

$$C_3 + o_n(1)\|v_n\| \geq C\Big(|\nabla v_n|_p + 1 + \mu\int_{\mathbb{R}^N} |f(v_n)|^p\Big) \geq C\|v_n\|, \tag{3.6}$$

which gives that (v_n) is bounded in W_μ. Since $(W_\mu, \|\ \|)$ is reflexive, there are a subsequence of (v_n), still denote by itself, and $v \in W_\mu$ such that $v_n \rightharpoonup v$ in W_μ.

ii) Since $C_0^\infty(\mathbb{R}^N)$ is dense in W_μ, we only have to show that $\langle E'_\mu(v), \psi\rangle = 0$ for all $\psi \in C_0^\infty(\mathbb{R}^N)$. Firstly, let us observe that

$$\langle E'(v_n), \psi\rangle - \langle E'(v), \psi\rangle = \int_{\mathbb{R}^N}\left(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v\right)\nabla\psi$$
$$+\mu\int_{\mathbb{R}^N}\left[\frac{|f(v_n)|^{p-2}f(v_n)}{(1+2^{p-1}|f(v_n)|^p)^{1/p}} - \frac{|f(v)|^{p-2}f(v)}{(1+2^{p-1}|f(v)|^p)^{1/p}}\right]\psi$$

$$+\int_{\mathbb{R}^N}\left[\frac{h(f(v))}{(1+2^{p-1}|f(v)|^p)^{1/p}}-\frac{h(f(v_n))}{(1+2^{p-1}|f(v_n)|^p)^{1/p}}\right]\psi.$$

Using the fact that $v_n \to v$ in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [1, p^*)$, the Lebesgue dominated convergence theorem and $(H_0)-(H_1)$, it follows that

$$\langle E'(v_n), \psi\rangle - \langle E'(v), \psi\rangle \to 0.$$

Since $\langle E'(v_n), \psi\rangle \to 0$, we conclude that $\langle E'(v), \psi\rangle = 0$ for all $\psi \in C_0^\infty(\mathbb{R}^N)$.

iii) From the boundedness of (v_n) in W_μ, we derive that (v_{n-}) is bounded in W_μ. Thus, by Lemma 2.1(6), we obtain

$$o_n(1) = \left\langle E'_\mu(v_n), -v_{n-}\right\rangle \geq \frac{1}{2}\Big(\int_{\mathbb{R}^N}|\nabla v_{n-}|^p + \mu\int_{\mathbb{R}^N}|f(v_{n-})|^p\Big) = \frac{1}{2}\mathcal{Q}(v_{n-}).$$

Hence, $\mathcal{Q}(v_{n-}) \to 0$, and by Proposition 2.3, we can conclude that $v_{n-} \to 0$ in W_μ. Thereby, we get $v_n = v_n^+ + o_n(1)$. □

Lemma 3.5. *Let (v_n) be a $(PS)_c$ sequence for E_μ with $v_n \rightharpoonup 0$ in W_μ. Then only one of the following conditions holds:*

(*a*) $\mathcal{Q}(v_n) \to 0$; *or*

(*b*) *there exist a sequence $(y_n) \subset \mathbb{R}^N$ and positive constants R, ζ such that*

$$\liminf_{n\to\infty}\int_{B_R(y_n)}|v_n|^p \geq \zeta.$$

Proof. Suppose that (b) does not hold. Then, for all $R > 0$,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|v_n|^p = 0.$$

Once (v_n) is bounded in W_μ, by [24, Lemma I.1], we can assume that $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for any $s \in (p, p^*)$. Hence, from $(H_0)-(H_1)$

$$\int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n = o_n(1).$$

From this,

$$0 \leq \mathcal{Q}(v_n) \leq 2\left\langle E'_\mu(v_n), v_n\right\rangle + o_n(1) \to 0$$

which leads to $\mathcal{Q}(v_n) \to 0$ as $n \to \infty$. □

Now, we are able to prove the existence of a positive ground state solution for (A_μ).

Proposition 3.6. *Under the hypotheses $(H_0)-(H_3)$, problem (A_μ) has a positive ground state solution $v \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying $v(x) \to 0$ as $|x| \to \infty$.*

Proof. We start by recalling that E_μ has the mountain pass geometry, hence E_μ possesses a Palais-Smale sequence at the level c_μ; that is, there exists $(v_n) \subset W_\mu$ satisfying (see [38]) $E_\mu(v_n) \to c_\mu$ and $E'_\mu(v_n) \to 0$. By Lemma 3.4, up to a subsequence, $v_n \rightharpoonup v$ in W_μ with $E'_\mu(v) = 0$. Without loss of generality, we can suppose that $v \neq 0$. In fact, if $v = 0$, we derive that $\mathcal{Q}(v_n)$ does not converge to 0, because otherwise, the limit $\mathcal{Q}(v_n) \to 0$ gives

$$\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} \mu |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n \ \to 0,$$

from which it follows that

$$\int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n \to 0.$$

This limit combined with Lemma 2.1(6) and (H_2) yields

$$\int_{\mathbb{R}^N} H(f(v_n)) \ \to 0,$$

showing that the limit

$$E_\mu(v_n) = \frac{1}{p}\mathcal{Q}(v_n) - \int_{\mathbb{R}^N} H(f(v_n)) \ \to 0$$

holds, which is a contradiction, because $E_\mu(v_n) \to c_\mu > 0$. Hence, by Lemma 3.5, there exist $\zeta, R > 0$ and (y_n) in $\mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n)} |v_n|^p \geq \zeta.$$

Now, let $\widetilde{v}_n$ be defined by $\widetilde{v}_n(x) = v_n(x + y_n)$. As $(\widetilde{v}_n)$ is also a Palais-Smale sequence for E_μ, up to a subsequence, $\widetilde{v}_n \rightharpoonup \widetilde{v}$ with $E'_\mu(\widetilde{v}) = 0$. As $\widetilde{v}_n \to \widetilde{v}$ in $L^p(B_R)$, we obtain

$$\int_{B_R} |\widetilde{v}|^p \ = \lim_{n\to\infty} \int_{B_R} |\widetilde{v}_n|^p \ = \lim_{n\to\infty} \int_{B_R(y_n)} |v_n|^p \ \geq \zeta$$

which shows that $\widetilde{v} \neq 0$. Therefore, we can suppose that $v \neq 0$. By Lemma 2.1(6), for all n we have

$$f^2(v_n) - f(v_n) f'(v_n) v_n \geq 0$$

and consequently

$$|f(v_n)|^p - |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n \geq 0.$$

Furthermore, from the assumption (H_2), we conclude that

$$\frac{1}{p} h(f(v_n)) f'(v_n) v_n - H(f(v_n)) \geq \frac{1}{2p} h(f(v_n)) f(v_n) - H(f(v_n)) \geq 0 \ \ \forall n \in \mathbb{N}.$$

This combined with Fatou's lemma implies that

$$\begin{aligned} c_\mu &= \lim_{n\to\infty}\left[E_\mu(v_n) - \frac{1}{p}\langle E'_\mu(v_n), v_n\rangle\right] \\ &= \liminf_{n\to\infty}\frac{1}{p}\int_{\mathbb{R}^N}\mu\left[|f(v_n)|^p - |f(v_n)|^{p-2}f(v_n)f'(v_n)v_n\right] \\ &\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{p}h(f(v_n))f'(v_n)v_n - H(f(v_n))\right] \\ &\geq \frac{1}{p}\int_{\mathbb{R}^N}\mu\left[|f(v)|^p - |f(v)|^{p-2}f(v)f'(v)v\right] \\ &\quad + \int_{\mathbb{R}^N}\left[\frac{1}{p}h(f(v))f'(v)v - H(f(v))\right] \\ &= E_\mu(v) - \frac{1}{p}\langle E'_\mu(v), v\rangle = E_\mu(v). \end{aligned}$$

Therefore, $v \neq 0$ is a critical point of E_μ satisfying $E_\mu(v) \leq c_\mu$. Since $v \in \mathcal{M}_\mu$, by (3.3), we deduce that $E_\mu(v) = c_\mu$. In order to prove that v is nonnegative, we recall that for all $w \in W_\mu$

$$\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla w + \int_{\mathbb{R}^N}\mu|f(v)|^{p-2}f(v)f'(v)w = \int_{\mathbb{R}^N}h(f(v))f'(v)w\,. \tag{3.7}$$

Then, taking $w = -v_-$ as a test function, we get

$$0 \leq \mathcal{Q}(v_-) \leq \int_{\mathbb{R}^N}|\nabla v_-|^2 + \int_{\mathbb{R}^N}\mu|f(v_-)|^{p-2}f(v_-)f'(v_-))(v_-) = 0$$

which yields $v_- = 0$ in W_μ, and therefore $v \geq 0$. Next, we show that $v \in L^\infty(\mathbb{R}^N)$ and $v(x) \to 0$ as $|x| \to \infty$. To this end, for each $k > 0$ let us consider the functions

$$v_k = \begin{cases} v & \text{if } v \leq k \\ k & \text{if } v \geq k, \end{cases} \qquad \vartheta_k = v_k^{p(\sigma-1)}v, \quad \text{and} \quad w_k = vv_k^{\sigma-1}$$

with $\sigma > 1$ to be determined later. Taking ϑ_k as a test function in (3.7) and using the estimate

$$h(f(v)) \leq \frac{\mu}{2}f(v)^{p-1} + Cf(v)^q,$$

we obtain

$$\int_{\mathbb{R}^N}v_k^{p(\sigma-1)}|\nabla v|^p + p(\sigma-1)\int_{\mathbb{R}^N}v_k^{p(\sigma-1)-1}v\nabla v_k\nabla v \leq C\int_{\mathbb{R}^N}f(v)^q f'(v)vv_k^{p(\sigma-1)}.$$

Because the second summand in the left side of the inequality above is not negative, using again Lemma 2.1, we see that

$$\int_{\mathbb{R}^N} v_k^{p(\sigma-1)}|\nabla v|^p \le C\int_{\mathbb{R}^N} v^{(q+1)/2}v_k^{p(\sigma-1)} = C\int_{\mathbb{R}^N} v^{\widetilde{q}-p}w_k^p \tag{3.8}$$

where $\widetilde{q} \doteq (q+1)/2$. By the Gagliardo-Nirenberg-Sobolev inequality and (3.8), we obtain

$$\begin{aligned}\Big(\int_{\mathbb{R}^N} w_k^{p^*}\Big)^{p/p^*} &\le C_1\int_{\mathbb{R}^N} |\nabla w_k|^p \\ &\le C_2\int_{\mathbb{R}^N} v_k^{p(\sigma-1)}|\nabla v|^p\,dx + C_3(\sigma-1)^p\int_{\mathbb{R}^N} v^p v_k^{p(\sigma-2)}|\nabla v_k|^p \\ &\le C_4\sigma^p\int_{\mathbb{R}^N} v_k^{p(\sigma-1)}|\nabla v|^p \le C_5\sigma^p\int_{\mathbb{R}^N} v^{\widetilde{q}-p}w_k^p,\end{aligned}$$

where we have used the fact that $v_k \le v$, $1 \le \sigma^p$ and $(\sigma-1)^p \le \sigma^p$. Using the Hölder inequality we get

$$\Big(\int_{\mathbb{R}^N} w_k^{p^*}\Big)^{p/p^*} \le \sigma^p C_5\Big(\int_{\mathbb{R}^N} v^{p^*}\Big)^{(\widetilde{q}-p)/p^*}\Big(\int_{\mathbb{R}^N} w_k^{pp^*/(p^*-\widetilde{q}+p)}\Big)^{(p^*-\widetilde{q}+p)/p^*}.$$

Since $|w_k| \le |v|^\sigma$, by continuity of the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ we obtain

$$\Big(\int_{\mathbb{R}^N} |vv_k^{\sigma-1}|^{p^*}\Big)^{p/p^*} \le \sigma^p C_6\|v\|^{\widetilde{q}-p}\Big(\int_{\mathbb{R}^N} v^{\sigma pp^*/(p^*-\widetilde{q}+p)}\Big)^{(p^*-\widetilde{q}+p)/p^*}.$$

Choosing $\sigma = 1 + \frac{p^*-\widetilde{q}}{p}$ we have $\sigma pp^*/(p^*-\widetilde{q}+p) = p^*$. Thus,

$$\Big(\int_{\mathbb{R}^N} |vv_k^{\sigma-1}|^{p^*}\Big)^{p/p^*} \le \sigma^p C_6\|v\|^{\widetilde{q}-p}|v|_{\sigma\alpha^*}^{p\sigma},$$

where $\alpha^* = pp^*/(p^*-\widetilde{r}+p)$. By Fatou's lemma, we obtain

$$|v|_{\sigma p^*} \le (\sigma^p C_6\|v\|^{\widetilde{q}-p})^{1/p\sigma}|v|_{\sigma\alpha^*}. \tag{3.9}$$

For each $m = 0, 1, 2, ...$ let us define $\sigma_{m+1}\alpha^* \doteq p^*\sigma_m$ with $\sigma_0 \doteq \sigma$. From (3.9), it follows that

$$\begin{aligned}|v|_{\sigma_1 p^*} &\le (\sigma_1^p C_6\|v\|^{\widetilde{q}-p})^{1/p\sigma_1} v|_{\sigma_1\alpha^*} \\ &\le (\sigma_1^p C_6\|v\|^{\widetilde{q}-p})^{1/p\sigma_1}(\sigma^p C_6\|v\|^{r-p})^{1/p\sigma}|v|_{\sigma\alpha^*} \\ &\le (C_6\|v\|^{\widetilde{q}-p})^{1/p\sigma+1/p\sigma_1}(\sigma)^{1/\sigma}(\sigma_1)^{1/\sigma_1}|v|_{p*}.\end{aligned}$$

Observing that $\sigma_m = \chi^m\sigma$ where $\chi = \frac{p^*}{\alpha^*}$, by iteration we obtain

$$|v|_{\sigma_m p^*} \le (C_6\|v\|^{\widetilde{q}-p})^{1/p\sigma\sum_{i=0}^{|m}\chi^{-i}}\sigma^{1/\sigma\sum_{i=0}^{m}\chi^{-i}}\chi^{1/\sigma\sum_{|}i=0^m i\chi^{-i}}|v|_{p*}.$$

Since $\chi > 1$ and $\lim_{m\to\infty} \frac{1}{p\sigma}\sum_{i=0}^{m} \chi^{-i} = \frac{1}{p^*-\widetilde{q}}$, we can take the limit as $m \to \infty$ to conclude that $v \in L^\infty(\mathbb{R}^N)$ and

$$|v|_\infty \leq C_7 \|v\|^{(p^*-p)/(p^*-\widetilde{q})}.$$

As a consequence of a result due to Tolksdorf [36] (see also [14]), we deduce that $v \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ with $\alpha \in (0,1)$. Furthermore, by a Harnack type inequality (see [37, Theorem 1]) and an argument of connectedness, we must have $v > 0$ in $\mathbb{R}^N$. Moreover, since $v \in L^\infty(\mathbb{R}^N)$, by Lemma 2.1(6) and (3.7), we conclude that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \varphi \ \leq C \int_{\mathbb{R}^N} v\varphi$$

for all nonnegative functions $\varphi \in C_0^\infty(\mathbb{R}^N)$. Thus, by [37, Theorem 1.3], for any $x \in \mathbb{R}^N$ the estimate below holds:

$$\sup_{y\in B_1(x)} v(y) \leq C|v|_{L^p(B_2(x))}.$$

In particular, $v(x) \leq C|v|_{L^p(B_2(x))}$ for all $x \in \mathbb{R}^N$ and since $|v|_{L^p(B_2(x))} \to 0$ as $|x| \to \infty$, we can conclude that $v(x) \to 0$ as $|x| \to \infty$. $\square$

3.2. **The nonautonomous problem.** Hereafter, we will work with the quasilinear problem

$$\begin{cases} -\Delta_p u - \Delta_p(u^2)u + V(\epsilon x)|u|^{p-2}u = h(u) \ \text{ in } \ \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N) \ \text{ with } \ 2 \leq p < N, \\ u(x) > 0, \forall\ x \in \mathbb{R}^N \end{cases} \qquad (P_\epsilon^*),$$

which is equivalent to (P_ϵ) by the change of variables $\epsilon z = x$. The weak solutions of (P_ϵ^*) are the critical points of the functional

$$I_\epsilon(v) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon x)|f(v)|^p - \int_{\mathbb{R}^N} H(f(v))$$

which is well defined on the Banach space X_ϵ defined by

$$X_\epsilon = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)|f(v)|^p < \infty \right\}$$

endowed with the norm

$$\|u\|_\epsilon = |\nabla u|_p + \inf_{\xi>0} \left[\frac{1}{\xi} + \int_{\mathbb{R}^N} V(\epsilon x)|f(\xi v)|^p\right].$$

A straightforward computation shows that $I_\epsilon \in C^1(X_\epsilon, \mathbb{R})$ and

$$\left\langle I'_\epsilon(v), \phi \right\rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla v \nabla \phi + \int_{\mathbb{R}^N} V(\epsilon x)|f(v)|^{p-2} f(v) f'(v)\phi$$

$$- \int_{\mathbb{R}^N} h(f(v))f'(v)\phi$$

for all $\phi \in X_\epsilon$. From now on, let us denote the Nehari manifold associated to I_ϵ by

$$\mathcal{N}_\epsilon = \Big\{ v \in X_\epsilon \setminus \{0\} : \big\langle I'_\epsilon(v), v \big\rangle = 0 \Big\}.$$

It is easy to check that there exists $r > 0$ such that

$$\|u\|_\epsilon \geq r > 0, \ \text{ for all } \ u \in \mathcal{N}_\epsilon. \tag{3.10}$$

Arguing as in the preceding lemmas, it is routine to verify that I_ϵ exhibits the mountain pass geometry, hence, by [38], there is a $(PS)_{c_\epsilon}$ sequence $(v_n) \subset X_\epsilon$, which we can suppose nonnegative, such that $v_n \rightharpoonup v$ in X_ϵ, for some $v \in X_\epsilon$ and $I'_\epsilon(v) = 0$. Moreover,

$$c_\epsilon = \inf_{\mathcal{N}_\epsilon} I_\epsilon = \inf_{u \in X_\epsilon \setminus \{0\}} \max_{t \geq 0} I_\epsilon(tu)$$

and there exists a unique $t = t_v$ such that $I_\epsilon(t_v v) = \max\limits_{t \geq 0} I_\epsilon(tv)$.

In what follows, we will prove some important lemmas in order to establish a compactness condition.

Lemma 3.7. *Let (v_n) be a Palais-Smale sequence for I_ϵ in X_ϵ such that $v_n \rightharpoonup v$ in X_ϵ for some $v \in X_\epsilon$. Then*

$$I_\epsilon(\widetilde{v}_n) = I_\epsilon(v_n) - I_\epsilon(v) + o_n(1)$$

and

$$I'_\epsilon(\widetilde{v}_n) = o_n(1)$$

where $\widetilde{v}_n = v_n - v$.

Proof. Firstly, we observe that the limits below hold:

$$\int_{\mathbb{R}^N} H(f(\widetilde{v}_n)) = \int_{\mathbb{R}^N} H(f(v_n)) - \int_{\mathbb{R}^N} H(f(v)) + o_n(1), \tag{3.11}$$

$$\int_{\mathbb{R}^N} V(\epsilon x)|f(\widetilde{v}_n)|^p = \int_{\mathbb{R}^N} V(\epsilon x)|f(v_n)|^p - \int_{\mathbb{R}^N} V(\epsilon x)|f(v)|^p + o_n(1), \tag{3.12}$$

$$\begin{aligned} \int_{\mathbb{R}^N} V(\epsilon x)\Big| |f(\widetilde{v}_n)|^{p-2} f(\widetilde{v_n}) f'(\widetilde{v}_n) - V(\epsilon x)|f(v_n)|^{p-2} f(v_n) f'(v_n) \\ + V(\epsilon x)|f(v)|^{p-2} f(v) f'(v)\Big|^{\frac{p}{p-1}} = o_n(1), \end{aligned} \tag{3.13}$$

and

$$\int_{\mathbb{R}^N} \Big| h(f(\widetilde{v}_n)) f'(\widetilde{v}_n) - h(f(v_n)) f'(v_n) + h(f(v)) f'(v) \Big|^\alpha = o_n(1), \tag{3.14}$$

for some $\alpha \in (p, p^*)$.

We will show only the first limit, because the same arguments can be used in the proofs of the other ones. We begin by remarking that

$$H(f(\widetilde{v}_n + v)) - H(f(\widetilde{v}_n)) = \int_0^1 \frac{d}{dt} H(f(\widetilde{v}_n + tv))dt.$$

Thus

$$H(f(v_n)) - H(f(\widetilde{v}_n)) = \int_0^1 h(f(\widetilde{v}_n + tv))f'(\widetilde{v}_n + tv)vdt.$$

Now, (H_0) and (H_1) combined give

$$|H(f(v_n)) - H(f(\widetilde{v}_n))| \leq |\int_0^1 (|\widetilde{v}_n + t|v|^{p-1}|v| + C|\widetilde{v}_n + tv|^{p^*-1}|v|)dt.$$

From this, for each $\delta > 0$, there exists $C_\delta > 0$ such that

$$|H(f(v_n)) - H(f(\widetilde{v}_n))| \leq |\delta|\widetilde{v}_n|^p + C_\delta|v|^p + |\delta|\widetilde{v}_n|^{p^*} + C_\delta|v|^{p^*},$$

and thus

$$|H(f(v_n)) - H(f(\widetilde{v}_n)) - H(f(v))| \leq |\delta|\widetilde{v}_n|^p + \widehat{C}_\delta|v|^p + |\delta|\widetilde{v}_n|^{p^*} + \widehat{C}_\delta|v|^{p^*}$$

for some positive constant $\widehat{C}_\delta$. Now, repeating the same arguments found in [1, Lemma 3.1], it follows that

$$\int_{\mathbb{R}^N} |H(f(v_n)) - H(f(\widetilde{v}_n)) - H(f(v))| \to 0$$

or equivalently,

$$\int_{\mathbb{R}^N} H(f(\widetilde{v}_n)) = \int_{\mathbb{R}^N} H(f(v_n)) - \int_{\mathbb{R}^N} H(f(v)) + o_n(1).$$

On the other hand, by Brezis-Lieb [6],

$$\int_{\mathbb{R}^N} |\nabla \widetilde{v}_n|^p = \int_{\mathbb{R}^N} |\nabla v_n|^p - \int_{\mathbb{R}^N} |\nabla |v|^p + o_n(1). \tag{3.15}$$

Now, using (3.11)-(3.15), we deduce that

$$I_\epsilon(\widetilde{v}_n) = I_\epsilon(v_n) - I_\epsilon(v) + o_n(1)$$

and

$$\|I'_\epsilon(\widetilde{v}_n)\|_{X^*_\epsilon} = o_n(1)$$

which completes the proof. □

Lemma 3.8. *Assume that $V_\infty < \infty$ and let (v_n) be a $(PS)_d$ sequence for I_ϵ in X_ϵ with $v_n \rightharpoonup 0$ in X_ϵ. If $\mathcal{Q}(v_n) \not\to 0$ in $\mathbb{R}$, then $d \geq c_{V_\infty}$, where c_{V_∞} is the minimax level of E_{V_∞}.*

Proof. Let $(t_n) \subset (0, +\infty)$ be a sequence such that $(t_n v_n) \subset \mathcal{M}_\infty$. We start showing the following claim.

Claim 1. *The sequence* (t_n) *satisfies* $\limsup_{n\to\infty} t_n \leq 1$.

Suppose, to derive a contradiction, that the claim above does not hold. Then, there exist $\delta > 0$ and a subsequence of (t_n), still denoted by itself, such that

$$t_n \geq 1 + \delta \ \text{ for all } \ n \in \mathbb{N}. \tag{3.16}$$

Since (v_n) is bounded in X_ϵ, $\langle I'_\epsilon(v_n), v_n\rangle = o_n(1)$; that is,

$$\int_{\mathbb{R}^N} \Big[|\nabla v_n|^p + V(\epsilon x)|f(v_n)|^{p-2} f(v_n) f'(v_n)\Big] = \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n + o_n(1).$$

Moreover, recalling that $(t_n v_n) \subset \mathcal{M}_\infty$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big[t_n^p|\nabla v_n|^p + V_\infty|f(t_n v_n)|^{p-2} f(t_n v_n) f'(t_n v_n) t_n v_n\Big]\\ &= \int_{\mathbb{R}^N} h(f(t_n v_n)) f'(t_n v_n) t_n v_n.\end{aligned}$$

The last two equalities imply that

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big[\frac{h(f(t_n v_n)) f'(t_n v_n) t_n v_n}{t_n^p} - h(f(v_n)) f'(v_n) v_n\Big]\\ &= \int_{\mathbb{R}^N} \frac{V_\infty|f(t_n v_n)|^{p-2} f(t_n v_n) f'(t_n v_n) t_n v_n}{t_n^p}\\ &- \int_{\mathbb{R}^N} V(\epsilon x)] \mid f(v_n \mid^{p-2} f(v_n) f'(v_n) v_n + o_n(1)\end{aligned}$$

or

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big[\frac{h(f(t_n v_n))}{[f(t_n v_n)]^{2p-1}} \frac{[f(t_n v_n)]^{2p-1} f'(t_n v_n)}{(t_n v_n)^{p-1}} - \frac{h(f(v_n))}{[f(v_n)]^{2p-1}} \frac{[f(v_n)]^{2p-1} f'(v_n)}{[v_n]^{p-1}}\Big] v_n^p\\ &= \int_{\mathbb{R}^N} [V_\infty - V(\epsilon x)]|f(v_n)|^{p-1} f'(v_n) v_n\\ &+ \int_{\mathbb{R}^N} V_\infty\Big[\frac{\mid f(t_n v_n) \mid^{p-1} f'(t_n v_n)}{t_n^p} - |f(v_n)|^{p-1} f'(v_n)\Big] v_n + o_n(1).\end{aligned}$$

Given $\xi > 0$, from condition (V_1) there exists $R = R(\xi) > 0$ such that

$$V(\epsilon x) \geq V_\infty - \xi \ \text{ for any } \ |\epsilon x| \geq R.$$

This together with Corollary 2.2 and the boundedness of (v_n) in X_ϵ give

$$\int_{\mathbb{R}^N} \Big[\frac{h(f(t_n v_n))}{[f(t_n v_n)]^{2p-1}} \frac{[f(t_n v_n)]^{2p-1} f'(t_n v_n)}{(t_n v_n)^{p-1}} - \frac{h(f(v_n))}{[f(v_n)]^{2p-1}} \frac{[f(v_n)]^{2p-1} f'(v_n)}{[v_n]^{p-1}}\Big] v_n^p$$

$$\leq \xi C + o_n(1) \tag{3.17}$$

for some positive constant C. Since $\mathcal{Q}(v_n) \not\to 0$ in $\mathbb{R}$, we may invoke Lemma 3.5 to obtain $(y_n) \subset \mathbb{R}^N$ and $\widetilde{R}, \zeta > 0$ satisfying

$$\int_{|y_n| \leq \widetilde{R}} |v_n|^p \geq \zeta. \tag{3.18}$$

If we define $\widetilde{v}_n(x) = v_n(x + y_n)$, we may suppose that, up to a subsequence, $\widetilde{v}_n \rightharpoonup \widetilde{v}$ in X_ϵ. Moreover, in view of (3.18), there exists a subset $\Omega \subset \mathbb{R}^N$ with positive measure such that $\widetilde{v} > 0$ in Ω. From (H_3) and Corollary 2.2, we can use (3.16) to rewrite (3.17) as

$$\int_{\mathbb{R}^N} \Big[\frac{h(f((1+\delta)\widetilde{v}_n))}{[f((1+\delta)\widetilde{v}_n)]^{2p-1}} \frac{[f((1+\delta)\widetilde{v}_n)]^{2p-1} f'((1+\delta)\widetilde{v}_n)}{((1+\delta)\widetilde{v}_n)^{p-1}} - \frac{h(f(\widetilde{v}_n))}{[f(\widetilde{v}_n)]^{2p-1}} \frac{[f(\widetilde{v}_n)]^{2p-1} f'(\widetilde{v}_n)}{[\widetilde{v}_n]^{p-1}} \Big] \widetilde{v}_n^p \leq \xi C + o_n(1), \quad \text{for any} \quad \xi > 0.$$

Letting $n \to \infty$ in the last inequality and applying Fatou's lemma, it follows that

$$0 < \int_{\Omega} \Big[\frac{h(f((1+\delta)\widetilde{v}))}{[f((1+\delta)\widetilde{v})]^{2p-1}} \frac{[f((1+\delta)\widetilde{v})]^{2p-1} f'((1+\delta)\widetilde{v})}{((1+\delta)\widetilde{v})^{p-1}} - \frac{h(f(\widetilde{v}))}{[f(\widetilde{v})]^{2p-1}} \frac{[f(\widetilde{v})]^{2p-1} f'(\widetilde{v})}{[\widetilde{v}_n]^{p-1}} \Big] \widetilde{v}^p \leq \xi C, \quad \text{for any} \quad \xi > 0,$$

which is absurd, and the claim is proved. Now, we will consider two more cases.

Case 1: $\limsup_{n\to\infty} t_n = 1$. In this case, there exists a subsequence, still denoted by (t_n), such that $t_n \to 1$. Thus,

$$d + o_n(1) = I_\epsilon(v_n) \geq c_\infty + I_\epsilon(v_n) - E_\infty(t_n v_n). \tag{3.19}$$

On the other hand, using the boundedness of (v_n) in X_ϵ, the condition (V_1) and the definitions of I_ϵ and E_∞, it follows that

$$I_\epsilon(v_n) - E_{V_\infty}(t_n v_n) \geq o_n(1) - C\xi + \frac{1}{p} \int_{\mathbb{R}^N} V_\infty [|f(v_n)|^p - |f(t_n v_n)|^p] + \int_{\mathbb{R}^N} [H(f(t_n v_n)) - H(f(v_n))].$$

Moreover, from the mean value theorem,

$$\int_{\mathbb{R}^N} [H(f(t_n v_n)) - H(f(v_n))] = o_n(1),$$

$$\int_{\mathbb{R}^N} V_\infty[|f(v_n)|^p - |f(t_n v_n)|^p] = o_n(1).$$

Putting this information together into (3.19), we get

$$d + o_n(1) \geq c_\infty - C\xi + o_n(1).$$

Taking the limit as $n \to \infty$ in the last inequality, we reach

$$d \geq c_\infty - C\xi \quad \forall \xi > 0,$$

which implies $d \geq c_\infty$.

Case 2: $\limsup_{n\to\infty} t_n = t_0 < 1$. In the present case, we may suppose that there exists a subsequence, still denoted by (t_n), satisfying

$$t_n \to t_0 \text{ and } t_n < 1 \;\; \forall n \in \mathbb{N}.$$

Now, since $[f(t)]^p - [f(t)]^{p-1} f'(t)t$ and $\frac{1}{p}h(f(t))f'(t)t - H(f(t))$ are nonincreasing for $t \geq 0$, we derive that

$$c_\infty \leq I_\epsilon(v_n) - \frac{1}{p}\left\langle I'_\epsilon(v_n), v_n \right\rangle = d + o_n(1).$$

Taking the limit as $n \to \infty$ in the last inequality, it follows again that $d \geq c_\infty$. □

Proposition 3.9. *The functional I_ϵ satisfies the $(PS)_c$ condition at any level $c < c_\infty$ if $V_\infty < \infty$ and at any level $c \in \mathbb{R}$ if $V_\infty = \infty$.*

Proof. Let $(v_n) \subset X_\epsilon$ be such that $I_\epsilon(u_n) \to c$ and $I'_\epsilon(u_n) \to 0$. Arguing as in Lemma 3.4, we can see that (v_n) is bounded in X_ϵ. Thus there exists $v \in X_\epsilon$ such that, up to a subsequence, $v_n \rightharpoonup v$ in X_ϵ and $I'_\epsilon(v) = 0$.

Considering $\widetilde{v}_n = v_n - v$ and using Lemma 3.7, we have $I'_\epsilon(\widetilde{v}_n) \to 0$ and

$$I_\epsilon(\widetilde{v}_n) = I_\epsilon(v_n) - I_\epsilon(v) + o_n(1) = c - I_\epsilon(v) + o_n(1) = d + o_n(1).$$

By (H_2) and Lemma 2.1(6), $I_\epsilon(v) \geq 0$, thus if $V_\infty < \infty$, we obtain

$$d \leq c < c_\infty.$$

It follows from Lemma 3.8 that $\mathcal{Q}(\widetilde{v}_n) \to 0$, consequently, by Proposition 2.3, $\widetilde{v}_n \to 0$ in X_ϵ, or equivalently, $v_n \to v$ in X_ϵ. If $V_\infty = \infty$, it follows that V is coercive and by [13], the continuous Sobolev imbedding $X_\epsilon \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $2p \leq s < 2p^*$. Hence, up to a subsequence, $\widetilde{v}_n \to 0$ in $L^s(\mathbb{R}^N)$ and $\mathcal{Q}(\widetilde{v}_n) \to 0$. Thus, $\widetilde{v}_n \to 0$ in X_ϵ, or equivalently, $v_n \to v$ in X_ϵ, and the proof is complete. □

Proposition 3.10. *The functional I_ϵ restricted to $\mathcal{N}_\epsilon$ satisfies the $(PS)_c$ condition at any level $c < c_\infty$ if $V_\infty < \infty$ and at any level $c \in \mathbb{R}$ if $V_\infty = \infty$.*

Proof. Let $(v_n) \subset \mathcal{N}_\epsilon$ be such that $I_\epsilon(v_n) \to c$ and $\|I'_\epsilon(v_n)\|_* = o_n(1)$. Then there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$I'_\epsilon(v_n) = \lambda_n J'_\epsilon(v_n) + o_n(1), \tag{3.20}$$

where $J_\epsilon : X_\epsilon \to \mathbb{R}$ is given by

$$J_\epsilon(v) = \int_{\mathbb{R}^N} |\nabla v|^p + \int_{\mathbb{R}^N} V(\epsilon x)|f(v)|^{p-2} f(v) f'(v) v - \int_{\mathbb{R}^N} h(f(v)) f'(v) v.$$

Note that

$$\begin{aligned}
&\left\langle J'_\epsilon(v_n), v_n \right\rangle = p \int_{\mathbb{R}^N} |\nabla v_n|^p + (p-1) \int_{\mathbb{R}^N} V(\epsilon x)|f(v_n)|^{p-2}[f'(v_n)]^2[v_n]^2 \\
&+ \int_{\mathbb{R}^N} V(\epsilon x)|f(v_n)|^{p-1} f''(v_n)[v_n]^2 + \int_{\mathbb{R}^N} V(\epsilon x)|f(v_n)|^{p-1} f'(v_n) v_n \\
&- \int_{\mathbb{R}^N} h'(f(v_n))[f'(v_n)]^2[v_n]^2 - \int_{\mathbb{R}^N} h(f(v_n)) f''(v_n)[v_n]^2 \\
&- \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n,
\end{aligned}$$

consequently,

$$\begin{aligned}
\left\langle J'_\epsilon(v_n), v_n \right\rangle \leq &\int_{\mathbb{R}^N} V(\epsilon x)|f(v_n)|^{p-1} f''(v_n)[v_n]^2 + (p-1) \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n \\
&- \int_{\mathbb{R}^N} h'(f(v_n))[f'(v_n)]^2[v_n]^2 - \int_{\mathbb{R}^N} h(f(v_n)) f''(v_n)[v_n]^2.
\end{aligned}$$

From Lemma 2.2 (iii), we know that the function $h(f(t))f'(t)t^{1-p}$ is increasing for $t > 0$; this fact implies that

$$(p-1)h(f(t))f'(t) - h'(f(t))(f'(t))^2 t - h(f(t))f''(t)t \leq 0 \quad \text{for} \quad t \geq 0,$$

from which it follows that

$$\left\langle J'_\epsilon(v_n), v_n \right\rangle \leq \int_{\mathbb{R}^N} V(\epsilon x)|f(v_n)|^{p-1} f''(v_n)[v_n]^2.$$

Since $f''(t) = -2^{p-1}(f(t))^{p-1}(f'(t))^{p+2}$ for $t \geq 0$, the last inequality combined with (V_1) yields

$$\left\langle J'_\epsilon(v_n), v_n \right\rangle \leq -2^{p-1} V_0 \int_{\mathbb{R}^N} |[f(v_n)]^2 f'(v_n)|^p.$$

From this, we can assume that $\langle J'_\epsilon(v_n), v_n \rangle \to l \leq 0$. If $l = 0$, the inequality above yields

$$\int_{\mathbb{R}^N} |[f(v_n)]^2 f'(v_n)|^p \to 0. \tag{3.21}$$

On the other hand, from (H_0) and (H_1), for each $\delta > 0$ there exists $C_\delta > 0$ such that

$$0 \leq h(t) \leq \delta(t^{p-1} + t^q) + C_\delta t^p \ \text{ for } \ t \geq 0$$

and so

$$0 \leq h(f(v_n))f'(v_n)v_n \leq \delta(|v_n|^p + |v_n|^{q+1}) + C_\delta |v_n|^{p-1}[f(v_n)]^2 f'(v_n).$$

This together with boundedness of (v_n) gives

$$0 \leq \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \leq \delta C + \widehat{C}_\delta \Big(\int_{\mathbb{R}^N} |[f(v_n)]^2 f'(v_n)|^p \Big)^{\frac{1}{p}}.$$

Now, using (3.21) we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \leq \delta C, \ \ \forall \delta > 0,$$

showing that

$$\int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \to 0.$$

This limit yields $\mathcal{Q}(v_n) \to 0$, from which it follows that $v_n \to 0$ in X_ϵ, which contradicts (3.10). Thus, $l \neq 0$ and $\lambda_n = o_n(1)$. From (3.20), $I'_\epsilon(v_n) = o_n(1)$, and so, (v_n) is a $(PS)_c$ sequence for I_ϵ and the result follows from Proposition 3.9. □

Corollary 3.11. *The critical points of I_ϵ on $\mathcal{N}_\epsilon$ are critical points of I_ϵ in W_ϵ.*

Proof. The proof follows by using similar arguments explored in the last proposition. □

4. Existence of ground state solution

In this section, we prove the existence of a ground state solution for (P_ϵ^*), that is, the existence of a nontrivial critical point u_ϵ of I_ϵ satisfying $I_\epsilon(u_\epsilon) = c_\epsilon$. We adapt some ideas found in [3]. The main result in this section is the following.

Theorem 4.1. *Suppose that h satisfies $(H_0) - (H_3)$ and V satisfies (V_1). Then there exists $\bar{\epsilon} > 0$ such that (P_ϵ^*) has a ground state solution $u_\epsilon \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for all $0 < \epsilon < \bar{\epsilon}$.*

Proof. Arguing as in the proofs of Lemmas 3.1 and 3.2, the functional I_ϵ satisfies the mountain pass geometry. Then using a version of the mountain pass theorem without the Palais-Smale condition, there exists $(v_n) \subset X_\epsilon$ satisfying (see [38])

$$I_\epsilon(v_n) \to c_\epsilon \text{ and } I'_\epsilon(v_n) \to 0.$$

If $V_\infty = \infty$, it follows that

$$I_\epsilon(v) = c_\epsilon \text{ and } I'_\epsilon(v) = 0,$$

where $v \in X_\epsilon$ is the weak limit of (v_n) in X_ϵ. Using Proposition 3.9, [14], [20] and [36], we have $v \in L^\infty(\mathbb{R}^N)$ and $v \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ with $0 < \alpha < 1$. By Harnack's inequality [37], $v(x) > 0$ for all $x \in \mathbb{R}^N$. If $V_\infty < \infty$, consider without loss of generality that

$$V(0) = V_0 = \inf_{x \in \mathbb{R}^N} V(x).$$

Let $\mu \in \mathbb{R}$ such that $V_0 < \mu < V_\infty$. Since $c_{V_0} < c_\mu < c_\infty$, there is a non-negative function $w \in W^{1,p}(\mathbb{R}^N)$ with compact support such that $E_\mu(w) = \max_{t \geq 0} E_\mu(tw)$ and $E_\mu(w) < c_\infty$. The condition (V_1) implies that for some $\overline{\epsilon} > 0$

$$\int_{\mathbb{R}^N} V(\epsilon x)|f(w)|^p \leq \int_{\mathbb{R}^N} \mu |f(w)|^p \text{ for all } \epsilon \in (0, \overline{\epsilon})$$

so that

$$I_\epsilon(tw) \leq E_\mu(tw) \leq E_\mu(w) \text{ for all } t > 0.$$

Therefore

$$\max_{t>0} I_\epsilon(tw) \leq E_\mu(w),$$

and $c_\epsilon < c_\infty$ for all $\epsilon \in (0, \overline{\epsilon})$ and the theorem follows immediately from Proposition 3.9. □

5. MULTIPLICITY OF SOLUTIONS TO (P_{ϵ^*})

In this section, our main goal is to show the existence of multiple positive solutions and study the behavior of their maximum points in relation to the set M. The main result in this section has the following statement.

Theorem 5.1. *Assume that (V_1) and $(H_0) - (H_3)$ hold. Then given $\delta > 0$ there exists $\overline{\varepsilon} = \overline{\varepsilon}(\delta) > 0$ such that the equation (P^*_ϵ) has at least $cat_{M_\delta}(M)$ positive weak solutions in $C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover, each solution decays to zero at infinity and if u_ϵ denotes one of these positive solutions and $z_\epsilon \in \mathbb{R}^N$ its global maximum, then*

$$\lim_{\epsilon \to 0} V(\epsilon z_\epsilon) = V_0.$$

In order to prove the theorem above, in the next subsection we fix some notation and prove some preliminary lemmas.

5.1. **Preliminary results.** Let $\delta > 0$ be fixed and w be a ground state solution of problem (P_{V_0}). Let η be a smooth nonincreasing cut-off function defined in $[0, \infty)$ such that $\eta(s) = 1$ if $0 \leq s \leq \frac{\delta}{2}$ and $\eta(s) = 0$ if $s \geq \delta$.

For any $\epsilon > 0$ and $y \in M$, let us define

$$\Psi_{\epsilon,y}(x) = \eta(|\ \epsilon x - y\ |)w\Big(\frac{\epsilon x - y}{\epsilon}\Big),$$

$t_\epsilon > 0$ satisfying

$$\max_{t \geq 0} I_\epsilon(t\Psi_{\epsilon,y}) = I_\epsilon(t_\epsilon \Psi_{\epsilon,y})$$

and $\Phi_\epsilon : M \to \mathcal{N}_\epsilon$ by

$$\Phi_\epsilon(y) = t_\epsilon \Psi_{\epsilon,y}.$$

By construction, $\Phi_\epsilon(y)$ has compact support for any $y \in M$.

Lemma 5.2. *The function Φ_ϵ satisfies the following limit:*

$$\lim_{\epsilon \to 0} I_\epsilon(\Phi_\epsilon(y)) = c_{V_0}, \ uniformly\ in\ y \in M.$$

Proof. The proof is similar to that presented in [2]. □

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) = x$ for $|x| \leq \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \geq \rho$. Finally, let us consider $\beta : \mathcal{N}_\epsilon \to \mathbb{R}^N$ given by

$$\beta(u) = \frac{\displaystyle\int_{\mathbb{R}^N} \chi(\epsilon x)|u(x)|^p}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^p}.$$

Lemma 5.3. *The function Φ_ϵ satisfies the following limit:*

$$\lim_{\epsilon \to 0} \beta(\Phi_\epsilon(y)) = y, \quad uniformly \quad in \quad y \in M.$$

Proof. The proof follows by using similar arguments found in [2]. □

Lemma 5.4. (A compactness lemma) *Let $(v_n) \subset \mathcal{M}_\mu$ be a sequence satisfying $E_\mu(v_n) \to c_\mu$. Then*

a) *(v_n) has a subsequence strongly convergent in $W^{1,p}(\mathbb{R}^N)$ or*
b) *there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $\overline{v}_n(x) = u_n(x + \tilde{y}_n)$ converges strongly in $W^{1,p}(\mathbb{R}^N)$.*

In particular, there exists a minimizer for c_μ.

Proof. Applying Ekeland's variational principle (see Theorem 8.5 in [38]), we may suppose that (v_n) is a $(PS)_{c_\mu}$ for E_μ. Thus, going to a subsequence if necessary, we have that $v_n \rightharpoonup v$ weakly in $W^{1,p}(\mathbb{R}^N)$ and $E'_\mu(v) = 0$.

If $v \neq 0$, we can use the fact that

$$[f(t)]^p - [f(t)]^{p-1} f'(t)t \ \text{ and } \ \frac{1}{p} h(f(t)) f'(t)t - H(f(t))$$

are nonnegative functions for $t \geq 0$ together with Fatou's lemma to conclude that v is a ground state solution of the autonomous problem (A_μ); that is, $E_\mu(v) = c_\mu$.

If $v \equiv 0$, applying the same arguments employed in the proof of Lemma 3.5, there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that $\overline{v}_n \rightharpoonup \overline{v}$ in $W^{1,p}(\mathbb{R}^N)$, where $\overline{v}_n = v_n(x + y_n)$. Therefore, $\overline{v}_n$ is also a $(PS)_{c_\mu}$ sequence of E_μ and $\overline{v} \not\equiv 0$, and so $\overline{v}$ is a ground state solution of the autonomous problem (A_μ). □

Proposition 5.5. *Let $\epsilon_n \to 0$ and $(v_n) \subset \mathcal{N}_{\epsilon_n}$ be such that $I_{\epsilon_n}(v_n) \to c_{V_0}$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $\overline{v}_n(x) = v_n(x + \tilde{y}_n)$ has a convergent subsequence in $W^{1,p}(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n \to y \in M$, where $y_n = \epsilon_n \tilde{y}_n$.*

Proof. Arguing as in the proof of Lemma 3.5, we obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and positive constants R and ζ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^p \geq \zeta > 0.$$

Thus, if $\overline{v}_n(x) = v_n(x + \tilde{y}_n)$, up to a subsequence, $\overline{v}_n \rightharpoonup \overline{v} \not\equiv 0$ in $W^{1,p}(\mathbb{R}^N)$. Let $t_n > 0$ be such that $\tilde{v}_n = t_n \overline{v}_n \in \mathcal{M}_{V_0}$. Then,

$$E_{V_0}(\tilde{v}_n) \to c_{V_0} \ \text{ and } \ (\tilde{v}_n) \subset \mathcal{M}_{V_0}.$$

Since (t_n) is bounded, the sequence $(\tilde{v}_n)$ also is bounded and for some subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in $W^{1,p}(\mathbb{R}^N)$. Moreover, reasoning as in [2], to some subsequence still denoted by (t_n), we can also assume that $t_n \to t_0 > 0$, and this limit implies that $\tilde{v} \not\equiv 0$. Due to Lemma 5.4, $\tilde{v_n} \to \tilde{v}$ in $W^{1,p}(\mathbb{R}^N)$, and so, $\overline{v}_n \to \overline{v}$ in $W^{1,p}(\mathbb{R}^N)$.

Now, we will show that $(y_n) = (\epsilon_n \tilde{y}_n)$ has a subsequence satisfying $y_n \to y \in M$.

Claim 2. *The sequence (y_n) is bounded in $\mathbb{R}^N$.* Suppose the claim were false. Then, we could find a subsequence of (y_n), still denoted by itself, such that $|y_n| \to \infty$. Considering firstly the case $V_\infty = \infty$, the inequality

$$\int_{\mathbb{R}^N} V(\epsilon_n x + y_n)|f(\overline{v}_n)|^p \leq \int_{\mathbb{R}^N} |\nabla \overline{v}_n|^p + \int_{\mathbb{R}^N} V(\epsilon_n x + y_n)|f(\overline{v}_n)|^p$$

$$= \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n,$$

together with Fatou's lemma imply

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n = \infty$$

which is absurd, because the sequence $\{h(f(v_n))f'(v_n)v_n\}$ is bounded in $L^1(\mathbb{R}^N)$.

Now, let us consider the case $V_\infty < \infty$. Since $\widetilde{v}_n \to \widetilde{v}$ in $W^{1,p}(\mathbb{R}^N)$ and $V_0 < V_\infty$, we have

$$\begin{aligned} c_{V_0} &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla \widetilde{v}|^p + \frac{1}{p}\int_{\mathbb{R}^N} V_0 |f(\widetilde{v})|^p - \int_{\mathbb{R}^N} H(f(\widetilde{v})) \\ &< \frac{1}{p}\int_{\mathbb{R}^N} |\nabla \widetilde{v}|^p + \frac{1}{p}\int_{\mathbb{R}^N} V_\infty |f(\widetilde{v})|^p - \int_{\mathbb{R}^N} H(f(\overline{v})) \\ &\le \liminf_{n\to\infty} \Big[\frac{1}{p}\int_{\mathbb{R}^N} |\nabla \widetilde{v}_n|^p + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n x + y_n)|f(\widetilde{v}_n)|^p - \int_{\mathbb{R}^N} H(f(\widetilde{v}_n))\Big], \end{aligned}$$

or equivalently

$$c_{V_0} < \liminf_{n\to\infty} \Big[\frac{t_n^p}{p}\int_{\mathbb{R}^N} |\nabla \overline{v}_n|^p + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n z)|f(t_n\overline{v}_n)|^p - \int_{\mathbb{R}^N} H(f(t_n\overline{v}_n))\Big].$$

The last inequality implies

$$c_{V_0} < \liminf_{n\to\infty} I_{\epsilon_n}(t_n\overline{v}_n) \le \liminf_{n\to\infty} I_{\epsilon_n}(v_n) = c_{V_0}$$

which does not make sense. Hence, (y_n) is bounded and, up to a subsequence, $y_n \to y \in \mathbb{R}^N$. If $y \notin M$, then $V(y) > V_0$ and we obtain a contradiction arguing as above. Thus, $y \in M$ and the lemma is proved. □

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive function tending to 0 such that $g(\epsilon) \to 0$ as $\epsilon \to 0$ and let

$$\widetilde{\mathcal{N}}_\epsilon = \{v \in \mathcal{N}_\epsilon : I_\epsilon(u) \le c_{V_0} + g(\epsilon)\}.$$

Lemma 5.6. *Let $\delta > 0$ and $M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \le \delta\}$. Then*

$$\lim_{\epsilon\to 0} \sup_{v\in\widetilde{\mathcal{N}}_\epsilon} \inf_{y\in M_\delta} \Big|\beta(v) - y\Big| = 0.$$

Proof. Let $(\epsilon_n) \subset \mathbb{R}$ be such that $\epsilon_n \to 0$. For each $n \in \mathbb{N}$, there exists $(v_n) \subset \widetilde{\mathcal{N}}_\epsilon$ satisfying

$$\inf_{y\in M_\delta} \Big|\beta(v_n) - y\Big| = \sup_{v\in\widetilde{\mathcal{N}}_{\epsilon_n}} \inf_{y\in M_\delta} \Big|\beta(v) - y\Big| + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$\lim_{n\to\infty} \Big|\beta(v_n) - y_n\Big| \quad = 0. \tag{5.1}$$

In order to obtain this sequence, we note that $(v_n) \subset \widetilde{\mathcal{N}}_{\epsilon_n} \subset \mathcal{N}_{\epsilon_n}$, and thus,

$$c_{V_0} \le c_{\epsilon_n} \le I_{\epsilon_n}(v_n) \le c_{V_0} + g(\epsilon_n),$$

so that $I_{\epsilon_n}(v_n) \to c_{V_0}$ and $(v_n) \subset \mathcal{N}_{\epsilon_n}$. From Proposition 5.5, we get a sequence $(\tilde{y_n}) \subset \mathbb{R}^N$ such that $(y_n) = (\epsilon_n \tilde{y_n}) \subset M_\delta$, for n sufficiently large. Thus,

$$\beta(v_n) = y_n + \frac{\displaystyle\int_{\mathbb{R}^N} [\chi(\epsilon_n z + y_n) - y_n] |\overline{v}_n(z)|^p}{\displaystyle\int_{\mathbb{R}^N} |\overline{v}_n(z)|^p}$$

where $\overline{v}_n(x) = v_n(x + \tilde{y}_n)$. Recalling that $\overline{v}_n \to \overline{v}$ in $W^{1,p}(\mathbb{R}^N)$, it can be easily seen that the sequence (y_n) satisfies (5.1). □

The next two lemmas play a role in the study of the behavior of the maximum points of the solutions. In the proof of the next lemma, we adapted some arguments found in [20], which are related with the Moser iteration method [29].

Lemma 5.7. *Let v_n be a solution of the following problem:*

$$\begin{cases} -\Delta_p v_n + V_n(x)|f(v_n)|^{p-2} f(v_n) f'(v_n) = h(f(v_n)) f'(v_n) \quad in \quad \mathbb{R}^N \\ v_n \in W^{1,p}(\mathbb{R}^N) \ \ with \ \ 2 \le p < N, \\ v_n(x) > 0, \forall \ x \in \mathbb{R}^N, \end{cases}$$

where $V_n(x) = V(\epsilon_n x + \epsilon_n \tilde{y_n})$. Assuming that the conditions (V_1) and $(H_1)-(H_3)$ hold and that $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$ with $v \not\equiv 0$, then $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $|v_n|_\infty \le C$ for all $n \in \mathbb{N}$. Furthermore

$$\lim_{|x|\to\infty} v_n(x) = 0 \ \ \textit{uniformly in } n.$$

Proof. The proof is similar to that presented in [2]. □

Lemma 5.8. *There exists $\delta > 0$ such that $|v_n|_\infty \ge \delta \ \ \forall n \in \mathbb{N}$.*

Proof. Suppose that $|v_n|_\infty \to 0$. For fixed $\epsilon_0 = \frac{V_0}{4}$, it follows from (H_1) that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{h(f((v_n(x))))}{(f(v_n(x)))^{p-1}} < \epsilon_0, \ \text{ a.e. in } \mathbb{R}^N \ \text{ for } n \ge n_0.$$

Hence,

$$\int_{\mathbb{R}^N} |\nabla v_n|^p + \frac{V_0}{2}\int_{\mathbb{R}^N} |f(v_n)|^p \leq \int_{\mathbb{R}^N} \frac{h(f(v_n(x)))}{(f(v_n(x)))^{p-1}}(f(v_n(x)))^{p-1} f'(v_n)v_n$$
$$\leq \epsilon_0 \int_{\mathbb{R}^N} |f(v_n)|^p,$$

thus,

$$\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} |f(v_n)|^p = 0 \quad \text{for} \quad n \geq n_0.$$

From Proposition 2.3, we have $v_n = 0$ in $W^{1,p}(\mathbb{R}^N)$ for all $n \geq n_0$, which is impossible, because by hypothesis $v_n \neq 0$, for all $n \in \mathbb{N}$. Thus, there exists $\delta > 0$ such that $|v_n|_\infty \geq \delta$, for all $n \in \mathbb{N}$. □

Using the lemmas and propositions proved up to now, we can prove Theorem 5.1.

5.2. **Proof of Theorem 5.1.**

Proof. We will divide the proof into two parts.

Part I: Multiplicity of solutions. For $\epsilon > 0$ small enough, by Lemmas 5.2 and 5.6, we have $\beta \circ \Phi_\epsilon$ homotopic to the inclusion map $id : M \to M_\delta$ and this fact implies that

$$cat_{\widetilde{\mathcal{N}}_\epsilon}(\widetilde{\mathcal{N}}_\epsilon) \geq cat_{M_\delta}(M).$$

Since I_ϵ satisfies the $(PS)_c$ condition for $c \in (c_{V_0}, c_{V_0} + g(\epsilon))$, by the Lusternik-Schnirelman theory of critical points (see [17], [38]) we can conclude that I_ϵ has at least $cat_{M_\delta}(M)$ critical points on $\mathcal{N}_\epsilon$. Consequently, by Corollary 3.11, I_ϵ has at least $cat_{M_\delta}(M)$ critical points in W_ϵ.

Part II: The behavior of maximum points. If v_{ϵ_n} is a solution of problem (P_{ϵ_n}), then $\overline{v}_n(x) = v_{\epsilon_n}(x + \tilde{y_n})$ is a solution of problem

$$\begin{cases} -\Delta_p \overline{v}_n + V_n(x)|f(\overline{v}_n)|^{p-2} f(\overline{v}_n) f'(\overline{v}_n) = h(f(\overline{v}_n)) f'(\overline{v}_n) \ \text{ in } \ \mathbb{R}^N \\ \overline{v}_n \in W^{1,p}(\mathbb{R}^N) \ \text{ with } \ 2 \leq p < N, \\ \overline{v}_n(x) > 0, \forall\, x \in \mathbb{R}^N, \end{cases}$$

with $V_n(x) = V(\epsilon_n x + \epsilon_n \tilde{y_n})$ and $(\tilde{y_n}) \subset \mathbb{R}^N$ given in Proposition 5.5. Moreover, up to a subsequence, $\overline{v}_n \to \overline{v}$ in $W^{1,p}(\mathbb{R}^N)$ and $y_n \to y$ in M, where $y_n = \epsilon_n \tilde{y_n}$. Considering p_n the global maximum of $\overline{v}_n$, by Lemmas 5.7 and 5.8, we have $p_n \in B_R(0)$ for some $R > 0$. Thus, the global maximum of v_{ϵ_n} is $z_\epsilon = p_n + \tilde{y_n}$ and therefore

$$\epsilon_n z_{\epsilon_n} = \epsilon_n p_n + \epsilon_n \tilde{y_n} = \epsilon_n p_n + y_n$$

Since (p_n) is bounded, we have

$$\lim_{n\to\infty} V(\epsilon_n z_{\epsilon_n}) = V_0. \qquad \square$$

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