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CONVERGENCE RESULTS FOR CRITICAL POINTS OF THE ONE-DIMENSIONAL AMBROSIO-TORTORELLI FUNCTIONAL WITH FIDELITY TERM

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Abstract. In this paper, we show that critical points of the one-dimensional Ambrosio-Tortorelli functional with fidelity term converge to those of the corresponding Mumford-Shah functional, a famous model for image segmentation. Equi-partition and convergence of the energy-density are also derived.

1. Introduction

In this paper, we continue our previous study [13] on the convergence of critical points of the Ambrosio-Tortorelli functional [3, 4] to those of the Mumford-Shah functional [22]. Here, as opposed to the Dirichlet case in [13], the functionals we study contain the fidelity term linking the approximate images to the original image. These functionals were proposed as models for image segmentation in computer vision. We will be more specific on these functionals in the next paragraphs.

The Mumford-Shah functional for image segmentation alluded to above was proposed by D. Mumford and J. Shah in their celebrated paper [22]. If $g \in L^\infty(\Omega; [0,1])$ represents a continuous interpolation of the collected pixelated data over the image domain $\Omega \subset \mathbb{R}^2$, then the proposed segmentation consists in minimizing

$$(u, K) \mapsto \mathcal{MS}(u, K) := \int_{\Omega \setminus K} |\nabla u|^2 \, dx + 2\mathcal{H}^1(K) + \lambda \int_\Omega (u - g)^2 \, dx,$$

among all subsets $K \subset \Omega$ which are unions of piecewise smooth curves and all $u \in H^1(\Omega \setminus K)$. In that functional, λ is a positive tuning parameter, K represents the contours of the image, and u the resulting grey contrast $(0 \le u(x) \le 1)$.

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Proving existence for minimizers of that functional was not a trivial task and it gave rise to an abundant literature spearheaded by the work of E. De Giorgi and that of L. Ambrosio on the space $SBV(\Omega)$; see e.g [1]. The underlying idea was to view $\mathcal{MS}(u, K)$ as a one-field functional

$$MS(u) = \int_\Omega |\nabla u|^2\, dx + 2\mathcal{H}^1(S(u)) + \lambda \int_\Omega (u-g)^2\, dx \tag{1.1}$$

over $SBV(\Omega)$, the space of functions $u \in L^1(\Omega)$ such that their distributional derivative is a Radon measure Du with finite total variation $|Du|(\Omega)$, with jump set $S(u)$ (the complement of the set of Lebesgue points for u), and no Cantor part. The next step was to prove existence of a minimizer u_m of MS in that space, and then show that the pair $(u_m, \overline{S(u_m)})$ was actually a minimizer for $\mathcal{MS}$. That program was successfully completed, culminating in De Giorgi-Carriero-Leaci [9] where the authors proved the existence of minimizers of $\mathcal{MS}$ within the class of closed sets $K \subset \overline{\Omega}$. A different approach to prove the existence of minimizers of $\mathcal{MS}$, based on a priori estimates and elimination techniques, was discovered by Dal Maso-Morel-Solimini [8]. See also [20, 15].

From a computational standpoint, the search for a minimizer of (1.1) is not easy, because the test fields exhibit discontinuities at unknown locations and the implementation of classical finite element methods becomes a perilous endeavor. A possible remedy consists in resorting to variational convergence, specifically Γ-convergence, so as to approximate MS by a more regular functional whose minimizers are easier to evaluate. For more information on Γ-convergence, we refer the interested reader to e.g. [7] and merely emphasize for now that an important property of Γ-convergence is that (approximate) minimizers of the approximating functionals that converge as $\varepsilon \searrow 0$ will converge to bona fide minimizers of MS.

There is by now an abundant literature on the approximation of the Mumford-Shah functional and many approximating sequences have been proposed. The most computationally efficient in our opinion is that originally proposed by L. Ambrosio and V. Tortorelli in [3], [4]– denoted henceforth by AT_ε – in the footstep of the functional proposed by L. Modica and S. Mortola for the approximation of the perimeter [17]. Consider

$$AT_\varepsilon(u, v) = \int_\Omega \Big((\eta_\varepsilon + v^2)|\nabla u|^2 + \varepsilon|\nabla v|^2 + \frac{(1-v)^2}{\varepsilon}\Big) dx + \lambda \int_\Omega (u-g)^2\, dx,$$

with $0 < \eta_\varepsilon \ll \varepsilon$. It is proved in [3], [4] that AT_ε $\Gamma(\mathcal{B}(\Omega) \times \mathcal{B}(\Omega))$-converges to MS, suitably extended to a two-field functional as

$$\mathcal{MS}(u,v) = \begin{cases} \mathcal{MS}(u) & \text{if } v \equiv 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Above, $\mathcal{B}(\Omega)$ stands for the set of all Borel functions on Ω, and the convergence is the convergence in measure. Actually, we can also view the convergence as taking place in $L^2(\Omega) \times L^2(\Omega)$.

The functional $\mathcal{AT}_\varepsilon$ is easily seen, through the direct method of the calculus of variations, to admit at least one minimizing pair $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega)\times H^1(\Omega)$, for any fixed value of ε. It follows from a truncation argument (see Lemma 3.1) that the associated minimizing sequence is bounded in $L^\infty(\Omega)\times L^\infty(\Omega)$, and a subsequence can be shown to converge in measure (and also strongly in $L^2(\Omega) \times L^2(\Omega)$) to $(u, v \equiv 1)$, which, by the already evoked property of Γ-convergence, will be a minimizer for $\mathcal{MS}$.

In practice, one often minimizes the $\mathcal{AT}_\varepsilon$ functional using the steepest descent method. This consideration naturally leads to the study of the gradient flow of the $\mathcal{AT}_\varepsilon$ functional. Numerical studies of the gradient flow of the Ambrosio-Tortorelli functional have been performed, for example, in [10, 16]; its analysis and fully discrete finite element approximations have been investigated in [12].

A good understanding of the gradient flow of the Ambrosio-Tortorelli functional paves the way to analyzing a reasonably-defined gradient flow of the Mumford-Shah functional. One may also hope that solutions to the gradient flow of the Ambrosio-Tortorelli functional will converge to a suitably defined solution of the gradient flow of the Mumford-Shah functional, as are the cases in other models using variational approximations. In any case, the first step in understanding the convergence properties of solutions to the gradient flow of $\mathcal{AT}_\varepsilon$ is to analyze its stationary solutions and this is the main purpose of this paper. Note that these stationary solutions correspond to the critical points of the Ambrosio-Tortorelli functional $\mathcal{AT}_\varepsilon$.

In the framework of Γ-convergence, there have been relatively few attempts at proving that critical points of the approximation functionals converge to those of the limiting functional. One may refer to such convergence results in other settings such as that of the Allen-Cahn functional in phase transitions, see [14, 24, 25], or that of the Ginzburg-Landau functional in superconductivity, see [5, 23], or that in the framework of thin elastic bodies, see [18, 19, 21]. We may also expect those convergence results in the framework of image segmentation.

This study is a first step in that direction. It investigates the one-dimensional case. Because real digital images are two dimensional, the analysis of these functionals in 2D will be of greater interest and applications. However, carrying out this task is quite a challenge for the time being, and we only limit ourselves to the one-dimensional case.

In the next section, the one-dimensional functional is introduced and the two main results are stated: the convergence of critical points of AT_ε to critical points of MS (see Theorem 2.1); finally, the convergence of the various terms in the energy AT_ε to their MS-analogues (see Theorem 2.3). As a preparation for the their proofs, we establish essential estimates in Section 3. The proof of Theorem 2.1 will be carried out in Section 4. Finally, in the final section, we prove Theorem 2.3.

We end this introduction by highlighting a few differences between the results and proofs of our paper and those in [13]. First of all, the presence of the fidelity terms in our functionals makes the analysis more involved than the Dirichlet case in [13]. There are no conservation laws. We compensate this by using a balance law stated in Proposition 3.3 at the expense of requiring higher regularity of the fidelity terms g_ε. Secondly, critical points of MS are no longer piecewise constant and there are no symmetry properties for critical points of AT_ε. Therefore, we need more refined analysis in our arguments such as in Proposition 3.12.

2. Statement of the results

Throughout, $\mathcal{C}$ stands for a generic positive constant (so that e.g. $\mathcal{C} = 2\mathcal{C}$) that may change from line to line but does not depend on the small parameter $\varepsilon > 0$ and L is the length of the interval under consideration. We now introduce the Ambrosio-Tortorelli and Mumford-Shah functionals used in the paper.

For $\varepsilon > 0$ and $\lambda > 0$, we consider the following ε-indexed one-dimensional Ambrosio-Tortorelli functional with fidelity term g_ε :

$$AT_\varepsilon(u,v) = \int_0^L \Big((\eta_\varepsilon + v^2)(u')^2 + \varepsilon(v')^2 + \frac{(1-v)^2}{\varepsilon} + \lambda(u-g_\varepsilon)^2\Big)dx. \quad (2.1)$$

In (2.1), η_ε is a positive number, and (u,v) belongs to the space Y defined by $Y := \{u, v \in H^1(0,L)\}$. We assume that, as $\varepsilon \searrow 0$,

$$\eta_\varepsilon/\varepsilon \to 0, \quad i.e., \quad \eta_\varepsilon \ll \varepsilon. \quad (2.2)$$

We assume that the fidelity term g_ε satisfies the following.

Assumption. g_ε converges weakly in $H^1(0,L)$ to a function $g \in H^1(0,L)$.

We also introduce, for $u \in SBV(\mathbb{R})$, the one-dimensional Mumford-Shah functional with fidelity term g

$$\mathcal{MS}(u,v) = \begin{cases} \int_0^L (u')^2\, dx + 2\#\left(S(u)\right) + \lambda(u-g)^2 & \text{if } v \equiv 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3}$$

In (2.3), u' denotes the approximate derivative of u, *i.e.*, the density of the absolutely continuous part of the measure Du with respect to the Lebesgue measure, while $S(u)$ denotes the jump set of u, defined as the complement in $\mathbb{R}$ of the set of Lebesgue points of u. Finally, $SBV(\mathbb{R})$ stands for the class of special functions of bounded variation (see [2]). They are $L^1(\mathbb{R})$ functions u whose distributional derivatives Du have finite total variation $|Du|(\mathbb{R})$ and have no Cantor parts.

Note that $\mathcal{AT}_\varepsilon$ serves as a variational approximation for $\mathcal{MS}$ in the sense that $\mathcal{AT}_\varepsilon$ Γ- converges to $\mathcal{MS}$. For other approximations of $\mathcal{MS}$, see [6]. Therefore, by the nature of Γ-convergence, global minimizers of $\mathcal{AT}_\varepsilon$ converge in a suitable topology to those of $\mathcal{MS}$. The question of interest is whether we have convergence results for intermediate states. This is the purpose of this paper. We propose to study the convergence properties of critical points that are not necessarily minimizers.

Let $(u_\varepsilon, v_\varepsilon)$ be critical points of the Ambrosio-Tortorelli functional (2.1), *i.e.*, pairs of functions $(u_\varepsilon, v_\varepsilon) \in Y$ that satisfy the Euler-Lagrange equations

$$\begin{cases} -\varepsilon v_\varepsilon'' + v_\varepsilon (u_\varepsilon')^2 + \frac{v_\varepsilon - 1}{\varepsilon} = 0 \\ [u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2)]' = \lambda(u_\varepsilon - g_\varepsilon) \\ u_\varepsilon'(0) = u_\varepsilon'(L) = 0, \quad v_\varepsilon'(0) = v_\varepsilon'(L) = 0. \end{cases} \tag{2.4}$$

Our main goal is to study the limit properties of $(u_\varepsilon, v_\varepsilon)$ as ε goes to 0, provided additionally that

$$\mathcal{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \le \mathcal{C} < \infty. \tag{2.5}$$

Our main result here is to show that indeed critical points of the Ambrosio-Tortorelli functional (2.1) converge to corresponding critical points of the Mumford-Shah functional (2.3). Recall, from Chapter 7 in [2], that a critical point of (2.3) is a couple $(u, 1)$ where u is discontinuous at a finite number of points, and between these points, u solves the equation $u'' = \lambda(u-g)$ with homogeneous Neumann boundary conditions.

We are now ready to state our main result.

Theorem 2.1. *After extracting a subsequence, v_ε converges strongly to* 1 *in $L^2((0,L))$ and u_ε converges strongly in $L^4((0,L))$ to a function $u \in BV((0,L))$. The pair $(u,1)$ is a critical point of the Mumford-Shah functional. More precisely, there is a finite number of points $X_0 < X_1 < \cdots < X_k < X_{k+1} \in [0,L]$ with $X_0 = 0$ and $X_{k+1} = L$ such that the limit function u of u_ε solves the following equation on each interval $(X_i, X_{i+1})(0 \leq i \leq k)$:*

$$u^{''} = \lambda(u-g) \quad u^{'}(X_i) = u^{'}(X_{i+1}) = 0.$$

Before stating our next result, it is convenient to introduce the following definition whose motivation will be given in Section 3.3.

Definition 2.2. *Let $x_\varepsilon \in [0,L]$ be a critical point of v_ε. We say that x_ε is a v- jump if $v_\varepsilon(x_\varepsilon) \leq \mathcal{C}\varepsilon^{1/2}$ for some positive constant $\mathcal{C}$ independent of ε. We will say that x is a limit of v-jumps (or a limiting v-jump) if it is a limit of a family (x_ε) of v-jumps as $\varepsilon \to 0$.*

The second result concerns the convergence of various terms of the Ambrosio-Tortorelli functional (2.1).

Theorem 2.3. *If $u_\varepsilon \to u$, a critical point for the Mumford-Shah functional, as given in Theorem* 2.1, *and if $x_1, \cdots, x_j$ are corresponding limits of v-jumps associated with $(v_\varepsilon, u_\varepsilon)$ (see also Definition* 3.11 *and Proposition* 3.12), *then the following hold.*

The limit measure of $(\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2dx$ is $(u')^2dx$ where u' denotes the approximate gradient of u; in fact, $(\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2$ converges weakly in $L^1((0,L))$ to $(u')^2$.

The limit measure of $\varepsilon(v_\varepsilon')^2dx$, which is also that of $(1-v_\varepsilon)^2/\varepsilon dx$ (i.e., there is equi-partition of the energy), is a finite sum of Dirac masses at the x_i's.

Remark 2.4. As can be seen from the proof of Theorem 2.1, $\{X_1, \cdots, X_k\} \subset \{x_1, \cdots, x_j\}$. It follows from the fact that u can be discontinuous only at limiting v-jumps, but it does not have to be.

2.1. **Ingredients of the proofs.** Let us say a few words about the ingredients of the proofs. The difficulties in establishing the convergence result in Theorem 2.1 come from the smallness of v_ε. In the regions where v_ε is small, the limit function u of u_ε can be discontinuous. We need to prove the vanishing of $u^{'}$ at both sides of the points of discontinuity of u. To do this, we introduce the quantity F_ε in (3.5), which is in some sense an approximation for the derivative $u_\varepsilon^{'}$ of u_ε. Its limit function F is smooth and the proof relates the smoothness of u to the value of F at each point in the interval

$[0, L]$. This will be done in Section 4. In the course of the proof, we also use the gradient estimates for v_ε and u_ε that can be derived from the balance law (3.3).

For the proof of the equi-partition of energy in Theorem 2.3, we exploit the positivity of the discrepancy measure $(\frac{(1-v_\varepsilon)^2}{\varepsilon} - \varepsilon|v_\varepsilon'|^2)$ and the closeness to 1 of v_ε away from the $v-$jumps. The latter ingredient, to be made precise in Proposition 3.12, is based on the maximum principle and precise estimates on the values of v_ε at its critical points established in Lemma 3.8.

2.2. **Open questions.** In this section, we list several open questions left from our study. In the Dirichlet case, we proved in Theorem 2.1 in [13] that critical points of Ambrosio-Tortorelli not only converge to critical points of the corresponding Mumford-Shah functional but also to very special ones: only critical points with jumps that are symmetrically located on the interval of study can be obtained through the limiting process. In other words, in the Dirichlet case, the Ambrosio-Tortorelli approximation acts as a selection mechanism for the Mumford-Shah functional. In our case with the fidelity term, it is not clear whether this is still true. If it is, then the next question is: what is the selection mechanism?

Question 1. Is there any selection mechanism for limiting critical points of the Ambrosio-Tortorelli functional with fidelity term? When u is discontinuous at $X \in [0, L]$, is it true that $\frac{u(X^-)+u(X^+)}{2} = g(X)$?

In Lemma 3.5, we prove that 0 and L are local maxima of v_ε on $[0, L]$ and the values $v_\varepsilon(0), v_\varepsilon(L)$ are nearly 1. It is not clear if v-jumps (critical points of v_ε at which v_ε is close to 0) can accumulate to these boundary points. This leads us to the following.

Question 2. Is there any limit of v-jumps at the boundary of the interval $[0, L]$?

More generally, one can ask: given the limiting fidelity term g and a uniform energy bound $\mathcal{C}$, does there exist a positive constant $\delta = \delta(g, \mathcal{C})$ such that the distance between any local minimum and local maximum of v_ε is at least δ? In the Dirichlet case in [13], due to the symmetry property of v_ε, the answer is positive.

In order to evaluate precisely the energy $AT_\varepsilon(u_\varepsilon, v_\varepsilon)$, one might also ask the following.

Question 3. Do we have single-multiplicity? i.e., does the limit of the weights of $\varepsilon(v_\varepsilon')^2dx$ at each x_i equal to 1?

Another question, which is related to Remark 2.6 in [13], is whether u is discontinuous at every limiting v-jump. In other words, we may ask the following.

Question 4. Is it true that $\{X_1, \cdots, X_k\} = \{x_1, \cdots, x_j\}$?

Finally, in the proof of equi-partition of energy in Theorem 2.3, we used the weak convergence of g_ε to g in $H^1((0, L))$. In general, we can ask the following.

Question 5. What are the minimal convergence hypotheses on $g_\varepsilon - g$ that are needed?

3. Preliminary estimates

3.1. **Classical a priori estimates.** In this section, we establish a few canonical estimates that will prove instrumental in the proof of Theorems 2.1, 2.3. These estimates are completely standard but we include their proofs for convenience of the reader. First, it follows from the assumptions on g_ε and the embedding $H^1(0, L) \hookrightarrow C([0, L])$ that

$$\|g_\varepsilon^{'}\|_{L^2(0,L)} \leq \mathcal{C}, \text{ and } \|g_\varepsilon\|_{L^\infty(0,L)} \leq \mathcal{C}. \tag{3.1}$$

For now, we prove some elementary estimates on the critical points $(u_\varepsilon, v_\varepsilon)$ of (2.1), which, by the way, are smooth by elliptic regularity.

A first result is a maximum principle for v_ε and u_ε, as follows.

Lemma 3.1. $0 \leq v_\varepsilon \leq 1, \ \|u_\varepsilon\|_\infty \leq \|g_\varepsilon\|_\infty \leq \mathcal{C}$.

Proof. The proof of the inequality $0 \leq v_\varepsilon \leq 1$ is exactly the same as that of Lemma 3.1 in [13]. Therefore, we only need to prove the inequality for u_ε.

Multiplying both sides of the second equation of (2.4) by $(u_\varepsilon - \|g_\varepsilon\|_\infty)^+ = \max(0, u_\varepsilon - \|g_\varepsilon\|_\infty)$ and recalling the Neumann boundary conditions on u_ε, we get

$$\int_0^L -u_\varepsilon^{'}(\eta_\varepsilon + v_\varepsilon^2)((u_\varepsilon - \|g_\varepsilon\|_\infty)^+)^{'} = \int_0^L \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon - \|g_\varepsilon\|_\infty)^+.$$

This equality can be rewritten as

$$\int_0^L -((u_\varepsilon - \|g_\varepsilon\|_\infty)^+)^{'}(\eta_\varepsilon + v_\varepsilon^2)((u_\varepsilon - \|g_\varepsilon\|_\infty)^+)^{'} = \int_0^L \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon - \|g_\varepsilon\|_\infty)^+. \tag{3.2}$$

The left-hand side of (3.2) is nonpositive. Thus

$$\int_0^L \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon - \|g_\varepsilon\|_\infty)^+ \leq 0,$$

showing that $(u_\varepsilon - \|g_\varepsilon\|_\infty)^+ = 0$.

Multiplication of the second equation of (2.4) by

$$(u_\varepsilon + \|g_\varepsilon\|_\infty)^- = \max(0, -(u_\varepsilon + \|g_\varepsilon\|_\infty))$$

would yield the other inequality for u_ε. □

Next, we establish the convergence properties of the pair $(u_\varepsilon, v_\varepsilon)$.

Lemma 3.2. $v_\varepsilon \to 1$, *strongly in* $L^2((0,L))$, *and, modulo extraction,*

$$u_\varepsilon \to u \in BV((0,L)), \text{ strongly in } L^p((0,L)), \text{ for any } p \in [1,\infty).$$

Proof. The energy bound (2.5) immediately implies the first convergence. Together with the bounds obtained in Lemma 3.1, it also implies that $u_\varepsilon(2v_\varepsilon - v_\varepsilon^2)$ is bounded in $BV((0,L))$. Now let $p \in [1,\infty)$. By the compactness of $BV((0,L))$ into $L^p((0,L))$ (see [11]), a subsequence of $u_\varepsilon(2v_\varepsilon - v_\varepsilon^2)$ converges in $L^p((0,L))$ to $u \in BV((0,L))$. Because $u_\varepsilon - u = (u_\varepsilon(2v_\varepsilon - v_\varepsilon^2) - u) + u_\varepsilon(1-v_\varepsilon)^2$, the strong convergence of u_ε to u in $L^p((0,L))$ follows. □

In the present context, we have the following Noether- type balance law.

Proposition 3.3.

$$\left\{\frac{1}{2}\Big(\frac{(1-v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - \varepsilon(v_\varepsilon')^2 + \lambda(u_\varepsilon - g_\varepsilon)^2\Big)\right\}' = -\lambda g_\varepsilon'(u_\varepsilon - g_\varepsilon). \quad (3.3)$$

Proof. The left-hand side of the previous expression also reads as

$$\begin{aligned} A_\varepsilon &:= \frac{(v_\varepsilon - 1)v_\varepsilon'}{\varepsilon} - \varepsilon v_\varepsilon' v_\varepsilon'' - v_\varepsilon v_\varepsilon'(u_\varepsilon')^2 - (v_\varepsilon^2 + \eta_\varepsilon)u_\varepsilon' u_\varepsilon'' + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon') \\ &= v_\varepsilon'(-\varepsilon v_\varepsilon'' - v_\varepsilon(u_\varepsilon')^2 + \frac{v_\varepsilon - 1}{\varepsilon}) - (v_\varepsilon^2 + \eta_\varepsilon)u_\varepsilon' u_\varepsilon'' + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon'). \end{aligned}$$

The first and second equation of (2.4) then imply that

$$\begin{aligned} A_\varepsilon &= -v_\varepsilon'(2v_\varepsilon(u_\varepsilon')^2) - (v_\varepsilon^2 + \eta_\varepsilon)u_\varepsilon' u_\varepsilon'' + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon') \\ &= -u_\varepsilon'[u_\varepsilon''(\eta_\varepsilon + v_\varepsilon^2) + 2v_\varepsilon v_\varepsilon' u_\varepsilon'] + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon') \\ &= -u_\varepsilon'[u_\varepsilon'(\eta_\varepsilon + v_\varepsilon^2)]' + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon') \\ &= -\lambda(u_\varepsilon - g_\varepsilon)u_\varepsilon' + \lambda(u_\varepsilon - g_\varepsilon)(u_\varepsilon' - g_\varepsilon') = -\lambda g_\varepsilon'(u_\varepsilon - g_\varepsilon). \end{aligned}$$

□

This proposition is key to the following gradient estimates.

Lemma 3.4. *There exists a positive constant* $\mathcal{C}$ *independent of* $\varepsilon > 0$ *such that for* $\varepsilon > 0$ *sufficiently small* $\|v_\varepsilon'\|_\infty \le \frac{\mathcal{C}}{\varepsilon}$.

The above lemma also plays a crucial role in the following boundary-value estimates whose proofs will be given in the next subsection.

Lemma 3.5. *For ε sufficiently small, we have $v_\varepsilon(0), v_\varepsilon(L) > 1/2$. Furthermore, 0 and L are local maxima of v_ε on $[0, L]$.*

Proof of Lemma 3.4. For any $x \in (0, L)$, integrating (3.3) from 0 to x, and using the last two equations of (2.4), we get

$$\begin{aligned}&\frac{1}{2}\left(\frac{(1-v_\varepsilon(x))^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2(x))(u_\varepsilon^{'}(x))^2 - \varepsilon(v_\varepsilon^{'}(x))^2 + \lambda(u_\varepsilon(x) - g_\varepsilon(x))^2\right)\\ &= \frac{1}{2}\left(\frac{(1-v_\varepsilon(0))^2}{\varepsilon} + \lambda(u_\varepsilon(0) - g_\varepsilon(0))^2\right) + \int_0^x -\lambda g_\varepsilon^{'}(t)(u_\varepsilon(t) - g_\varepsilon(t))dt.\end{aligned} \tag{3.4}$$

Rearranging, we obtain

$$\begin{aligned}\frac{1}{2}\varepsilon(v_\varepsilon^{'}(x))^2 &\leq \frac{1}{2}\left(\frac{(1-v_\varepsilon(x))^2}{\varepsilon} + \lambda(u_\varepsilon(x) - g_\varepsilon(x))^2\right)\\ &\quad + \int_0^x \lambda g_\varepsilon^{'}(t)(u_\varepsilon(t) - g_\varepsilon(t))dt\\ &\leq \frac{1}{2}(\frac{1}{\varepsilon} + 4\lambda \left\|g_\varepsilon\right\|_\infty^2) + \lambda \left\|g_\varepsilon^{'}\right\|_{L^2(0,L)} \left\|u_\varepsilon - g_\varepsilon\right\|_{L^2(0,L)} \leq \frac{\mathcal{C}}{\varepsilon},\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. Therefore, the L^∞-estimate of $v_\varepsilon^{'}$ follows. □

Remark 3.6. Note that, on a compact set K away from the limiting v-jumps, we have the uniform estimates $\|v_\varepsilon^{'}\|_{L^\infty(K)} \leq \mathcal{C}_K$, where $\mathcal{C}_K$ is a constant depending only on K. This estimate follows from Proposition 3.12 and Lemma 5.6.

3.2. **Approximate slopes.** In the sequel, it is convenient to introduce the following.

Notation. Let

$$F_\varepsilon(x) := u_\varepsilon^{'}(x)(\eta_\varepsilon + v_\varepsilon(x)^2) = \int_0^x \lambda(u_\varepsilon(t) - g_\varepsilon(t))dt. \tag{3.5}$$

From (3.1) and the bounds in Lemma 3.1, we have the uniform bound on F_ε

$$\left\|F_\varepsilon\right\|_{L^\infty} \leq \mathcal{C} \tag{3.6}$$

and the convergence results in Lemma 3.2 imply that F_ε converges uniformly to F on $[0, L]$, where

$$F(x) = \int_0^x \lambda(u(t) - g(t))dt. \tag{3.7}$$

Remark 3.7. F_ε is a function analogous to the constant c_ε in our previous paper [13]. It is formally the slope $u_\varepsilon^{'}$ of the function u_ε. The limit values of F_ε at v-jumps are closely related to the limit values of c_ε. On the other hand, if x is a limiting v-jump and u is not continuous there, then Theorem 2.1 requires that $u'(x) = 0$, which suggests that we prove that $F(x) = 0$.

Proof of Lemma 3.5. Assume that $v_\varepsilon(0) \leq 1/2$. Let $x > 0$ be a critical point of v_ε. This set of x's contains L and is thus not empty. Then, rewriting (3.4) and using the definition of F_ε, we have

$$\frac{(1-v_\varepsilon(x))^2-(1-v_\varepsilon(0))^2}{\varepsilon}-\frac{F_\varepsilon(x)^2}{\eta_\varepsilon+v_\varepsilon^2(x)} \tag{3.8}$$
$$= -\lambda(u_\varepsilon(x)-g_\varepsilon(x))^2+\lambda(u_\varepsilon(0)-g_\varepsilon(0))^2-\int_0^x 2\lambda g_\varepsilon^{'}(t)(u_\varepsilon(t)-g_\varepsilon(t))dt.$$

The bounds (3.1) imply that the left-hand side is bounded. This fact, together with the uniform bound on F_ε, shows that, for ε sufficiently small, $v_\varepsilon(x) < 2/3$. This, in turn, implies that

$$M_\varepsilon = \max_{x\in[0,L]} v_\varepsilon(x) \leq 2/3. \tag{3.9}$$

However, without any assumption except the energy bound (2.5), we can estimate M_ε from below via the following inequalities:

$$\mathcal{C} \geq \int_0^L \frac{(1-v_\varepsilon)^2}{\varepsilon} \geq \int_0^L \frac{(1-M_\varepsilon)^2}{\varepsilon} = \frac{L(1-M_\varepsilon)^2}{\varepsilon}.$$

Thus, $M_\varepsilon \geq 1-\mathcal{C}\sqrt{\varepsilon}$. This is a contradiction with (3.9). Therefore $v_\varepsilon(0) > 1/2$. Similarly, $v_\varepsilon(L) > 1/2$.

Because 0 is a critical point of v_ε on $[0, L]$ with $v_\varepsilon^{''}(0) = \frac{v_\varepsilon(0)-1}{\varepsilon} \leq 0$ by the maximum principle from Lemma 3.1, it is a local maximum for v_ε on $[0, L]$ and, similarly, so is L. ☐

The next lemma gives more precise estimates on the values of v_ε at its critical points. Though rather simple, they will turn out to be very useful for the proofs of Proposition 3.12 and Theorem 2.3.

Lemma 3.8. *There exists a positive constant $\mathcal{C}$ independent of $\varepsilon > 0$ such that, if x is a critical point of v_ε, then either*

$$v_\varepsilon(x) \leq \mathcal{C}\sqrt{\varepsilon} \tag{3.10}$$

or

$$v_\varepsilon(x) = v_\varepsilon(0) + O(\sqrt{\varepsilon}) \geq 1-\mathcal{C}\sqrt{\varepsilon}. \tag{3.11}$$

Proof. Let x be a critical point of v_ε. Then (3.8) holds. Multiplying this identity by $(\eta_\varepsilon + (v_\varepsilon(x))^2)$ and using (3.6), we see that $\frac{\eta_\varepsilon + v_\varepsilon(x)^2}{\varepsilon}((1-v_\varepsilon(x))^2 - (1-v_\varepsilon(0))^2)$ is bounded; i.e.,

$$\frac{\eta_\varepsilon + v_\varepsilon(x)^2}{\varepsilon} \left|(1 - v_\varepsilon(x))^2 - (1 - v_\varepsilon(0))^2\right| \leq \mathcal{C}. \tag{3.12}$$

If $v_\varepsilon(x) \leq 1/4$ then, from the inequality $v_\varepsilon(0) \geq 1/2$ in Lemma 3.5, we deduce that $\frac{\eta_\varepsilon + v_\varepsilon(x)^2}{\varepsilon}$ is bounded. Therefore, $v_\varepsilon(x) \leq \mathcal{C}\sqrt{\varepsilon}$.

On the other hand, if $v_\varepsilon(x) \geq 1/4$ then $[(1 - v_\varepsilon(x))^2 - (1 - v_\varepsilon(0))^2]/\varepsilon$ is bounded (by $\mathcal{C}$). Using the inequality $\left|a^2 - b^2\right| \geq (a - b)^2$ for $a, b > 0$, we find that

$$(v_\varepsilon(x) - v_\varepsilon(0))^2 \leq \left|(1 - v_\varepsilon(x))^2 - (1 - v_\varepsilon(0))^2\right| \leq \mathcal{C}\varepsilon.$$

Consequently, $v_\varepsilon(x) = v_\varepsilon(0) + O(\sqrt{\varepsilon})$. This shows, by choosing x to be the maximum point of v_ε, that $v_\varepsilon(0)$ is nearly maximal: it is the maximum value of v_ε up to $\mathcal{C}\sqrt{\varepsilon}$. Now, recalling the estimate for the maximum value M_ε of v_ε in the proof of Lemma 3.5, we have

$$v_\varepsilon(0) + O(\sqrt{\varepsilon}) = M_\varepsilon \geq 1 - \mathcal{C}\sqrt{\varepsilon},$$

and the proof of (3.11) is complete. □

Relation (3.11) is crucial in establishing the following result.

Lemma 3.9. *For all* $x \in [0, L]$*, we have*

$$\left|\frac{(1 - v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - \varepsilon(v_\varepsilon')^2 + \lambda(u_\varepsilon - g_\varepsilon)^2\right| \leq \mathcal{C} \tag{3.13}$$

and

$$|F_\varepsilon u_\varepsilon'| \leq \frac{\mathcal{C}}{\varepsilon}. \tag{3.14}$$

Proof. From (3.4), it suffices to show that $\frac{(1-v_\varepsilon(0))^2}{\varepsilon}$ is bounded. From the relation (3.11), we find that $1 \geq v_\varepsilon(0) \geq 1 - \mathcal{C}\sqrt{\varepsilon}$ and hence the boundedness of $\frac{(1-v_\varepsilon(0))^2}{\varepsilon}$ follows.

From (3.13) and the gradient bound of Lemma 3.4, the inequality (3.14) follows easily. □

Remark 3.10. Inequality (3.13) gives a uniform bound for the generalized discrepancy measure

$$d_\varepsilon(x) = \frac{(1 - v_\varepsilon)^2}{\varepsilon} - (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - \varepsilon(v_\varepsilon')^2 + \lambda(u_\varepsilon - g_\varepsilon)^2.$$

Its analogue in [13] is a constant, uniformly bounded in ε.

3.3. **Definition of v-jump.** Inequality (3.10) motivates the following.

Definition 3.11. *Let $x_\varepsilon \in [0,L]$ be a critical point of v_ε. We say that x_ε is a v- jump if $v_\varepsilon(x_\varepsilon) \le \mathcal{C}\varepsilon^{1/2}$ for some positive constant $\mathcal{C}$ independent of ε. We will say that x is a limit of v-jumps (or a limiting v-jump) if it is a limit of a family (x_ε) of v-jumps as $\varepsilon \to 0$.*

An important fact in the proofs of the main results is that v-jumps accumulate at only a finite number of points, as stated in the following.

Proposition 3.12. *There are a finite number j of points on $[0,L]$, denoted by $x_1, ..., x_j$, where $0 \le x_1 \le \cdots \le x_j \le L$, that are limits of v-jumps. For any compact set $K \subset [0,L] \backslash \{x_1,..,x_j\}$, there is a positive constant $\mathcal{C}_K$ depending only on K such that, for all ε small, we have $v_\varepsilon(x) \ge 1 - \mathcal{C}_K\varepsilon$ for all $x \in K$.*

Proof. Let x be a limit of v-jumps; i.e., there exists x_ε such that $x_\varepsilon \to x$ and $v_\varepsilon(x_\varepsilon) \le \mathcal{C}\sqrt{\varepsilon}$. Fix $\delta > 0$. Then for ε sufficiently small, $x_\varepsilon \in I_\delta = [x-\delta, x+\delta]$. Let T_ε be the maximum value of v_ε on I_δ. Then, the easy estimate

$$\mathcal{C} \ge \int_{I_\delta} \frac{(1-v)^2}{\varepsilon} \ge \int_{I_\delta} \frac{(1-T_\varepsilon)^2}{\varepsilon} = \frac{2\delta}{\varepsilon}(1-T_\varepsilon)^2$$

gives

$$T_\varepsilon \ge 1 - \mathcal{C}\sqrt{\varepsilon/\delta}.$$

Let $y_\varepsilon \in I_\delta$ be such that $v_\varepsilon(y_\varepsilon) = T_\varepsilon$ and, without loss of generality, assume that $y_\varepsilon > x_\varepsilon$. We can estimate

$$\begin{aligned}\int_{I_\delta} \varepsilon(v_\varepsilon^{'})^2 + \frac{(1-v_\varepsilon)^2}{\varepsilon} &\ge \int_{x_\varepsilon}^{y_\varepsilon} 2\left|v_\varepsilon^{'}(1-v_\varepsilon)\right| \ge \int_{x_\varepsilon}^{y_\varepsilon} (2v_\varepsilon - v_\varepsilon^2)' \\ &= [2v_\varepsilon(y_\varepsilon) - (v_\varepsilon(y_\varepsilon))^2] - [2v_\varepsilon(x_\varepsilon) - (v_\varepsilon(x_\varepsilon))^2] \ge 1 - \mathcal{C}\sqrt{\varepsilon}.\end{aligned}$$

This inequality in a small neighborhood of a limiting v-jump is crucial in proving the finiteness of limiting v-jumps. Suppose that there is an infinite number of them $x_1, x_2, \cdots$. Then, given a uniform bound $\mathcal{C} < \infty$ on the energies $AT_\varepsilon(u_\varepsilon, v_\varepsilon)$, we choose $N > 2\mathcal{C} + 10$. We cover each point x_i ($i = 1, \cdots, N$) by a small neighborhood $I_{i,\delta}$ for δ sufficiently small such that $I_{i,\delta} \cap I_{j,\delta} = \emptyset$ if $i \ne j$. Thus,

$$\mathcal{C} \ge AT_\varepsilon(u_\varepsilon, v_\varepsilon) \ge \sum_{i=1}^{N} \int_{I_{i,\delta}} \varepsilon(v_\varepsilon^{'})^2 + \frac{(1-v_\varepsilon)^2}{\varepsilon} \ge N(1-\mathcal{C}\sqrt{\varepsilon}) > N/2 > \mathcal{C}$$

for ε sufficiently small. This is a contradiction and hence concludes the proof of the first statement.

Now, we prove the second statement. We start with the following.

Claim. Fix a positive number $\alpha < 1/2$. Then, for any compact set $K \subset [0,L]\backslash\{x_1, .., x_j\}$, we have $v_\varepsilon(x) \geq 1 - \varepsilon^\alpha$ for all $x \in K$ when ε is small.

We argue by contradiction. Suppose that there exists a compact set $K \subset [0,L]\backslash\{x_1, \cdots, x_j\}$ such that for all $n \in N$, there exist $\varepsilon_n \leq 1/n$ and $x_n \in K$ with the property that $v_{\varepsilon_n}(x_n) \leq 1 - \varepsilon_n^\alpha$. After extracting a subsequence, we can assume that $x_n \to x \in K$. Let $\delta = \frac{1}{4}\min\{|x - x_i| : 1 \leq i \leq j\}$. Consider the intervals $I_\delta = [x - \delta, x + \delta]$ and $I_{2\delta} = [x - 2\delta, x + 2\delta]$. Then, $I_{2\delta} \cap \{x_1, \cdots, x_k\} = \emptyset$. For n sufficiently large, $x_n \in I_\delta$. As in the proof of the first statement, there exist $y_n \in [x_n - \delta, x_n)$ and $z_n \in (x_n, x_n + \delta]$ such that $v_{\varepsilon_n}(y_n), v_{\varepsilon_n}(z_n) \geq 1 - \mathcal{C}\sqrt{\varepsilon_n/\delta}$. We were able to choose $y_n \neq x_n$ because $v_{\varepsilon_n}(x_n) \leq 1 - \varepsilon_n^\alpha \leq 1 - \mathcal{C}\sqrt{\varepsilon_n/\delta}$, due to the fact that $\alpha < 1/2$. Similar arguments give the inequality $z_n \neq x_n$. For n sufficiently large, $[y_n, z_n] \subset [x_n - \delta, x_n + \delta] \subset I_{2\delta}$. In each interval $I_n = [y_n, z_n]$, there must be a minimum point t_n of v_{ε_n} such that $t_n \notin \{y_n, z_n\}$ and $v_{\varepsilon_n}(t_n) \leq v_{\varepsilon_n}(x_n) \leq 1 - \varepsilon_n^\alpha < 1 - \mathcal{C}\sqrt{\varepsilon_n}$ for n sufficiently large. Therefore, by Lemma 3.8, $v_{\varepsilon_n}(t_n) \leq \mathcal{C}\sqrt{\varepsilon_n}$. In other words, t_n is a v-jump. Because $t_n \in [x - 2\delta, x + 2\delta] \subset K$, we can assume, after extraction of a subsequence, that $t_n \to t \in K$. This means that K contains a limiting v-jump. However, this is a contradiction with the definition of K and the x_i's, completing the proof of the Claim.

Now, we employ the maximum principle together with the lower bounds on v_ε established in the Claim to complete the proof of the second statement. Indeed, choose larger compact sets $K_1 \subset K_2 \subset [0,L]\backslash\{x_1, \cdots, x_j\}$ such that $K \subset K_1$, $\mathrm{dist}(K, \partial K_1) > 0$ and $\mathrm{dist}(K_1, \partial K_2) > 0$. Choose ε sufficiently small such that $v_\varepsilon(x) \geq 1 - \varepsilon^{1/4}$ for all $x \in K$. Now, consider a smooth function $\xi \in C^\infty(K_2)$ such that $1 - \varepsilon^{1/4} \leq \xi \leq 1 - c_1\varepsilon/2$ in K_2, $\xi = 1 - c_1\varepsilon/2$ in K, and $\xi = 1 - \varepsilon^{1/4}$ in $K_2\backslash K_1$, where c_1 is a positive constant to be chosen later. Assume that $\inf_K v_\varepsilon \leq 1 - c_1\varepsilon$. Let h_ε be a function defined on K_2 by $h_\varepsilon = v_\varepsilon - \xi$. Then, the function h_ε satisfies $h_\varepsilon \geq 0$ on ∂K_2, $\inf_K h_\varepsilon \leq -c_1\varepsilon/2$. Thus, h_ε has an interior minimum point at, say, $x_0 \in K_2$. At x_0, one has $v_\varepsilon(x_0) \geq 1 - \varepsilon^{1/4}$, $h_\varepsilon = v_\varepsilon(x_0) - \xi(x_0) \leq -c_1\varepsilon/2$ and

$$\begin{aligned} 0 \leq \varepsilon\Delta h_\varepsilon(x_0) &= \varepsilon\Delta v_\varepsilon(x_0) - \varepsilon\Delta\xi(x_0) \\ &= v_\varepsilon(x_0)(u_\varepsilon'(x_0))^2 + \frac{v_\varepsilon(x_0) - 1}{\varepsilon} - \varepsilon\Delta\xi(x_0) \\ &= \frac{v_\varepsilon(x_0)}{(\eta_\varepsilon + v_\varepsilon^2(x_0))^2}(F_\varepsilon(x_0))^2 + \frac{h(x_0) + \xi(x_0) - 1}{\varepsilon} - \varepsilon\Delta\xi(x_0) \end{aligned}$$

$$\begin{aligned} &\leq \mathcal{C} + \frac{-c_1\varepsilon/2 + 1 - c_1\varepsilon/2 - 1}{\varepsilon} + \varepsilon\,|\Delta\xi(x_0)| \\ &\leq \mathcal{C} - c_1 + \varepsilon \sup_{K_2} |\Delta\xi|\,. \end{aligned}$$

However, the last term is negative provided that c_1 is large enough. Thus, one must have $v_\varepsilon(x) \geq 1 - \mathcal{C}_K\varepsilon$ for all $x \in K$ as desired. □

A simple consequence of the bound (3.6) and Proposition 3.12 is the following.

Lemma 3.13. *Introduce, if necessary, $x_0 = 0$ and $x_{j+1} = L$. Then, on each interval $(x_i, x_{i+1})(0 \leq i \leq j)$, the limit function u solves the equation*

$$u'' = \lambda(u - g).$$

If $v_\varepsilon(x) \geq \delta > 0$ for all $x \in [0, L]$ and $\varepsilon > 0$, then the limit function u solves the following equation on $(0, L)$:

$$u'' = \lambda(u - g), \qquad u'(0) = u'(L) = 0.$$

Proof. We first prove the second statement. Suppose that $v_\varepsilon(x) \geq \delta > 0$ for all $x \in [0, L]$ and $\varepsilon > 0$. For all $\varphi \in C^1(0, L)$, one has from the second and third equations of (2.4)

$$\int_0^L -u_\varepsilon'(\eta_\varepsilon + v_\varepsilon(x)^2)\varphi' = \int_0^L \lambda(u_\varepsilon - g_\varepsilon)\varphi. \tag{3.15}$$

From (3.6), we deduce that u_ε' is bounded. Thus u_ε is bounded in $H^1(0, L)$ and, up to extraction, weakly converges to $u \in H^1(0, L)$. Now, letting $\varepsilon \to 0$ in (3.15) yields

$$\int_0^L -u'\varphi' = \int_0^L \lambda(u - g)\varphi \ \forall\varphi \in C^1(0, L).$$

This is equivalent to what needs to be established.

We now prove the first statement of the lemma. Let K be any compact set of (x_i, x_{i+1}). Then, by Proposition 3.12, there is $\delta_K > 0$ such that $v_\varepsilon(x) \geq \delta_K > 0$ for all $x \in K$ and $\varepsilon > 0$. Now, we argue similarly as above, with $\varphi \in C_0^1(K)$ to conclude that u satisfies the equation $u'' = \lambda(u - g)$ on (x_i, x_{i+1}). □

4. PROOF OF THEOREM 2.1

In this section, we prove Theorem 2.1. It will follow from the characterization of points of discontinuity of u in Lemma 4.1 and the relation between u and F when F does not vanish, as stated in Lemma 4.2.

Lemma 4.1. *Let x_0 be an arbitrary point in $[0, L]$. If u is discontinuous at x_0 then $F(x_0) = 0$.*

Lemma 4.2. *Let x_0 be an arbitrary point in $[0, L]$. If $F(x_0) \neq 0$ then u is differentiable at x_0 and $u^{'}(x_0) = F(x_0)$.*

Proof of Theorem 2.1. By Lemma 3.2, after extracting a subsequence, v_ε converges strongly to 1 in $L^2((0, L))$ and u_ε converges strongly in $L^4((0, L))$ to a function $u \in BV((0, L))$. We now prove that the pair $(u, 1)$ is a critical point of the Mumford-Shah functional.

Let $X_1 < \cdots < X_k$ be all points of discontinuity of u in $(0, L)$. By Lemma 3.13, we know that k is finite and $X_i \in \{x_1, \cdots, x_j\}$ for all $i \in \{1, \cdots, k\}$.

Now, fix $i \in \{0, \cdots, k\}$. We have to prove that $u^{''} = \lambda(u-g)$ in (X_i, X_{i+1}) with Neumann boundary conditions. First, we claim that $F(X_l) = 0$ for all $l \in \{i, i+1\}$. Indeed, if X_l is a discontinuity point of u in $(0, L)$, then, from Lemma 4.1, we know that $F(x_l) = 0$. Otherwise, X_j is a boundary point of $[0, L]$ and hence $F(X_j) = 0$ because $F(0) = F(L) = 0$.

Consider the case where F does not vanish in (X_i, X_{i+1}). Then, by Lemma 4.2, $u^{'} = F$ in (X_i, X_{i+1}), but $F' = \lambda(u - g)$. Thus $u^{''} = \lambda(u - g)$ in (X_i, X_{i+1}). The Neumann boundary conditions follow from the fact that $F(X_i) = F(x_{i+1}) = 0$ and the continuity up to the boundary of u'.

The case where F vanishes inside (X_i, X_{i+1}) is more involved. It is rather easy in the case where F vanishes at only a finite number of points inside (X_i, X_{i+1}). In this case, the argument is as follows. Without loss of generality, we can assume that F only vanishes at Y. Then, we can prove that u solves $u^{''} = \lambda(u - g)$ with Neumann boundary conditions on each interval (X_i, Y) and (Y, X_{i+1}). Because u is continuous at Y with zero derivative there, it solves the desired equation on the whole interval (X_i, X_{i+1}).

The general argument makes use of Lemma 3.13 and the assumption, which will be justified below, that F is not identically zero on any subinterval in $I = (X_i, X_{i+1})$. There are two cases.

Consider the first case where there are no limiting v-jumps in (X_i, X_{i+1}). Then Lemma 3.13 tells us that, in I, $u^{''} = \lambda(u - g)$. Thus, in I, $u^{'} = F + c$ for some constant c. Because F must be different from zero at some point X in I, we have, by Lemma 4.2, $u^{'}(X) = F(X)$ and hence $c = 0$. It follows that $u^{'} = F$ in I and, as above, we must have $u(X_i) = u^{'}(X_{i+1}) = 0$.

Now, we consider the second case where there are some limiting v-jumps in I. We only need to consider the case where there is only one limiting jump point X in I. Arguing as in the first case, one has $u^{'} = F$ in (X_i, X) and $u^{'} = F$ in (X, X_{i+1}). Because u is continuous at X with the same derivative on both sides of X, it solve the equation $u^{''} = \lambda(u-g)$ on the whole interval (X_i, X_{i+1}). Now, the Neumann boundary conditions easily follow.

Now, we indicate how to remove the assumption that F is not identically zero on any subinterval in $I = (X_i, X_{i+1})$. Note that, in the previous arguments, what is needed is the fact that $u^{'} = F$ between two consecutive limiting jump points x_i and x_{i+1}. We show how to get this identity in the case where F vanishes identically on $J = (x_i, x_{i+1})$. In this case, we must have $u = g$ on J and by Lemma 3.13, we have $u^{''} = 0$. Therefore, u, and hence g, is a linear function on J. On the other hand, on any interval $K = [y, z] \subset J$, we have $v_\varepsilon \geq \frac{1}{2}$, by Proposition 3.12. Because $F_\varepsilon(x) = u^{'}_\varepsilon(x)(\eta_\varepsilon + v^2_\varepsilon(x)) \rightrightarrows F(x) = 0$ on K, we find that $u^{'}_\varepsilon(x) \rightrightarrows 0$ on K. Thus, by lower-semicontinuity,

$$0 = \liminf_{\varepsilon\to 0} \int_K \left| u^{'}_\varepsilon(x) \right| dx \geq \int_K |Du|\,,$$

implying that u is a constant on K. By its linearity on J, u is also a constant on J. Therefore, on J, we have $u^{'} = 0 = F$, as desired. □

Remark 4.3. The above proof also shows that, for all $x \in [0, L]$, one has $F(x) = u'(x)$.

The remaining part of this section is devoted to the proof of Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. Assume by contradiction that u is discontinuous at x_0 but $|F(x_0)| = \alpha > 0$. Note that, by Lemma 3.13, x_0 must be a limit of v-jumps. This means that there are $x_\varepsilon' s$ converging to x_0 such that $v_\varepsilon(x_\varepsilon) \leq \mathcal{C}\sqrt{\varepsilon}$. Our proof, as in [13], is then based on the estimate on the size of the set $\{v_\varepsilon \leq M\sqrt{\varepsilon}\}$, for M large enough. Because the F_ε's are Lipschitz and converge uniformly to F, we can choose positive numbers ε_0 and δ_0 sufficiently small so as to have, for all $\delta \leq \delta_0$,

$$|F_\varepsilon(x)| \geq \frac{\alpha}{2}\ \forall\varepsilon \leq \varepsilon_0, \forall x \in I_\delta := (x_0 - \delta, x_0 + \delta). \tag{4.1}$$

Recalling $u^{'}_\varepsilon = F_\varepsilon/(\eta_\varepsilon + v^2_\varepsilon)$, we rewrite the first equation of (2.4) as

$$-\varepsilon v^{''}_\varepsilon + \frac{v_\varepsilon F^2_\varepsilon}{(\eta_\varepsilon + v^2_\varepsilon)^2} + \frac{v_\varepsilon - 1}{\varepsilon} = 0.$$

Integrate this equation over $K_{\varepsilon,\delta} := \{v_\varepsilon \leq M\sqrt{\varepsilon}\} \cap I_\delta$ to obtain

$$\int_{K_{\varepsilon,\delta}} \varepsilon v^{''}_\varepsilon dx = \int_{K_{\varepsilon,\delta}} \frac{v_\varepsilon F^2_\varepsilon}{(\eta_\varepsilon + v^2_\varepsilon)^2} dx + \int_{K_{\varepsilon,\delta}} \frac{v_\varepsilon - 1}{\varepsilon} dx. \tag{4.2}$$

We now claim the following.

$$\text{The number of connected components of } K_{\varepsilon,\delta} \text{ is bounded by } \mathcal{C}\varepsilon^{-1/3}. \tag{4.3}$$

On each connected component (a_i, b_i) of $K_{\varepsilon,\delta}$, we obtain, by virtue of the gradient bound of Lemma 3.4,

$$\Big|\int_{a_i}^{b_i} \varepsilon v_\varepsilon^{''}\Big| = \Big|\varepsilon v_\varepsilon^{'}(b_i) - \varepsilon v_\varepsilon^{'}(a_i)\Big| \leq \mathcal{C}.$$

Then, with (4.3), the left-hand side of (4.2) is bounded from above by $\mathcal{C}\varepsilon^{-1/3}$.

Next, we estimate the size of $K_{\varepsilon,\delta}$. Recall from Lemma 3.9 that $|F_\varepsilon u_\varepsilon^{'}| \leq \frac{\mathcal{C}}{\varepsilon}$ on $[0, L]$. This gives

$$\eta_\varepsilon + v_\varepsilon^2(x) = \frac{F_\varepsilon^2(x)}{F_\varepsilon(x)u_\varepsilon^{'}(x)} \geq \frac{(\alpha/2)^2}{\mathcal{C}/\varepsilon} = \frac{\alpha^2\varepsilon}{\mathcal{C}} \ \forall x \in K_{\varepsilon,\delta}. \tag{4.4}$$

It follows that, for ε sufficiently small, $v_\varepsilon^2(x) \gg \eta_\varepsilon$, for all $x \in K_{\varepsilon,\delta}$. Thus, by (4.1), the right-hand side of (4.2) is bounded from below by

$$\int_{K_{\varepsilon,\delta}} \frac{\mathcal{C}(\alpha/2)^2 v_\varepsilon}{v_\varepsilon^4} dx - \frac{|K_{\varepsilon,\delta}|}{\varepsilon} \quad \geq \quad \frac{\mathcal{C}\alpha^2\,|K_{\varepsilon,\delta}|}{M^3\varepsilon^{3/2}} - \frac{|K_{\varepsilon,\delta}|}{\varepsilon} \geq \frac{\mathcal{C}\alpha^2\,|K_{\varepsilon,\delta}|}{M^3\varepsilon^{3/2}}.$$

Therefore, for ε sufficiently small, $\mathcal{C}\varepsilon^{-1/3} \geq \frac{\mathcal{C}\alpha^2\left|K_{\varepsilon,\delta}\right|}{M^3\varepsilon^{3/2}}$, implying in turn that

$$|K_{\varepsilon,\delta}| \leq \frac{\mathcal{C}M^3\varepsilon^{7/6}}{\alpha^2}. \tag{4.5}$$

These estimates combined with (4.1) give

$$\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0}\int_{x_0-\delta}^{x_0+\delta}\left|u_\varepsilon^{'}\right| = 0. \tag{4.6}$$

Indeed, we split

$$\begin{aligned}\int_{x_0-\delta}^{x_0+\delta}\left|u_\varepsilon^{'}\right| &= \int_{K_{\varepsilon,\delta}}\left|u_\varepsilon^{'}\right| + \int_{I_\delta\cap\{M\sqrt{\varepsilon}\leq v_\varepsilon\leq 1/2\}}\left|u_\varepsilon^{'}\right| + \int_{I_\delta\cap\{v_\varepsilon\geq 1/2\}}\left|u_\varepsilon^{'}\right| \\ &:= J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon.\end{aligned}$$

We bound each of the terms on the right-hand side from above.

The first term J_1^ε. From Lemma 3.9 and the inequality (4.1), we note that, on $K_{\varepsilon,\delta}$,

$$\left|u_\varepsilon^{'}\right| \leq \frac{\mathcal{C}}{\varepsilon F_\varepsilon} \leq \frac{\mathcal{C}}{\varepsilon\alpha/2} = \frac{\mathcal{C}}{\varepsilon\alpha}.$$

So, by (4.5), the first term is bounded by

$$J_1^\varepsilon \le |K_{\varepsilon,\delta}| \left\| u_\varepsilon' \right\|_{L^\infty(K_{\varepsilon,\delta})} \le \frac{\mathcal{C}M^3\varepsilon^{7/6}}{\alpha^2}\frac{\mathcal{C}}{\varepsilon\alpha} = \frac{\mathcal{C}M^3\varepsilon^{1/6}}{\alpha^3}. \tag{4.7}$$

The second term J_2^ε. From the energy bound (2.5), it follows that

$$\begin{aligned}\mathcal{C} &\ge \int_0^L \frac{(1-v_\varepsilon)^2}{\varepsilon}dx \ge \int_{\{v_\varepsilon \le 1/2\}} \frac{(1-v_\varepsilon)^2}{\varepsilon}dx \\ &\ge \int_{\{v_\varepsilon\le 1/2\}} \frac{1}{4\varepsilon}dx = \frac{1}{4\varepsilon}\left|\{v_\varepsilon \le 1/2\}\right|,\end{aligned}$$

yielding the estimate

$$\left|\{M\sqrt{\varepsilon} \le v_\varepsilon \le 1/2\}\right| \le \left|\{v_\varepsilon \le 1/2\}\right| \le \mathcal{C}\varepsilon. \tag{4.8}$$

On $\{M\sqrt{\varepsilon} \le v_\varepsilon \le 1/2\}$, we find that

$$\left|u_\varepsilon'\right| = \frac{|F_\varepsilon|}{\eta_\varepsilon + v_\varepsilon^2(x)} \le \frac{|F_\varepsilon|}{v_\varepsilon^2} \le \frac{\mathcal{C}}{M^2\varepsilon}. \tag{4.9}$$

Inserting inequalities (4.8), (4.9) into the expression for the second term produces the following uniform upper bound:

$$J_1^\varepsilon \le \int_{\{M\sqrt{\varepsilon}\le v_\varepsilon \le \frac{1}{2}\}} \left|u_\varepsilon'\right| \le \frac{\mathcal{C}}{M^2\varepsilon}\mathcal{C}\varepsilon = \frac{\mathcal{C}}{M^2}. \tag{4.10}$$

The third term J_3^ε. It is easy to see that the third term is bounded by

$$J_1^\varepsilon \le \int_{I_\delta \cap \{v_\varepsilon \ge 1/2\}} \left|u_\varepsilon'\right| \le \delta\mathcal{C}. \tag{4.11}$$

Coalescing (4.7), (4.11) and (4.10) and letting ε tend to 0 finally leads to

$$\liminf_{\varepsilon\to 0} \int_{x_0-\delta}^{x_0+\delta} \left|u_\varepsilon'\right| \le \liminf_{\varepsilon \to 0}\left(\frac{\mathcal{C}M^3\varepsilon^{1/2}}{\alpha^3} + \frac{\mathcal{C}}{M^2} + \delta\mathcal{C}\right) = \frac{\mathcal{C}}{M^2} + \delta\mathcal{C}.$$

Letting M tend to ∞ and δ tend to 0, we obtain (4.6).

We can now complete our proof. Recall from Lemma 3.2 that $u_\varepsilon \to u$ in $L^4((0,L))$. Thus, by lower semicontinuity, one has

$$0 = \liminf_{\delta\to 0}\liminf_{\varepsilon\to 0} \int_{x_0-\delta}^{x_0+\delta} |u_\varepsilon'| \ge \liminf_{\delta \to 0}\int_{x_0-\delta}^{x_0+\delta} |Du|.$$

This is, however, a contradiction with the assumption that u is discontinuous at x_0. Therefore, we must have $F(x_0) = 0$ as desired.

It remains to prove (4.3). We will use the lower bound on v_ε in (4.4). Then, for ε sufficiently small, we have on K

$$\varepsilon v_\varepsilon^{''} = \frac{v_\varepsilon F_\varepsilon^2}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{\varepsilon} \geq \frac{\mathcal{C}\alpha^2}{v_\varepsilon^3} - \frac{1}{\varepsilon}. \tag{4.12}$$

Thus $v_\varepsilon^{''}(x) \geq 0$ if $v_\varepsilon(x) \leq \mathcal{C}\varepsilon^{1/3}$. Note that, for ε small, $M\sqrt{\varepsilon} \leq \mathcal{C}\varepsilon^{1/3}$. By (4.12), v_ε is convex in each connected component $D_\varepsilon^i = (a_i, b_i)$ of $K_{\varepsilon,\delta}$. So, once v_ε goes above $M\sqrt{\varepsilon}$, it cannot go back down below that value without reaching $\mathcal{C}\varepsilon^{1/3}$ first. Thus, the number of such components is certainly no greater than the number of connected components E_ε^i of $E_\varepsilon := \{M\sqrt{\varepsilon} \leq v_\varepsilon \leq \mathcal{C}\varepsilon^{1/3}\}$. To complete the proof of (4.3), it suffices to show that the number of connected components of E_ε is bounded by $\mathcal{C}\varepsilon^{-1/3}$.

Denote by s_ε^i the length of E_ε^i. On each E_ε^i, there exist $c_i, d_i \in E_\varepsilon^i$ such that $c_i < d_i$, $v_\varepsilon(c_i) = M\sqrt{\varepsilon}$, and $v_\varepsilon(d_i) = C\varepsilon^{1/3}$. Then, application of the mean value theorem yields, in view of the gradient bound on v_ε in Lemma 3.4, that, for some $z_i \in (c_i, d_i)$, $v_\varepsilon^{'}(z_i) = \frac{\mathcal{C}\varepsilon^{1/3} - M\sqrt{\varepsilon}}{d_i - c_i} \leq \frac{\mathcal{C}}{\varepsilon}$. Therefore,

$$s_\varepsilon^i \geq d_i - c_i \geq \mathcal{C}\varepsilon(\varepsilon^{1/3} - \varepsilon^{1/2}). \tag{4.13}$$

Next, observe that, for ε sufficiently small, $E_\varepsilon \subset \{v_\varepsilon \leq 1/2\}$. Recalling (4.8), we obtain

$$\sum_i s_\varepsilon^i \leq |\{v_\varepsilon \leq 1/2\}| \leq \mathcal{C}\varepsilon. \tag{4.14}$$

Combining (4.13) and (4.14) yields the desired bound on the number of connected components of E_ε. $\square$

Proof of Lemma 4.2. We will use the same notation as in Lemma 4.1. Because F is continuous, there is $\gamma > 0$ such that F does not vanish on $(x_0 - \gamma, x_0 + \gamma)$. From Lemma 4.1, we deduce that u is continuous on $(x_0 - \gamma, x_0 + \gamma)$. Because $u_\varepsilon(x) \to u(x)$ for almost every $x \in (0, L)$, we can choose $\delta_i \to 0$ such that $u_\varepsilon(x_0 + \delta_i) \to u(x_0 + \delta_i)$ and $u_\varepsilon(x_0 - \delta_i) \to u(x_0 - \delta_i)$. Denote $I_i = (x_0 - \delta_i, x_0 + \delta_i)$. As above, assuming $|F(x_0)| = \alpha > 0$, we have

$$\begin{aligned} &u_\varepsilon(x_0 + \delta_i) - u_\varepsilon(x_0 - \delta_i) = \int_{x_0 - \delta_i}^{x_0 + \delta_i} u_\varepsilon^{'} \\ &= \int_{K_{\varepsilon,\delta_i}} u_\varepsilon^{'} + \int_{I_i \cap \{M\sqrt{\varepsilon} \leq v_\varepsilon \leq 1/2\}} u_\varepsilon^{'} + \int_{I_i \cap \{v_\varepsilon \geq 1/2\}} u_\varepsilon^{'} \\ &= O(\frac{\mathcal{C}M^3\varepsilon^{1/6}}{\alpha^3}) + O(\frac{\mathcal{C}}{M^2}) + \int_{I_i} u_\varepsilon^{'} \chi_{\{v_\varepsilon \geq 1/2\}}. \end{aligned}$$

Because $u_\varepsilon^{'}(x) \to F(x)$ for almost every $x \in (0,L)$ and $\chi_{\{v_\varepsilon \geq \frac{1}{2}\}}(x) \to \chi_{(0,L)}(x)$ for almost every $x \in (0,L)$, it follows that $w_\varepsilon(x) := u_\varepsilon^{'}(x)\chi_{\{v_\varepsilon \geq \frac{1}{2}\}}(x) \to F(x)\chi_{(0,L)}(x)$ for almost every $x \in (0,L)$. On the other hand, for all $x \in (0,L)$,

$$|w_\varepsilon(x)| = \frac{|F_\varepsilon(x)|}{\eta_\varepsilon + v_\varepsilon^2(x)}\chi_{\{v_\varepsilon \geq \frac{1}{2}\}}(x) \leq 4\,|F_\varepsilon(x)| \leq \mathcal{C}.$$

Hence, by Lebesgue's dominated convergence theorem,

$$\int_{I_i} u_\varepsilon^{'}\chi_{\{v_\varepsilon \geq 1/2\}} = \int_{I_i} w_\varepsilon dx \to \int_{I_i} F\chi_{(0,L)}dx.$$

We let ε tend to 0 and M tend to ∞ and obtain

$$u(x_0+\delta_i) - u(x_0-\delta_i) = \int_{x_0-\delta_i}^{x_0+\delta_i} F$$

and the result follows. □

5. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. Its proof is divided into several subsections for clarity of presentation. In subsection 5.1, we prove the first part of Theorem 2.3. The proof of the second part will follow from subsections 5.2 and 5.4.

5.1. **Convergence of the Dirichlet energy.** We first prove that the convergence result in Theorem 2.1 implies the nonconcentration of the energy density $(\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2$ in the limit.

Lemma 5.1. *The limit measure of* $(\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 dx$ *is* $\left((u')^2\right)dx$ *where* u' *is the approximate gradient of* u.

Proof. Let $\varphi \in C^\infty(0,L)$. Consider

$$I_\varepsilon(\varphi) = \int_0^L (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2\varphi \equiv \int_0^L F_\varepsilon u_\varepsilon^{'}\varphi dx. \tag{5.1}$$

Integrating by parts and using the zero boundary conditions on F_ε, we get

$$\begin{aligned} I_\varepsilon(\varphi) &= -\int_0^L (F_\varepsilon\varphi)^{'}u_\varepsilon = -\int_0^L (F_\varepsilon^{'}\varphi + F_\varepsilon\varphi^{'})u_\varepsilon dx \\ &= -\int_0^L \left(\lambda(u_\varepsilon - g_\varepsilon)\varphi + F_\varepsilon\varphi^{'}\right)u_\varepsilon. \end{aligned}$$

Now, letting $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} I_\varepsilon(\varphi) = -\int_0^L \left(\lambda(u-g)\varphi + F\varphi^{'}\right) u. \tag{5.2}$$

Now, using Theorem 2.1, we will show that the right-hand side, $I(\varphi)$, of the above equation is $\int_0^L \left((u')^2\right)\varphi dx$. Indeed, we decompose

$$\begin{aligned} I(\varphi) &= -\int_0^L \lambda(u-g)u(x)\varphi(x)dx + \sum_{i=0}^k I_i \\ &\equiv -\int_0^L \lambda(u-g)u(x)\varphi(x)dx + \sum_{i=0}^k \int_{X_i}^{X_{i+1}} -Fu\varphi^{'}dx. \end{aligned}$$

For each $i \in \{0, 1, \cdots, k\}$, we use integration by parts again, recalling that $u^{''} = \lambda(u-g)$, $u^{'} = F$ on (X_i, X_{i+1}), and $u^{'}(X_i) = u^{'}(X_{i+1}) = F(X_i) = F(X_{i+1}) = 0$. Therefore,

$$\begin{aligned} I_i &= \int_{X_i}^{X_{i+1}} (Fu)^{'}\varphi dx = \int_{X_i}^{X_{i+1}} (F^{'}u + Fu^{'})\varphi dx \\ &= \int_{X_i}^{X_{i+1}} \left(\lambda(u-g)u + (u^{'})^2\right)\varphi dx. \end{aligned}$$

Thus,

$$I(\varphi) = \sum_{i=0}^k \int_{X_i}^{X_{i+1}} ((u^{'})^2)\varphi = \int_0^L (u^{'})^2\varphi. \tag{5.3}$$

Note that $(u^{'})^2$ makes sense in each interval (X_i, X_{i+1}) and equals zero on end points $X_i^{\prime}$s by Theorem 2.1. □

Now, we improve the above convergence result to that of weak $L^1(\Omega)$.

Lemma 5.2. *$(\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2$ converges weakly in $L^1((0, L))$ to $(u')^2$.*

Proof. For each positive integer k, set $E_k = \cup_{i=1}^j [x_i - \frac{1}{k}, x_i + \frac{1}{k}]$. Then $|E_k| = \frac{2j}{k} \to 0$, as $k \to \infty$. From the convergence result in Lemma 5.1, we deduce that

$$\sup_\varepsilon \int_{E_k} (\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 dx \le \int_{E_k} (u^{'})^2 dx \tag{5.4}$$

for each k fixed (cf. Theorem 1, page 54, [11]). On the other hand, Proposition 3.12 and Remark 4.3 show that

$$(\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 = \frac{F_\varepsilon^2}{\eta_\varepsilon + v_\varepsilon^2} \rightrightarrows F^2 = (u^{'})^2 \text{ on } [0, L]\backslash E_k. \tag{5.5}$$

Now, let $\varphi \in L^\infty((0,L))$, $\|\varphi\|_{L^\infty((0,L))} \leq 1$. Using (5.4) and (5.5), we have, for each fixed k,

$$\begin{aligned}
&\limsup_{\varepsilon} \Big| \int_0^L \Big((\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - (u')^2 \Big) \varphi dx \Big| \\
&\leq \limsup_{\varepsilon} \Big| \int_{[0,L]\setminus E_k} \Big((\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - (u')^2 \Big) \varphi dx \Big| \\
&\qquad + \limsup_{\varepsilon} \Big| \int_{E_k} \Big((\eta_\varepsilon + v_\varepsilon^2)(u_\varepsilon')^2 - (u')^2 \Big) \varphi dx \Big| \leq 2 \int_{E_k} (u')^2 dx.
\end{aligned}$$

Now, letting $k \to \infty$, we see that the right-hand side of the above inequality goes to zero and the assertion follows. □

5.2. **Nonconcentration of energy away from v-jumps.** We next establish that there is no concentration of energy for the two terms $\varepsilon(v_\varepsilon'(x))^2$ and $(1 - v_\varepsilon(x))^2/\varepsilon$ away from the limiting jump points $x_1, \cdots, x_j$.

Lemma 5.3. *For any compact subset $K \subset [0,L] \setminus \cup_{k=1,\ldots,j}\{x_k\}$, we have*

$$\int_K \left(\varepsilon(v_\varepsilon'(x))^2 + (1 - v_\varepsilon(x))^2/\varepsilon \right) dx \leq \mathcal{C}_K \varepsilon^{1/4},$$

where $\mathcal{C}_K$ may depend only on K.

Proof. For a given compact subset K of $[0,L] \setminus \cup_{k=1,\ldots j}\{x_k\}$, set $\delta := 1/4 \min_{k=1,\ldots,j} \operatorname{dist}(x_k, K)$. Let $U_\delta := \cup_{k=1,\ldots,j}(x_k - \delta, x_k + \delta)$ and let us denote $V_\delta = [0,L] \setminus U_\delta$. Consider $A_\varepsilon := \{x \in [0,L] : v_\varepsilon(x) \leq 1 - \varepsilon^{1/4}\}$. Then, for ε sufficiently small, we have by Proposition 3.12

$$V_\delta \cap A_\varepsilon = \emptyset. \tag{5.6}$$

Because $K \subset [0,L] \setminus U_\delta = V_\delta$, it suffices to prove that

$$\int_{V_\delta} \left(\varepsilon(v_\varepsilon'(x))^2 + (1 - v_\varepsilon(x))^2/\varepsilon \right) dx \leq \mathcal{C}_K \varepsilon^{1/4}. \tag{5.7}$$

Multiplying both sides of the first equation of (2.4) by $v_\varepsilon - 1$ and integrating over V_δ, we get

$$\begin{aligned}
&\int_{V_\delta} -\varepsilon v_\varepsilon''(x)(v_\varepsilon(x) - 1)dx + \int_{V_\delta} v_\varepsilon(x)(u_\varepsilon'(x))^2(v_\varepsilon(x) - 1)dx \\
&\qquad + \int_{V_\delta} \frac{(v_\varepsilon(x) - 1)^2}{\varepsilon} dx = 0.
\end{aligned} \tag{5.8}$$

Note that V_δ is a union of a finite ε-independent number J $(\leq j+1)$ of intervals on $[0,L]$: $V_\delta = \cup_{k=1,\dots,J}[a^k, b^k]$. Now, integrating by parts the first term of (5.8), and rearranging, one obtains

$$\begin{aligned}\int_{V_\delta} &\Big(\varepsilon(v_\varepsilon^{'}(x))^2 + \frac{(v_\varepsilon(x)-1)^2}{\varepsilon}\Big)dx \\ &= \sum_{k=1}^{J} \varepsilon\Big(v_\varepsilon^{'}(b^k)(v_\varepsilon(b^k)-1) - v_\varepsilon^{'}(a^k)(v_\varepsilon(a^k)-1)\Big) \\ &\quad + \int_{V_\delta} (u_\varepsilon^{'}(x))^2 v_\varepsilon(x)(1-v_\varepsilon(x))dx. \end{aligned} \tag{5.9}$$

By the definitions of A_ε and V_δ and (5.6), we have $|1-v_\varepsilon| \leq \varepsilon^{1/4}$ on V_δ. Furthermore, from (3.6), we deduce that $u_\varepsilon^{'}$ is bounded on V_δ. Combining this fact with the gradient bound for v_ε in Lemma 3.4 yields that the right-hand side of (5.9) is bounded from above by $\mathcal{C}_K \varepsilon^{1/4}$ for some constant $\mathcal{C}_K$ which may depend only on K. Hence the desired result stated in (5.7) follows. □

Remark 5.4. The previous lemma shows that the measure limits of $\varepsilon(v_\varepsilon'(x))^2\ dx$, and of $(v_\varepsilon(x)-1)^2/\varepsilon\ dx$ are Dirac masses concentrated at $x_1, \dots, x_j$.

Remark 5.5. The above proof is very similar to that of Lemma 6.1 in [13]. If we use the positivity of the discrepancy measure $\frac{(1-v_\varepsilon(x))^2}{\varepsilon} - \varepsilon(v_\varepsilon^{'}(x))^2$ in Lemma 5.6, then, also from Proposition 3.12, we get a stronger estimate

$$\int_K \left(\varepsilon(v_\varepsilon'(x))^2 + (1-v_\varepsilon(x))^2/\varepsilon\right) dx \leq 2\int_K (1-v_\varepsilon(x))^2/\varepsilon dx \leq 2\,|K|\,\mathcal{C}_K\varepsilon.$$

5.3. **Positivity of the discrepancy measure.** We now prove the positivity of the discrepancy measure $\frac{(1-v_\varepsilon(x))^2}{\varepsilon} - \varepsilon(v_\varepsilon^{'}(x))^2$ which is crucial in the proof of the equi-partition of energy in Lemma 5.7. Its positivity played a central role in establishing the equi-partition of energy in our previous paper [13].

Lemma 5.6. *For all* $x \in [0,L]$, $D_\varepsilon(x) := \frac{(1-v_\varepsilon(x))^2}{\varepsilon} - \varepsilon(v_\varepsilon^{'}(x))^2 \geq 0.$

Proof. If x is a critical point of v_ε then clearly $D_\varepsilon(x) \geq 0$. Suppose that x is not a critical point of v_ε. Then x lies between two consecutive critical points x^0 and x^1 of v_ε. Because there is no other critical point between x^0

and x^1, $v_\varepsilon(x)$ must lie between the two values $v_\varepsilon(x^0)$ and $v_\varepsilon(x^1)$. We can assume that $v_\varepsilon(x^0) \geq v_\varepsilon(x) \geq v_\varepsilon(x^1)$. Because

$$(D_\varepsilon(x))' = 2v_\varepsilon^{'}(x)(\frac{v_\varepsilon(x)-1}{\varepsilon} - \varepsilon v_\varepsilon^{"}(x)) = -2v_\varepsilon^{'}(x)v_\varepsilon(x)(u_\varepsilon^{'}(x))^2,$$

we can integrate (3.3) from x^0 to x to obtain

$$D_\varepsilon(x) = D_\varepsilon(x^0) + \int_{x^0}^{x} (D_\varepsilon(t))^{'} dt = \frac{(1-v_\varepsilon(x^0))^2}{\varepsilon} + \int_{x_0}^{x} -2(u_\varepsilon^{'})^2 v_\varepsilon^{'} v_\varepsilon.$$

If $x^0 < x$ then v_ε is decreasing in $[x^0, x]$. Thus $v_\varepsilon^{'}(t) \leq 0$ for all $t \in (x^0, x)$ and hence the positivity of the right-hand side of the above equation follows. If $x^0 > x$ then v_ε is increasing in $[x, x^0]$, and because

$$D_\varepsilon(x) = \frac{(1-v_\varepsilon(x^0))^2}{\varepsilon} + \int_{x}^{x^0} 2(u_\varepsilon^{'})^2 v_\varepsilon^{'} v_\varepsilon,$$

we also have the positivity of D_ε. □

5.4. **Equi-partition of energy.** We are now ready to establish the equi-partition property of energy.

Lemma 5.7.

$$\lim_{\varepsilon\to 0} \int_0^L |\varepsilon(v_\varepsilon'(x))^2 - (v_\varepsilon(x)-1)^2/\varepsilon| \, dx = 0. \tag{5.10}$$

Proof. Let K be a connected compact subset of $[0, L]\backslash\{x_1, \cdots, x_j\}$ with $|K| > 0$. Let x_0^ε be a global maximum of v_ε over K. Then, from Proposition 3.12, one has for ε small

$$v_\varepsilon(x_0^\varepsilon) \geq 1 - \varepsilon^{1/4}. \tag{5.11}$$

After extracting a subsequence of ε's, x_0^ε converges to $x_0 \in K$. Set $\delta := 1/4 \min_{k=1,\ldots,j} \operatorname{dist}(x_k, K)$. Then

$$K_1 = [x_0 - \delta, x_0 + \delta] \subset [0, L]\backslash\{x_1, \cdots, x_j\}. \tag{5.12}$$

Using the conservation law (3.3), we find that for all $x \in (0, L)$

$$\begin{aligned} D_\varepsilon(x) &= F_\varepsilon(x)u_\varepsilon^{'}(x) - \lambda(u_\varepsilon(x) - g_\varepsilon(x))^2 + \frac{(1-v_\varepsilon(x_0^2))^2}{\varepsilon} - \varepsilon(v_\varepsilon^{'}(x_0^\varepsilon))^2 \\ &+ \lambda(u_\varepsilon(x_0^\varepsilon) - g_\varepsilon(x_0^\varepsilon))^2 - F_\varepsilon(x_0^\varepsilon)u_\varepsilon^{'}(x_0^\varepsilon) + \int_{x_0^\varepsilon}^{x} -\lambda g_\varepsilon^{'}(t)[u_\varepsilon(t) - g_\varepsilon(t)]dt. \end{aligned} \tag{5.13}$$

Observe that, by Lemma 3.9, we also have

$$D_\varepsilon(x) = F_\varepsilon(x)u_\varepsilon^{'}(x) - \lambda(u_\varepsilon(x) - g_\varepsilon(x))^2 + d_\varepsilon(x)$$

where $\|d_\varepsilon\|_{L^\infty((0,L))} \leq \mathcal{C}$. It follows from Lemmata 5.1 and 3.2 that $D_\varepsilon(x)$ converges in the weak sense of Radon measures to $D(x)$. We are going to evaluate $D(x)$ by passing to the limit in the right-hand side of (5.13) in the weak sense of Radon measures.

* The first term $F_\varepsilon(x)u_\varepsilon^{'}(x)$, by Lemma 5.1, converges to $(u^{'})^2$.

* The second term $-\lambda(u_\varepsilon(x) - g_\varepsilon(x))^2$, by Lemma 3.2, converges to $-\lambda(u(x) - g(x))^2$ in $L^2(0, L)$.

* The third term $\frac{(1-v_\varepsilon(x_0^2))^2}{\varepsilon}$, by Proposition 3.12, converges to 0.

* The fifth term $\lambda(u_\varepsilon(x_0^\varepsilon) - g_\varepsilon(x_0^\varepsilon))^2$, by the local maximality of v_ε at x_0^ε, converges to $\lambda(u(x_0) - g(x_0))^2$. Indeed, we write $g_\varepsilon(x_0^\varepsilon) - g(x_0) = g_\varepsilon(x_0^\varepsilon) - g_\varepsilon(x_0) + g_\varepsilon(x_0) - g(x_0)$. The last term $g_\varepsilon(x_0) - g(x_0)$, by the weak convergence $g_\varepsilon \rightharpoonup g$ in $H^1((0, L))$, converges to 0. The same is true of the first term, because it is bounded by $\|g_\varepsilon^{'}\|_{L^2([x_0^\varepsilon, x_0])} |x_0^\varepsilon - x_0|^{1/2} \leq \mathcal{C}\, |x_0^\varepsilon - x_0|^{1/2}$. On the other hand, for the function u, we also write $u_\varepsilon(x_0^\varepsilon) - u(x_0) = u_\varepsilon(x_0^\varepsilon) - u_\varepsilon(x_0) + u_\varepsilon(x_0) - u(x_0)$. Note that, for ε sufficiently small, $[x_0^\varepsilon - \delta, x_0^\varepsilon + \delta] \subset K_1 \subset [0, L]\backslash\{x_1, \cdots, x_j\}$ and $v_\varepsilon(x) \geq 1 - \varepsilon^{1/4}$ in $[x_0^\varepsilon - \delta, x_0^\varepsilon + \delta]$ by Proposition 3.12. Thus, from (3.6), we deduce that $u_\varepsilon^{'}$ is bounded in $[x_0^\varepsilon - \delta, x_0^\varepsilon + \delta]$. Now the arguments are similar to those for g.

* The sixth term $-F_\varepsilon(x_0^\varepsilon)u_\varepsilon^{'}(x_0^\varepsilon)$ converges to $-(u^{'}(x_0))^2$. Indeed, from (5.11), we get

$$-F_\varepsilon(x_0^\varepsilon)u_\varepsilon^{'}(x_0^\varepsilon) = -\frac{(F_\varepsilon(x_0^\varepsilon))^2}{\eta_\varepsilon + v_\varepsilon^2(x_0^\varepsilon)} \to -(F(x_0))^2 = -(u^{'}(x_0))^2,$$

by Remark 4.3.

* The final term $\int_{x_0{}^\varepsilon}^{x} -\lambda g_\varepsilon^{'}(t)[u_\varepsilon(t) - g_\varepsilon(t)]dt$ converges to

$$\begin{aligned}
&\int_{x_0}^{x} -2\lambda g^{'}(t)(u(t) - g(t)) = \int_{x_0}^{x} (\lambda[u-g]^2)^{'} - 2\int_{x_0}^{x} \lambda u^{'}(t)(u(t) - g(t)) \\
&= \lambda(u(x) - g(x))^2 - \lambda(u(x_0) - g(x_0))^2 - 2\int_{x_0}^{x} \lambda u^{'}(t)(u(t) - g(t)).
\end{aligned}$$

Therefore, keeping in mind that the fourth term $-\varepsilon(v_\varepsilon^{'}(x_0^\varepsilon))^2$ of (5.13) is nonpositive, D_ε converges, in the sense of Radon measures, to

$$D(x) \quad \leq \quad (u^{'}(x))^2 - (u^{'}(x_0))^2 - 2\int_{x_0}^{x} \lambda u^{'}(t)(u(t) - g(t))$$

$$= \quad (u'(x))^2 - (u'(x_0))^2 - 2\int_{x_0}^{x} u'(t)u''(t) = 0.$$

We indicate how to obtain rigorously the last line of the above equation. It is obvious in the absence of X_i's in the interval $[x_0, x]$. Now, it suffices to consider the case that $x_0 < x$ and that there is only one, say X_1, in $[x_0, x]$. Then, $u'' = \lambda(u - g)$ in (x_0, X_1) and (X_1, x). Therefore,

$$\begin{aligned}
&2\int_{x_0}^{x} \lambda u'(t)(u(t) - g(t)) \\
&= 2\int_{x_0}^{X_1} \lambda u'(t)(u(t) - g(t)) + 2\int_{X_1}^{x} \lambda u'(t)(u(t) - g(t)) \\
&= 2\int_{x_0}^{X_1} u'(t)u''(t) + 2\int_{X_1}^{x} u'(t)u''(t) \\
&= (u'(X_1))^2 - (u'(x_0))^2 + (u'(x))^2 - (u'(X_1))^2 = u'(x))^2 - (u'(x_0))^2.
\end{aligned}$$

Because $D(x) \leq 0$ and by the positivity of D_ε from Lemma 5.6, we must have $D(x) = 0$. This completes the proof of equi-partition of energy. □

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