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SOLVABILITY OF REDUCED POSSIO INTEGRAL EQUATION IN THEORETICAL AEROELASTICITY

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Abstract. This paper is concerned with the solvability of the reduced version of the Possio singular integral equation, which plays a fundamental role in theoretical aeroelasticity. This equation relates the pressure distribution over a typical section of a long slender wing in subsonic compressible air flow to the normal velocity of the points of a wing (down-wash). In spite of the importance of the Possio equation, the question of the existence of its solution has not yet been settled. In this paper, we provide a rigorous study of the reduced version of the Possio equation and prove its solvability. The reduced Possio equation is important in its own right for two reasons: (a) it can be used as an approximate equation for the numerical analysis of the coupled system describing a vibrating wing in a surrounding air flow; (b) its analysis is essential for identifying analytical difficulties associated with the general Possio equation.

1. Introduction

This research is devoted to the study of a well-known problem in theoretical aeroelasticity. This problem is concerned with the solvability of a specific singular integral equation that plays a key role in identifying an air load exerted on an aircraft wing in flight by an air flow. This is the *Possio integral equation*, which relates the pressure distribution over a typical section of a slender wing in subsonic compressible air flow to the normal velocity of the points on a wing surface (down-wash). First derived by C. Possio [13], this integral equation is an essential tool in stability (wing flutter) analysis. Despite the fact that an extensive literature exists on numerical treatment of the Possio (or modified Possio) equation, its solvability has never rigorously been proved. In our work, we focus on subsonic *compressible* flows. The problem of a pressure distribution around a flying wing can be reduced to the study of a velocity potential dynamics. Having a pressure distribution, one can

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calculate forces and moments exerted on a wing due to the air flow, which is an extremely important component of wing modeling. The central problem in aeroelasticity is *stability* of a wing in an air flow. This stability problem can be formulated in the form of a system of nonlinear integro–differential equations in a state space of a system, the evolution–convolution equations. (For an illustrating example of an *incompressible* air flow, see papers [2, 16].) So, the problem of stability must involve a nonlinear equation governing the air flow. Assuming that the flow is subsonic, *compressible*, and inviscid, we will work with the linearized version of the above-mentioned equation together with a Neumann-type boundary condition on a part of the boundary. The above problem can be cast equivalently as an integral equation, which is exactly *the Possio integral equation*. Our main goal is to prove solvability of this equation using analytical tools. We should mention here that most of the research is currently done by different numerical methods, e.g., [14, 20], and very few papers on analytical treatment of the Possio equation are available [1, 3]. In essence, for the numerical simulations partial differential equations are approximated by ordinary differential equations for both the structural dynamics and aerodynamics. However, it is important to retain full continuous models. The Possio integral equation is the bridge between Lagrangian structural dynamics and Eulerian aero-dynamics. In particular, using the solution of the Possio equation, one can calculate the aerodynamic loading for the structural equations.

Using the simplest structural model, the Goland model [7, 9], let us consider a uniform rectangular beam with two degrees of freedom, plunge and pitch. Let the flow velocity be along the positive x-axis with x denoting the cord variable, $-b \leqslant x \leqslant b$; let y be the span or length variable along the y-axis, $0 \leqslant y \leqslant l$. Let h be the plunge, or bending, along the z-axis; let θ be the pitch, or torsion angle, about the elastic axis located at $x = ab$, where $0 < a < 1$ and ab is the distance between the center line and the pitch axis (see Figure 1).

Let $X(y,t) = (h(y,t), \theta(y,t))^T$ (the superscript "T" means the transposition). Then the structural dynamics equation is

$$\mathbf{M}X_{tt}(y,t) + \mathbf{K}X(y,t) = (\mathbb{L}(y,t), \mathbb{M}(y,t))^T, \tag{1.1}$$

where $\mathbf{M}$ is the mass–inertia matrix and $\mathbf{K}$ is the stiffness differential operator, i.e., $\mathbf{K} = diag\left\{EI\frac{\partial^4}{\partial y^4}, -GJ\frac{\partial^2}{\partial y^2}\right\}$, with EI and GJ being the bending and torsion stiffness respectively. $\mathbb{L}(y,t)$ and $\mathbb{M}(y,t)$ are the aerodynamic lift and moment about the elastic axis. Let the boundary conditions for system (1.1) correspond to a cantilever beam model: $h(0,t) = h'(0,t) = \theta(0,t) = 0$;

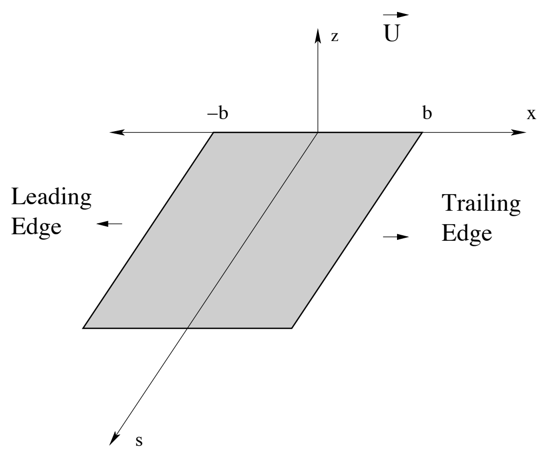

Figure 1. Wing Structure Beam Model

$h''(l,t) = h'''(l,t) = \theta'(l,t) = 0$. In general, a structure's model can be extended to several degrees of freedom as well as to nonlinear models [6, 10]; one can also add a control term as in [15, 17]. However, the emphasis in this paper is on aerodynamics—by far the most complicated part—and the structure interaction, specifically, on force and moment terms in Equation (1.1).

In what follows we assume that air flow is nonviscous, which means that the governing aerodynamic field equation is no longer the Navier–Stokes equation but the Euler Full Potential equation [3], and the main assumption is that the entropy is constant. As a result, the flow is curl-free and can be described in terms of a velocity potential. The unknown variable in the Possio equation is the velocity potential φ. The velocity potential yields explicit expressions for the lift and moment from (1.1) (see, e.g., [7]). Namely, the aerodynamic lift and moment (per unit length) are given by the formulas

$$\mathbb{L}(y,t) = \int_{-b}^{b} \delta p \, dx; \qquad \mathbb{M}(y,t) = \int_{-b}^{b} (x-a)\delta p \, dx, \tag{1.2}$$

where p is a pressure and

$$\delta p = p(x,y,0+,t) - p(x,y,0-,t), \quad 0 \leqslant y \leqslant l, |x| < b. \tag{1.3}$$

If one can derive a representation for the potential φ, then the following formula can be used for a pressure calculation:

$$p(x,y,z,t) = p_\infty \Big(1 + \frac{\gamma - 1}{a_\infty^2}\Big(\frac{u^2}{2} - \frac{\partial \varphi}{\partial t} - \frac{\nabla\varphi \cdot \nabla\varphi}{2}\Big)\Big)^{\frac{\gamma}{\gamma-1}}, \tag{1.4}$$

where $p_\infty = \rho^2 a_\infty^2 \gamma^{-1}$, a_∞ is the far-field speed of sound, ρ is the air density, and $\gamma > 1$ is the adiabatic constant; u is a speed of a moving wing. Knowing the velocity potential φ, we obtain δp and thus lift $\mathbb{L}$ and moment $\mathbb{M}$, which play the role of a forcing term in Equation (1.1).

Any aeroelastic problem consists of two parts. The first part involves a field equation (for a linearized version of the field equation for a velocity potential φ; see Equation (1.6) below) with the following boundary conditions: (a) Flow tangency condition (see (1.7) below), (b) Kutta–Joukowski condition (see (1.8) below), and (c) Far-field conditions (see (1.9) below). The Flow tangency condition establishes the connection between the structural and aerodynamical parts, since the down-wash function w_a of (1.7) is expressed explicitly in terms of the plunge h and pitch θ. We point out that the Neumann condition holds only on a part of the boundary $z = 0$, i.e., only on the structure; a different condition holds on the rest of the boundary, and this condition is purely aerodynamic (not involving the structure dynamics). The second part of an aeroelastic problem involves the structure state variables via Flow tangency condition (1.7) and more importantly lift, $\mathbb{L}$, and moment, $\mathbb{M}$.

Summarizing all of the above, one can see that due to the expressions for the lift and moment, the entire aeroelastic problem (structure and aerodynamics) becomes a nonlinear evolution–convolution equation in terms of the structure state variables. Stability with respect to the parameter u is the "flutter problem" one has to resolve.

Now we present the initial-boundary-value problem for a partial differential equation, known as the "small disturbance potential field equation" *for subsonic, inviscid, compressible flow* [3]. We assume that the air flow is around a large aspect ratio planar wing, which means that the dependence on the span variable along the wing is neglected. The wing is then reduced to a "typical section" or a "chord." We will use the following notation:

$$\begin{aligned} &u - \text{free stream velocity}; \qquad a_\infty - \text{sound speed}; \\ &M = u/a_\infty - \text{Mach number}, \quad 0 < M \leqslant 1. \end{aligned} \tag{1.5}$$

The velocity potential of the airflow is given by the expression

$$ux + \varphi(x, z, t), \quad -\infty < x < \infty, \quad 0 \leqslant z < \infty, \quad t > 0,$$

where ux is the free-stream velocity potential and φ is a small perturbation of the velocity potential. The disturbance potential φ satisfies the following linearized field equation [1]:

$$\varphi_{tt}(x,z,t) + 2Ma_\infty\varphi_{tx}(x,z,t) = a_\infty^2(1 - M^2)\varphi_{xx}(x,z,t) + a_\infty^2\varphi_{zx}(x,z,t),$$

$$-\infty < x < \infty, \quad 0 \leqslant z < \infty, \quad t > 0. \tag{1.6}$$

Together with this equation, we introduce the following boundary conditions.

(1) **Flow tangency** (or nonseparable flow) condition

$$\frac{\partial}{\partial z}\varphi(x,z,t)\Big|_{z=0} = w_a(x,t), \quad |x| < b, \tag{1.7}$$

where $w_a(x,t)$ is the given normal velocity of the wing (the down-wash); b is a size of a half-chord. We note, that condition (1.7) is a nonhomogeneous Neumann condition prescribed only on a *part of the boundary* $z = 0$.

(2) **The Kutta-Joukowski** conditions. To formulate these conditions, we introduce *an acceleration potential* defined by

$$\psi(x,z,t) = \varphi_t(x,z,t) + u\varphi_x(x,z,t). \tag{1.8}$$

The conditions below reflect the following physical situation: pressure off the wing and at the trailing edge must be equal to zero.

$$\psi(x,0,t) = 0 \text{ for } |x| > b, \qquad \lim_{x\to b-0}\psi(x,0,t) = 0. \tag{1.9}$$

(3) **Far-field** conditions. The disturbance potential and velocity tend to zero at large distances from the wing; i.e.,

$$\varphi(x,z,t) \to 0, \quad \nabla\varphi(x,z,t) \to 0, \quad \text{as } |x| \to \infty \text{ or } z \to \infty. \tag{1.10}$$

Assuming that the initial conditions are trivial, i.e., $\varphi(x,z,0) = \varphi_t(x,z,0) = 0$, we consider the initial-boundary-value problem (1.6)–(1.10) in the space $L_p(-\infty,\infty)$, $1 < p < 2$. It will be clear from the analysis below that it is essential to use $p \neq 2$. We assume that the function $\varphi(x,z,t)$ is absolutely continuous with absolutely continuous first derivatives with respect to the variables z and t. Regarding the properties of φ as a function of x, we make the following assumption. Let D_x be a closed linear operator in $L_p(-\infty,\infty)$ corresponding to the partial derivative $\frac{\partial}{\partial x}$. Then we require that

$$\begin{aligned} &\varphi(\cdot,z,t) \in \mathcal{D}(D_x^2), \text{ with } \mathcal{D} \text{ being the domain of } D_x^2, \text{ and} \\ &D_x\varphi(\cdot,z,t) \text{ being absolutely continuous with respect to } t. \end{aligned} \tag{1.11}$$

We require that Equation (1.6) is satisfied in the sense that

$$\frac{\partial^2}{\partial t^2}\varphi(\cdot,z,t) + 2Ma_\infty\frac{\partial}{\partial t}D_x\varphi(\cdot,z,t) = \Big(a_\infty^2(1-M^2)D_x^2 + a_\infty^2\frac{\partial^2}{\partial z^2}\Big)\varphi(\cdot,z,t). \tag{1.12}$$

The Flow tangency condition (1.7) will be understood in the sense that

$$\int_{-b}^{b}\Big|\frac{\partial}{\partial z}\varphi(x,z,t) - w_a(x,t)\Big|^p dx \to 0 \text{ as } z \to 0. \tag{1.13}$$

The Kutta–Joukowski conditions (1.9) will be written in integral form as well. We note that due to our assumptions on φ, the acceleration potential is well-defined. Conditions (1.9) are replaced with the following requirements:

$$\begin{aligned} &\int_{-\infty}^{\infty} |\psi(x,z,t) - \psi(x,0,t)|^p dx \to 0 \text{ as } z \to 0, \\ &\int_{-\infty}^{-b} |\psi(x,0,t)|^p dx + \int_{b}^{\infty} |\psi(x,0,t)|^p dx = 0. \end{aligned} \tag{1.14}$$

Regarding the initial conditions, we require

$$\left\|\varphi(\cdot,z,t)\right\|_{L^p(-\infty,\infty)} \longrightarrow 0, \quad \left\|\frac{\partial}{\partial t}\varphi(\cdot,z,t)\right\|_{L^p(-\infty,\infty)} \longrightarrow 0 \tag{1.15}$$

as $t \to 0$, $0 < z < \infty$.

In our paper [4], we have reduced the initial-boundary-value problem (1.6)–(1.10) to a specific singular integral equation, *the Possio integral equation.* To make this reduction, we first rewrote problem (1.6)–(1.10) by applying the two integral transformations to the unknown function φ. Namely, we have obtained the equation for the Laplace transform in time-variable and the Fourier transform in x-variable of φ. Such a double transform allowed us to give the first version of the Possio equation (see Equation (3.19) of [4]). To reduce that equation to a more convenient form, we have applied the Mikhlin multiplier theory [11] and obtained a singular integral equation, which is singular in more than one sense (see Section 6 of [4]). In our paper [18], we have shown that the aforementioned integral equation can be split up into two parts in such a way that the first part is not small with respect to a complex parameter λ appearing in the equation due to the Laplace transformation, while the second part is asymptotically small and tends to zero as the above parameter λ tends to infinity in the right half-plane of the complex plane. In the present paper, we focus on the first part of the Possio equation that is not asymptotically small. By ignoring asymptotically small terms, we obtain an equation that we call *the reduced Possio equation* (see Equation (2.1) below). It is the goal of this paper to prove the unique solvability in $L_p(-b,b)$ of the reduced equation. Our final step will be to show that addition of the asymptotically small terms cannot destroy unique solvability obtained for the reduced equation.

The present paper is organized as follows. Section 2 states the problem and formulation of the main result of the paper on the unique solvability of the reduced Possio integral equation. It also contains the proof demonstrating the Possio equation to be of Fredholm type. Section 3 is of a technical nature; i.e., it is devoted to the proof of the asymptotical representations for specific singular integral operations that are used in the following section. Section 4 is devoted to the proof of solvability of the equation obtained from the main one by omitting the term containing the function $h_-(\cdot,\lambda)$ (see Equation (3.2) below). In Section 5, the proof of the main result is finalized.

Remark 1.1. In abstract form, we study the unique solvability in $L_p(-b,b)$, $0<b<\infty, 1<p<2$, of the following integral equation:

$$G=\mathcal{A}_1[F]+\mathcal{A}_2[f]+\mathcal{A}_3[F]+\mathcal{A}_4[F], \tag{1.16}$$

where $G\in L_p(-b,b)$, and $\mathcal{A}_1$ is a finite Hilbert transformation, $\mathcal{A}_2$ is a "specific" inverse to $\mathcal{A}_1$, $\mathcal{A}_3$ is a Volterra integral operator, and $\mathcal{A}_4$ is an integral operator with a degenerate kernel. We would like to emphasize that mathematically such a problem is nontrivial for two reasons: (a) it contains a finite Hilbert integral transformation, which is a singular integral operator, a specific "inverse" to the above transformation, and also (b) it contains a Volterra integral operator and an operator with the degenerate kernel. It is exactly the second reason, i.e., a combination of different types of integral transformations in one equation, that makes the problem nonstandard.

2. Statement of Problem—Formulation of Main Result

In this paper we focus our attention on the following singular equation, derived in [16], that we call the reduced Possio equation:

$$\begin{aligned}\frac{2}{\sqrt{1-M^2}}\mathcal{T}W_a(\cdot,\lambda)&=\Big[1-\frac{\lambda}{\sqrt{1-M^2}}\mathcal{L}(\lambda)\Big]F(\cdot,\lambda)\\&+\frac{\lambda}{\sqrt{1-M^2}}h_-(x,\lambda)e^{-b\lambda}L(\lambda,F)-\int_0^{\alpha_1}a(s)\lambda[\mathcal{L}(\lambda s)F(\cdot,\lambda)]ds\\&+\int_0^{\alpha_2}a(-s)\lambda[\mathcal{L}^*(\lambda s)F(\cdot,\lambda)]ds+\frac{M}{\sqrt{1-M^2}}\left\{\mathcal{H}_b[F]-\mathcal{T}[F]\right\}.\end{aligned}\tag{2.1}$$

Let us recall all the notation in Equation (2.1). We begin with $W_a(\cdot,\lambda)$, which stands at the left-hand side of the equation, and is the Laplace transform of the down-wash function $w_a(x,t)$. The function $h_-(x,\lambda)\in L_p(-b,b)$, $1<p<2$, is given explicitly by the formula

$$h_-(x,\lambda)=\frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_0^\infty\frac{1}{b-x-\sigma}\sqrt{\frac{2b+\sigma}{\sigma}}e^{-\lambda\sigma}d\sigma,\quad |x|<b. \tag{2.2}$$

The function $a(\cdot)$ and the limits of integrations α_1 and α_2 are given by

$$a(s)=\frac{\sqrt{(\alpha_1-s)(\alpha_2+s)}}{1-s},\quad -\alpha_2\le s\le\alpha_1,\quad \alpha_1=\frac{M}{1+M},\quad \alpha_2=\frac{M}{1-M}, \tag{2.3}$$

with M being a Mach number (1.5). Now we describe all the operations in Equation (2.1).

- $\mathcal{L}(\lambda)$ is a Volterra integral operator in $L_p(-b,b)$, $1<p<2$, defined by the formula

$$\mathcal{L}(\lambda)f = g, \quad g(x) = \int_{-b}^{x} e^{-\lambda(x-\xi)} f(\xi)d\xi, \quad |x|<b. \tag{2.4}$$

- $\mathcal{L}^*(\lambda)$ is an operator in $L_p(-b,b)$ defined by the formula

$$\mathcal{L}^*(\lambda)f = g, \quad g(x) = \int_{x}^{b} e^{-\lambda(\xi-x)} f(\xi)d\xi, \quad |x|<b. \tag{2.5}$$

- $L(\lambda,\cdot)$ is a linear functional on $L_p(-b,b)$ defined by the formula

$$L(\lambda,f) = \int_{-b}^{b} e^{\lambda\sigma} f(\sigma)d\sigma. \tag{2.6}$$

- $\mathcal{H}_b$ is a finite Hilbert transformation in $L_p(-b,b)$, i.e., a singular integral operator defined by the formula

$$\mathcal{H}_b[f] = g, \quad g(x) = \frac{1}{\pi} P.V.\int_{-b}^{b} \frac{f(\xi)}{x-\xi} d\xi, \tag{2.7}$$

  where "*P.V.*" means that the integral is understood as a principal value integral. We find it convenient (following [19]) to use the notation "$*$" in the upper limit of the integration instead of "*P.V.*". Thus, we will write the formula (2.7) as

$$\mathcal{H}_b[f] = g, \quad g(x) = \frac{1}{\pi}\int_{-b}^{b*} \frac{f(\xi)}{x-\xi} d\xi, \quad |x|<b.$$

- $\mathcal{T}$ is an integral operator that plays the role of an "inverse" operator to $\mathcal{H}_b$. This statement requires an explanation.

As is well-known, the finite Hilbert transform $\mathcal{H}_b$ as an operator in $L_p(-b,b)$ does not have an inverse operator. Indeed, if one applies $\mathcal{H}_b$ to the function $\phi = \left(b^2-x^2\right)^{-1/2}$, then the result is the following:

$$\mathcal{H}_b[\phi] = 0 \quad \text{for} \quad |x|<b.$$

(However, $\mathcal{H}_b[\phi] \neq 0$ for $|x|>b$.) This example demonstrates that there is a nontrivial function from $L_p(-b,b)$, $1<p<2$, which belongs to $Ker\{\mathcal{H}_b\}$. However, if the equation $\{\mathcal{H}_b\}[f] = \psi$ is considered on a special class of functions, then it has a unique solution. More precisely, the following result holds [8, 12, 19].

Theorem 2.1. *Let* $\rho(x) = \sqrt{\frac{b-x}{b+x}}$, $|x| < b$, *and* $L_{2,\rho}(-b,b)$ *and* $L_{2,\rho^{-1}}(-b,b)$ *be the two weighted Hilbert spaces with the weights* ρ *and* ρ^{-1} *respectively. Then the "airfoil equation" can be given in the form*

$$f(x) = \frac{1}{\pi}\int_{-b}^{b_*} \frac{w(\xi)}{x-\xi} d\xi = \frac{1}{\pi}\int_{-b}^{b_*} \frac{v(\xi)}{x-\xi} \rho(\xi) d\xi \equiv \boldsymbol{H}[v]. \tag{2.8}$$

The operator $\boldsymbol{H}$ *defined in* (2.8) *is a Hilbert-space isomorphism of* $L_{2,\rho}(-b,b)$ *onto* $L_{2,\rho^{-1}}(-b,b)$. *The inverse operator* $\boldsymbol{H}^{-1}$ *is given by*

$$v(x) = \boldsymbol{H}^{-1}[f](x) = \frac{1}{\pi}\int_{-b}^{b_*} \frac{f(\xi)}{x-\xi}\rho^{-1}(\xi) d\xi, \quad f \in \ L_{2,\rho^{-1}}(-b,b). \tag{2.9}$$

Equation (2.9) represents the classical Söhngen inversion formula for the airfoil equation in the presence of the Kutta condition. Multiplying both sides of Equation (2.9) by ρ, we obtain, for $f \in \ L_{2,\rho^{-1}}(-b,b)$,

$$w(x) = v(x)\rho(x) = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}}\, \frac{f(\xi)}{x-\xi}\, d\xi \equiv \mathcal{T}[f]. \tag{2.10}$$

In what follows, we need a statement from [12, 19] on the operator $\mathcal{H}_b$ considered as a mapping of $L_2(-b,b)$ into itself.

Theorem 2.2.

(1) *The operator* $\mathcal{H}_b$ *is bounded in* $L_2(-b,b)$ *and such that* $\|\mathcal{H}_b\| \leqslant 1$.
(2) $\mathcal{H}_b$ *is skew-self-adjoint in* $L_2(-b,b)$; *i.e.,* $\mathcal{H}_b^* = -\mathcal{H}_b$.
(3) *For any* $p > 1$, $\mathcal{H}_b$ *is a bounded operator in* $L_p(-b,b)$.

Now we return to Equation (2.1) and formulate the main result of the paper.

Theorem 2.3. *The nonhomogeneous Equation* (2.1) *considered in the space* $L_p(-b,b)$, $1 < p < 2$, *has a solution and this solution is unique.*

In the conclusion of this section we prove the following statement.

Theorem 2.4. *Equation* (2.1) *is of a Fredholm type; i.e., a nonhomogeneous equation has a unique solution if and only if the homogeneous equation has only the trivial solution.*

Proof. Let us assume that a homogeneous equation has only a trivial solution and show that the nonhomogeneous Equation (2.1) has a unique solution, which is constructed explicitly. First, we observe that the operator $\mathcal{H}_b$ can be applied to both sides of this equation, which yields a new equation:

$$2W_a(x,\lambda) = \sqrt{1-M^2}\mathcal{H}_b[F] + \lambda\mathcal{H}_b\Big\{ -\mathcal{L}(\lambda)F + h_-(x,\lambda)e^{-b\lambda}L(\lambda,F)$$

$$-\sqrt{1-M^2}\int_0^{\alpha_1} a(s)\mathcal{L}(\lambda s)F(\cdot,\lambda)ds \tag{2.11}$$
$$+\sqrt{1-M^2}\int_0^{\alpha_2} a(-s)\mathcal{L}^*(\lambda s)F(\cdot,\lambda)ds\Big\} + M\left\{\mathcal{H}_b^2[F]-F\right\}.$$

Note that operator $\mathcal{H}_b^2$ makes sense since $\mathcal{H}_b$ maps $L_p(-b,b)$ into itself. Let $\mathbb{H}$ be the following operator:

$$\mathbb{H} = \sqrt{1-M^2}\mathcal{H}_b + M\mathcal{H}_b^2 - M\mathbb{I}, \tag{2.12}$$

with $\mathbb{I}$ being the identity operator. Using $\mathbb{H}$, we rewrite Equation (2.11) in the form

$$2W_a(x,\lambda) = \mathbb{H}[F] + \lambda\mathcal{H}_b\Big\{-\mathcal{L}(\lambda)F(\cdot,\lambda) + h_-(x,\lambda)e^{-b\lambda}L(\lambda,F) \tag{2.13}$$
$$-\sqrt{1-M^2}\int_0^{\alpha_1} a(s)[\mathcal{L}(\lambda s)F(\cdot,\lambda)]ds$$
$$+\sqrt{1-M^2}\int_0^{\alpha_2} a(-s)[\mathcal{L}^*(\lambda s)F(\cdot,\lambda)]ds\Big\}.$$

By direct calculations, we verify that $\mathbb{H}$ can be factored out as follows:

$$\mathbb{H} = M\left(\mathcal{H}_b - z_1\mathbb{I}\right)\left(\mathcal{H}_b - z_2\mathbb{I}\right), \tag{2.14}$$

where z_1 and z_2 are the roots of the quadratic polynomial $(Mz^2+\sqrt{1-M^2}z-M)$ given by

$$z_{1,2} = -\frac{\sqrt{1-M^2}}{2M} \pm \frac{\sqrt{1+3M^2}}{2M}, \quad z_1>0,\ z_2<0. \tag{2.15}$$

To proceed, we need the following statement from [12].

Proposition 2.1. *The integral equation*

$$\mathcal{H}_b[F] - \mu F = G \tag{2.16}$$

has a unique solution in $L_p(-b,b)$, $1<p<2$, *as long as* μ *is not a purely imaginary number.*

Due to Proposition 2.1, $\mathbb{H}^{-1}$ exists and is a bounded operator in $L_p(-b,b)$. If

$$\mathcal{R}_M \equiv \mathbb{H}^{-1}, \tag{2.17}$$

then applying the operator $\mathcal{R}_M$ to both sides of Equation (2.13), we obtain

$$2(\mathcal{R}_M W_a)(x,\lambda) = F(x,\lambda) + \lambda\mathcal{R}_M\mathcal{H}_b\Big\{-\mathcal{L}(\lambda)F + h_-(x,\lambda)e^{-b\lambda}L(\lambda,F) \tag{2.18}$$

$$- \sqrt{1-M^2} \int_0^{\alpha_1} a(s)[\mathcal{L}(\lambda s)F(\cdot,\lambda)]ds + \int_0^{\alpha_2} a(-s)[\mathcal{L}^*(\lambda s)F(\cdot,\lambda)]ds\Big\}.$$

We notice that the operators $\mathcal{L}(\lambda)$, $\int_0^{\alpha_1} a(s)\mathcal{L}(\lambda s)\cdot ds$, and $\int_0^{\alpha_2} a(-s)\mathcal{L}^*(\lambda s)\cdot ds$ are Volterra integral operators, and $h_-(x,\lambda)e^{-\lambda b}L(\lambda,\cdot)$ is a one-dimensional operator. Therefore, their linear combination from Equation (2.18) is a compact operator. The operators $\mathcal{R}_M$ and $\mathcal{H}_b$ are bounded in $L_p(-b,b)$. Therefore the operator

$$\begin{aligned}\mathcal{M} =& \lambda\mathcal{R}_M\mathcal{H}_b\Big\{\mathcal{L}(\lambda) - h_-(x,\lambda)e^{-b\lambda}L(\lambda,\cdot) \\ &+ \sqrt{1-M^2}\int_0^{\alpha_1} a(s)\mathcal{L}(\lambda s,\cdot)ds - \sqrt{1-M^2}\int_0^{\alpha_2} a(-s)\mathcal{L}^*(\lambda s,\cdot)ds\Big\}\end{aligned} \tag{2.19}$$

is compact and may have only a normal eigenvalue at point 1 . (Recall that an eigenvalue is normal if it is an isolated point of the spectrum and has a finite algebraic multiplicity.) Let us show that this is not the case. Arguing by contradiction, assume that there exists a nontrivial function φ such that the following equation holds:

$$\begin{aligned}&\varphi(x) - \lambda\mathcal{R}_M\mathcal{H}_b\Big\{\mathcal{L}(\lambda)\varphi - h_-(x,\lambda)e^{-b\lambda}L(\lambda,\varphi) \\ &+ \sqrt{1-M^2}\int_0^{\alpha_1} a(s)\mathcal{L}(\lambda s)\varphi\, ds - \sqrt{1-M^2}\int_0^{\alpha_2} a(-s)\mathcal{L}^*(\lambda s)\varphi\, ds\Big\} = 0.\end{aligned} \tag{2.20}$$

Applying $\mathbb{H}$ to both sides of Equation (2.20) we obtain that the homogeneous equation corresponding to (2.13) has a nontrivial solution, which contradicts our assumption.

Therefore, the operator $(\mathbb{I} - \mathcal{M})$ ($\mathcal{M}$ is defined in (2.19)) has a bounded inverse and the solution of Equation (2.13) can be given by the formula

$$F(x,\lambda) = 2\left(\mathbb{I} - \mathcal{M}\right)^{-1}\left[\mathcal{R}_M W_a(\cdot,\lambda)\right]. \tag{2.21}$$

The uniqueness of this solution can easily be seen. Assuming that there exist two linearly independent solutions f_1 and f_2, we obtain that a nontrivial function $f = f_1 - f_2$ is a solution to the homogeneous equation. This contradiction yields the proof.

The inverse statement (that from the existence and uniqueness of the solution of the nonhomogeneous equation follows the existence only of the trivial solution of a homogeneous equation) does not require any specific proof. The theorem is completely proved. □

3. Technical Results

In what follows, we focus on a simpler equation obtained from Equation (2.1). Namely, we start with the proof of unique solvability of the equation corresponding to the case $\alpha_1 = \alpha_2 = 0$,

$$\begin{aligned} 2\mathcal{T}\left[W_a(\cdot,\lambda)\right] =& \sqrt{1-M^2}F(x,\lambda) - \lambda\mathcal{L}(\lambda)F(\cdot,\lambda) \\ &+ \lambda h_-(x,\lambda)e^{-b\lambda}L(\lambda,F) - M\left\{\mathcal{H}_b[F] - \mathcal{T}[F]\right\}. \end{aligned} \tag{3.1}$$

Generalization of the proof to the case $\alpha_{1,2} > 0$ can be shown without difficulties. Since Equation (3.1) is a particular case of Equation (2.1) ($\alpha_1 = \alpha_2 = 0$), the Fredholm alternative is valid for this equation as well. Solvability of Equation (3.1) will be shown in two steps. First we prove the solvability of the equation obtained from (3.1) by omitting the term $\lambda h_-(x,\lambda)e^{-b\lambda}L(\lambda,F)$, i.e., of the following equation:

$$2\mathcal{T}\left[W_a(\cdot,\lambda)\right] = \sqrt{1-M^2}F(x,\lambda) - \lambda\mathcal{L}(\lambda)F(\cdot,\lambda) + M\left\{\mathcal{H}_b[F] - \mathcal{T}[F]\right\}, \tag{3.2}$$

and then we derive the formula for the solution of the complete Equation (3.1). Note, Equation (3.2) is not standard: it contains two singular integral operators $\mathcal{H}_b$ and $\mathcal{T}$ and a Volterra integral operator $\mathcal{L}(\lambda)$. Our main result concerning Equation (3.2) is the following statement.

Theorem 3.1. *The homogeneous equation corresponding to Equation* (3.2) *has only the trivial solution.*

Proof of this theorem requires several technical results that will be formulated and proved in the present section in the form of two lemmas and a corollary. The proof of Theorem 3.1 will be shown in Section 4. To formulate our first technical result, we rewrite the homogeneous integral equation corresponding to (3.2),

$$\sqrt{1-M^2}F(x,\lambda) - \lambda\mathcal{L}(\lambda)F(\cdot,\lambda) + M\left\{\mathcal{H}_b[F] - \mathcal{T}[F]\right\} = 0, \tag{3.3}$$

in the following form:

$$\left(\sqrt{1-M^2}-1\right)F(x,\lambda) + \left(F(x,\lambda) - \lambda\mathcal{L}(\lambda)F(\cdot,\lambda)\right) + M\left\{\mathcal{H}_b[F] - \mathcal{T}[F]\right\} = 0. \tag{3.4}$$

Before proceeding, we observe that if

$$g(x) = f(x) - \lambda\int_{-b}^{x} e^{-\lambda(x-\xi)}f(\xi)d\xi,$$

then

$$f(x) = g(x) + \lambda\int_{-b}^{x} g(\xi)d\xi = \left[\mathbb{I} + \lambda\mathcal{L}(0)\right]g. \tag{3.5}$$

Indeed, to prove (3.5), it suffices to show that two integral operators $\mathcal{I}_1$ and $\mathcal{I}_2$ defined by

$$\mathcal{I}_1 = \mathbb{I} - \lambda \int_{-b}^{x} e^{-\lambda(x-\xi)} \cdot d\xi, \quad \mathcal{I}_2 = \mathbb{I} + \lambda \int_{-b}^{x} \cdot d\xi$$

are inverses of each other. To check directly that $\mathcal{I}_1\mathcal{I}_2 = \mathbb{I}$, we have

$$\mathcal{I}_1(\mathcal{I}_2 f) = \Big(\mathbb{I} - \lambda \int_{-b}^{x} e^{-\lambda(x-\xi)} \cdot d\xi\Big)\Big(f + \lambda \int_{-b}^{x} f(\xi)d\xi\Big) \tag{3.6}$$

$$= f(x) - \lambda \int_{-b}^{x} e^{-\lambda(x-\xi)} f(\xi)d\xi + \lambda \int_{-b}^{x} f(\xi)d\xi - \lambda^2 \int_{-b}^{x} e^{-\lambda(x-\xi)}d\xi \int_{-b}^{\xi} f(\eta)d\eta.$$

Integrating by parts in the double integral, we get

$$\lambda^2 \int_{-b}^{x} e^{-\lambda(x-\xi)}d\xi \int_{-b}^{\xi} f(\eta)d\eta = \lambda \int_{-b}^{x} f(\eta)d\eta - \lambda\mathcal{L}(\lambda)f. \tag{3.7}$$

Substituting (3.7) into (3.6) yields $\mathcal{I}_1\mathcal{I}_2 = \mathbb{I}$. The relation $\mathcal{I}_1\mathcal{I}_2 = \mathbb{I}$ can be shown in a similar manner.

Now let us rewrite Equation (3.4) in terms of a function g given by

$$g(x) = F(x,\lambda) - \lambda \int_{-b}^{x} e^{-\lambda(x-\xi)} F(\xi,\lambda)d\xi. \tag{3.8}$$

We have

$$\big(\sqrt{1-M^2} - 1\big)\Big(g(x) + \lambda \int_{-b}^{x} g(\xi)d\xi\Big) + g(x) + M\left\{\mathcal{H}_g - \mathcal{T}\right\}[g]$$
$$+ M\left\{\mathcal{H}_b - \mathcal{T}\right\}\Big[\lambda \int_{-b}^{x} g(\xi)d\xi\Big] = 0.$$

Letting

$$M' = \sqrt{1-M^2}\,, \qquad M'' = 1 - \sqrt{1-M^2}\,, \tag{3.9}$$

we reduce the last equation to the desired form:

$$M'g(x) - M''\lambda \int_{-b}^{x} g(\xi)d\xi + M\left\{\mathcal{H}_b - \mathcal{T}\right\}[g] + M\lambda\left\{\mathcal{H}_b - \mathcal{T}\right\}\Big[\int_{-b}^{x} g(\xi)d\xi\Big] = 0. \tag{3.10}$$

In what follows we need detailed information on the behavior of the functions $\mathcal{T}[g]$ and $\mathcal{T}^2[g]$ at the vicinity of $x = b$. The next two statements describe such behavior.

Lemma 3.1. *Assume that $g \in L_p(-b,b)$ and in a small vicinity of $x = b$ has the following representation:*

$$g(x) = \gamma(b-x)^\beta + \tilde{g}(x), \quad x \in (b-\delta, b], \ 0 < \beta < 1/2, \ \delta > 0, \ \gamma > 0. \tag{3.11}$$

Assume that $\tilde{g}$ *is such that the function*

$$\tilde{\tilde{g}}(x) = \tilde{g}(x)\sqrt{\frac{b+x}{b-x}} \tag{3.12}$$

satisfies the Hölder condition of some positive order ϵ *for* $(b-\delta, b]$. *Then the following asymptotic representation holds:*

$$\mathcal{T}[g] = \hat{\gamma}\frac{(b-x)^\beta}{\sqrt{b+x}} + O\Big(\sqrt{\frac{b-x}{b+x}}\Big), \tag{3.13}$$

with $\hat{\gamma}$ *being an absolute constant whose precise value is immaterial for us.*

Proof. Let

$$\mathcal{T}[g] = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{(b-\delta)_*}\sqrt{\frac{b+\xi}{b-\xi}}g(\xi)\frac{d\xi}{x-\xi} \tag{3.14}$$
$$+\frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{b-\delta}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}g(\xi)\frac{d\xi}{x-\xi} \equiv I_1 + I_2.$$

To analyze (3.14) we consider two cases: (a) $x \in (-b, b-\delta]$ and (b) $x \in (b-\delta, b)$. We provide the proof for the more difficult case (b). The proof for case (a) can be easily constructed. We observe that in case (b), the integral of I_1 from (3.14) represents a real part of an analytic function of z given by

$$G(z) = \int_{-b}^{(b-\delta)_*}\sqrt{\frac{b+\xi}{b-\xi}}g(\xi)\frac{d\xi}{z-\xi}.$$

This function is regular in the whole z-plane with the branch-cut along the segment $(-b, b-\delta)$; hence it is regular even at the vicinity of $z = b$. So, when $x \to b - 0$, we have the estimate

$$I_1 = O\Big(\sqrt{\frac{b-x}{b+x}}\Big). \tag{3.15}$$

Now we turn to the integral in the term I_2. Using (3.11) and (3.12), we obtain

$$\int_{-b}^{(b-\delta)_*}\sqrt{\frac{b+\xi}{b-\xi}}g(\xi)\frac{d\xi}{x-\xi} = \int_{b-\delta}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}\left[\gamma(b-\xi)^\beta + \tilde{g}(\xi)\right]\frac{d\xi}{x-\xi}$$
$$\equiv I_2^{(1)} + I_2^{(2)}. \tag{3.16}$$

For $I_2^{(1)}$, we obtain the following estimate:

$$\gamma\int_{-b}^{(b-\delta)_*}\sqrt{\frac{b+\xi}{b-\xi}}(b-\xi)^{\beta}\frac{d\xi}{x-\xi}\leqslant\gamma_1(b-x)^{\beta-1/2}+O(1). \tag{3.17}$$

This result follows from *the asymptotic theorem* [17, p. 182] and the fact that $\beta<1/2$. To evaluate $I_2^{(2)}$, we use (3.11) and (3.12) to have

$$\begin{aligned}
&\int_{b-\delta}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}\tilde{g}(\xi)\frac{d\xi}{x-\xi}=\int_{b-\delta}^{b_*}\tilde{\tilde{g}}(\xi)\frac{d\xi}{x-\xi}\\
&=\int_{b-\delta}^{b_*}\frac{\tilde{\tilde{g}}(\xi)-\tilde{\tilde{g}}(x)}{x-\xi}d\xi+\tilde{\tilde{g}}(x)\int_{b-\delta}^{b_*}\frac{d\xi}{x-\xi}\\
&\leqslant\int_{b-\delta}^{b_*}\frac{|x-\xi|^{\epsilon}}{|x-\xi|}d\xi+\left|\tilde{\tilde{g}}(x)\right|\left|\lim_{\omega\to0}\left[\int_{b-\delta}^{x-\omega}\frac{d\xi}{x-\xi}+\int_{x+\omega}^{b}\frac{d\xi}{x-\xi}\right]\right|\\
&\leqslant C_2+\left|\tilde{\tilde{g}}(x)\right|\left|\lim_{\omega\to0}\left[\ln(x-\xi)\Big|_{b-\delta}^{x-\omega}+\ln(\xi-x)\Big|_{x+\omega}^{b}\right]\right|\\
&=C_2+\left|\tilde{\tilde{g}}(x)\right|\left|\ln\frac{b-x}{x-b+\delta}\right|.
\end{aligned} \tag{3.18}$$

Combining (3.15), (3.17), and (3.18), we arrive at (3.13).
The lemma is shown. □

The next statement follows immediately from this lemma.

Corollary 3.1. *The following asymptotic representation is valid for* $\mathcal{T}^2[g]$*:*

$$\mathcal{T}^2[g]=\tilde{\gamma}\frac{(b-x)^{\beta}}{\sqrt{b+x}}+O\left(\sqrt{\frac{b-x}{b+x}}\ln\frac{b+x}{b-x}\right)\quad as\quad x\to b-0. \tag{3.19}$$

Proof. Let $G=\mathcal{T}[g]$; then

$$\begin{aligned}
\mathcal{T}^2[g]=\frac{1}{\pi}\mathcal{T}[G]=\sqrt{\frac{b-x}{b+x}}\int_{-b}^{(b-\delta)_*}\sqrt{\frac{b+\xi}{b-\xi}}G(\xi)\frac{d\xi}{x-\xi}\\
+\frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{b-\delta}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}G(\xi)\frac{d\xi}{x-\xi}.
\end{aligned} \tag{3.20}$$

Reasoning as in the proof of Lemma 3.1, we focus on the more difficult case, $x\in(b-\delta,b]$. In this case, the first term in (3.20) has the estimate

$O\left((b-x)^{1/2}(b+x)^{-1/2}\right)$. The second integral can be estimated as follows:

$$\hat{\gamma}\sqrt{\frac{b-x}{b+x}}\int_{b-\delta}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}\frac{(b-\xi)^\beta}{\sqrt{b+\xi}}\frac{d\xi}{x-\xi}+\sqrt{\frac{b-x}{b+x}}\int_{b-\delta}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}g_1(\xi)\frac{d\xi}{x-\xi}$$
$$=\tilde{\gamma}\frac{(b-x)^\beta}{\sqrt{b+x}}+O\Big(\sqrt{\frac{b-x}{b+x}}\ln\Big(\frac{b+x}{b-x}\Big)\Big).$$

In the last estimate, we have taken into account that

$$g_1(x)=O(\sqrt{b-x}/\sqrt{b+x}).$$

The corollary is shown. □

In the conclusion of this section, we prove that if Equation (3.10) has a solution in a specific linear subset of functions from $L_p(-b,b)$, then its average value must be zero. Functions from the above subset have the properties described in Lemma 3.1 and Corollary 3.1. Clearly this subset is dense in $L_p(-b,b)$.

Lemma 3.2. *Let g be such that $g\in L_p(-b,b)$ and let it satisfy the conditions of Lemma* 3.1 *and Corollary* 3.1. *If g is a solution of Equation* (3.10), *then g has a zero average value, i.e.,*

$$\int_{-b}^{b}g(\xi)d\xi=0. \tag{3.21}$$

Proof. Applying the operator $\mathcal{T}$ to both sides of Equation (3.10), we have

$$M'\mathcal{T}[g]-M''\lambda\mathcal{T}\Big[\int_{-b}^{x}g(\xi)d\xi\Big]+M\left(\mathbb{I}-\mathcal{T}^2\right)[g]+M\lambda\left(\mathbb{I}-\mathcal{T}^2\right)\Big[\int_{-b}^{x}g(\xi)d\xi\Big]=0. \tag{3.22}$$

1) Let us consider the two terms in (3.22) containing $G(x)=\int_{-b}^{x}g(\xi)d\xi$. We start with $\mathcal{T}[G]$ and have

$$\mathcal{T}[G]=\mathcal{T}\Big[\int_{-b}^{x}g(\xi)d\xi\Big]=\frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}\frac{d\xi}{x-\xi}\int_{-b}^{\xi}g(\tau)d\tau$$
$$=\frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}\left\{\int_{-b}^{\xi}g(\tau)d\tau-\int_{-b}^{b}g(\tau)d\tau+\int_{-b}^{b}g(\tau)d\tau\right\}\frac{d\xi}{x-\xi}$$
$$\equiv\frac{1}{\pi}(i^{(1)}+i^{(2)}). \tag{3.23}$$

For the integral $i^{(2)}$, we immediately get

$$i^{(2)} = \sqrt{\frac{b-x}{b+x}} \int_{-b}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} \int_{-b}^{b} g(\tau)d\tau = -\pi \int_{-b}^{b} g(\tau)d\tau. \tag{3.24}$$

To evaluate $i^{(1)}$, we split the interval of integration into two subintervals, i.e., $(-b, b-\delta]$ and $(b-\delta, b)$, and denote the corresponding integrals by $\hat{i}^{(1)}$ and $\hat{i}^{(2)}$, ($i^{(1)} = \hat{i}^{(1)} + \hat{i}^{(2)}$). It is clear that when $x \longrightarrow b-0$, the integral along $(-b, b-\delta)$ can be estimated as $O(\sqrt{b-x})$. Let us evaluate the integral along $(b-\delta, b)$ and have

$$\hat{i}^{(2)} = \sqrt{\frac{b-x}{b+x}} \int_{-b}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \left(\int_{\xi}^{b} g(\tau)d\tau \right) \frac{d\xi}{x-\xi}. \tag{3.25}$$

Using the representation for $g(x) = \gamma(b-x)^\beta + \tilde{g}(x)$ from Lemma 3.1, we have

$$\hat{i}^{(2)} = \sqrt{\frac{b-x}{b+x}} \int_{b-\delta}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} \left[\gamma \int_{\xi}^{b} (b-\eta)^\beta d\eta + \int_{\xi}^{b} \tilde{g}(\eta)d\eta \right]. \tag{3.26}$$

Evaluating the first integral from (3.26) we get

$$\begin{aligned} \int_{b-\delta}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} \int_{\xi}^{b} (b-\eta)^\beta d\eta &= C_1 \int_{b-\delta}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} (b-\xi)^{\beta+1} \frac{d\xi}{x-\xi} \\ &= C_1 \int_{b-\delta}^{b_*} \sqrt{b+\xi} (b-\xi)^{1/2+\beta} \frac{d\xi}{x-\xi}. \end{aligned} \tag{3.27}$$

To estimate (3.27), we notice that if $\alpha = 1/2 + \beta > 0$, then the function $(b-x)^\alpha$ belongs to the Hölder class of the order α, and proceeding as in [17, p. 183] we obtain that

$$\int_{b-\delta}^{b_*} \sqrt{b+\xi} \frac{d\xi}{x-\xi} (b-\xi)^\alpha = O(1), \tag{3.28}$$

and thus

$$\sqrt{\frac{b-x}{b+x}} \int_{b-\delta}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} \int_{\xi}^{b} \gamma(b-\eta)^\beta d\eta \asymp \sqrt{b-x}. \tag{3.29}$$

Consider the second integral from (3.26) and have

$$\int_{b-\delta}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} \int_{\xi}^{b} \tilde{g}(\eta)d\eta = \int_{b-\delta}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} \int_{\xi}^{b} \tilde{\tilde{g}}(\eta)\sqrt{b-\eta}\, d\eta. \tag{3.30}$$

Using the fact that $\tilde{\tilde{g}}$ is a Hölder-continuous function with index ϵ, we obtain

$$\left|\tilde{\tilde{g}}(\eta)\right| \leq \left|\tilde{\tilde{g}}(b) - \tilde{\tilde{g}}(\eta)\right| + \left|\tilde{\tilde{g}}(b)\right| \leqslant K|b-\eta|^{\epsilon} + \left|\tilde{\tilde{g}}(b)\right|. \tag{3.31}$$

Using (3.31), we represent (3.30) in the form

$$\int_{b-\delta}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} \int_{\xi}^{b} \left[\tilde{\tilde{g}}(\eta) - \tilde{\tilde{g}}(b)\right] \sqrt{b-\eta}\, d\eta$$

$$+ \tilde{\tilde{g}}(b) \int_{b-\delta}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} \int_{\xi}^{b} \sqrt{b-\eta}\, d\eta \equiv \tilde{i_1} + \tilde{i_2}. \tag{3.32}$$

Obviously, $\tilde{i_2}$ can be estimated similar to (3.28), which yields the result

$$\tilde{i_2} = O(1). \tag{3.33}$$

Evaluating $\tilde{i_1}$, we use the estimate

$$\left| \int_{\xi}^{b} \left(\tilde{\tilde{g}}(b) - \tilde{\tilde{g}}(\eta)\right) \sqrt{b-\eta}\, d\eta \right| \leqslant K \int_{\xi}^{b} (b-\eta)^{\epsilon+1/2} d\eta = K(b-\eta)^{\epsilon+3/2},$$

to obtain

$$\sqrt{\frac{b+\xi}{b-\xi}} \int_{\xi}^{b} \left[\tilde{\tilde{g}}(\eta) - \tilde{\tilde{g}}(b)\right] \sqrt{b-\eta}\, d\eta \asymp (b-\xi)^{\epsilon+1}, \quad \xi \to b-0. \tag{3.34}$$

Having (3.34), we obtain that

$$\tilde{i_1} = O(1) \quad \text{as} \quad x \to b-0. \tag{3.35}$$

Combining (3.33) and (3.35), we obtain the following estimate for (3.32) (and thus for (3.30)):

$$\int_{b-\delta}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} \int_{\xi}^{b} \tilde{g}(\ )d\eta = O(1), \quad x \to b-0.$$

This estimate together with (3.28) yields the following estimate for $i^{(1)}$ from (3.23):

$$i^{(1)} = O(\sqrt{b-x}), \quad x \to b-0. \tag{3.36}$$

Combining (3.24) with (3.36), we obtain the desired result,

$$\mathcal{T}[G] = -\int_{-b}^{b} g(\xi)d\xi + G_1(x), \quad G_1(x) = O(\sqrt{b-x}) \quad \text{as} \quad x \to b-0. \tag{3.37}$$

2) Now we consider $\mathcal{T}^2[G]$, and we have

$$\mathcal{T}\left[\mathcal{T}[G]\right] = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}} \int_{-b}^{b_*} \sqrt{\frac{b+\xi}{b-\xi}} \frac{d\xi}{x-\xi} (\mathcal{T}G)(\xi)$$

$$= -\frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}\frac{d\xi}{x-\xi}\int_{-b}^{b} g(\eta)d\eta$$

$$+ \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}\frac{d\xi}{x-\xi}G_1(\xi)$$

$$\equiv \int_{-b}^{b} g(\eta)d\eta + G_2(x). \tag{3.38}$$

To show that $G_2(x) \to 0$ as $x \to b-0$, it suffices to notice that G_1 from (3.37) behaves as $G_1(x) \asymp (b-x)^{1/2}$ for $x \to b-0$, and therefore

$$\int_{-b}^{b_*}\sqrt{\frac{b+\xi}{b-\xi}}\frac{d\xi}{x-\xi}G_1(\xi) = O(1). \tag{3.39}$$

Collecting together (3.38) and (3.39), we obtain

$$\lim_{x\to b-0}\mathcal{T}^2[G] = \int_{-b}^{b} g(\xi)d\xi. \tag{3.40}$$

To complete the proof of the lemma we return to Equation (3.22). We notice that if a function g satisfies this equation and the conditions of Lemma 3.1, then we get, as $x \to b-0$,

$$M' \lim_{x\to b-0}\mathcal{T}[g] - M''\lambda \lim_{x\to b-0}\mathcal{T}\Big[\int_{-b}^{x} g(\xi)d\xi\Big] \tag{3.41}$$

$$+ M \lim_{x\to b-0}(\mathbb{I}-\mathcal{T}^2)[g] + M\lambda \lim_{x\to b-0}[\mathbb{I}-\mathcal{T}^2]\Big[\int_{-b}^{x} g(\xi)d\xi\Big] = 0.$$

To evaluate the limits of $\mathcal{T}[g]$ and $\mathcal{T}^2[g]$, we use Lemma 3.1, and to evaluate the limits of $\mathcal{T}[G]$ and $\mathcal{T}^2[G]$, $G(x) = \int_{-b}^{x} g(\xi)d\xi$, we use estimates (3.37) and (3.38). Finally we obtain the following equation: $M''\lambda\int_{-b}^{b} g(\xi)d\xi = 0$, which means that $\int_{-b}^{b} g(\xi)d\xi = 0$. The lemma is shown. □

4. Unique Solvability of Equation (3.10)

In Section 3, we have shown that if a solution of Equation (3.10) tends to zero as $x \to b$, then this solution must have an average value equal to zero.

Let us introduce a proper subspace of $L_p(-b,b)$ whose elements have zero average values and denote it by $L_p^0(-b,b)$. Note,

$$\dim L_p(-b,b)(\text{mod } L_p^0(-b,b)) = 1.$$

It suffices to consider solvability of Equation (3.10) only in $L_p^0(-b,b)$. We recall that our goal is to find a solution to the nonhomogeneous equation in the space–time domain. This means that after proving that Equation (3.2) has a unique solution, we must reconstruct a function of t, whose Laplace transform is the aforementioned solution of Equation (3.2). However, for the reconstruction we need only the solution of Equation (3.2) for real λ and such that $\lambda \geqslant \sigma_a \gg 1$. To this end, we can use the *Post–Widder Inversion Theorem.* To keep this paper self-contained, we provide a convenient formulation [5].

Post–Widder Inversion Theorem. *Let $f \in L^1_{loc}(0,\infty)$. Assume that there exists $\omega \in \mathbb{R}$ such that $r(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt$ exists for all $\lambda > \omega$. Then for all continuity points $t > 0$ of a function f,*

$$f(t) = \lim_{k\to\infty} (-1)^k \frac{1}{k!}\Big(\frac{k}{t}\Big)^{k+1} r^k\Big(\frac{k}{t}\Big).$$

So, without loss of generality we can restrict our study to the case $\lambda > 0$. The first statement in this section is an immediate consequence of the definitions of the operators $\mathcal{H}_b$ and $\mathcal{T}$.

Proposition 4.1. *For $\lambda > 0$, the solution of Equation* (3.10) *can be considered real-valued.*

In what follows we need the statement below.

Lemma 4.1. *Let A and B be two commuting bounded linear operators in a Hilbert space $\mathcal{H}$. Let*

$$A^* = -A, \qquad B^* = B. \tag{4.1}$$

Assume that one of the operators has a bounded inverse; let B^{-1} exist and $b = \|B^{-1}\|$. Then the sum $A+B$ has a bounded inverse and $\|(A+B)^{-1}\| \leqslant b$.

Proof. Let f be an arbitrary vector from $\mathcal{H}$; then

$$\|(A+B)f\|^2 = \|Af\|^2 + \|Bf\|^2 + (Af, Bf) + (Bf, Af). \tag{4.2}$$

Due to (4.1), we have

$$\begin{aligned}(Af, Bf) &= (f, A^*Bf) = -(f, ABf),\\ (Bf, Af) &= (f, B^*Af) = (f, BAf) = (f, ABf).\end{aligned} \tag{4.3}$$

Thus, $\|(A+B)f\|^2 = \|Af\|^2 + \|Bf\|^2 \geqslant \|Bf\|^2$, which means that the operator $(A+B)^{-1}$ exists. Denoting $\mathcal{F} = Bf$, we obtain that $\|(A+B)B^{-1}\mathcal{F}\| \geqslant \|\mathcal{F}\|$, or $\|A+B\|\|B^{-1}\| \geqslant \|(A+B)B^{-1}\| \geqslant 1$. The last estimate immediately yields the result. The lemma is shown. □

Now the main result of this section for $\lambda > 0$ may be formulated.

Theorem 4.1. *Equation* (3.10) *has only the trivial solution in* $L_p^0(-b,b)$.

Proof. Let us apply the operator $\mathcal{H}_b$ to both sides of Equation (3.10) and then rearrange terms in the new equation as follows:

$$\left(M\mathcal{H}_b^2 + M'\mathcal{H}_b - M\mathbb{I}\right)[g] = \left(\lambda M''\mathcal{H}_b - \lambda M(\mathcal{H}_b^2 - \mathbb{I})\right)\Big[\int_{-b}^{x} g(\xi)d\xi\Big]. \quad (4.4)$$

First we prove that if Equation (4.4) has a nontrivial solution in $L_p(-b,b)$, then this solution, in fact, belongs to $L_2(-b,b)$. Let $\mathcal{A}$ be the following integral operator in $L_p(-b,b)$:

$$\mathcal{A} = M\mathcal{H}_b^2 + M'\mathcal{H}_b - M\mathbb{I}. \quad (4.5)$$

Equation (4.4) can be written as

$$\mathcal{A}[g] = \lambda\left(M''\mathcal{H}_b - M(\mathcal{H}_b^2 - \mathbb{I})\right)\Big[\int_{-b}^{x} g(\xi)d\xi\Big]. \quad (4.6)$$

Using Equation (4.6), we show that if Equation (3.10) has a nontrivial solution from $L_p(-b,b)$, then this solution belongs, in fact, to the space $L_2(-b,b)$. Indeed, with any $g \in L_p(-b,b)$ the function $G(x) = \int_{-b}^{x} g(\xi)d\xi$ is absolutely continuous and, thus, belongs to $L_2(-b,b)$.

The operator $\mathcal{H}_b$ considered as an operator in $L_2(-b,b)$ is bounded and skew self-adjoint ($\mathcal{H}_b^* = -\mathcal{H}_b$) [18]. Therefore, the right-hand side of Equation (4.6) belongs to $L_2(-b,b)$. The operator $\mathcal{A}$ of (4.5) can be factored out as follows:

$$\mathcal{A} = M(\mathcal{H}_b - z_3\mathbb{I})(\mathcal{H}_b - z_4\mathbb{I}), \quad (4.7)$$

with $z_{3,4}$ being real numbers given by

$$z_{3,4} = -\frac{M'}{2M} \pm \sqrt{\frac{(M')^2}{4M^2} + 1}. \quad (4.8)$$

Since $\mathcal{H}_b$ is skew self-adjoint one can use Lemma 4.1 and claim that each operator $(\mathcal{H}_b - z_3\mathbb{I})$ and $(\mathcal{H}_b - z_4\mathbb{I})$ has bounded inverse in $L_2(-b,b)$. Thus,

$$\mathcal{A}^{-1} = M^{-1}(\mathcal{H}_b - z_3\mathbb{I})^{-1}(\mathcal{H}_b - z_4\mathbb{I})^{-1} \equiv M^{-1}\mathcal{R}_{z_3}(\mathcal{H}_b)\mathcal{R}_{z_4}(\mathcal{H}_b). \quad (4.9)$$

Applying $\mathcal{A}^{-1}$ to both sides of Equation (4.6), we obtain

$$g(x) = \lambda M''M^{-1}\mathcal{R}_{z_3}(\mathcal{H}_b)\mathcal{R}_{z_4}(\mathcal{H}_b)\mathcal{H}_b\Big[\int_{-b}^{x} g(\xi)d\xi\Big] \quad (4.10)$$
$$- \lambda\mathcal{R}_{z_3}(\mathcal{H}_b)\mathcal{R}_{z_4}(\mathcal{H}_b)(\mathcal{H}_b^2 - \mathbb{I})\Big[\int_{-b}^{x} g(\xi)d\xi\Big].$$

Since every operator in the right-hand side of Equation (4.10) is bounded in $L_2(-b,b)$, we obtain that $g \in L_2(-b,b)$. Now we are in a position to complete the proof of the theorem, i.e., to show that Equation (3.10) has only the trivial solution. Arguing by contradiction, let us assume that Equation (3.10) has a nontrivial solution $g \in L_2^0(-b,b)$. Let us multiply Equation (3.10) by g and integrate over the interval $(-b,b)$. We have

$$\begin{aligned} M'\|g\|^2_{L_2(-b,b)} - M'' \langle G, g\rangle_{L_2(-b,b)} + M \langle \mathcal{H}_b[g], g\rangle_{L_2(-b,b)} - M \langle \mathcal{T}[g], g\rangle_{L_2(-b,b)} \\ + \lambda M \langle \mathcal{H}_b[G], g\rangle_{L_2(-b,b)} - \lambda M \langle \mathcal{T}[G], g\rangle_{L_2(-b,b)} = 0, \end{aligned} \tag{4.11}$$

where $G(x) = \int_{-b}^x g(\xi)d\xi$ and $\langle\cdot,\cdot\rangle_{L_2(-b,b)}$ is the notation for a scalar product in $L_2(-b,b)$.

In what follows, we will omit the subindex "$L_2(-b,b)$" and by angular brackets we will denote a scalar product in $L_2(-b,b)$. Consider each term in Equation (4.11). We have

$$\begin{aligned} \langle G, g\rangle = \int_{-b}^b dx\Big(\int_{-b}^x g(\xi)d\xi\Big)g(x) &= \frac{1}{2}\int_{-b}^b \frac{d}{dx}\Big(\int_{-b}^x g(\xi)d\xi\Big)^2 dx \\ &= \frac{1}{2}\Big(\int_{-b}^x g(\xi)d\xi\Big)\Big|_{-b}^b = 0, \end{aligned} \tag{4.12}$$

since $g \in L_2(-b,b) \cap L_p^0(-b,b)$. Taking into account that $\mathcal{H}_b$ is skew self-adjoint and g is real-valued, we have

$$\langle \mathcal{H}_b g, g\rangle = \langle g, \mathcal{H}_b^* g\rangle = -\langle g, \mathcal{H}_b g\rangle = -\overline{\langle \mathcal{H}_b g, g\rangle} = -\langle \mathcal{H}_b g, g\rangle. \tag{4.13}$$

Equation (4.13) is valid if and only if $\langle \mathcal{H}_b g, g\rangle = 0$.

To evaluate the next integral from (4.11) involving $\mathcal{T}$, let $f = \mathcal{T}[g]$ and get

$$\begin{aligned} \langle \mathcal{T}[g], g\rangle = \langle f, \mathcal{H}_b[f]\rangle = \overline{\langle \mathcal{H}_b[f], f\rangle} = -\overline{\langle f, \mathcal{H}_b[f], g\rangle} \\ = -\langle f, \mathcal{H}[f]\rangle = -\langle \mathcal{T}[g], g\rangle = 0. \end{aligned} \tag{4.14}$$

Counting (4.12)–(4.14), we rewrite Equation (4.11) as

$$M'\|g\|^2 + \lambda M \langle \mathcal{H}_b[G], g\rangle - \lambda M \langle \mathcal{T}[G], g\rangle = 0. \tag{4.15}$$

Next, we return to Equation (3.10), multiply both sides by $G = \int_{-b}^x g(\xi)d\xi$ and integrate:

$$\begin{aligned} M'\langle g, G\rangle - M''\lambda\|G\|^2 + M\langle \mathcal{H}_b[g], G\rangle - M\langle \mathcal{T}[g], G\rangle \\ + M\lambda\langle \mathcal{H}_b[G], G\rangle - M\lambda\langle \mathcal{T}[G], G\rangle = 0. \end{aligned} \tag{4.16}$$

Completing the steps in a way similar to (4.12)–(4.14), we obtain

$$\langle g, G\rangle = \langle \mathcal{H}_b[G], G\rangle = \langle \mathcal{T}[G], G\rangle = 0,$$

and thus, Equation (4.16) becomes

$$-M''\lambda \left\|G\right\|^2 + M \left\langle \mathcal{H}_b[g], G\right\rangle - M \left\langle \mathcal{T}[g], G\right\rangle = 0. \tag{4.17}$$

Combining (4.15) and (4.17), we obtain the following important relation:

$$\sqrt{M''}\lambda \left\|G\right\| = \sqrt{M'} \left\|g\right\|. \tag{4.18}$$

Let us return to Equation (4.4) and rewrite it in the form

$$\mathcal{H}_b\Big[M'g - M''\lambda \int_{-b}^{x} g(\xi)d\xi\Big] = -M\left(\mathcal{H}_b^2 - \mathbb{I}\right)\Big[g + \lambda \int_{-b}^{x} g(\xi)d\xi\Big]. \tag{4.19}$$

We will use relation (4.18) to prove that Equation (4.19) has only a trivial solution. We distinguish two cases, $M = 0$ and $M > 0$. If $M = 0$, then Equation (4.19) becomes ($M' = 1, M'' = 0$): $\mathcal{H}_b[g] = 0$, $g \in L_2(-b, b)$. This equation does not have nontrivial solutions. Consider the case $M > 0$. Let us estimate the $L_2(-b, b)$-norm of the right-hand side of Equation (4.19). Since $\mathcal{H}_b$ is skew-selfadjoint (Lemma 4.1), we have $\|(\mathcal{H}_b \pm \mathbb{I})f\| \geqslant \|f\|$ for any $f \in L_2(-b, b)$. We also know that g and G are orthogonal in $L_2(-b, b)$. Therefore,

$$\begin{aligned}\|(\mathcal{H}_b^2 - \mathbb{I})(g + \lambda G)\|^2 &= \|(\mathcal{H}_b + \mathbb{I})(\mathcal{H}_b - \mathbb{I})(g + \lambda G)\|^2 \\ &\geqslant \|g + \lambda G\|^2 = \|g\|^2 + \lambda^2\|G\|^2.\end{aligned} \tag{4.20}$$

Counting (4.18), estimate (4.20) can be reduced to

$$\|(\mathcal{H}_b^2 - \mathbb{I})(g + \lambda G)\|^2 \geqslant \|g\|^2 + (M'/M'')\|g\|^2 = (M'')^{-1}\|g\|^2. \tag{4.21}$$

Now we use (4.18) and estimate the norm of the left-hand side of Equation (4.19) as

$$\begin{aligned}&\left\|\mathcal{H}_b\left[M'g - M''\lambda G\right]\right\|^2 \leqslant \left\|M'g - M''\lambda G\right\|^2 = (M')^2\left\|g\right\|^2 + (M'')^2\lambda^2\left\|G\right\|^2 \\ &= (M')^2\left\|g\right\|^2 + (M'')^2\frac{M'}{M''}\left\|g\right\|^2 = M'\left\|g\right\|^2\left(M' + M''\right) = M'\left\|g\right\|^2.\end{aligned} \tag{4.22}$$

Using estimate (4.22) and the assumption that $\|g\| \neq 0$, we obtain that the following inequality must be valid:

$$\frac{\sqrt{M'M''}}{M} \geqslant 1. \tag{4.23}$$

However,

$$\frac{M'M''}{M^2} = \frac{\sqrt{1-M^2}(1-\sqrt{1-M^2})}{(1+\sqrt{1-M^2})(1-\sqrt{1-M^2})} = \frac{\sqrt{1-M^2}}{(1+\sqrt{1-M^2})} < 1 \tag{4.24}$$

for $0 < M \leqslant 1$. Estimate (4.24) means that (4.23) is not valid and, therefore, $\|g\| = 0$ and so $g = 0$. The theorem is completely proved. □

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We recall that our goal is to show that the homogeneous equation

$$M'f - \lambda\mathcal{L}(\lambda)f + M\{\mathcal{H}_b - \mathcal{T}\}[f] = 0 \tag{4.25}$$

does not have nontrivial solutions in $L_p(-b,b)$. As we have already shown, if f is a solution of Equation (4.25) from $L_p(-b,b)$, $1 < p < 2$, then this solution must be from $L_2(-b,b)$. We have introduced an operator

$$\mathbb{T} = M'\mathcal{H}_b - \lambda\mathcal{H}_b\mathcal{L}(\lambda) + M\{\mathcal{H}_b^2 - \mathbb{I}\} = \mathbb{H} - \lambda\mathcal{H}_b\mathcal{L}(\lambda), \tag{4.26}$$

with $\mathbb{H}$ being defined in (2.13), and have shown that $\mathbb{T}$ is a Fredholm-type operator in $L_p(-b,b)$. This means that to show that a nonhomogeneous equation $\mathbb{T}f = v$, $v \in L_p(-b,b)$, has a unique solution, it suffices to prove that the homogeneous equation has only the trivial solution in $L_p(-b,b)$. However, we have already proved that the equation $\mathbb{T}f = 0$ considered on a closed subspace of functions implies $f = 0$. So our remaining task is to extend the zero solvability result to the case when g is an arbitrary function from $L_2(-b,b)$. First, let us rewrite the condition $\int_{-b}^{b} g(\xi)d\xi = 0$ in terms of f, where $g = f - \lambda\mathcal{L}(\lambda)f$. We have

$$\begin{aligned} 0 = \int_{-b}^{b} g(\xi)d\xi &= \int_{-b}^{b} f(\xi)d\xi - \lambda\int_{-b}^{b} d\xi e^{-\lambda\xi}\int_{-b}^{\xi} e^{\lambda\sigma} f(\sigma)d\sigma \\ &= e^{-\lambda b}\int_{-b}^{b} e^{\lambda\sigma} f(\sigma)d\sigma. \end{aligned} \tag{4.27}$$

Thus, we have shown that the equation $\mathbb{T}f = 0$ implies $f = 0$ on the subspace of function from $L_2(-b,b)$ satisfying the following condition:

$$\int_{-b}^{b} e^{\lambda\sigma} f(\sigma)d\sigma = 0. \tag{4.28}$$

Let $\mathfrak{M}$ be the aforementioned subspace of $L_2(-b,b)$, functions of which satisfy (4.28); obviously codim $\mathfrak{M} = 1$. Due to this fact, there might be only one linearly independent function that satisfies the homogeneous equation $\mathbb{T}f = 0$ and such that $\int_{-b}^{b} e^{\lambda\sigma} f(\sigma)d\sigma \neq 0$. Let us represent this possible solution in the form $f(x) = f_0(x) + \gamma e^{\lambda x}$, $f_0 \in \mathfrak{M}$. Without loss of generality, we can assume that $\gamma = 1$ and thus

$$f(x) = f_0(x) + e^{\lambda x}, \quad x \in (-b,b), \quad f_0 \in \mathfrak{M}. \tag{4.29}$$

From our assumption about f, it follows that

$$\mathbb{T}[f_0] = -\mathbb{T}[e^{\lambda\cdot}] \equiv F, \tag{4.30}$$

where $F = -\mathbb{T}\left[e^{\lambda\cdot}\right]$. Let us show that $F \neq 0$. We have

$$\mathbb{T}[e^{\lambda\cdot}] = \mathbb{H}[e^{\lambda\cdot}] - [\lambda\mathcal{H}_b\mathcal{L}(\lambda)][e^{\lambda\cdot}]. \tag{4.31}$$

Consider $\mathcal{L}(\lambda)][e^{\lambda\cdot}]$. We have

$$\lambda\mathcal{L}(\lambda)][e^{\lambda\cdot}] = \lambda\int_{-b}^{x} e^{-\lambda x}e^{2\lambda\xi}d\xi = 1/2[e^{\lambda x} - e^{-\lambda(x+2b)}],$$

and thus $\mathbb{T}[e^{\lambda\cdot}] = \mathbb{H}[e^{\lambda\cdot}] - 1/2\mathcal{H}_b[e^{\lambda\cdot} - e^{-2b\lambda-\lambda\cdot}]$. Arguing by contradiction, we assume that $F = 0$, and this means that

$$(\mathbb{H} - 1/2\mathcal{H}_b)[e^{\lambda\cdot}] = -\mathcal{H}_b[e^{-2b\lambda-\lambda\cdot}]. \tag{4.32}$$

Taking into account definition (2.12) of the operator $\mathbb{H}$, we observe that $(\mathbb{H} - 1/2\mathcal{H}_b)$ can be factored out as follows:

$$\mathbb{H} - 1/2\mathcal{H}_b = M\mathcal{H}_b^2 + (\sqrt{1-M^2} - 1/2)\mathcal{H}_b - M\mathbb{I} = M(\mathcal{H}_b - z_5\mathbb{I})(\mathcal{H}_b - z_6\mathbb{I}), \tag{4.33}$$

$0 < M \leqslant 1$, where

$$z_{5,6} = (2M)^{-1}\left[\sqrt{1-M^2} - 1/2 \pm \sqrt{5/4 - \sqrt{1-M^2} + 3M^2}\right].$$

Since $5/4 + 3M^2 - \sqrt{1-M^2} > 1/4$, the roots z_5 and z_6 are real numbers, which means that for every M, the operators $(\mathcal{H}_b - z_i\mathbb{I})^{-1}$, $i = 5, 6$, exist and are bounded. Therefore, Equation (4.32) can be rewritten in the form

$$e^{\lambda x} = -M^{-1}e^{-2b\lambda}(\mathcal{H}_b - z_5\mathbb{I})^{-1}(\mathcal{H}_b - z_6\mathbb{I})^{-1}\mathcal{H}_b[e^{-\lambda\cdot}]. \tag{4.34}$$

Based on Lemma 4.1, we obtain the following estimate for the norm of the operator in Equation (4.34):

$$\|(\mathcal{H}_b - z_5\mathbb{I})^{-1}(\mathcal{H}_b - z_6\mathbb{I})^{-1}\mathcal{H}_b\| \leqslant |z_5z_6| \leqslant 2M^2. \tag{4.35}$$

Substituting (4.35) into Equation (4.34) and taking into account that

$$\|e^{\lambda\cdot}\|_{L_2(-b,b)} = \|e^{-\lambda\cdot}\|_{L_2(-b,b)} = \lambda^{-1}\sinh\lambda b,$$

we reduce (4.34) to the following inequality: $e^{2\lambda b} \leqslant 2M$. Obviously for large $\lambda > 0$, this inequality is not valid. This obtained contradiction shows that the right-hand side of Equation (4.30) is not equal to zero.

Let $\mathfrak{N}$ be a closed subspace in $L_2(-b,b)$ which is an image of $\mathfrak{M}$ under the mapping $\mathbb{T}$, i.e., $G \in \mathfrak{N}$ implies that there exists $g \in \mathfrak{M}$ such that $G = \mathbb{T}[g]$.

Based on the fact that the operator $\mathbb{T}$ is of a Fredholm type in $L_2(-b,b)$ and that equation $\mathbb{T}[g] = 0$ implies $g = 0$, we obtain that mapping defined by $\mathfrak{M} \xrightarrow{\mathbb{T}} \mathfrak{N}$ is one-to-one. In particular, it means that if $G \in \mathfrak{N}$, then the equation $\mathbb{T}[g] = G$ has a unique solution from $\mathfrak{M}$. Now we return to Equation (4.33), which can be written as

$$\mathbb{T}[f_0] = F, \quad F \in \mathfrak{N}. \tag{4.36}$$

Since $F \in \mathfrak{N}$, we can apply $\mathbb{T}^{-1}$ to both sides of Equation (4.36) and get

$$f_0(x) = \mathbb{T}^{-1}[F] = -\mathbb{T}^{-1}\big[\mathbb{T}[e^{\lambda\cdot}]\big] = -e^{\lambda x}.$$

The last equation cannot hold since $e^{\lambda\cdot} \notin \mathfrak{M}$. Theorem 3.1 is completely proved. □

5. Concluding Remarks

In this section we briefly outline the proof of the unique solvability of Equation (2.1). We recall that in Section 4 we have shown that the following equation has a unique solution in $L_p(-b,b)$:

$$\begin{aligned}&\frac{2}{\sqrt{1-M^2}}\mathcal{T}W_a(\cdot,\lambda) \\ &= \Big[1 - \frac{\lambda}{\sqrt{1-M^2}}\mathcal{L}(\lambda)\Big]F(\cdot,\lambda) + \frac{M}{\sqrt{1-M^2}}\left\{\mathcal{H}_b[F] - \mathcal{T}[F]\right\}.\end{aligned} \tag{5.1}$$

Now we add to Equation (5.1) the term from Equation (2.1) containing the function $h_-(x,\lambda)$. If we apply $\mathcal{H}_b$ to both sides of that equation, we have the following result:

$$\begin{aligned}\sqrt{1-M^2}W_a(x,\lambda) &= (M\mathcal{H}_b^2 + \sqrt{1-M^2}\mathcal{H}_b - M\mathbb{I})[F] \\ &\quad + \lambda e^{-b\lambda}\mathcal{H}_b[h_-(\cdot,\lambda)]L(\lambda, F(\cdot,\tilde{\lambda})) - \lambda\mathcal{H}_b[\mathcal{L}(\lambda)F(\cdot,\tilde{\lambda})].\end{aligned} \tag{5.2}$$

In (2.13) we have introduced the bounded operator $\mathbb{H}$, which has a bounded inverse in $L_p(-b,b)$, $\;1 < p < 2$, $\mathbb{R}_M \equiv \mathbb{H}^{-1}$. Applying $\mathbb{R}_M$ to both sides of Equation (5.2), we obtain

$$\begin{aligned}&\sqrt{1-M^2}\mathbb{R}_M\left[W_a(\cdot,\lambda)\right] \\ &= F(x,\tilde{\lambda}) - \lambda\mathbb{R}_M\mathcal{H}_b[\mathcal{L}(\lambda)F(\cdot,\tilde{\lambda})] + \lambda e^{-b\lambda}\mathbb{R}_M\mathcal{H}_b[h_-]L(\lambda, F(\cdot,\tilde{\lambda})) \\ &= (\mathbb{I} - \mathbb{R}_M\mathcal{H}_b\mathcal{L}(\lambda))[F(\cdot,\tilde{\lambda})] + \lambda e^{-b\lambda}L[\lambda, F(\cdot,\tilde{\lambda})]\mathbb{R}_M\mathcal{H}_b[h_-].\end{aligned} \tag{5.3}$$

Let

$$\widetilde{W}_a(x,\lambda) = \sqrt{1-M^2}\mathbb{R}_M[W_a(\cdot,\lambda)], \qquad \tilde{h}_- = \mathbb{R}_M\mathcal{H}_b[h_-]. \tag{5.4}$$

Based on the results of Section 4 and using (5.4), we can see that Equation (5.3) is equivalent to the following one:

$$\begin{aligned}(\mathbb{I} - \mathbb{R}_M \mathcal{H}_b \mathcal{L}(\lambda))^{-1}[\widetilde{W}_a(x,\lambda)] \\ = F(x,\tilde{\lambda}) + \lambda e^{-b\lambda} L(\lambda, F(\cdot,\tilde{\lambda}))(\mathbb{R}_M - \lambda \mathcal{H}_b \mathcal{L}(\lambda))^{-1}[\tilde{h}_-].\end{aligned} \tag{5.5}$$

Let

$$G_1 = (\mathbb{I} - \mathbb{R}_M \mathcal{H}_b \mathcal{L}(\lambda))^{-1}[\widetilde{W}_a(x,\lambda)], \quad G_2 = \lambda e^{-b\lambda}(\mathbb{R}_M - \lambda \mathcal{H}_b \mathcal{L}(\lambda))^{-1}[\tilde{h}_-]. \tag{5.6}$$

In terms of (5.6), Equation (5.5) becomes

$$G_1 = F(x,\tilde{\lambda}) + L(\lambda, F(\cdot,\tilde{\lambda}))G_2. \tag{5.7}$$

Multiplying both sides of Equation (5.7) by $e^{\lambda x}$ and integrating over $(-b,b)$, we have

$$L(\lambda, G_1) = L(\lambda, F)(1 + L(\lambda, G_2)). \tag{5.8}$$

One can verify directly that neither $L(\lambda, G_1)$ nor $(1 + L(\lambda, G_2))$ is equal to zero identically. Therefore,

$$L(\lambda, F) = \frac{L(\lambda, G_1)}{1 + L(\lambda, G_2)}, \quad \lambda \geqslant \sigma_a \gg 1. \tag{5.9}$$

Substituting (5.9) into (5.5) yields the desired formula for the solution of the problem with $\alpha_1 = \alpha_2 = 0$.

As has already been mentioned, the addition of the two remaining integral terms of Equation (2.1) cannot destroy the unique solvability of Equation (2.1) in $L_p(-b,b)$.

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