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GENERALIZED EIGENVALUES FOR FULLY NONLINEAR SINGULAR OR DEGENERATE OPERATORS IN THE RADIAL CASE

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Abstract. In this paper we extend some existence results concerning generalized eigenvalues for fully nonlinear operators, singular or degenerate. We consider the radial case and we prove the existence of an infinite number of eigenvalues, simple and isolated. This completes the results obtained by the author with Isabeau Birindelli for the first eigenvalues in the radial case, and the results obtained for the Pucci's operator by Busca Esteban and Quaas and for the p-Laplace operator by Del Pino and Manasevich.

1. Introduction

The extension of the concept of eigenvalue for fully nonlinear operators has seen a remarkable development in these last years, let us mention the works of Quaas, Sirakov [30], Ishii, Yoshimura [23], Juutinen [24], Patrizi [26], Armstong [1], and previous papers of the author with Isabeau Birindelli [3, 4] which all deal with the existence of eigenvalues and corresponding eigenfunctions for different fully-nonlinear operators in bounded domains.

In [3] we defined the concept of first eigenvalue on the model of [2] and we proved some existence results for Dirichlet problems, and for the eigenvalue problem.

The simplicity of the first eigenvalue which is known in the case of the p-Laplacian, for Pucci's operators, and for operators related but homogeneous of degree 1, remains an open problem for general operators fully nonlinear singular or degenerate homogeneous of degree $1+\alpha$ with $\alpha > -1$. However, in [7] we proved some uniqueness results in the case where the domain is a ball or an annulus and when the operator is radial.

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Concerning the "other eigenvalues," little is known about them, except for the Pucci's operators and, for the p-Laplacian, in the radial case.

More precisely in [18] the authors prove that in the radial case for the p-Laplace operator, there exists an infinite denumerable set of eigenvalues, which are simple and isolated; in [11] the authors prove the same result for Pucci's operators. Moreover, in each of these papers, the authors establish some bifurcation results of positive (respectively negative) solutions for some related partial differential equations.

Here we consider also the radial case for the model operator

$$F(Du, D^2u) = |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2u)$$

where a and A are two positive numbers, $a \leq A$, $\alpha > -1$ and $\mathcal{M}_{a,A}$ is the Pucci's operator $\mathcal{M}_{a,A}(M) = Atr(M^+) - atr(M^-)$.

We prove the existence of a denumerable set of eigenvalues $(\mu_k)_k$ which are simple and isolated, and some continuity results for the eigenvalues with respect to the parameters α, a, A.

2. Assumptions, notation and previous results in the general case

We begin with some generalities about the operators that we consider.

Let Ω be some bounded domain in $\mathbb{R}^N$. Let S be the space of symmetric matrices on $\mathbb{R}^N$. For $\alpha > -1$, F_α satisfies:

(H1) $F_\alpha : \Omega \times \mathbb{R}^N \setminus \{0\} \times S \to \mathbb{R}$, is continuous and for all $t \in \mathbb{R}^\star$, $\mu \geq 0$, for any $x \in \Omega$, $p \in \mathbb{R}^N \setminus \{0\}$, $X \in S$, $F_\alpha(x, tp, \mu X) = |t|^\alpha \mu F_\alpha(x, p, X)$;

(H2) there exist $0 \leq a \leq A$, such that for any $x \in \Omega$, $p \in \mathbb{R}^N \setminus \{0\}$, $M \in S$, $N \in S$, $N \geq 0$

$$a|p|^\alpha tr(N) \leq F(x, p, M + N) - F(x, p, M) \leq A|p|^\alpha tr(N); \tag{2.1}$$

(H3) there exists a continuous function ω with $\omega(0) = 0$, such that if $(X, Y) \in S^2$ and $\zeta \in \mathbb{R}^+$ satisfy

$$-\zeta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 4\zeta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the identity matrix in $\mathbb{R}^N$, then for all $(x, y) \in \mathbb{R}^N$, $x \neq y$

$$F(x, \zeta(x - y), X) - F(y, \zeta(x - y), -Y) \leq \omega(\zeta|x - y|^2).$$

Let us now recall the definition of viscosity solutions.

Definition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ and suppose that f is continuous on $\Omega \times \mathbb{R}$; then v, continuous in Ω, is called a viscosity super* *solution (respectively sub-solution) of* $F(x,\nabla u, D^2u) = f(x,u)$ *if for all* $x_0 \in \Omega$*:*

-Either there exists an open ball $B(x_0,\delta)$*,* $\delta > 0$ *in* Ω *on which* $v = cte = c$ *and* $0 \le f(x,c)$*, for all* $x \in B(x_0,\delta)$ *(respectively* $0 \ge f(x,c)$*).*

-Or for all $\varphi \in \mathcal{C}^2(\Omega)$*, such that* $v - \varphi$ *has a local minimum on* x_0 *(respectively a local maximum) and* $\nabla\varphi(x_0) \neq 0$*, one has*

$$F(x_0,\nabla\varphi(x_0), D^2\varphi(x_0)) \le f(x_0, v(x_0))$$

(respectively

$$F(x_0,\nabla\varphi(x_0), D^2\varphi(x_0)) \ge f(x_0, v(x_0))).$$

One can also extend the definition of viscosity solution to upper semicontinuous sub-solutions and lower semicontinuous super solutions, as is done in the paper of Ishii [19].

We shall consider in the sequel radial solutions, which will be solutions of differential equations of order two. These solutions will be $\mathcal{C}^1$ everywhere and $\mathcal{C}^2$ on each point where their gradient is not zero, so it is easy to see that these solutions are viscosity solutions.

We now recall the definition of the first eigenvalue and first eigenfunction adapted to this context, on the model of [2].

We define

$$\lambda^+(\Omega) = \sup\{\lambda : \exists\ \varphi > 0,\ F(x,\nabla\varphi, D^2\varphi) + \lambda\varphi^{1+\alpha} \le 0 \text{ in } \Omega\},$$

$$\lambda^-(\Omega) = \sup\{\lambda : \exists\ \varphi < 0,\ F(x,\nabla\varphi, D^2\varphi) + \lambda|\varphi|^{\alpha}\varphi \ge 0 \text{ in } \Omega\}.$$

Remark 2.2. Let us observe that in this definition, for λ^+ (respectively λ^-), the supremum can be taken over either continuous and bounded functions, or lower semicontinuous and bounded functions (respectively continuous and bounded functions, or upper semicontinuous and bounded).

We proved in [3] the following existence result of "eigenfunctions."

Theorem 2.3. *Suppose that* Ω *is a bounded regular domain. There exists* $\varphi \ge 0$ *such that*

$$\begin{cases} F(x,\nabla\varphi, D^2\varphi) + \lambda^+(\Omega)\varphi^{1+\alpha} = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\varphi > 0$ *inside* Ω *is bounded and continuous.*

Symmetrically there exists $\varphi \le 0$ *such that*

$$\begin{cases} F(x,\nabla\varphi, D^2\varphi) + \lambda^-(\Omega)|\varphi|^{\alpha}\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\varphi < 0$ *inside* Ω *is bounded and continuous.*

These eigenvalues have the following properties, called maximum and minimum principles.

Theorem 2.4. *Suppose that Ω is a bounded regular domain. If $\lambda < \lambda^+$, every upper semicontinuous and bounded sub-solution of*

$$F(x, \nabla u, D^2 u) + \lambda |u|^\alpha u \geq 0$$

which is ≤ 0 on the boundary is ≤ 0 inside Ω. If $\lambda < \lambda^-$, every lower semicontinuous and bounded super-solution of

$$F(x, \nabla u, D^2 u) + \lambda |u|^\alpha u \leq 0$$

which is ≥ 0 on the boundary is ≥ 0 inside Ω.

The maximum and minimum principle and an iterative process permits us to prove the existence of solutions for the Dirichlet problem

$$\begin{cases} F(x, \nabla u, D^2 u) + \lambda |u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is supposed to be continuous and bounded, and $\lambda < \inf(\lambda^+, \lambda^-)$. Moreover, if $f \leq 0$ and $\lambda < \lambda^+$, (respectively $f \geq 0$ and $\lambda < \lambda^-$), there exists a nonnegative (respectively nonpositive) solution.

We now give an increasing property of the eigenvalues $\lambda^\pm$ with respect to the domain.

Proposition 2.5. *Suppose that Ω and Ω' are some regular bounded domains such that $\Omega' \subset\subset \Omega$. Then $\lambda^\pm(\Omega') > \lambda^\pm(\Omega)$.*

For the convenience of the reader we give a short proof here; we do it for λ^+. Let φ be an eigenfunction for $\lambda^+(\Omega)$. Then by the strict maximum principle there exists $\epsilon > 0$ such that $\varphi \geq 2\epsilon$ on Ω'. Define $\lambda' = \lambda^+(\Omega) \inf_{\Omega'} \frac{\varphi^{1+\alpha}}{(\varphi - \epsilon)^{1+\alpha}} > \lambda^+(\Omega)$. Then the function $\varphi - \epsilon$ is some positive function which satisfies in Ω'

$$F(x, \nabla(\varphi - \epsilon), \nabla\nabla(\varphi - \epsilon)) + \lambda'(\varphi - \epsilon)^{1+\alpha} \leq 0$$

which implies by the definition of $\lambda^+(\Omega')$ that $\lambda^+(\Omega) < \lambda' \leq \lambda^+(\Omega')$.

The following property of eigenvalues will be needed in Section 4.

Proposition 2.6. *Suppose that there exists $\mu \in \mathbb{R}$, and u continuous and bounded such that*

$$\begin{cases} F(x, \nabla u, D^2 u) + \mu |u|^\alpha u = 0 & , \; u \geq 0, \; u \not\equiv 0 \text{ in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\mu = \lambda^+$. Symmetrically suppose that there exists $\mu \in \mathbb{R}$, and u continuous and bounded such that

$$\begin{cases} F(x, \nabla u, D^2 u) + \mu |u|^\alpha u = 0, & u \le 0,\ u \not\equiv 0 \text{ in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\mu = \lambda^-$.

Proof of Proposition 2.6. We consider only the first case; the other can be treated in the same manner.

By the strict maximum principle, [3], [10], $u > 0$ in Ω. By the definition of the first eigenvalue, $\mu \le \lambda^+$. If $\mu < \lambda^+$, then the minimum principle would imply that $u \le 0$ in Ω, a contradiction.

We now recall some regularity and compactness results which will be used in the last section (see [5]).

Proposition 2.7. *Suppose that Ω is a bounded regular domain. Suppose that F satisfies the previous assumptions. Let f be a continuous and bounded function in Ω. Let u be a continuous and bounded viscosity solution of*

$$\begin{cases} F(x, \nabla u, D^2 u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Then for any $\gamma < 1$ there exists some constant C, which depends only on $|f|_\infty$, γ, a, A, and N, such that for any $(x, y) \in \bar{\Omega}^2$

$$|u(x) - u(y)| \le C|x - y|^\gamma.$$

Corollary 2.8. *Suppose that Ω is a bounded regular domain. Suppose that F satisfies the previous assumptions. Suppose that (f_n) is a sequence of continuous and uniformly bounded functions, and (u_n) is a sequence of continuous and bounded viscosity solutions of*

$$\begin{cases} F(x, \nabla u_n,\ D^2 u_n) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the sequence (u_n) is relatively compact in $\mathcal{C}(\overline{\Omega})$. Moreover, if f_n converges, even simply, to some continuous and bounded function f, and if for a subsequence $\sigma(n)$, $u_{\sigma(n)} \to u$, then u is a solution of the equation with the right-hand side f.

(See [5]).

Remark 2.9. Under some additional assumptions on the regularity of F, one has some Lipschitz regularity of the solutions. This assumption is satisfied in the case of the operator considered in the following sections.

We end this section by giving some properties of the first demi-eigenvalues for some particular operators related to Pucci's operators: Let $0 < a < A$ and the Pucci's operator be

$$\mathcal{M}_{a,A}(D^2u) = Atr((D^2u)^+) - atr((D^2u)^-),$$

where $(D^2u)^{\pm}$ denote the positive and negative part of the symmetric matrix D^2u. For $\alpha > -1$ the operator

$$F(\nabla u, D^2u) = |\nabla u|^\alpha \mathcal{M}_{a,A}(D^2u)$$

satisfies the assumption $(H1), (H2)$. We denote by $\lambda^{\pm}_{a,A,\alpha}$ its corresponding first eigenvalues. Then we have the following.

Proposition 2.10. *If $a < A$, one has $\lambda^+_{a,A,\alpha}(\Omega) < \lambda^-_{a,A,\alpha}(\Omega)$. Moreover, if λ_{eq} is the first eigenvalue for the operator $|\nabla u|^\alpha \Delta u$,*

$$\lambda^+_{a,A,\alpha} \le a\lambda_{eq} < A\lambda_{eq} \le \lambda^-_{a,A,\alpha}.$$

Proof of Proposition 2.10. Let $\phi > 0$ be some eigenfunction for the eigenvalue $\lambda^+_{a,A,\alpha}(\Omega)$. We observe that

$$a\Delta\phi \le Atr(D^2\phi)^+ - atr(D^2\phi)^- = \mathcal{M}_{a,A}(D^2\phi).$$

This implies that

$$a\Delta\phi|\nabla\phi|^\alpha + \lambda^+_{a,A,\alpha}|\phi|^\alpha\phi \le 0$$

and then, by the definition of λ_{eq}, $a\lambda_{eq} \ge \lambda^+_{a,A,\alpha}$.

In the same manner let $\phi < 0$ be such that $\Delta\phi|\nabla\phi|^\alpha = -\lambda_{eq}|\phi|^\alpha\phi$, then

$$|\nabla\phi|^\alpha \left(Atr((D^2\phi)^+) - atr((D^2\phi)^-)\right) \ge |\nabla\phi|^\alpha A\Delta\phi = -A\lambda_{eq}|\phi|^\alpha\phi$$

and by the definition of $\lambda^-_{a,A,\alpha}$ this implies that

$$A\lambda_{eq} \le \lambda^-_{a,A,\alpha}.$$

The question of the simplicity of the first eigenvalues for general operators satisfying $(H1)$,.. $(H3)$, is an open problem. The difficulty resides in the fact that one cannot establish some strict comparison principle. More precisely we should need the following result:

If $u \ge v$ and $F(x, \nabla u, D^2u) = f \le F(x, \nabla v, D^2v) = g$, then either $u > v$ everywhere, or $u \equiv v$.

The difficulty when one wants to prove this result resides on the points where test functions have their gradient equal to zero.

However, we proved in [7] the simplicity result in the radial case. It will be precised in the forthcoming section; this will be an argument for the existence and the properties of the other eigenvalues in the case of the operator $|\nabla u|^\alpha \mathcal{M}_{a,A}(D^2u)$.

3. The radial case

Let Ω be a ball $B(0,1)$ or an annulus $B(0,1)\setminus\overline{B(0,\rho)}$ for some $\rho\in(0,1)$.

We suppose that there exists $\tilde F$ such that, for any radial function $u(x)=g(|x|)$, $F(x,\nabla u,D^2u)=\tilde F(r,g',g'')$. In that case the conditions on F imply that

$$|g'|^\alpha\Big(\gamma_1 g''+\frac{\gamma_2(N-1)}{|x|}g'\Big)\le F(x,\nabla\phi,D^2\phi)\le |g'|^\alpha\Big(\Gamma_1 g''+\frac{\Gamma_2(N-1)}{|x|}g'\Big),$$

where

$$\gamma_1=\begin{cases} a & \text{if } g''>0\\ A & \text{if } g''<0\end{cases},\quad \gamma_2=\begin{cases} a & \text{if } g'>0\\ A & \text{if } g'<0,\end{cases}$$

$$\Gamma_1=\begin{cases} A & \text{if } g''>0\\ a & \text{if } g''<0\end{cases},\quad \Gamma_2=\begin{cases} A & \text{if } g'>0\\ a & \text{if } g'<0.\end{cases}$$

In this situation one can define the first radial eigenvalues $\lambda^\pm_{rad}(\Omega)$

$$\lambda^+_{rad}(\Omega)=\sup\{\lambda:\exists\varphi>0,\ \text{radial},\ \tilde F(r,\varphi',\varphi'')+\lambda\varphi^{1+\alpha}\le 0 \text{ in } \Omega\},$$

$$\lambda^-_{rad}(\Omega)=\sup\{\lambda:\exists\varphi<0,\ \text{radial},\ \tilde F(r,\varphi',\varphi'')+\lambda|\varphi|^\alpha\varphi\ge 0 \text{ in } \Omega\}.$$

Acting as in the general case, one can prove the existence of eigenfunctions for each of these eigenvalues, and using the maximum and minimum principle one derives that $\lambda^\pm_{rad}(\Omega)=\lambda^\pm(\Omega)$ in the sense given in theorem 2.3 for the operator $F(x,\nabla u,D^2u)$.

Remark 3.1. In the case of the ball, for any constant sign viscosity solution of

$$\begin{cases} \tilde F(r,u',u'')+\lambda^\pm|u|^\alpha u=0 & \text{in } B(0,1)\\ u=0 & \text{on } \{r=1\},\end{cases}$$

u is decreasing from $r=0$ for λ^+ and increasing from $r=0$ for λ^-. In particular if u is $\mathcal{C}^1$, 0 is the unique point where u' is zero.

In the case of an annulus $B(0,1)\setminus\overline{B(0,\rho)}$, if u is a positive (respectively negative) viscosity solution of

$$\begin{cases} \tilde F(r,u',u'')+\lambda^\pm|u|^\alpha u=0 & \text{in } B(0,1)\setminus\overline{B(0,\rho)}\\ u=0 & \text{on } \{r=1\} \text{ and } \{r=\rho\},\end{cases}\tag{3.1}$$

then there exists a unique point $r=r_u$ such that u is increasing (respectively decreasing) on $[\rho,r_u]$, and decreasing (respectively increasing) on $[r_u,1]$. In particular if u is $\mathcal{C}^1$, r_u is the unique point where u' is zero.

The uniqueness result obtained in [7] is the following.

Proposition 3.2. *Suppose that Ω is a ball or an annulus. Suppose that φ and ψ are two positive radial eigenfunctions in the viscosity sense, for the eigenvalue λ^+, which are zero on the boundary; then there exists some positive constant c such that $\varphi = c\psi$.*

Remark 3.3. Of course the same result holds for the negative eigenfunctions corresponding to λ^-.

From now on we shall denote by an abuse of notation by $\mathcal{M}_{a,A}(r, g', g")$ the operator $g \mapsto \Gamma_1 g'' + \frac{\Gamma_2(N-1)}{r} g'$ and $\tilde{F}$ will be

$$\tilde{F}(r, g', g'') = |g'|^\alpha \Big(\Gamma_1 g'' + \frac{\Gamma_2(N-1)}{r} g'\Big), \tag{3.2}$$

where Γ_1 and Γ_2 are the multivalued functions defined at the beginning of Section 3.

Remark 3.4. We shall most of the time use more correctly the definition which is valid when g is Lipshitz, and when Γ_1 and Γ_2 are determined:

$$\tilde{F}(r, g', g'') = \Gamma_1 \frac{d}{dr}\Big(\frac{|g'|^\alpha g'}{1+\alpha}\Big) + \Gamma_2 \frac{(N-1)}{r} |g'|^\alpha g',$$

the derivative $\frac{d}{dr}(\frac{|g'|^\alpha g')}{1+\alpha})$ being taken in the distributional sense.

We end this section by giving one consequence of the Hopf principle in the case of the operator $\tilde{F}$.

Remark 3.5. Suppose that u is a nonnegative solution in the viscosity sense of $\tilde{F}(r, u', u'') = f$ on $[0, R)$ for some $R \leq \infty$, with f continuous and nonpositive; then either $u > 0$ everywhere, or $u \equiv 0$. In particular if u satisfies $\tilde{F}(r, u', u'') = -\lambda |u|^\alpha u$ with $\lambda > 0$, and if $u(r_o) = 0$ then u must change sign on r_o.

4. The functions w^+ and w^-.

In this section we prove the existence and uniqueness of some radial solutions of

$$\begin{cases} |w'|^\alpha \mathcal{M}_{a,A}(r, w', w'') = -|w|^\alpha w & \text{in } \mathbb{R}^+, \\ w(0) = 1, w'(0) = 0. \end{cases}$$

This will permit us as in [11], [18] to prove the existence of an infinite numerable set of radial eigenvalues for the operator $|\nabla w|^\alpha \mathcal{M}_{a,A}(D^2 w)$ in the ball.

Proposition 4.1. *There exists a unique $\mathcal{C}^1$ solution of the equation*

$$|w'|^\alpha(\mathcal{M}_{a,A}(r,w',w'')) = -|w|^\alpha w \text{ in } \mathbb{R}^+, w(0)=1, w'(0)=0. \tag{4.1}$$

Moreover, w is $\mathcal{C}^2$ around each point where $w' \neq 0$.

This proposition will be a consequence of the following three results.

Proposition 4.2. *For all $r_o \geq 0$, and for all $k_o \neq 0$ there exists some $\delta > 0$ such that there is existence and uniqueness of a solution to*

$$a\Big(|k'|^\alpha k'\Big(\frac{N-1}{r}\Big) + \frac{d}{dr}(\frac{|k'|^\alpha k'}{1+\alpha})\Big) = -|k|^\alpha k$$
$$\text{for } r \in (r_o, r_o+\delta),\ \text{ or } r \in (r_o-\delta, r_o)\cap\mathbb{R}^+,\ k(r_o)=k_o, k'(r_o)=0, \tag{4.2}$$

$$A|k'|^\alpha k'\Big(\frac{N-1}{r}\Big) + a\frac{d}{dr}(\frac{|k'|^\alpha k'}{1+\alpha}) = -|k|^\alpha k$$
$$\text{for } r \in (r_o, r_o+\delta), \text{or } r \in (r_o-\delta, r_o)\cap\mathbb{R}^+,\ k(r_o)=k_o, k'(r_o)=0, \tag{4.3}$$

$$A\Big(|k'|^\alpha k'\Big(\frac{N-1}{r}\Big) + \frac{d}{dr}(\frac{|k'|^\alpha k'}{1+\alpha})\Big) = -|k|^\alpha k$$
$$\text{for } r \in (r_o, r_o+\delta),\ \text{ or } r \in (r_o-\delta, r_o)\cap\mathbb{R}^+,\ k(r_o)=k_o, k'(r_o)=0, \tag{4.4}$$

$$a|k'|^\alpha k'\Big(\frac{N-1}{r}\Big) + A\frac{d}{dr}\Big(\frac{|k'|^\alpha k'}{1+\alpha}\Big) = -|k|^\alpha k$$
$$\text{for } \ r \in (r_o, r_o+\delta),\ \text{ or } r \in (r_o-\delta, r_o)\cap\mathbb{R}^+\ k(r_o)=k_o, k'(r_o)=0. \tag{4.5}$$

Moreover, k is $\mathcal{C}^2$ around each point where $k' \neq 0$.

In a second step we shall prove the following existence and uniqueness result.

Proposition 4.3. *If $w'_o \neq 0$, for all w_o, there exists a local unique solution to*

$$\mathcal{M}_{a,A}(r,w',w'') = -\frac{|w|^\alpha w}{|w'|^\alpha}, \quad (w(r_o), w'(r_o)) = (w_o, w'_o).$$

Moreover, if on $(r_1, r_2) \subset (0,\infty)$ w is a maximal solution,

$$\lim_{r\to r_i, r\in(r_1,r_2)} w'(r) = 0,$$

w is $\mathcal{C}^2$ on (r_1,r_2), $\frac{d}{dr}(|w'|^\alpha w'(r))$ exists everywhere on (r_1,r_2) and

$$\frac{d}{dr}(|w'|^\alpha w')(r_1^+)w(r_1^+) < 0, \quad \textit{and} \quad \frac{d}{dr}(|w'|^\alpha w')(r_2^-)w(r_2^-) < 0.$$

Proposition 4.4. *Let δ be such that, on $\mathcal{C}([0,\delta])$, k in (4.2) with $r_o = 0$ and $k_o = 1$ is well defined and $|k-1|_{\mathcal{C}([0,\delta])} < \frac{1}{2}$. Then there exists some constant c_1 depending on a, A, N such that $|k'| \le c_1$. Moreover, there exists $r_1 > 0$ which depends only on a, A, and N such that k' and k'' are < 0 on $(0, r_1)$.*

Remark 4.5. The analogous result holds for the situations in (4.3), (4.4), (4.5).

We postpone the proof of these three propositions, and we conclude the local existence and uniqueness result, arguing as follows.

Let $r_o = 0$, k be the solution of (4.2) with $k_o = 1$, and, according to Proposition 4.4, let r_1 be such that on $(0, r_1]$, k' and k'' are negative. Let w be the solution given by Proposition 4.3 of

$$\mathcal{M}_{a,A}(r, w', w'') = -\frac{|w|^\alpha w}{|w'|^\alpha} \text{ in } \mathbb{R}^+, w(r_1) = k(r_1), w'(r_1) = k'(r_1) \neq 0 \quad (4.6)$$

on some neighborhood $(r_1 - \delta_1, r_1)$. By the equation one must have $w''(r_1) < 0$. Then by uniqueness $w = k$ on $(r_1 - \delta_1, r_1)$. We can continue replacing r_1 by $r_1 - \delta_1$ and finally obtain that $w = k$ on the left of r_1 as long as $w' \neq 0$, i.e., until 0. So we have obtained the existence and uniqueness of a solution on a neighborhood on the right of zero.

We can extend the solution on the right of r_1. If $w'(r) \neq 0$ for all $r \ge r_1$, the result is given by Proposition 4.3. Suppose now that $r_o \ge r_1$ is the first point after r_1 such that $w'(r_o) = 0$. By Remark 3.5 in Section 3, $w(r_o)$ cannot be zero. If $w(r_o) < 0$, anticipating the behaviour of the possible solutions on the right of r_o, we know by using the conclusion in Proposition 4.3 that one must have $\lim_{r \to r_o, r > r_o} \frac{d}{dr}(|w'|^\alpha w'(r)) > 0$, so the equation to solve on the right of r_o is (4.4), and we get a local solution on the right of r_o. The situation $w(r_o) > 0$ cannot occur, since this would imply that $\lim_{r \to r_o, r > r_o} \frac{d}{dr}(|w'|^\alpha w'(r)) < 0$ and w' could not be ≤ 0 on the left of r_o and $= 0$ on r_o.

Proof of Proposition 4.2. We prove the result for equation (4.2), with $k_o = 1$ and $r_o = 0$; the changes to bring in the other cases are given shortly at the end of the proof.

The equation can also be written as

$$\begin{cases} \dfrac{d}{dr}(r^{(N-1)(1+\alpha)}|k'|^\alpha k')(r) = -\dfrac{(\alpha+1) r^{(N-1)(1+\alpha)}|k|^\alpha k(r)}{a} \quad \text{in } \mathbb{R}^+ \\ k(0) = 1, \quad k'(0) = 0, \end{cases} \quad (4.7)$$

or equivalently, defining $\phi_{p'}(u) = |u|^{p'-2}u$ and $p' = \frac{\alpha+2}{\alpha+1}$, as

$$k(r) = 1 - \int_0^r \phi_{p'}\Big(\frac{\alpha+1}{as^{(N-1)(1+\alpha)}}\int_0^s t^{(N-1)(1+\alpha)}|k|^\alpha k(t)\ dt\Big)ds. \tag{4.7}$$

We use the properties of the operator T defined by

$$T(k)(r) = 1 - \int_0^r \phi_{p'}\Big(\frac{\alpha+1}{as^{(N-1)(1+\alpha)}}\int_0^s t^{(N-1)(1+\alpha)}|k|^\alpha k(t)\ dt\Big)ds \tag{4.8}$$

which satisfies on $[0,\delta]$

$$\begin{aligned}||T(k)-1||_\infty &\le \delta\Big|\phi_{p'}\Big(\frac{(\alpha+1)\delta||k||_\infty^{\alpha+1}}{a((N-1)(1+\alpha)+1)}\Big)\Big|\\ &\le c_1\delta^{p'}||k||_\infty \le c_1\delta^{p'}(||k-1||_\infty+1),\end{aligned}$$

where $c_1 = \Big(\frac{(\alpha+1)}{a((N-1)(1+\alpha)+1)}\Big)^{p'-1}$.

If $\delta < \big(\frac{1}{3^{|\alpha|+1}c_1}\big)^{\frac{1}{p'}}$, T sends the ball $\{u \in \mathcal{C}([0,\delta]) : ||u-1||_{\mathcal{C}([0,\delta])} \le \frac{1}{2}\}$ into itself. We now prove that it is contracting. We observe that for k with values in $[\frac{1}{2}, \frac{3}{2}]$

$$\begin{aligned}\frac{(\alpha+1)}{a((N-1)(1+\alpha)+1)}\Big(\frac{1}{2}\Big)^{\alpha+1}s &\le \frac{\alpha+1}{as^{(N-1)(1+\alpha)}}\int_0^s t^{(N-1)(1+\alpha)}|k|^\alpha k(t)\ dt\\ &\le \frac{(\alpha+1)}{a((N-1)(1+\alpha)+1)}\Big(\frac{3}{2}\Big)^{\alpha+1}s,\end{aligned}$$

and then by the mean value theorem for $(u,v) \in B_{\mathcal{C}([0,\delta])}(1,\frac{1}{2})$

$$\begin{aligned}&\Big|\phi_p'\Big((\frac{\alpha+1}{as^{(N-1)(1+\alpha)}}\int_0^s t^{(N-1)(1+\alpha)}u^{1+\alpha}(t)dt\Big)\\ &\qquad - \phi_{p'}\Big((\frac{\alpha+1}{as^{(N-1)(1+\alpha)}}\int_0^s t^{(N-1)(1+\alpha)}v^{1+\alpha}(t)dt\Big)\Big|\\ &\le c_1 s^{p'-1}|u^{\alpha+1}-v^{\alpha+1}|_{L^\infty([0,s])}\sup\Big((\frac{3}{2})^{-\alpha},(\frac{1}{2})^{-\alpha}\Big)\\ &\le c_1 s^{p'-1}|u-v|_{L^\infty([0,s])}\sup\left(\left(\frac{3}{2}\right)^{-\alpha},\left(\frac{1}{2}\right)^{-\alpha}\right)\sup\left(\left(\frac{3}{2}\right)^{\alpha},\left(\frac{1}{2}\right)^{\alpha}\right)\\ &\le c_1 s^{p'-1}|u-v|_{L^\infty([0,s])}3^{|\alpha|}.\end{aligned}$$

This implies that

$$|T(u)-T(v)| \le c_1\frac{\delta^{p'}}{p'}|u-v|_{L^\infty([0,\delta])}3^{|\alpha|} \le \frac{1}{3}|u-v|_{L^\infty([0,\delta])}.$$

Then the fixed-point theorem implies that there exists a unique fixed point in $\mathcal{C}([0,\delta])$.

In the case of equation (4.3) one is led to consider

$$T(k)(r) = k_o - \int_{r_o}^{r} \phi_{p'}\left(\frac{\alpha+1}{as^{N^+}}\int_{r_o}^{s} t^{N^+}|k|^{\alpha}k(t)\,dt\right)ds$$

with $N^+ = \frac{(N-1)(1+\alpha)A}{a}$.

For equation (4.5) we shall consider

$$T(k)(r) = k_o - \int_{r_o}^{r} \phi_{p'}\left(\frac{\alpha+1}{As^{N^-}}\int_{r_o}^{s} t^{N^-}|k|^{\alpha}k(t)\,dt\right)ds$$

with $N^- = \frac{(N-1)(1+\alpha)a}{A}$. Finally for equation (4.4)

$$T(k)(r) = k_o - \int_{r_o}^{r} \phi_{p'}\left(\frac{\alpha+1}{As^{(N-1)(1+\alpha)}}\int_{r_o}^{s} t^{(N-1)(1+\alpha)}|k|^{\alpha}k(t)\,dt\right)ds.$$

Proof of Proposition 4.3. We prove the local existence by proving that, for each (w_o, w_o') with $w_o' \neq 0$ and for all $r_o > 0$, there exists a neighborhood around r_o and a solution to the equation which satisfies the condition $(w(r_o), w'(r_o)) = (w_o, w_o')$. We suppose that $w_o' \neq 0$ and we introduce the function

$$f_2(r, y_1, y_2) = M\Big(-\frac{m(y_2)(N-1)}{r} - \frac{|y_1|^{\alpha}y_1}{|y_2|^{\alpha}}\Big), \tag{4.9}$$

where M and m are respectively the functions

$$M(x) = \begin{cases} \frac{x}{A} & \text{if } x > 0 \\ \frac{x}{a} & \text{if } x < 0 \end{cases} \tag{4.10}$$

and

$$m(x) = \begin{cases} Ax & \text{if } x > 0 \\ ax & \text{if } x < 0. \end{cases} \tag{4.11}$$

The functions M and m are Lipschitzian, hence f_2 is Lipschitzian with respect to $y = (y_1, y_2)$ around (w_o, w_o') when $w_o' \neq 0$. Let $f_1(r, y_1, y_2) = y_2$, and $f(y_1, y_2) = (f_1(y_1, y_2), f_2(y_1, y_2))$; then the standard theory of ordinary differential equations implies that

$$(y_1', y_2') = f(y_1, y_2), \qquad (y_1, y_2)(r_o) = (w_o, w_o')$$

has a unique solution around (w_o, w_o') when $w_o' \neq 0$. Then $w = y_1$ is a local solution of

$$w'' = M\Big(-\frac{m(w')(N-1)}{r} - \frac{|w|^{\alpha}w}{|w'|^{\alpha}}\Big) \tag{4.12}$$

with the initial condition $w(r_o) = w_o$, $w'(r_o) = w'_o$.

If w is a solution on (r_1, r_2) and $\lim_{r\to r_2, r<r_2} w'(r)$ exists and is $\neq 0$, w'' has also a finite limit from the equation, thus $\lim_{r\to r_2, r<r_2}(y'_1, y'_2)(r)$ exists and is finite and one can continue, replacing r_o by r_2 and (w_o, w'_o) by $(w(r_2), \lim_{r\to r_2, r<r_2} w'(r))$. If $\lim_{r\to r_2, r<r_2} w'(r))$ is zero, one gets at least when $\alpha > 0$, $\lim_{r\to r_2} w''(r) = \pm\infty$, so that one cannot get a continuation, since the solutions of $(y'_1, y'_2) = f(y_1, y_2)$ must be $\mathcal{C}^1$.

We prove the last facts concerning $\frac{d}{dr}(|w'|^\alpha w')$. Suppose that $w(r_2) > 0$ and assume by contradiction that $\lim_{r\to r_2} \frac{d}{dr}(|w'|^\alpha w'(r)) \geq 0$. Then the equation on the left of r_2 is, since it is clear from the equation that w' cannot be nonnegative,

$$A\frac{d}{dr}\Big(\frac{|w'|^\alpha w'}{1+\alpha}\Big) + \frac{a(N-1)}{r}|w'|^\alpha w' = -|w|^\alpha w$$

which yields a contradiction when $r \to r_2$.

Suppose now that $w(r_2) < 0$ and $\lim_{r\to r_2} \frac{d}{dr}(|w'|^\alpha w'(r)) \leq 0$; then from the equation w' cannot be ≥ 0 on the left of r_2, and one is led to solve on the left of r_2:

$$a\frac{d}{dr}\Big(\frac{|w'|^\alpha w'}{1+\alpha}\Big) + \frac{A(N-1)}{r}|w'|^\alpha w' = -|w|^\alpha w.$$

This is absurd by passing to the limit when $r \to r_2$. Suppose that $w(r_1) > 0$ and assume by contradiction that $\lim_{r\to r_1, r>r_1} \frac{d}{dr}(|w'|^\alpha w')(r) \geq 0$; by the equation w' cannot be ≥ 0 thus this equation is on the right of r_1 :

$$A\frac{d}{dr}\Big(\frac{|w'|^\alpha w'}{1+\alpha}\Big) + \frac{a(N-1)}{r}|w'|^\alpha w' = -|w|^\alpha w.$$

This is absurd by passing to the limit when $r \to r_1$.

Suppose that $w(r_1) < 0$ and that $\lim_{r\to r_1, r>r_1} \frac{d}{dr}(|w'|^\alpha w')(r) \leq 0$; then the equation on the right of r_1 is

$$a\frac{d}{dr}\Big(\frac{|w'|^\alpha w'}{1+\alpha}\Big) + \frac{A(N-1)}{r}|w'|^\alpha w' = -|w|^\alpha w.$$

This is absurd letting r go to r_1.

Proof of Proposition 4.4. We can observe that $|k'|^\alpha k'$ is differentiable for $r > 0$ and has a limit < 0 for $r \to 0$. Moreover we shall give some constant δ_1 which depends only on a, A, α, N such that $k' \neq 0$ and $\frac{d}{dr}\,(|k'|^\alpha k')$ remains < 0 on $(0, \delta_1)$.

We begin to prove that $\frac{d}{dr}(|k'|^\alpha k') < 0$ around zero. One has for $r > 0$

$$(|k'|^\alpha k')(r) = -\frac{1+\alpha}{ar^{N_o}} \int_0^r (|k|^\alpha k)(s) s^{N_o} ds,$$

where $N_o = (N-1)(1+\alpha)$, and thus $(|k'|^\alpha k')$ is continuously differentiable for $r \neq 0$, as the primitive of some continuous function, and

$$\frac{d}{dr}(|k'|^\alpha k')(r) = \frac{N_o(1+\alpha)}{ar^{N_o+1}} \int_0^r (|k|^\alpha k)(s) s^{N_o} ds - \frac{1+\alpha}{a}(|k|^\alpha k)(r).$$

For the point 0, one has

$$\lim_{r\to 0} \frac{|k'|^\alpha k'(r)}{r} = -\lim_{r\to 0} \frac{1+\alpha}{ar^{N_o+1}} \int_0^r |k|^\alpha k(s) s^{N_o} ds = -\frac{1+\alpha}{a(N_o+1)} < 0.$$

Using the fact that k tends to 1 when r goes to zero we get that

$$\lim_{r\to 0} \frac{d}{dr}(|k'|^\alpha k')(r) = \frac{1+\alpha}{a}(\frac{N_o}{N_o+1} - 1) = -\frac{(1+\alpha)}{A(N_o+1)} < 0$$

and then $|k'|^\alpha k'$ is $\mathcal{C}^1$ on 0.

Moreover we prove that there exists a neighborhood on the right of zero which depends only on the data, such that $\frac{d}{dr}(|k'|^\alpha k') < 0$ on it. For that aim we begin to establish some Lipschitz estimate on the solution with some constant which depends only on the data.

We have chosen δ (which depends only on a, A, α, and N) such that, for $r \in [0, \delta]$, $k(r) \in [\frac{1}{2}, \frac{3}{2}]$. We now observe that k' is then bounded by

$$|k'|^{\alpha+1}(r) \leq \frac{1+\alpha}{a(N_o+1)} \left(\frac{3}{2}\right)^{\alpha+1} r.$$

We have obtained the fact that there exists some constant c_2 which depends only on the constant a, N, A such that $|k'| \leq c_2$ on $(0, \delta)$. We derive from this that on $[0, \delta]$ $|k(r) - 1| \leq c_2 r$, and also that

$$|(|k|^\alpha k)(r) - 1| \leq (1+\alpha) \sup \left(\left(\frac{3}{2}\right)^\alpha, \left(\frac{1}{2}\right)^\alpha \right) c_2 r = c_3 r,$$

and thus

$$\begin{aligned}
&\Big| \frac{d}{dr}(|k'|^\alpha k')(r) + \frac{1+\alpha}{a(N_o+1)} \Big| \\
&\leq \frac{(1+\alpha)N_o}{ar^{N_o+1}} \int_0^r ||k|^\alpha k - 1|(s) s^{N_o} ds + \frac{1+\alpha}{a} |(|k|^\alpha k)(r) - 1| \\
&\leq \frac{c_3(1+\alpha)r}{a} \Big(\frac{N_o}{N_o+2} + 1 \Big).
\end{aligned}$$

We have obtained the fact that as long as $r < \frac{N_o+2}{2(N_o+1)^2 c_3} \equiv r_1$, $\frac{d}{dr}(|k'|^\alpha k')$ remains negative (and thus so does k'). This ends the proof of Proposition 4.4.

To finish the proof of Proposition 4.1, i.e., to prove global existence result, suppose that w is a solution on $[0, r_1)$. If $w'(r_1) \neq 0$ we use Proposition 4.3, if $w'(r_1) = 0$, using $\lim_{r\to r_1,\ r>r_1} \frac{d}{dr}(|w'|^\alpha w')(r) w(r_1) < 0$ we consider on the right of r_1, equation (4.2) if $w(r_1) > 0$, and equation (4.4) if $w(r_1) < 0$. We have obtained a solution on $\mathbb{R}^+$.

We now prove that the solution w is oscillatory.

Proposition 4.6. *The solution of* (4.1) *is oscillatory; i.e., for all* $r > 0$ *there exists* $\tau > r$ *such that* $w(\tau) = 0$.

Proof of Proposition 4.6. First step. We suppose that $a = A$. We follow the arguments in [18], but this process is classical in ODE oscillation theory, [21].

We assume by contradiction that there exists r_o such that w does not vanish on $[r_o, \infty)$. Then one can consider the function

$$y(r) = r^{(N-1)(1+\alpha)} \frac{|w'|^\alpha w'(r)}{|w|^\alpha w(r)},$$

which satisfies the equation

$$y'(r) = -\frac{(\alpha+1) r^{(N-1)(1+\alpha)}}{a} - \frac{(\alpha+1)|y|^{\alpha+2}(r)}{r^{(N-1)(1+\alpha)^2}}.$$

Integrating between r_0 and t one gets that

$$y(t) + (\alpha+1)\int_{r_0}^{t} \frac{|y|^{\alpha+2}(r)}{r^{(N-1)(1+\alpha)^2}} dr$$
$$= -\frac{(\alpha+1) t^{(N-1)(1+\alpha)+1}}{a\left((N-1)(1+\alpha)+1\right)} + y(r_0) + \frac{(\alpha+1) r_o^{(N-1)(1+\alpha)+1}}{a\left((N-1)(1+\alpha)+1\right)}.$$

In particular we obtain that $y(t) \leq 0$ for t large enough.

For the next step it will be useful to remark that if, in place of the equation, we had the inequality

$$\frac{d}{dr}\left(r^{(N-1)(1+\alpha)}|w'|^\alpha w'(r)\right) \leq \frac{-r^{(N-1)(1+\alpha)}|w|^\alpha w}{a},$$

the conclusion would be the same.

We obtain that $-y(t) = |y(t)| \geq Ct^{(N-1)(1+\alpha)+1}$ for some constant $C > 0$, as soon as t is large enough. Let

$$k(t) = \int_{r_0}^{t} \frac{|y|^{\alpha+2}(r)}{r^{(N-1)(1+\alpha)^2}} dr;$$

then using the previous considerations $k(t) \geq c_1 t^{N(1+\alpha)+2}$ for some positive constant c_1.

Coming back to the equation, always for t large,

$$(\alpha+1)k(t) \leq |y(t)| = (k'(t)t^{(N-1)(1+\alpha)^2})^{\frac{1}{\alpha+2}}$$

and thus

$$(1+\alpha)^{\alpha+2} k^{\alpha+2}(t) \leq k'(t)t^{(N-1)(1+\alpha)^2}.$$

Integrating between t and s, $s > t$, we obtain the fact that for some positive constant c_2

$$\frac{1}{k^{\alpha+1}(t)} - \frac{1}{k^{\alpha+1}(s)} \geq c_2\Big(\frac{1}{t^{(N-1)(1+\alpha)^2-1}} - \frac{1}{s^{(N-1)(1+\alpha)^2-1}}\Big).$$

Letting s go to infinity,

$$\frac{1}{k^{\alpha+1}}(t) \geq c_2 \frac{1}{t^{(N-1)(1+\alpha)^2-1}}.$$

From this one gets a contradiction with $k(t) \geq c_1 t^{N(1+\alpha)+2}$. This ends the proof of the first step.

Second step: $a < A$. We argue along the model of [11]. We suppose as in the first step that there exists r_o such that w does not vanish on $[r_o, \infty)$. We begin by proving that, if $w > 0$ for $r \geq r_o$, then for $r \geq r_o$

$$\frac{d}{dr}\left(r^{(N-1)(1+\alpha)}|w'|^\alpha w'(r)\right) \leq \frac{-r^{(N-1)(1+\alpha)}|w|^\alpha w(r)}{a},$$

and then following the previous arguments in the first step we obtain that if $y(r) = r^{(N-1)(1+\alpha)}\frac{|w'|^\alpha w'(r)}{|w|^\alpha w(r)}$, then

$$\begin{aligned} &y(t) + (\alpha+1)\int_{r_o}^{t} \frac{|y|^{\alpha+2}(r)}{r^{(N-1)(1+\alpha)^2}} dr \\ &\leq -\frac{t^{(N-1)(1+\alpha)+1}}{a\left((N-1)(1+\alpha)+1\right)} + y(r_o) + \frac{r_o^{(N-1)(1+\alpha)+1}}{a\left((N-1)(1+\alpha)+1\right)}, \end{aligned}$$

a contradiction if $y > 0$ for t large enough.

To prove that

$$\frac{d}{dr}\left(r^{(N-1)(1+\alpha)}|w'|^\alpha w'(r)\right) \leq \frac{-r^{N-1)(1+\alpha)}|w|^\alpha w(r)}{a},$$

let us note that in the case $w' \leq 0$ and $\frac{d}{dr}(\frac{|w'|^\alpha w'}{1+\alpha}) \leq 0$ equality holds in the previous inequality; if $w' \geq 0$ and $\frac{d}{dr}(\frac{|w'|^\alpha w'}{1+\alpha}) \geq 0$ the equation is impossible. For the other cases, we assume first that $w' \leq 0$; this implies if $\frac{d}{dr}\left(\frac{|w'|^\alpha w'}{1+\alpha}\right) \geq 0$ that

$$\begin{aligned} &a\frac{d}{dr}\Big(\frac{|w'|^\alpha w'}{1+\alpha}\Big) + \frac{a(N-1)}{r}|w'|^\alpha w' \\ &\leq \Big(A\frac{d}{dr}(\frac{|w'|^\alpha w'}{1+\alpha}) + \frac{a(N-1)}{r}|w'|^\alpha w'\Big) = -|w|^\alpha w, \end{aligned}$$

which implies the result.

If $w' \geq 0$ and $\frac{d}{dr}(\frac{|w'|^\alpha w'}{1+\alpha}) \leq 0$

$$\begin{aligned} &a\frac{d}{dr}\Big(\frac{|w'|^\alpha w'}{1+\alpha}\Big) + \frac{a(N-1)}{r}|w'|^\alpha w' \\ &\leq a\frac{d}{dr}(\frac{|w'|^\alpha w'}{1+\alpha}) + \frac{A(N-1)}{r}|w'|^\alpha w' = -|w|^\alpha w. \end{aligned}$$

This also implies the result.

We now assume that $w < 0$ on $[r_o, \infty)$. Then we prove that there exists $r^\star$ such that $w'(r^\star) = 0$ and $w' > 0$ on $(r^\star, \infty)$. Indeed, by the equation, if $w'(r^\star) = 0$, by Proposition 4.3

$$\lim_{r \to r^\star, r > r^\star} \frac{d}{dr}(|w'|^\alpha w')(r) > 0.$$

This implies that w' is increasing on $r^\star$, thus w' is > 0 on a neighborhood on the right of $r^\star$. Moreover, if there exists $r' > r^\star$ such that $w'(r') = 0$, we argue as before and then $w' > 0$ after r'.

From these remarks, it is sufficient to discard $w' < 0$ on $[r_o, \infty)$. Then in that case necessarily $\frac{d}{dr}(|w'|^\alpha w') > 0$ on $[r_o, \infty)$ by the equation, and then w satisfies

$$\frac{d}{dr}(|w'|^\alpha w'(r) r^{N^-}) = -(1+\alpha)\frac{r^{N^-}|w|^\alpha w(r)}{a} > 0.$$

Let $g(r) \equiv (|w'|^\alpha w'(r) r^{N^-})$; g is monotone increasing, and since $w' < 0$, it has a limit $c_1 \leq 0$ at $+\infty$. On the other hand, since $w' < 0$ there exists $c_2 \in [-\infty, 0)$ such that $\lim_{r\to+\infty} w(r) = c_2$, thus from the equation satisfied by w, $\lim_{r\to+\infty} g'(r) = +\infty$, which is a contradiction with $\lim_{r\to+\infty} g(r) = c_1 \leq 0$. Finally, $w' > 0$ after r_o.

We recall that $N^+ = \frac{A(1+\alpha)(N-1)}{a}$. Distinguishing the cases $\frac{d}{dr}(|w'|^\alpha w') > 0$ and $\frac{d}{dr}(|w'|^\alpha w') < 0$ on the right of r_o and arguing as we already did before we obtain that w satisfies

$$\frac{d}{dr}(|w'|^\alpha w'(r)r^{N^+}) \leq -\frac{(1+\alpha)r^{N^+}|w|^\alpha w(r)}{a}.$$

Then defining

$$y(r) = r^{N^+}\frac{|w'|^\alpha w'(r)}{|w|^\alpha w(r)}$$

one has

$$y'(t) + \frac{(\alpha+1)|y(r)|^{\alpha+2}}{r^{(N^+)(\alpha+1)}} + \frac{(\alpha+1)r^{N^+}}{a} \leq 0. \tag{4.13}$$

Hence integrating between r_o and t one gets for some constant $c_1 > 0$

$$|y(t)| = -y(t) \geq c_1 t^{N^+ + 1}.$$

Let

$$k(t) = \int_{r_o}^t \frac{|y|^{\alpha+2}(r)}{r^{N^+(\alpha+1)}}dr \geq ct^{N^+ + \alpha + 3}.$$

From the equation (4.13) integrated between r_o and t, using

$$k'(t) = \frac{|y|^{\alpha+2}(t)}{t^{N^+(\alpha+1)}},$$

we get

$$(\alpha+1)^{\alpha+2}k^{\alpha+2}(t) \leq k'(t)t^{N^+(\alpha+1)},$$

hence, for some positive constant c_2

$$k^{-(\alpha+1)}(t) - k^{-(\alpha+1)}(s) \geq c_2(t^{-N^+(\alpha+1)+1} - s^{-N^+(\alpha+1)+1})$$

for $s > t$. Letting s go to infinity and using $\lim k(t) = +\infty$, one derives that $k^{-(\alpha+1)}(t) \geq c_2 t^{-N^+(\alpha+1)+1}$, which is a contradiction with $k(t) \geq c_1 t^{N^+ + \alpha + 3}$. We have obtained that w is oscillatory. This ends the proof of Proposition 4.6.

For the sake of completeness, we give some properties of the function w inherited from the property of the eigenfunctions in the viscosity sense [7].

Lemma 4.7. *Between two successive zeros of w, there exists a unique zero of w'.*

Proof. Suppose that w is of constant sign on $B(0,t) \setminus \overline{B(0,s)}$, $s < t$ and $w(s) = w(t) = 0$; then $w_1(x) = w(\mu^{\frac{1}{2+\alpha}}x)$ is an eigenfunction for one of the first demi-eigenvalues $\mu = \lambda^+(B(0,t) \setminus \overline{B(0,s)})$ if $w > 0$, or $\mu = \lambda^-(B(0,t) \setminus \overline{B(0,s)})$ if $w < 0$. Then by the uniqueness of the first eigenfunction in the radial case, if $w > 0$, by Remark 3.1, w is increasing on $[s, r_w]$ and decreasing on $[r_w, t]$ and r_w is the unique point on which $w' = 0$. We argue in the same manner when $w < 0$, using the fact that in that case w is decreasing on $[s, r_w]$ and increasing on $[r_w, t]$.

In the sequel we shall denote by w^+ the radial solution given by Proposition 4.1 of

$$|w'|^\alpha \mathcal{M}_{a,A}(r, w', w'') = -|w|^\alpha w, \qquad w(0) = 1,\ w'(0) = 0.$$

We also denote by w^- the radial solution of

$$|w'|^\alpha \mathcal{M}_{a,A}(r, w', w'') = -|w|^\alpha w, \qquad w(0) = -1,\ w'(0) = 0.$$

The proof of the existence and uniqueness of w^- is obtained by the same arguments used for w^+. The results in Proposition 4.6 can be adapted to the case of w^-, and then we also get that w^- is oscillatory.

5. Eigenvalues and eigenfunctions

In this section we prove the existence of an infinite denumerable set of eigenvalues for the radial operator defined in equation (3.2). These eigenvalues are simple and isolated. We begin with some properties of the eigenfuntions.

Proposition 5.1. *Suppose that u is a radial viscosity solution of*

$$\begin{cases} \tilde{F}(r, u', u'') = -\mu |u|^\alpha u & \text{in } B(0,1) \\ u(1) = 0, u(0) > 0. \end{cases}$$

Then 0 is a local maximum for u, u is $\mathcal{C}^2$ on a neighborhood $(0, r_o)$ of zero, is $\mathcal{C}^1$ on $[0, r_o]$ and $u'(0) = 0$.

Proof of Proposition 5.1. First let us note that $\mu > 0$, because if not the maximum principle would imply that $u \le 0$.

Since u is continuous there exists some neighborhood $B(0, r_o)$ on which $\tilde{F}(r, u', u'') < 0$. Then, using the comparison principle for such operators, and remarking that positive constants are sub-solutions, one gets that $u(r) \ge u(r_1)$ on $B(0, r_1)$, if $r_1 < r_o$. This implies in particular that u is decreasing from zero, and 0 is a local maximum. We now prove that u is $\mathcal{C}^1$ around zero and $\mathcal{C}^2$ on a neighborhood of 0, except on 0.

Let r_1 be the first zero of u. Then $u > 0$ on $B(0, r_1)$ and $\lambda^+(B(0, r_1)) = \mu$, by Proposition 2.6. Let w^+ be the $\mathcal{C}^1$ solution in Proposition 4.1 and β_1^+ its first zero (it exists according to Proposition 4.6). Define

$$v(r) = w^+\left(\frac{\beta_1^+ r}{r_1}\right).$$

Then $v > 0$ on $B(0, r_1)$ and v is an eigenfunction in $B(0, r_1)$ for the eigenvalue $(\frac{\beta_1^+}{r_1})^{2+\alpha}$; in particular, $\lambda^+(B(0, r_1)) = \mu = (\frac{\beta_1^+}{r_1})^{2+\alpha}$, and by the uniqueness of the first radial eigenfunction > 0 in Proposition 3.2, there exists some constant $c > 0$ such that $u = cv$ on $B(0, r_1)$. In particular u is $\mathcal{C}^2$ on each point where u' is different from zero and $\mathcal{C}^1$ everywhere on $B(0, r_1)$. This proves in particular, since u is $\mathcal{C}^1$ on $B(0, r_1)$ and u has a maximum on 0, that $u'(0) = 0$.

Of course the symmetric result holds for u such that $u(0) < 0$.

We now present an improvement of Proposition 2.5 which will be used in the proof of Corollary 5.4

Proposition 5.2. *Suppose that $s < t < 1$. Suppose that there exist some eigenfunctions for the annulus $B(0,1)\setminus\overline{B(0,s)}$ and for $B(0,1)\setminus\overline{B(0,t)}$, which are $\mathcal{C}^2$ on each point where their first derivative is different from 0, and $\mathcal{C}^1$ anywhere, then $\lambda^\pm(B(0,1) \setminus \overline{B(0,s)}) < \lambda^\pm(B(0,1) \setminus \overline{B(0,t)})$.*

Proof. Suppose by contradiction that $\lambda^\pm(B(0,1) \setminus \overline{B(0,s)}) = \lambda^\pm(B(0,1) \setminus \overline{B(0,t)})$, that we shall denote for simplicity by $\lambda^\pm$. Let φ and u be solutions of the equation

$$\tilde{F}(r, \varphi', \varphi'') + \lambda^\pm |\varphi|^\alpha \varphi = 0$$

which are $\mathcal{C}^2$ on each point where their first derivative is different from 0, and $\mathcal{C}^1$ anywhere, with $\varphi = 0$ on $\{r = 1\}$ and $\{r = s\}$, and $u = 0$ on $\{r = 1\}$ and $\{r = t\}$. To fix the ideas we also assume that φ and u are positive (and then we replace $\lambda^\pm$ by λ^+).

Using the same arguments as in Propositions 4.2 and 4.3, since $\varphi(1) = u(1) = 0$ and $u'(1) < 0$, $\varphi'(1) < 0$, by uniqueness there exists some constant $c > 0$ such that $\varphi = cu$ as long as φ' or u' is different from zero. By Remark 3.1 there exists exactly one point r_u on $(t, 1)$ for which $u'(r_u) = 0$ and it is a global strict maximum for u on $(t, 1)$. By uniqueness, $\varphi'(r_u) = 0$ and r_u must also be a global strict maximum for φ on $(t, 1)$. Then the equation satisfied by u and φ on the left of r_u is equation (4.3). By local uniqueness of solutions to (4.3) one gets that $u = c\varphi$ on the left of r_u and this is true as long as u' or φ' is different from 0, hence at least on $(t, 1)$. We get a contradiction since $u = 0$ on $\{r = t\}$ and $\varphi(t) \neq 0$.

We now prove the existence of a denumerable set of eigenvalues.

The result in Proposition 4.6 implies that there exists a sequence $\beta_k^\pm$ of increasing zeros of $w^\pm$.

We now consider $u_k^\pm(r) = w^\pm(\beta_k^\pm r)$. Then $u_k^\pm$ is an eigenfunction on $B(0, 1)$ for the eigenvalue $\mu_k^\pm := (\beta_k^\pm)^{\alpha+2}$ and it has $k - 1$ zeros inside the ball, say $r_i := \frac{\beta_i^\pm}{\beta_k^\pm}$, $i \in [1, k-1]$. We need to prove that they are the only eigenvalues.

Proposition 5.3. *The set of eigenvalues of the radial operator is the set* $\{\mu_k^\pm : k \geq 1\}$. *These eigenvalues are simple in the following sense: Suppose that* v *is some eigenfunction for the eigenvalue* $\mu_k^\pm$, *which is* $\mathcal{C}^1$ *and* $\mathcal{C}^2$ *on each point where the first derivative is different from* 0, *then there exists some constant* $c > 0$ *such that* $v = cw^\pm((\mu_k^\pm)^{\frac{1}{2+\alpha}} \cdot)$.

Proof of Proposition 5.3. Let μ be an eigenvalue. Let v be a corresponding eigenfunction, that we suppose to fix the ideas such that $v(0) > 0$. Necessarily, since v is radial and $\mathcal{C}^1$, $v'(0) = 0$. Let $z(\cdot) = \frac{v(\mu^{\frac{-1}{2+\alpha}} \cdot)}{v(0)}$. Then z satisfies equation (4.2) and by uniqueness $z = w^+$ on $[0, \mu^{\frac{1}{2+\alpha}})$. This implies that $\mu^{\frac{1}{2+\alpha}}$ is one of the zeros of w. This proves also the simplicity of the eigenvalue μ. The fact that the eigenvalues are isolated is a consequence of the properties of the zeros of w^+.

The following corollary is not necessary for the present paper; they will be useful for the bifurcation results announced in the final concluding section.

Corollary 5.4. *There is uniqueness* (*up to a positive multiplicative constant*) *of the* k*-th eigenfunction. As a consequence one has* $\mu_k^- < \mu_{k+1}^+$ *and* $\mu_k^+ < \mu_{k+1}^-$.

Proof. It is sufficient to prove that $\beta_k^+ < \beta_{k+1}^-$ and $\beta_k^- < \beta_{k+1}^+$.
We begin to prove that $\beta_1^+ < \beta_2^-$. One has

$$\lambda^-((\beta_1^+, \beta_2^+)) = 1 = \lambda^-((0, \beta_1^-)).$$

Suppose first that $\beta_2^+ < \beta_1^-$; this contradicts Proposition 2.5. If $\beta_2^+ = \beta_1^-$, one has a contradiction with Proposition 5.2.

We consider the case $k \geq 2$. Suppose by contradiction that $\beta_k^- < \beta_{k+1}^+ < \beta_{k+2}^+ \leq \beta_{k+1}^-$, and in a first time we assume that $\beta_{k+2}^+ < \beta_{k+1}^-$. In that case one would have

$$\lambda^\epsilon((\beta_{k+1}^+, \beta_{k+2}^+)) = \lambda^\epsilon((\beta_k^-, \beta_{k+1}^-)) = 1,$$

where $\epsilon = sign(-1)^{k+1}$; this would then contradict Proposition 2.5. In a second time if we assume that $\beta_{k+2}^+ = \beta_{k+1}^-$, this contradicts Proposition 5.2 In the same manner we should prove that $\beta_{k+1}^+ < \beta_{k+2}^-$.

For the sake of completeness we finish this section with some additional properties of the eigenvalues. This result is an analogue of one result in [11].

Proposition 5.5. *The gap between the two first half eigenvalues is larger than between the second ones:*

$$\frac{\mu_1^-}{\mu_1^+} \geq \frac{\mu_2^-}{\mu_2^+}.$$

Proof of Proposition 5.5. Let $\varphi_i^\pm$ $i = 1, 2$ be the eigenfunctions associated with $\mu_i^\pm$ with $\varphi_i^\pm(0) = \pm 1$. Let r^+ be the first zero of φ_2^+, r^- the first zero of φ_2^-. We prove that $r^- \geq r^+$. Indeed, suppose by contradiction that $r^- < r^+$, and define $A^+ = \{r : r^+ < r < 1\}$ and $A^- = \{r : r^- < r < 1\}$, then $A^+ \subset A^-$ and then

$$\lambda^-(A^+) = \mu_2^+ \geq \lambda^-(A^-) > \lambda^+(A^-) = \mu_2^-$$

and

$$\lambda^+(B_{r^+}) = \mu_2^+ < \lambda^+(B_{r^-}) < \lambda^-(B_{r^-}) = \mu_2^-.$$

We have obtained a contradiction.

Moreover, let us consider $\psi(x) = \varphi_2^+(r^+x)$. Then ψ is a radial solution on $B(0, 1)$ of

$$|\psi'|^\alpha \mathcal{M}_{a,A}(r, \psi', \psi'') = -(r^+)^{2+\alpha} \mu_2^+ |\psi|^\alpha \psi,$$

which implies, since $\psi(1) = 0$, that $(r^+)^{2+\alpha}\mu_2^+ = \mu_1^+$, by the definition of the first half eigenvalue. In the same manner $(r^-)^{2+\alpha}\mu_2^- = \mu_1^-$, and then

$$\frac{\mu_1^-}{\mu_2^-} \geq \frac{\mu_1^+}{\mu_2^+};$$

this yields the result.

6. The continuity of the spectrum with respect to the parameters.

In this section we let $\alpha \in (-1, \infty)$ and $a \in [0, A]$ vary and for that reason we denote by $\tilde{F}_{\alpha,a}$ the operator $\tilde{F}$ defined before. We denote by $\mu_k^\pm(\alpha, a)$ the corresponding eigenvalues. In order to prove the continuity of the map $(\alpha, a) \mapsto \mu_k^\pm(\alpha, a)$, we begin by establishing the boundedness of the eigenvalues $\mu_k^\pm(\alpha, a)$ when α belongs to some compact set of $(-1, \infty)$ and $a \in [0, A]$.

Proposition 6.1. *We suppose that $a = A = 1$. Let $\lambda_{eq,\alpha}((c, b))$ be the first "radial" eigenvalue for the set $B(0, b) \setminus \overline{B(0, c)}$ and for the operator $u \mapsto -\frac{d}{dr}\frac{|u'|^\alpha u'}{1+\alpha} - \frac{N-1}{r}u'$. Then there exists some continuous function $\varphi(\alpha)$, bounded on every compact set of $[-1, \infty)$, such that*

$$\lambda_{eq,\alpha}((c, b)) \leq \varphi(\alpha)(b - c)^{-2-\alpha}.$$

Corollary 6.2. *We assume that $a < A$. Then*

$$\lambda^+_{a,A,\alpha}((c,b)) \le a\varphi(\alpha)(b-c)^{-2-\alpha}.$$

Corollary 6.3. *For all $k \ge 1$*

$$\mu^+_k(\alpha,a)(B(0,1)) \le a\varphi(\alpha)k^{2+\alpha},$$

and

$$\mu^-_k(\alpha,a)(B(0,1)) \le a\varphi(\alpha)(k+1)^{2+\alpha}.$$

Proof of Proposition 6.1. Let us note that one can also use the following result for general operators satisfying the hypothesis in Section 2, proved in [5]: *There exists some constant C which depends on a, A, N such that, if R is the radius of some ball included in Ω, then* $\lambda^\pm(\Omega) \le \frac{C}{R^{\alpha+2}}$. But we shall give a more precise estimate here. For the radial case, one can easily see that

$$\lambda_{eq,\alpha} = \inf_{u\in W_0^{1,2+\alpha}((c,b))} \frac{\int_c^b |u'|^{2+\alpha}(r)r^{(N-1)(1+\alpha)}dr}{\int_c^b |u|^{2+\alpha}(r)r^{(N-1)(1+\alpha)}dr}.$$

Let us consider the function $u(r) = (r-c)(b-r)$. We need to get an upper bound for

$$I = \int_c^b |2r-(c+b)|^{2+\alpha}r^{(N-1)(1+\alpha)}dr,$$

and to get a lower bound for

$$J = \int_c^b (r-c)^{2+\alpha}(b-r)^{2+\alpha}r^{(N-1)(1+\alpha)}dr.$$

For the first integral we use the inequality $r^{(N-1)(1+\alpha)} \le 2^{|1-(N-1)(1+\alpha)|}(r-\left(\frac{c+b}{2}\right))^{(N-1)(1+\alpha)} + \left(\frac{c+b}{2}\right)^{(N-1)(1+\alpha)})$.

In the following $c(\alpha,N)$ is some constant which can vary from one line to another but is bounded for $\alpha \in [-1,M]$. We obtain that

$$\begin{aligned} J &\le c(\alpha,N)\Big(\int_c^b |r-\Big(\frac{c+b}{2}\Big)|^{2+\alpha+(N-1)(1+\alpha)}dr \\ &+ \Big(\frac{c+b}{2}\Big)^{(N-1)(1+\alpha)}\int_c^b |r-\Big(\frac{c+b}{2}\Big)|^{2+\alpha}dr\Big) \\ &\le c(\alpha,N)\Big((b-c)^{3+\alpha+(N-1)(1+\alpha)} + (c+b)^{(N-1)(1+\alpha)}(b-c)^{3+\alpha}\Big) \\ &\le c(\alpha,N)(b-c)^{3+\alpha}(c+b)^{(N-1)(1+\alpha)}. \end{aligned}$$

To minorize I we use

$$r^{(N-1)(1+\alpha)} \ge 2^{-|1-(N-1)(1+\alpha)|}\left((r-c)^{(N-1)(1+\alpha)} + c^{(N-1)(1+\alpha)}\right)$$

and thus

$$I \geq c(\alpha, N) \int_c^b \Big((r-c)^{2+\alpha+(N-1)(1+\alpha)} (b-r)^{2+\alpha} + c^{(N-1)(1+\alpha)} (r-c)^{2+\alpha} (b-r)^{2+\alpha} dr \Big)$$
$$\geq c(\alpha, N)(b-c)^{5+2\alpha+(N-1)(1+\alpha)} B(N(1+\alpha)+2, 3+\alpha) + c^{(N-1)(1+\alpha)} (b-c)^{5+2\alpha} B(3+\alpha, 3+\alpha)$$
$$\geq c(\alpha, N)(b-c)^{5+2\alpha} b^{(N-1)(1+\alpha)},$$

where, in the previous lines, B denotes the Euler function. We have obtained the result.

Proof of Corollary 6.2. We use the inequality in Proposition 2.10

$$\lambda^+(B(0,b) \setminus \overline{B(0,c)}) \leq a\lambda_{eq}(B(0,b) \setminus \overline{B(0,c)}).$$

Proof of Corollary 5.4. Let us recall that we have denoted by $(r_i)_i$ the zeros of the eigenfunction φ_k^+. $\mu_k^+(B(0,1))$ coincides with $\lambda^+(B(0,r_1))$ and with $\lambda^+(B(0,r_{i+1}) \setminus \overline{B(0,r_i)}) = \mu_1^+(B(0,r_{i+1}) \setminus \overline{B(0,r_i)})$, for all $i \in [1,k]$. Now, either $r_1 \geq \frac{1}{k}$, or there exists $i_o \geq 2$ such that $r_{i_o+1} - r_{i_o} \geq \frac{1}{k}$. In each of the cases we get the result. Concerning μ_k^- we use the inequality $\mu_k^- \leq \mu_{k+1}^+$ in Corollary 5.4.

Proposition 6.4. *Let $M > 0$ be given. Suppose that $(\alpha_n, a_n) \to (\alpha, a) \in (-1, M) \times [0, A]$; then $\mu_k^\pm(\alpha_n, a_n) \to \mu_k^\pm(\alpha, a)$.*

Proof of Proposition 6.4. By Corollary 5.4, the sequence $(\mu_k^\pm(\alpha_n, a_n))_n$ is bounded, so we can extract from it a subsequence, denoted in the same manner for simplicity, such that $\mu_k^\pm(\alpha_n, a_n) \to \mu$, for some $\mu \in \mathbb{R}^+$. We fix the integer k. Let φ_n be such that $\varphi_n(0) = 1$, and

$$\begin{cases} \tilde{F}_{\alpha_n, a_n}(r, \varphi_n', \varphi_n'') + \mu_k^+(\alpha_n, a_n)|\varphi_n|^{\alpha_n}\varphi_n = 0 \quad \text{in } B(0,1) \\ \varphi_n(1) = 0. \end{cases}$$

Using the compactness results in Corollary 2.8 one can extract from (φ_n) a subsequence, which will be denoted in the same manner for simplicity, which converges uniformly to a viscosity solution φ of

$$\begin{cases} \tilde{F}_{\alpha, a}(r, \varphi', \varphi'') + \mu|\varphi|^{\alpha}\varphi = 0 \quad \text{in } B(0,1) \\ \varphi(1) = 0. \end{cases}$$

By the uniform convergence, φ is not identically zero and $\varphi(0) = 1$. Thus μ is some eigenvalue. We must prove first that φ has $k-1$ zeros, secondly that φ is $\mathcal{C}^1$ and $\mathcal{C}^2$ on every point where the first derivative is different from zero.

Let j be such that $(r_i)_{1\leq i\leq j-1}$ are the zeros of φ. By Remark 3.5 in Section 3, φ changes sign on each of them. As a consequence there exists $\delta > 0$ such that, for all $i \in [1, j-1]$, on $[r_i - \delta, r_i + \delta]$, φ has no other zero than r_i and on $[r_{i-1} + \delta, r_i - \delta]$ φ has no zero. From the fact that $\varphi(r_i - \delta)\varphi(r_i + \delta) < 0$, one has for n large enough $\varphi_n(r_i - \delta)\varphi_n(r_i + \delta) < 0$, and thus φ_n has at least one zero in $(r_i - \delta, r_i + \delta)$. In the same manner there exists $m > 0$ such that $|\varphi| > m$ on every interval $[r_{i-1} + \delta, r_i - \delta]$, which implies by the uniform convergence of φ_n towards φ that φ_n cannot have a zero in this interval. As a consequence $k \geq j$. Moreover, by the strict monotonicity of φ on $[r_i - \delta, r_i + \delta]$, φ_n is also monotone for n large enough. This implies in particular the uniqueness of the zero of φ_n on that interval. Finally, $j = k$.

There remains to prove that φ is "regular;" i.e., that φ is $\mathcal{C}^2$ on each point where the first derivative is different from zero, and $\mathcal{C}^1$ anywhere.
Suppose that $\bar{r} < \bar{t}$ are two successive zeros of φ; then for n large enough, there exists $r_n < t_n$ two successive zeros of φ_n which converge respectively to $\bar{r}, \bar{t}$. Moreover φ_n (respectively φ) has constant sign on (r_n, t_n) (respectively $(\bar{r}, \bar{t})$). One can assume without loss of generality that this sign is negative.

We need to prove that φ is "regular " on $[\bar{r}, \bar{t}]$. Let r'_n be the unique zero of φ'_n on (r_n, t_n). Then φ_n is the unique fixed point on (r_n, r'_n) of the operator T_n defined as

$$T_n(w)(r) = \varphi_n(r'_n) - \int_{r'_n}^{r} \phi_{p'}\Big(\frac{(1+\alpha_n)\mu_k^+(\alpha_n, a_n, A)}{As^{N_n^-}} \int_{r'_n}^{s} |w|^{\alpha_n} w(t) t^{N_n^-}\, dt\Big) ds,$$

where $N_n^- = \frac{a_n(N-1)(1+\alpha_n)}{A}$. One can prove as is done in the proof of Proposition 4.4 that there exists some neighborhood $(r'_n - \delta, r'_n)$ of δ which does not depend on n such that, on the left of r_n, $\varphi'_n < 0$ and $\varphi''_n > 0$.

In the same manner φ_n is the unique fixed point of T_n on $(r'_n, r'_n + \delta)$ defined as

$$T_n(w)(r) = \varphi_n(r'_n) - \int_{r'_n}^{r} \phi_{p'}\Big((1+\alpha_n)\frac{\mu_k^+(\alpha_n, a_n, A)}{As^{N_{0,n}}} \int_{r'_n}^{s} |w|^{\alpha_n} w(t) t^{N_{0,n}} dt\Big) ds,$$

where $N_{0,n} = (N-1)(1+\alpha_n)$ and there exists some $\delta > 0$ which does not depend on n such that, on $(r'_n, r'_n + \delta)$, $\varphi'_n > 0$, and $\varphi''_n > 0$.

Using Remark 3.1 there exists exactly one point r' such that φ is decreasing on $(\bar{r}, r')$ and increasing on $(r', \bar{t})$, hence, since φ_n converges uniformly to φ, one gets that r'_n converges to r'.

From the definition of T_n one sees that φ_n converges uniformly on $(r'-\frac{3\delta}{4}, r')$ to the solution ψ on that interval of $T(\psi)=\psi$, where

$$T(w)(r) = \varphi(r') - \int_{r'}^{r} \phi_{p'}\Big(\frac{\mu(1+\alpha)}{As^{N^-}} \int_{r'}^{s} |w|^\alpha w(t) t^{N^-} dt\Big) ds.$$

This implies that φ is a $\mathcal{C}^2$ solution on $(r'-\frac{3\delta}{4}, r')$. We do the same on $(r', r'+\frac{3\delta}{4})$.

We now consider the equation on $(\bar{r}, r'-\frac{\delta}{2})$. As soon as n is large enough in order that $\bar{r} > r'_{n-1}$, on that interval φ_n satisfies

$$(\varphi_n', \varphi_n'') = f_n(\varphi_n, \varphi_n'),$$

where $f_n = (f_{1,n}, f_{2,n})$, $f_{1,n}(r, y_1, y_2) = y_2$, and

$$f_{2,n}(r, y_1, y_2) = M_n\Big(-\frac{m_n(y_2)(N-1)}{r} - \frac{|y_1|^{\alpha_n} y_1}{|y_2|^{\alpha_n}}\Big),$$

where M_n and m_n are respectively the functions

$$M_n(x) = \begin{cases} \frac{x}{A} \text{ if } x>0 \\ \frac{x}{a_n} \text{ if } x<0, \end{cases} \quad \text{and} \quad m_n(x) = \begin{cases} Ax \text{ if } x>0 \\ a_n x \text{ if } x<0. \end{cases}$$

It is clear that f_n is uniformly Lipschitzian on $(\varphi(r'), \varphi(\bar{r})) \times (\varphi'(\bar{r}), \varphi'(r'-\frac{\delta}{2}))$. Thus φ_n converges in $\mathcal{C}^1$ (even $\mathcal{C}^2$) to some solution ψ on $(\bar{r}, r'-\frac{\delta}{2})$ of $(\psi', \psi'') = f(\psi, \psi')$ with $f=(f_1, f_2)$, where f_1 and f_2 have been defined in 4.9 in Section 4. with the condition $\psi(\bar{r}) = 0$, $\psi'(\bar{r}) = \varphi'(\bar{r})$.

This implies that φ is $\mathcal{C}^2$ on $(\bar{r}, r'-\frac{\delta}{2})$. We can do the same on $(r'+\frac{\delta}{2}, \bar{t})$ and get in that way the regularity of φ on $[r', \bar{t})$. In fact the proof contains the regularity of φ on an open neighborhood of $[\bar{r}, \bar{t}]$. Since this can be repeated on each interval delimited by two zeros of φ one gets the regularity of φ on $B(0,1)$. As a consequence of Proposition 5.3 we have obtained that $\mu = \mu_k^+$. Since $\mu_k^+(\alpha_n, a_n)$ has a unique cluster point we get that the entire sequence converges to μ_k^+.

7. Conclusion and supplementary results

Let $K_{\alpha,a}$ be the operator defined on $\mathcal{C}(\Omega)$ by: For $f \in \mathcal{C}(\overline{\Omega})$, $K_{\alpha,a}(f)$ is the unique $v \in \mathcal{C}(\overline{\Omega})$ solution of

$$\begin{cases} \tilde{F}_{\alpha,a}(r, v', v'') - |v|^\alpha v = -f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The operator $K_{\alpha,a}$ is well defined since $\alpha > -1$, and defining for μ positive given $K_{\alpha,a,\mu}(u) = K_{\alpha,a}((\mu+1)|u|^\alpha u)$, one can note that the fixed points of $K_{\alpha,a,\mu}$ exist if μ is an eigenvalue, as some associated eigenfunction.

We will be able to derive from the continuity results in the last section some results about the degree of the operator $K_{\alpha,a,\mu}$ as a function of the position of μ with respect to the eigengalues $\mu_k^{\pm}$. Next we shall establish some bifurcation results for the equations defined as follows.

Let f be defined as $(\mu, s) \mapsto f(\mu, s)$ which is "super-linear" in s uniformly with respect to μ in the sense that

$$\lim_{s\to 0} \frac{f(\mu, s)}{|s|^{1+\alpha}} = 0.$$

We also assume that f is locally bounded and continuous in all its variables.

Then we shall consider the problem

$$\begin{cases} \tilde{F}_{\alpha,a}(r, u', u'') + \mu |u|^{\alpha} u + f(\mu, u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

for which we shall prove bifurcation results, completing the results already obtained in [9].

This will be the object of a forthcoming paper.

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