CONVERGENCE TO SELF-SIMILAR SOLUTIONS
FOR A PARABOLIC-ELLIPTIC SYSTEM
OF DRIFT-DIFFUSION TYPE IN $\mathbb{R}^2$

TOSHIKATA NAGAI
Department of Mathematics, Graduate School of Science
Hiroshima University, Higashi-Hiroshima 739-8526, Japan

(Submitted by: Yoshikazu Giga)

Abstract. We consider the Cauchy problem for a parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$, modeling chemotaxis and self-attracting particles, with $L^1$-initial data. Under the assumption that the total mass of nonnegative initial data is less than $8\pi$, by using similarity arguments, it is shown that the nonnegative solution converges to a radially symmetric self-similar solution at rate $o(t^{-1+1/p})$ in the $L^p$-norm ($1 \leq p \leq \infty$) as time goes to infinity.

1. Introduction

In this paper we are concerned with the large-time behavior of nonnegative solutions of the Cauchy problem for the following nonlinear equation:

$$
\begin{align*}
\partial_t u &= \Delta u - \nabla \cdot (u \nabla \psi), & t > 0, & x \in \mathbb{R}^2, \\
-\Delta \psi &= u, & t > 0, & x \in \mathbb{R}^2, \\
u(0, x) &= u_0(x), & x \in \mathbb{R}^2,
\end{align*}
$$

where $\psi$ is defined by

$$
\psi(t, x) := (N * u)(t, x) = \int_{\mathbb{R}^2} N(x - y) u(t, y) dy
$$

and $N(x)$ is the logarithmic potential in $\mathbb{R}^2$, namely

$$
N(x) = \frac{1}{2\pi} \log \frac{1}{|x|}.
$$

This system is a simplified version of a chemotaxis system obtained from the original Keller-Segel model [25] (see also Childress-Percus [13]), and also a model of self-attracting particles in $\mathbb{R}^2$ (see [7, 45]).
In the subcritical case $\int_{\mathbb{R}^2} u_0 \, dx < 8\pi$ for the nonnegative initial data $u_0 \in L^1(\mathbb{R}^2)$, the global existence of nonnegative solutions to the Cauchy problem (1.1)–(1.3) have been studied in [10] under the assumption

$$u_0 \log u_0, \ u_0|x|^2 \in L^1(\mathbb{R}^2),$$

and in [34] under $u_0 \log(1 + |x|) \in L^1(\mathbb{R}^2)$. On the other hand, in the supercritical case $\int_{\mathbb{R}^2} u_0 \, dx > 8\pi$, the nonnegative solutions with initial data of finite second moment blow up in finite time (see [7, 10, 26]). The critical case $\int_{\mathbb{R}^2} u_0 \, dx = 8\pi$ has been studied in [6, 42] for radially symmetric solutions, and without symmetry assumptions, in [9] for the initial data of finite second moment and in [8, 38] for the initial data of infinite second moment. For related results for chemotaxis models, see [16, 19, 21, 22, 31, 35, 36, 43], and for models of self-attracting particles, see [4, 5], for example. We also refer to [20, 44] in which we can find related results for chemotaxis models.

Since $\nabla \psi$ may be rewritten as

$$\nabla \psi(t, x) = (\nabla N * u)(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} u(t, y) \, dy,$$

the Cauchy problem (1.1)–(1.3) leads to

$$\partial_t u = \Delta u - \nabla \cdot (u(\nabla N * u)), \quad t > 0, \ x \in \mathbb{R}^2,$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^2.$$  \hspace{1cm} (1.5)

Only under the assumption

$$u_0 \geq 0 \text{ on } \mathbb{R}^2, \quad u_0 \in L^1(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} u_0 \, dx < 8\pi, \hspace{1cm} (1.7)$$

the Cauchy problem (1.5)–(1.6) has a unique nonnegative mild solution $u$ globally in time, and the $L^p$-norms of $u(t)$ decay to zero with the exponents $t^{-1+1/p}$ for every $1 < p \leq \infty$ as time goes to infinity (see [32]).

In the study of the large-time behavior of nonnegative global solutions, radially symmetric self-similar solutions play an important role. Equation (1.5) has a scaling invariant property such that for a solution $u$ of (1.5), the function $u_\lambda$ for $\lambda > 0$ defined by

$$u_\lambda(t, x) = \lambda^2 u(\lambda^2 t, \lambda x), \quad t > 0, \ x \in \mathbb{R}^2$$

is also a solution of (1.5). If $u_\lambda = u$ for all $\lambda > 0$, the solution $u$ is called a self-similar solution. Given $\hat{M} > 0$, consider a radially symmetric self-similar solution $U_{\hat{M}}$ of (1.5) such that

$$U_{\hat{M}}(t, x) = \frac{1}{t} \Phi_{\hat{M}} \left( \frac{|x|}{\sqrt{t}} \right), \quad \int_{\mathbb{R}^2} U_{\hat{M}}(t, x) \, dx = \hat{M},$$

\hspace{1cm} (1.8)
where $\Phi$ is nonnegative, integrable, and bounded on $[0, \infty)$. The existence of such a radially symmetric self-similar solution has been studied in [3] by ODE methods and in [39] by PDE methods, and uniqueness has been studied in [6]. The existence result reads as follows: For every $\hat{M} \in (0, 8\pi)$, there exists uniquely a radially symmetric self-similar solution $U_{\hat{M}}$ satisfying (1.8), and if $U_{\hat{M}}$ exists, then $\hat{M} \in (0, 8\pi)$. As for nonnegative self-similar solutions of (1.5) without symmetry assumptions, in [39] it was proved that if $V = V(t, x)$ is a nonnegative self-similar solution of (1.5) satisfying $V(t, x) = \frac{1}{t} \Psi \left( \frac{x}{\sqrt{t}} \right)$, where $\Psi$ is nonnegative and belongs to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then $\Psi$ is radially symmetric about a point $x_0 \in \mathbb{R}^2$. Hence, if $\hat{M} := \int_{\mathbb{R}^2} V(t, x) \, dx < 8\pi$, then $V(t, x) = U_{\hat{M}}(t, x - x_0)$ by the uniqueness of nonnegative radially symmetric self-similar solutions satisfying (1.8).

The aim of this paper is to show that under assumption (1.7) on $u_0$, the nonnegative mild solution $u$ of (1.5)–(1.6) satisfies

$$\lim_{t \to \infty} t^{1-1/p} \| u(t) - U_{\hat{M}}(t) \|_p = 0$$

for every $1 \leq p \leq \infty$, where $\hat{M} = \int_{\mathbb{R}^2} u_0 \, dx$ (see Theorem 4.2). No restrictions on $u_0$ are assumed except assumption (1.7). The result (1.9) for $p = 1$ was obtained in [10] under the additional assumption (1.4) on $u_0$, and the proof relies on entropy methods. The entropy method requires

$$u(t) \log u(t), \, |x|^2 u(t) \in L^1(\mathbb{R}^2),$$

which is not expected only under assumption (1.7). For the radial case, the convergence of the mass distribution function $\int_{|x| < r} u(t, x) \, dx$ to self-similarity was obtained in [6] under assumption (1.7) on radially symmetric initial data by using a different method from that in [10] and this paper. We prove (1.9) by showing

$$\lim_{\lambda \to \infty} \| u_{\lambda}(1) - U_{\hat{M}}(1) \|_p = 0.$$ 

This rescaling method goes back to Carpio [12], studying the large-time behavior of solutions to the vorticity equations for incompressible viscous fluids, and we also refer to the book of Giga-Giga-Saal [17] for this method and related topics for the vorticity equations.

We remark that the self-similar property is also satisfied for the following chemotaxis system,

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla \psi), \quad \partial_t \psi = \Delta \psi + u, \quad t > 0, \, x \in \mathbb{R}^2, \quad (1.10)$$

and the convergence of solutions to self-similar solutions has been studied in [23, 37] for small initial data by using different methods. For the global
existence of nonnegative solutions to the Cauchy problem for (1.10), see [29, 33] for example. We also mention that the large-time behaviors for the Cauchy problem related to Keller-Segel systems have been studied in [28, 40, 46].

This paper is organized as follows. In Section 2 we mention known results on the global existence of nonnegative solutions to (1.5)–(1.6). In Section 3 we give the estimates on the derivatives of nonnegative solutions. Section 4 is devoted to the proof of (1.9).

2. Global solutions with subcritical $L^1$-initial data

Throughout this paper, we use the following notation: $L^p(\mathbb{R}^d)$ is the Lebesgue space on $\mathbb{R}^d$ with the usual norm $\| \cdot \|_{L^p}$ for $1 \leq p \leq \infty$. In the case $d = 2$, for simplicity, we denote $L^p(\mathbb{R}^2)$ and $\| \cdot \|_{L^p}$ by $L^p$ and $\| \cdot \|_p$, respectively. For $Q \subset \mathbb{R}^d$ and a Banach space $X$, we denote the set of all continuous functions from $Q$ to $X$ by $C(Q; X)$ and the set of all bounded continuous functions by $BC(Q; X)$. If $X = \mathbb{R}$, then we denote $C(Q)$ and $BC(Q)$, respectively. Denote by $\mathbb{Z}_+$ the set of all nonnegative integers. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}_+^d$, put $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and $\partial_\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$, $\partial_j = \partial_{\partial x_j}$. For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, we denote by $\partial_x^m$ any partial derivative of order $m$ with respect to the space variables and put

$$\|\partial_x^m f\|_{L^p} = \sum_{|\alpha|=m} \|\partial_\alpha f\|_{L^p}.$$ 

For a function $f = f(t, x)$, $(t, x) \in (a, b) \times \Omega$, where $-\infty \leq a < b \leq \infty$, $\Omega \subset \mathbb{R}^d$, we denote by $f(t) : \Omega \to \mathbb{R}$ for $t \in (a, b)$ the function $f(t)(x) = f(t, x)$.

We give the definition of mild solutions of the Cauchy problem (1.5)–(1.6).

**Definition 2.1.** Given $u_0 \in L^1$, a function $u$ on $[0, T) \times \mathbb{R}^2$ is said to be a mild solution of (1.5)–(1.6) on $[0, T)$ if

(i) $u \in C([0, T); L^1) \cap C((0, T); L^{4/3})$,

(ii) $\sup_{0 \leq t \leq T} t^{1/4} \|u(t)\|_{4/3} < \infty$,

(iii) $u$ satisfies the integral equation

$$u(t) = e^{t\Delta} u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta} (u(s)(\nabla N * u)(s)) \, ds, \quad 0 < t < T,$$

where $e^{t\Delta}$ is the heat semigroup defined by

$$(e^{t\Delta} f)(x) = \int_{\mathbb{R}^2} G(t, x-y)f(y) \, dy, \quad G(t, x) = \frac{1}{4\pi t} \exp(-\frac{|x|^2}{4t}). \quad (2.1)$$
A function $u$ on $[0, \infty) \times \mathbb{R}^2$ is a global mild solution of (1.5)–(1.6) with initial data $u_0$ if $u$ is a mild solution of (1.5)–(1.6) on $[0, T)$ for any $0 < T < \infty$.

In order to get local existence, uniqueness, and regularity for the Cauchy problem (1.5)–(1.6) in $\mathbb{R}^2$ (see [2, 11, 18, 24], for example), combined with the following estimate of $f(\nabla N \ast g)$ involved in the nonlinear term of (1.5), namely, for $4/3 \leq q < 2$,

$$
\| f(\nabla N \ast g) \|_{2q/(4-q)} \leq C_q \| f \|_q \| g \|_q \quad \text{for all } f, g \in L^q,
$$

(2.2)

where $C_q$ is a positive constant depending only on $q$. This inequality is deduced from the Hardy-Littlewood-Sobolev inequality in $\mathbb{R}^2$: For $1 < q < 2$,

$$
\left\| \frac{1}{|x|} \ast g \right\|_{2q/(2-q)} \leq C_q \| g \|_q \quad \text{for all } g \in L^q,
$$

where $C_q$ is a positive constant depending only on $q$.

To mention local existence, uniqueness, and regularity, following Kato [24], we introduce function spaces. Let $T > 0$. For $1 \leq p \leq \infty$ and $\gamma \geq 0$, define the Banach space $C_{\gamma,T}(L^p)$ with norm $\| \cdot \|_{p,\gamma,T}$ by

$$
C_{\gamma,T}(L^p) = \{ u : u \in C((0,T); L^p), \sup_{0 < t < T} t^\gamma \| u(t) \|_p < \infty \},
$$

$$
\| u \|_{p,\gamma,T} = \sup_{0 < t < T} t^\gamma \| u(t) \|_p \quad \text{for } u \in C_{\gamma,T}(L^p).
$$

For $\gamma > 0$, define $\dot{C}_{\gamma,T}(L^p) = \{ u \in C_{\gamma,T}(L^p) : \lim_{t \to 0} t^\gamma \| u(t) \|_p = 0 \}$, and for $\gamma = 0$, $\dot{C}_{0,T}(L^p) = BC([0,T); L^p)$, $\dot{C}_{\gamma,T}(L^p)$ is a closed subspace of $C_{\alpha,T}(L^p)$.

**Theorem 2.1.** For the initial data $u_0 \in L^1$, there exists $T \in (0, \infty)$ such that the Cauchy problem (1.5)–(1.6) has uniquely a mild solution $u$ on $[0,T)$.

Moreover, $u$ satisfies the following:

(i) $u(t) \to u_0$ in $L^1$ as $t \to 0$.

(ii) For $1 \leq q \leq \infty$, $u \in \dot{C}_{1-1/q,T}(L^q)$.

(iii) For $\ell \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^2$, $1 < q < \infty$, $\partial_\ell \partial_\alpha u \in \dot{C}_{1-1/q+|\alpha|/2+\ell,T}(L^q)$.

(iv) For $\ell \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^2$. For $2 < q < \infty$ if $|\alpha| = 0$, and for $1 < q < \infty$ if $|\alpha| \geq 1$, $\partial_\ell \partial_\alpha (\nabla N \ast u) \in \dot{C}_{1/2-1/q+|\alpha|/2+\ell,T}(L^q)$.

(v) $u$ is a classical solution of $\partial_t u = \Delta u - \nabla \cdot (u(\nabla N \ast u))$ in $(0,T) \times \mathbb{R}^2$.

(vi) $\int_{\mathbb{R}^2} u(t,x) \, dx = \int_{\mathbb{R}^2} u_0(x) \, dx$ for all $0 < t < T$.

(vii) If $u_0 \geq 0$ and $u_0 \not\equiv 0$ on $\mathbb{R}^2$, then $u(t,x) > 0$ on $(0,T) \times \mathbb{R}^2$.

For the proof of this theorem, see [32, 34].

Under the additional assumption $u_0 \log(1 + |x|) \in L^1$ on the initial data $u_0 \in L^1$, Proposition 2.1 below ensures that $\psi(t) := N \ast u(t)$ is well-defined in
Proposition 2.1. Let the initial data $u_0 \in L^1$ satisfy $u_0 \log(1 + |x|) \in L^1$. Then the mild solution $u$ to (1.5)–(1.6) on $[0, T)$ satisfies that for every $0 < t < T$,

$$\int_{|x| \geq 2} |u(t, x)| \log |x| \, dx \leq \int_{\mathbb{R}^2} |u_0(x)| \log(1 + |x|) \, dx + C,$$

where $C > 0$ is a constant depending only on $\sup_{0 < t < T} (\|u(t)\|_1 + t^{\frac{1}{2}} \|u(t)\|_{4/3})$ and $T$. Hence, $u(t) \log(1 + |x|) \in L^1$ for any $0 < t < T$.

For the nonnegative initial data $u_0 \in L^1$ of finite second moment, the second moment identity is described in the following proposition. For the proof, see [9, 10].

Proposition 2.2. Let the nonnegative initial data $u_0 \in L^1$ satisfy $|x|^2 u_0 \in L^1$. Then for the nonnegative mild solution $u$ to (1.5)–(1.6) on $[0, T)$, it holds that for every $0 < t < T$,

$$\int_{\mathbb{R}^2} |x|^2 u(t, x) \, dx = \int_{\mathbb{R}^2} |x|^2 u_0(x) \, dx + 4\hat{M} \left( 1 - \frac{\hat{M}}{8\pi} \right) t,$$

where $\hat{M} = \int_{\mathbb{R}^2} u_0 \, dx$.

In order to mention the global existence and decay estimates of the nonnegative solution $u$ for the nonnegative initial data $u_0 \in L^1$ with $\hat{M} := \int_{\mathbb{R}^2} u_0 \, dx < 8\pi$, we consider radially symmetric self-similar solutions $U_{\hat{M}}(t, x)$ satisfying (1.8). As mentioned in the Introduction, for every $\hat{M} \in (0, 8\pi)$, there exists uniquely such a radially symmetric self-similar solution $U_{\hat{M}}(t, x)$.

Following [3, 6], we introduce the mass distribution function

$$M(t, s) = \int_{|x| \leq \sqrt{s}} U_{\hat{M}}(t, x) \, dx, \quad t > 0, \ s \geq 0,$$

and see that the function $M(t, s)$ satisfies the following:

$$\begin{cases}
\partial_t M = 4\partial_s^2 M + \frac{1}{2} M \partial_s M, & t > 0, \ s > 0, \\
M(t, 0) = 0, \ M(t, +\infty) = \hat{M}, & t > 0, \\
\lim_{t \to 0} M(t, s) = \hat{M}, & s > 0.
\end{cases}$$
Convergence to self-similar solutions 845

Since \( M(t, s) \) has the property that for each \( \lambda > 0 \), \( M(\lambda t, \lambda s) = \lambda M(t, s) \), \( t > 0 \), \( s \geq 0 \), \( M(t, s) \) has the form \( M(t, s) = m(s/t) \), \( t > 0 \), \( 0 \leq s < \infty \) for some function \( m(y) \). The nonnegative function \( m(y) \) satisfies

\[
\begin{cases}
4 \frac{d^2 m}{dy^2}(y) + \frac{dm}{dy}(y) + \frac{1}{\pi y} m(y) \frac{dm}{dy}(y) = 0, & y > 0, \\
m(0) = 0, & m(+\infty) = \hat{M},
\end{cases}
\]

and it was shown in Lemma 4.1 of [6] that

\[
\begin{cases}
m \in C^1([0, \infty)), & \frac{dm}{dy}(y) > 0, \quad \frac{d^2 m}{dy^2}(y) < 0, \quad y > 0, \\
\hat{M}(1 - e^{-y/4}) \leq m(y) \leq \min \left\{ 4 \frac{dm}{dy}(0)(1 - e^{-y/4}), \hat{M} \right\}, & y > 0, \\
\frac{dm}{dy}(y) \leq \frac{dm}{dy}(0)e^{-y/4}, & y > 0.
\end{cases}
\]

The relation between \( \Phi_{\hat{M}}(y) \) in (1.8) and \( m(y) \) is given by

\[ \Phi_{\hat{M}}(y) = \pi^{-1} \frac{dm}{dy}(y^2), \]

and hence \( U_{\hat{M}}(t) \) is decreasing with respect to \( |x| \) and

\[ 0 < U_{\hat{M}}(t, x) \leq C \exp \left( -\frac{|x|^2}{4t} \right), \quad t > 0, \quad x \in \mathbb{R}^2. \quad (2.3) \]

The following theorem on the global existence and decay estimates of nonnegative mild solutions to (1.5)–(1.6) was proved in [32] by using rearrangement techniques.

**Theorem 2.2.** Assume \( \hat{M} := \int_{\mathbb{R}^2} u_0 dx < 8\pi \) for the nonnegative initial data \( u_0 \in L^1 \). Then the nonnegative mild solution \( u \) of (1.5)–(1.6) exists globally in time. Moreover it holds that for every \( 1 \leq p \leq \infty \),

\[ \| u(t) \|_p \leq \left( \frac{\pi t}{4} \right)^{-1/p} \| dm/dy \|_{L^p(0, \infty)} \text{ for } t > 0. \quad (2.4) \]

3. Estimates on derivatives of solutions

For a nonnegative initial data \( u_0 \in L^1 \) satisfying \( \hat{M} := \int_{\mathbb{R}^2} u_0 dx < 8\pi \), let \( u \) be the nonnegative mild solution of the Cauchy problem (1.5)–(1.6) on \([0, \infty)\) mentioned in Theorem 2.2. In what follows, we denote by \( C(*, \ldots, *) \) a positive constant depending only on the quantities appearing in the parentheses. Since \( \| dm/dy \|_{L^p(0, \infty)} \) in Theorem 2.2 depends only on \( \hat{M} \) and \( p \), we may write (2.4) as

\[
\sup_{t > 0} t^{1-1/p} \| u(t) \|_p \leq C(\hat{M}, p), \quad 1 \leq p \leq \infty.
\]
For the estimates on the derivatives of \( u \), we have the following.

**Theorem 3.1.** Let \( 1 \leq p \leq \infty \). Then it holds that for all \( \ell, n \in \mathbb{Z}_+ \),

\[
\sup_{t>0} t^{1-1/p+\ell/2+n} \| \partial_t^n \partial_x^\ell u(t) \|_p \leq C(\hat{M}, p, \ell, n),
\]

(3.2)

where \( \hat{M} = \int_{\mathbb{R}^2} u_0 \, dx \).

In order to prove this theorem, we need several lemmas. First we give the following lemma based on Lemma 4.2 of [24]. For the proof, see [24].

**Lemma 3.1.** For any \( \delta > 0 \), the solution \( u \) satisfies the integral equation

\[
t^\delta u(t) = \delta \int_0^t e^{(t-s)\Delta} (s^{\delta-1} u(s)) \, ds - \int_0^t \nabla \cdot e^{(t-s)\Delta} (s^{\delta} u(s) (\nabla N^* u)(s)) \, ds, \quad t > 0.
\]

The second lemma is about the well-known \( L^p-L^q \) estimates for the heat semigroup \( e^{t\Delta} \).

**Lemma 3.2.** Let \( 1 \leq q \leq p \leq \infty \), \( n \in \mathbb{N} \), and \( \alpha \in \mathbb{Z}_+^2 \). Then, for all \( f \in L^q \),

\[
\| \partial_t^n \partial_x^\alpha e^{t\Delta} f \|_p \leq C t^{-1/q+1/p-|\alpha|/2-2n} \| f \|_q,
\]

where \( C \) is a constant depending only on \( p, q, n, \text{ and } \alpha \).

**Lemma 3.3.** For all \( f \in L^1 \cap L^\infty \),

\[
\| \nabla N^* f \|_\infty \leq \kappa (\| f \|_1 \| f \|_\infty)^{1/2},
\]

(3.3)

where \( \kappa = (2/\pi)^{1/2} \).

**Proof.** For every \( A > 0 \),

\[
2\pi |(\nabla N^* f)(x)| \leq \left( \int_{|x-y| \leq A} + \int_{|x-y| > A} \right) \frac{|f(y)|}{|x-y|} \, dy \leq 2\pi \| f \|_\infty A + \| f \|_1 A^{-1}.
\]

From this we have

\[
2\pi |(\nabla N^* f)(x)| \leq 2(2\pi \| f \|_\infty \| f \|_1)^{1/2},
\]

which implies (3.3). \( \square \)

We introduce the following notation: For \( \ell, n \in \mathbb{Z}_+ \),

\[
\phi_p(t) = \sup_{0<s<t} s^{1-1/p} \| u(s) \|_p,
\]

\[
\phi_{p,(\ell)}(t) = \sup_{0<s<t} s^{1-1/p+\ell/2} \| \partial_x^\ell u(s) \|_p,
\]

\[
\phi_{p,(n,\ell)}(t) = \sup_{0<s<t} s^{1-1/p+\ell/2+n} \| \partial_t^n \partial_x^\ell u(s) \|_p.
\]
Lemma 3.4. Let $1 \leq p \leq \infty$ and $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| = \ell \in \mathbb{Z}_+$. Then, for $0 < s \leq t$,

$$s^{1-1/p+\ell/2+1}||\partial_x^\alpha \nabla \cdot (u(s)(\nabla N * u)(s))||_p \leq s^{1-1/p+\ell/2+1}||\nabla \partial_x^\alpha u(s) \cdot (\nabla N * u)(s)||_p + \phi_p(t)\phi_\ell^*(t)$$

$$+ \sum_{k=1}^\ell C_k\phi_p(k)(t)\{\phi_1^{(\ell-k+1)}(t)\phi_\ell^{(\ell-k)}(t)\}^{1/2} + \phi_\ell^{(\ell-k)}(t),$$

where $C_k$ are positive constants depending only on $\ell$ and $k$.

**Proof.** We observe that

$$\nabla \cdot (u(\nabla N * u)) = \nabla u \cdot (\nabla N * u) - u^2$$

because of $\nabla \cdot (\nabla N * u) = -u$, and that for $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| = \ell$,

$$\partial_x^\alpha \nabla u \cdot (\nabla N * u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \nabla \partial_x^{\alpha - \beta} u \cdot (\nabla N * \partial_x^{\beta} u).$$

Applying Lemma 3.3, for $0 < s \leq t$, we have

$$||\partial_x^\alpha (\nabla u(s) \cdot (\nabla N * u))(s)||_p \leq ||\nabla \partial_x^\alpha u(s) \cdot (\nabla N * u)(s)||_p$$

$$+ \sum_{k=0}^{\ell-1} C_k||\partial_x^{k+1} u(s)||_p(||\partial_x^{\ell-k} u(s)||_1||\partial_x^{\ell-k} u(s)||_\infty^{1/2}$$

$$\leq ||\nabla \partial_x^\alpha u(s) \cdot (\nabla N * u)(s)||_p$$

$$+ s^{-1+1/p-\ell/2-1} \sum_{k=1}^\ell C_{\ell-k}\phi_p(k)(t)(\phi_{1+1}(t)\phi_{\ell}\phi_\ell^{(\ell-k+1)}(t))^{1/2},$$

where $C_0 = C_\ell = 1$. Similarly,

$$||\partial_x^\alpha u^2(s)||_p \leq \sum_{k=0}^\ell C_k||\partial_x^k u(s)||_p||\partial_x^{\ell-k} u(s)||_\infty$$

$$\leq s^{-1+1/p-\ell/2-1} \sum_{k=1}^\ell C_k\phi_p(k)(t)\phi_\ell^{(\ell-k)}(t).$$

Hence,

$$s^{1-1/p+\ell/2+1}||\partial_x^\alpha \nabla \cdot (u(s)(\nabla N * u))(s)||_p \leq s^{1-1/p+\ell/2+1}||\nabla \partial_x^\alpha u(s) \cdot (\nabla N * u)(s)||_p + \phi_p(t)\phi_\ell^*(t)$$
where \( \delta > \phi \)

By Lemma 3.3 and

Thus (3.4) is deduced. \( \square \)

**Proof of Theorem 3.1.** First we prove (3.2) for the case \( n = 0 \) by induction on \( \ell \), namely, for all \( \ell \in \mathbb{Z}_+ \),

\[
\sup_{t>0} t^{1-1/p+\ell/2} \| \partial_x^\ell u(t) \|_p \leq C(M, p, \ell). \tag{3.5}
\]

By virtue of (3.1), (3.5) is true for \( \ell = 0 \).

Assume that (3.5) is true for all nonnegative integers less than or equal to \( \ell \). To prove (3.5) for \( \ell + 1 \), we use Lemma 3.1 to get

\[
t^\delta u(t) = \delta \int_0^t e^{(t-s)\Delta} (s^{\delta-1} u(s)) ds - \int_0^t \nabla \cdot e^{(t-s)\Delta} (s^{\delta} u(s)(\nabla N \ast u)(s)) ds
\]

\[
- \int_{t(1-\epsilon)}^t \nabla \cdot e^{(t-s)\Delta} (s^{\delta} u(s)(\nabla N \ast u)(s)) ds
\]

\[
= \delta I(t) + II(t) + III(t), \quad t > 0,
\]

where \( \delta > 0 \) and \( 0 < \epsilon < 1 \). We take \( \delta \) such that \( \delta > 1 + \ell/2 \) and fix it.

Let \( 1 \leq p \leq \infty \). Applying Lemma 3.2, we then have

\[
\| \partial_x^{\ell+1} I(t) \|_p \leq \int_0^t \| \partial_x e^{(t-s)\Delta} (s^{\delta-1} \partial_x^\ell u(s)) \|_p ds
\]

\[
\leq C_p \int_0^t (t-s)^{-1/2} s^{\delta-1} \| \partial_x^\ell u(s) \|_p ds
\]

\[
\leq C_p \int_0^t (t-s)^{-1/2} s^{\delta-2+1/p-\ell/2} ds \phi_p^{(\ell)}(t)
\]

\[
= C_p s^{\delta-1+1/p-(\ell+1)/2} \int_0^1 (1-s)^{-1/2} s^{\delta-2+1/p-\ell/2} ds \phi_p^{(\ell)}(t).
\]

By Lemma 3.3 and \( \phi_1(t) = M \), we get

\[
\| \partial_x^{\ell+1} II(t) \|_p \leq \int_0^t \| \partial_x^{\ell+1} \nabla \cdot e^{(t-s)\Delta} (s^{\delta} u(s)(\nabla N \ast u)(s)) \|_p ds
\]

\[
\leq C_p \int_0^t (t-s)^{-(\ell+2)/2} s^{\delta} \| u(s)(\nabla N \ast u)(s) \|_p ds
\]

\[
\leq C_p \int_0^t (t-s)^{-(\ell+2)/2} s^{\delta} \| u(s) \|_p (\| u(s) \|_1 \| u(s) \|_\infty)^{1/2} ds
\]
due to  
\[
\int t^{-(\ell+1)/2}(1-s)^{-(\ell+2)/2}s^{\delta-1+1/p-1/2}ds \phi_p(t)(\tilde{M}\phi_\infty(t))^{1/2}.
\]
Hence, by the induction assumption,
\[
t^{1-1/p+(\ell+1)/2}||\partial_x^{\ell+1}I(t)||_p + ||\partial_x^{\ell+1}II(t)||_p \leq C(\tilde{M},p,\ell,\varepsilon)t^\delta. \tag{3.6}
\]
For 1 \leq p \leq \infty, let 1 \leq q \leq p with q < \infty, 1/q - 1/p < 1/2. By Lemma 3.4 and the fact that \(\phi_1(t) = \tilde{M}\), we obtain
\[
||\partial_x^{\ell+1}III(t)||_p = \left\| \int_{t(1-\varepsilon)}^t \partial_x e^{(t-s)\Delta} \partial_x^\ell \nabla \cdot (s^\delta u(s)(\nabla N * u)(s)) ds \right\|_p
\]
\[
\leq C_{p,q} \int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^{\delta}||\partial_x^\ell \nabla \cdot (u(s)(\nabla N * u)(s))||_q ds
\]
\[
\leq C_{p,q} \int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^{\delta}\nabla \partial_x^\ell u(s) \cdot (\nabla N * u)(s)||_q ds
\]
\[
+ C_{p,q} \int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}ds \phi_1(t)\partial_x^{\ell-1}(t)
\]
\[
\phi_1(t)\partial_x^{\ell-1}(t)
\]
\[
+ C_{p,q} \sum_{k=1}^q C_k \int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}ds
\]
\[
\times \phi_1(t)\{(\phi_1(t)\partial_x^{\ell-1}(t))^{1/2} + \phi_1(t)\}
\]
Noting that
\[
\int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}ds
\]
\[
= t^{\delta-1+1/p-(\ell+1)/2} \int_{1-\varepsilon}^1 (1-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}ds
\]
and
\[
\int_0^1 (1-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}ds < \infty
\]
due to 1/q - 1/p < 1/2 and \(\delta > 1 + \ell/2\), we have
\[
t^{1-1/p+(\ell+1)/2}||\partial_x^{\ell+1}III(t)||_p \leq C_{p,q} t^{1-1/p+(\ell+1)/2} \int_{t(1-\varepsilon)}^t (t-s)^{-1/q+1/p-1/2}s^{\delta}\nabla \partial_x^\ell u(s) \cdot (\nabla N * u)(s)||_q ds
\]
\[
+ C_{p,q} t^\delta \int_0^1 (1-s)^{-1/q+1/p-1/2}s^{\delta-2+1/q-\ell/2}ds
\]
Consider the case $1 < p < \infty$. Take $q = p$ in (3.7). Then, by Lemma 3.3, the first term on the right-hand side of (3.7) is estimated as follows:

$$\begin{align*}
C_p t^{1-1/p+(\ell+1)/2} &\int_{(1-\varepsilon)}^{t} (t-s)^{-1/2} s^\delta \| \nabla \partial^k_x u(s) \cdot (\nabla N \ast u)(s) \|_p ds \\
&\leq C_p t^{1-1/p+(\ell+1)/2} \int_{(1-\varepsilon)}^{t} (t-s)^{-1/2} s^\delta \| \nabla \partial^k_x u(s) \|_p (\| u(s) \|_1 \| u(s) \|_\infty)^{1/2} ds \\
&\leq C_p t^\delta \int_{1-\varepsilon}^{1} (1-s)^{-1/2} s^{\delta-2+1/p-\ell/2} ds (\hat{M} \varphi_{\infty}(t))^{1/2} \phi_p^{(\ell+1)}(t).
\end{align*}$$

Hence,

$$t^{1-1/p+(\ell+1)/2} \| \partial^\ell_x III(t) \|_p \leq C_p t^\delta \int_{1-\varepsilon}^{1} (1-s)^{-1/2} s^{\delta-2+1/p-\ell/2} ds (\hat{M} \varphi_{\infty}(t))^{1/2} \phi_p^{(\ell+1)}(t) + C_p t^\delta \int_{0}^{1} (1-s)^{-1/2} s^{\delta-2+1/p-\ell/2} ds \times \left\{ \phi_p(t) \phi_{\infty}^{(\ell)}(t) + \sum_{k=1}^{\ell} C_k \phi_p^{(k)}(t) \left( (\phi_1^{(\ell-k+1)}(t) \phi_{\infty}^{(\ell-k+1)}(t))^{1/2} + \phi_{\infty}^{(\ell-k)}(t) \right) \right\}. \tag{3.8}$$

Therefore, by the induction assumption, it follows from (3.6) and (3.8) that

$$t^{1-1/p+(\ell+1)/2} \| \partial^\ell_x u(t) \|_p \leq C(\hat{M}, p) \int_{1-\varepsilon}^{1} (1-s)^{-1/2} s^{\delta-2+1/p-\ell/2} ds \phi_p^{(\ell+1)}(t) + C(\hat{M}, p, \ell, \varepsilon).$$

From this it follows that

$$\phi_p^{(\ell+1)}(t) \leq C(\hat{M}, p) \int_{1-\varepsilon}^{1} (1-s)^{-1/2} s^{\delta-2+1/p-\ell/2} ds \phi_p^{(\ell+1)}(t) + C(\hat{M}, p, \ell, \varepsilon).$$

Taking $0 < \varepsilon < 1$ such that

$$C(\hat{M}, p) \int_{1-\varepsilon}^{1} (1-s)^{-1/2} s^{\delta-2+1/p-\ell/2} ds \leq \frac{1}{2},$$

we have

$$\phi_p^{(\ell+1)}(t) \leq C(\hat{M}, p, \ell + 1). \tag{3.9}$$
Consider the case $p = \infty$. Take $p = \infty$ and $2 < q < \infty$ in (3.7) and fix $q$. Then, using the induction assumption and the fact that $\phi_{q}^{(\ell+1)}(t) \leq C(\hat{M}, q, \ell + 1)$ by (3.9), we estimate the first term on the right-hand side of (3.7) as follows:
\[
\begin{align*}
&\int_{t(1-\varepsilon)}^{t} (t - s)^{-1/2} \|\nabla \partial_{x}^{\ell} u(s) \cdot (\nabla N \ast u)(s)\|_{q} ds \\
&\leq C(\hat{M}, q, \ell + 1) t^{\delta}.
\end{align*}
\]

By this estimate and (3.6) for $q = 1$, we deduce $\phi_{q}^{(\ell+1)}(t) \leq C(\hat{M}, \ell + 1)$.

Consider the case $p = 1$. Take $q = 1$ in (3.7). Then the first term on the right-hand side of (3.7) is estimated by using (2.2) with $q = 4/3$:
\[
\begin{align*}
&\int_{t(1-\varepsilon)}^{t} (t - s)^{-1/2} \|\nabla \partial_{x}^{\ell} u(s) \cdot (\nabla N \ast u)(s)\|_{1} ds \\
&\leq C(\hat{M}, \ell + 1) t^{\delta}.
\end{align*}
\]

Here we used $\phi_{4/3}^{(\ell+1)}(t) \leq C(\hat{M}, \ell + 1)$ by (3.9). Hence, as in the case $p = \infty$, we have $\phi_{1}^{(\ell+1)}(t) \leq C(\hat{M}, \ell + 1)$. Since we establish that (3.5) is true for $\ell + 1$, (3.5) is true for all $\ell$.

It remains to prove (3.2) by induction on $n \in \mathbb{N}$:
\[
\sup_{t > 0} t^{1-p+\ell/2+n} \|\partial_{t}^{p} \partial_{x}^{\alpha} u(t)\|_{p} \leq C(\hat{M}, \ell, n) \quad \text{for all } \ell \in \mathbb{Z}_{+}. \tag{3.10}
\]

To this aim we observe that by equation (1.5),
\[
\begin{align*}
\partial_{t}^{n+1} \partial_{x}^{\alpha} u &= \partial_{t}^{n} \partial_{x}^{\alpha} \Delta u - \partial_{t}^{n} \partial_{x}^{\alpha} (\nabla u \cdot (\nabla N \ast u)) + \partial_{t}^{n} \partial_{x}^{\alpha} u_{2}.
\end{align*}
\tag{3.11}
\]
From (3.11) for \( n = 0 \), we deduce (3.10) for \( n = 1 \) by using the Leibniz formula for the second and third terms on the right-hand side of (3.11) and applying Lemma 3.3 and the induction assumption. Similarly, we can deduce that if (3.10) holds for all natural numbers less than or equal to \( n \), then (3.10) holds for \( n + 1 \). Therefore, we complete the proof of Theorem 3.1.

4. LARGE-TIME BEHAVIOR OF SOLUTIONS

For the nonnegative initial data \( u_0 \in L^1 \) with \( \hat{M} := \int_{\mathbb{R}^2} u_0 \, dx < 8\pi \), let \( u \) be the nonnegative mild solution of (1.5)–(1.6) on \([0, \infty)\); namely, \( u \) satisfies the following:

\[
  u \in C([0, \infty); L^1) \cap C((0, \infty); L^{4/3}),
\]

\[
  \sup_{0 < t < T} t^{1/4} \| u(t) \|_{4/3} < \infty \quad \text{for every } T > 0,
\]

\[
  u(t) = e^{t\Delta} u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s)) \, ds, \quad t > 0. \tag{4.1}
\]

By Theorem 2.1, \( u \) is smooth on \((0, \infty) \times \mathbb{R}^2\) and a classical solution of (1.5).

For \( \lambda > 0 \), define \( u_{0\lambda}(x) \) and \( u_\lambda(t, x) \) by

\[
  u_{0\lambda}(x) = \lambda^2 u_0(\lambda x), \quad u_\lambda(t, x) = \lambda^2 u(\lambda^2 t, \lambda x), \quad t > 0, \quad x \in \mathbb{R}^2.
\]

**Lemma 4.1.** \( u_\lambda \) satisfies

\[
  u_\lambda(t) = e^{t\Delta} u_{0\lambda} - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u_\lambda(s)(\nabla N * u_\lambda)(s)) \, ds, \quad t > 0. \tag{4.2}
\]

**Proof.** The integral equation (4.1) is rewritten as

\[
  u(t, x) = \int_{\mathbb{R}^2} G(t, x - y) u_0(y) \, dy
\]

\[
  - \int_0^t \, ds \int_{\mathbb{R}^2} (\nabla G)(t - s, x - y) \cdot u(s, y)(\nabla N * u)(s, y) \, dy,
\]

where \( G(t, x) \) is the heat kernel given in (2.1). Take \( (\lambda^2 t, \lambda x) \) as \((t, x)\) in this integral equation. Then, by using \( \lambda^2 G(\lambda^2 t, \lambda x) = G(t, x) \) and \( \nabla G(t, x) = \lambda^3 (\nabla G)(\lambda^2 t, \lambda x) \) and observing \( (\nabla N * u)(\lambda^2 t, \lambda x) = \lambda^{-1} (\nabla N * u_\lambda)(t, x) \), direct calculations give (4.2). \qed

By Lemma 4.1, we have the following.

**Proposition 4.1.** For each \( \lambda > 0 \), \( u_\lambda \) is a nonnegative mild solution of (1.5)–(1.6) on \([0, \infty)\) with the nonnegative initial data \( u_{0\lambda} \).
Since \( u_{0\lambda} \) satisfies
\[
\int_{\mathbb{R}^2} u_{0\lambda}(x) \, dx = \int_{\mathbb{R}^2} u_0(x) \, dx = \hat{M} < 8\pi,
\]
Theorem 3.1 ensures that for all \( 1 \leq p \leq \infty \) and \( \ell, n \in \mathbb{Z}_+ \),
\[
\sup_{t > 0} t^{1-1/p+\ell/2+n} \| \partial_t^n \partial_x^\ell u_\lambda(t) \|_p \leq C(\hat{M}, p, \ell, n). \tag{4.3}
\]
We remark that \( C(\hat{M}, p, \ell, n) \) is independent of \( \lambda \). Therefore, by Ascoli-Arzela’s theorem, for any sequence \( \{\lambda_j\}_{j=1}^\infty \) satisfying \( \lambda_j \nearrow \infty \) as \( j \nearrow \infty \), there exist a subsequence of \( \{\lambda_j\}_{j=1}^\infty \), denote it by \( \{\lambda_j\}_{j=1}^\infty \) again, and a nonnegative function \( U \in C^\infty((0, \infty) \times \mathbb{R}^2) \) such that
\[
\lim_{j \to \infty} \partial_t^n \partial_x^\alpha u_{\lambda_j} = \partial_t^n \partial_x^\alpha U \text{ locally uniformly in } (0, \infty) \times \mathbb{R}^2
\]
for all \( n \in \mathbb{Z}_+ \) and \( \alpha \in \mathbb{Z}_2^+ \). Since
\[
\int_{\mathbb{R}^2} u_\lambda(t, x) \, dx = \int_{\mathbb{R}^2} u_0(x) \, dx = \hat{M},
\]
by Fatou’s lemma,
\[
\int_{\mathbb{R}^2} U(t, x) \, dx \leq \hat{M} \text{ for all } t > 0.
\]
Moreover, by (4.3), we see that for all \( 1 \leq p \leq \infty \) and \( \ell, n \in \mathbb{Z}_+ \), \( U \) satisfies
\[
\sup_{t > 0} t^{1-1/p+\ell/2+n} \| \partial_t^n \partial_x^\ell U(t) \|_p \leq C(\hat{M}, p, \ell, n). \tag{4.4}
\]
As for the nonlinear term \( \nabla \cdot (u_{\lambda_j} (\nabla N \ast u_{\lambda_j})) \), we have the following.

**Lemma 4.2.** It holds that
\[
\lim_{j \to \infty} \nabla \cdot (u_{\lambda_j} (\nabla N \ast u_{\lambda_j})) = \nabla \cdot (U (\nabla N \ast U)) \text{ locally uniformly in } (0, \infty) \times \mathbb{R}^2.
\tag{4.5}
\]

**Proof.** To prove this lemma, we claim
\[
\lim_{j \to \infty} \nabla N \ast u_{\lambda_j} = \nabla N \ast U \text{ locally uniformly in } (0, \infty) \times \mathbb{R}^2. \tag{4.6}
\]
Once we get this claim, we deduce (4.5) by observing
\[
\nabla \cdot (u_{\lambda_j} (\nabla N \ast u_{\lambda_j})) = \nabla u_{\lambda_j} \cdot (\nabla N \ast u_{\lambda_j}) - u_{\lambda_j}^2
\]
and
\[
\nabla \cdot (\nabla N \ast u_{\lambda_j}) = -u_{\lambda_j}, \quad \nabla \cdot (\nabla N \ast U) = -U.
\]
We prove (4.6). For any fixed \( R_1 > 0 \) we take any \( R_2 > 2R_1 \). Then, for \( |x| \leq R_1 \) and \( |y| > R_2 \), we have \( |x - y| \geq R_2/2 \) and

\[
|\nabla N \ast u_{\lambda_j}(t, x) - (\nabla N \ast U)(t, x)| \leq \frac{1}{2\pi} \int_{|x-y|} \frac{1}{|x-y|} |u_{\lambda_j}(t, y) - U(t, y)| dy
\]

\[
\leq \frac{1}{2\pi} \left( \int_{|y| \leq R_2} + \int_{|y| > R_2} \right) \frac{1}{|x-y|} |u_{\lambda_j}(t, y) - U(t, y)| dy
\]

\[
\leq R_2 \sup_{|y| \leq R_2} |u_{\lambda_j}(t, y) - U(t, y)| + \frac{2}{\pi R_2} \hat{M}.
\]

From this, for any \( 0 < t_1 < t_2 \) it follows that

\[
\limsup_{j \to \infty} \left( \sup_{t_1 \leq t \leq t_2} \left| (\nabla N \ast u_{\lambda_j})(t, x) - (\nabla N \ast U)(t, x) \right| \right) \leq \frac{2}{\pi R_2} \hat{M},
\]

and hence, by letting \( R_2 \to \infty \), (4.5) is deduced.

Since \( u_{\lambda} \) is a classical solution of (1.5), by Lemma 4.2, we see that \( U \) is a classical solution of (1.5) on \((0, \infty) \times \mathbb{R}^2\), namely

\[
\partial_t U = \Delta U - \nabla \cdot (U(\nabla N \ast U)) \text{ in } (0, \infty) \times \mathbb{R}^2.
\]

We also see that \( U \) is a weak solution of (1.5)--(1.6) with initial data \( \hat{M} \delta_0(x) \), where \( \delta_0(x) \) is the Dirac measure supported at the origin. To say it precisely, we define a weak solution of (1.5)--(1.6) with initial data \( M \delta_0(x) \), where \( M \in \mathbb{R} \).

**Definition 4.1.** A function \( v \) on \((0, \infty) \times \mathbb{R}^2\) is said to be a weak solution of (1.5)--(1.6) with initial data \( M \delta_0(x) \), where \( M \in \mathbb{R} \), if

(i) \( v \in C((0, \infty); \mathcal{L}^1 \cap L^{4/3}) \),

(ii) \( \sup_{0 < t < 1} t^{1/4} \|v(t)\|_{4/3} < \infty \),

(iii) for any \( \varphi \in C_0^\infty((0, \infty) \times \mathbb{R}^2) \), \( v \) satisfies

\[
0 = M \varphi(0, 0) + \int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + \Delta \varphi) v \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot (v(\nabla N \ast v)) \, dx \, dt.
\]

**Proposition 4.2.** For the limit function \( U \) mentioned above, it holds that

(i) \( U \) is a classical solution of (1.5),

(ii) \( U \) is a weak solution of (1.5)--(1.6) with initial data \( \hat{M} \delta_0(x) \), where \( \hat{M} = \int_{\mathbb{R}^2} u_0 \, dx \).

**Proof.** We only prove (ii) because (i) has already been shown above.
$U$ satisfies (i) and (ii) of Definition 4.1 by virtue of (4.4). To prove (iii) of Definition 4.1, we multiply
\[ \partial_t u_{\lambda_j} = \Delta u_{\lambda_j} - \nabla \cdot (u_{\lambda_j}(\nabla N \ast u_{\lambda_j})) \]
by $\varphi$ and integrate on $(0, \infty) \times \mathbb{R}^2$. Then by integration by parts, we have
\[ 0 = \int_{\mathbb{R}^2} \varphi(0, x) u_{0\lambda_j}(x) \, dx + \int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + \Delta \varphi) u_{\lambda_j} \, dx \, dt \]
\[ + \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot u_{\lambda_j}(\nabla N \ast u_{\lambda_j}) \, dx \, dt. \]
It is easily obtained that
\[ \lim_{j \to \infty} \int_{\mathbb{R}^2} \varphi(0, x) u_{0\lambda_j}(x) \, dx = \varphi(0, 0) \int_{\mathbb{R}^2} u_0(x) \, dx = \varphi(0, 0) \hat{M}. \]
By Lemma 4.2, for each $t > 0$,
\[ \lim_{j \to \infty} \int_{\mathbb{R}^2} \nabla \varphi(t, x) \cdot (u_{\lambda_j}(t, x)(\nabla N \ast u_{\lambda_j})(t, x)) \, dx \]
\[ = \int_{\mathbb{R}^2} \nabla \varphi(t, x) \cdot (U(t, x)(\nabla N \ast U)(t, x)) \, dx, \]
and by (2.2) and (4.3),
\[ \left| \int_{\mathbb{R}^2} \nabla \varphi(t, x) \cdot (u_{\lambda_j}(t, x)(\nabla N \ast u_{\lambda_j})(t, x)) \, dx \right| \]
\[ \leq \| u_{\lambda_j}(t)(\nabla N \ast u_{\lambda_j})(t) \|_1 \| \nabla \varphi \|_\infty \]
\[ \leq C \| u_{\lambda_j}(t) \|_{1,3}^2 \| \nabla \varphi \|_\infty \leq C(\hat{M}) \| \nabla \varphi \|_\infty t^{-1/2}. \]
Hence, by letting $j \to \infty$ in (4.7), the Lebesgue convergence theorem ensures
\[ 0 = \hat{M} \varphi(0, 0) + \int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + \Delta \varphi) U \, dx \, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot (U(\nabla N \ast U)) \, dx \, dt. \]
This implies (iii) of Definition 4.1. \hfill \Box

4.1. **Uniqueness of nonnegative weak solutions with a Dirac measure.** From the definition of weak solutions the following lemma follows.

**Lemma 4.3.** Let $v$ be a weak solution of (1.5)–(1.6) with initial data $M\delta_0(x)$, where $M \in \mathbb{R}$. Then for all $\varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^2)$ and for all $T > 0$,
\[ 0 = M \varphi(0, 0) - \int_{\mathbb{R}^2} \varphi(T, x) v(T, x) \, dx + \int_0^T \int_{\mathbb{R}^2} (\partial_t \varphi + \Delta \varphi) v \, dx \, dt \quad (4.8) \]
\[
+ \int_0^T \int_{\mathbb{R}^2} \nabla \varphi \cdot (v(\nabla N * v)) \, dx \, dt.
\]

**Proof.** Take any positive number \( T \) and fix it. For \( 0 < h < 1 \), let \( \eta_h \in C_0^\infty((0,\infty)) \) be such that \( 0 \leq \eta_h(t) \leq 1, \eta'_h(t) \leq 0 \) \((t \geq 0)\), \( \eta_h(t) = 1 \) \((0 \leq t \leq T)\), and \( \eta_h(t) = 0 \) \((t \geq T + h)\), where \( \eta'_h = d\eta_h/dt \). Take \( \eta_h \varphi \in C_0^\infty((0,\infty) \times \mathbb{R}^2) \) as \( \varphi \) in Definition 4.1. Then

\[
0 = M\varphi(0,0) + \int_T^{T+h} \eta'_h(t) \left( \int_{\mathbb{R}^2} \varphi(t)v(t) \, dx \right) dt \quad (4.9)
\]

\[
+ \int_0^{T+h} \eta_h(t) \left( \int_{\mathbb{R}^2} (\partial_t \varphi(t) + \Delta \varphi(t))v(t) \, dx \right) dt
\]

\[
+ \int_0^{T+h} \eta_h(t) \left( \int_{\mathbb{R}^2} \nabla \varphi(t) \cdot (v(t)(\nabla N * v(t))) \, dx \right) dt
\]

\[
= M\varphi(0,0) + I_h + II_h + III_h.
\]

Since \( t \mapsto \int_{\mathbb{R}^2} \varphi(t,x)v(t,x) \, dx \) is continuous on \((0,\infty)\) and \( \eta'_h \leq 0 \), we deduce

\[
\lim_{h \to 0} I_h = -\int_{\mathbb{R}^2} \varphi(T,x)v(T,x) \, dx.
\]

By (2.2) and (i) and (ii) of Definition 4.1, for \( 0 < t < T \), we have

\[
\int_{\mathbb{R}^2} \left| \nabla \varphi(t) \cdot (v(t)(\nabla N * v(t))) \right| dx \leq C \| \nabla \varphi(t) \|_\infty \| v(t) \|^2 \leq C t^{-1/2}, \quad (4.10)
\]

where \( C \) is a positive constant independent of \( t \). By (4.10), the Lebesgue convergence theorem ensures that

\[
\lim_{h \to 0} III_h = \int_0^T \int_{\mathbb{R}^2} \nabla \varphi \cdot (v(\nabla N * v)) \, dx \, dt.
\]

Similarly,

\[
\lim_{h \to 0} II_h = \int_0^T \int_{\mathbb{R}^2} (\partial_t \varphi + \Delta \varphi)v \, dx \, dt.
\]

Hence, letting \( h \to 0 \) in (4.9), we deduce (4.8). \( \square \)

For later use, we give some properties of nonnegative weak solutions of (1.5)–(1.6) with initial data \( M\delta_0(x) \), where \( M \geq 0 \).

**Proposition 4.3.** Let \( v \) be a nonnegative weak solution of (1.5)–(1.6) with initial data \( M\delta_0(x) \), where \( M \geq 0 \). Then the following hold:
Convergence to self-similar solutions 857

(i) For all \( t > 0 \),
\[
\int_{\mathbb{R}^2} v(t, x) \, dx = M. \tag{4.11}
\]

(ii) For all \( \varphi \in C_0^\infty(\mathbb{R}^2) \),
\[
\lim_{t \to 0} \int_{\mathbb{R}^2} \varphi(x)v(t, x) \, dx = M\varphi(0). \tag{4.12}
\]

**Proof.** Take any \( T > 0 \) and fix it. Let \( \eta \in C_0^\infty([0, \infty)) \) be such that \( \eta(t) = 1 \) for \( 0 \leq t \leq T \), and let \( \varphi \in C_0^\infty(\mathbb{R}^2) \). Take \( \eta(t) \varphi(x) \) as \( \varphi(t, x) \) in (4.8). Then
\[
\int_{\mathbb{R}^2} \varphi(x)v(T, x) \, dx = M\varphi(0) + \int_0^T \int_{\mathbb{R}^2} \Delta \varphi v \, dx \, dt \tag{4.13}
\]
\[+ \int_0^T \int_{\mathbb{R}^2} \nabla \varphi \cdot (v(\nabla N * v)) \, dx \, dt. \]

By (4.10), we have
\[
\int_{\mathbb{R}^2} |\nabla \varphi(x) \cdot (v(t, x)(\nabla N * v)(t, x))| \, dx \leq C\|\nabla \varphi\|_\infty t^{-1/2}, \quad 0 < t < T. \tag{4.14}
\]

For \( R > 1 \), let \( \varphi_R \in C_0^\infty(\mathbb{R}^2) \) be a cutoff function such that
\[
0 \leq \varphi_R \leq 1, \quad \varphi_R(x) = 1 (|x| \leq R), \quad \varphi_R(x) = 0 (|x| \geq 2R),
\]
\[
|\nabla \varphi_R(x)| \leq \frac{C}{R}, \quad |\nabla^2 \varphi_R(x)| \leq \frac{C}{R^2},
\]
where \( C \) is a positive constant independent of \( R \). Take \( \varphi_R \) as \( \varphi \) in (4.13). Then
\[
\int_{\mathbb{R}^2} \varphi_R(x)v(T, x) \, dx = M + \int_0^T \int_{\mathbb{R}^2} \Delta \varphi_R v \, dx \, dt \tag{4.15}
\]
\[+ \int_0^T \int_{\mathbb{R}^2} \nabla \varphi_R \cdot (v(\nabla N * v)) \, dx \, dt. \]

By (i) and (ii) of Definition 4.1,
\[
\int_{\mathbb{R}^2} |\Delta \varphi_R(x)v(t, x)| \, dx \leq \|\Delta \varphi_R\|_4\|v(t)\|_{4/3} \leq \frac{C}{R^2} t^{-1/4}, \quad 0 < t < T.
\]

Putting \( \varphi = \varphi_R \) in (4.14) we have
\[
\int_{\mathbb{R}^2} |\nabla \varphi_R(x) \cdot (v(t, x)(\nabla N * v)(t, x))| \, dx \leq \frac{C}{R^{3/2}} t^{-1/2}.
\]

Hence, letting \( R \to \infty \) in (4.15), we deduce \( \int_{\mathbb{R}^2} v(T, x) \, dx = M \) by the Lebesgue convergence theorem.

(4.12) is obtained by letting \( T \to 0 \) in (4.13) \( \square \)
Proposition 4.4. Let $v$ be a nonnegative weak solution of (1.5)–(1.6) with initial data $M\delta_0(x)$, where $M \geq 0$. Then, for all $t > 0$, $|x|^2v(t) \in L^1$ and

$$
\int_{\mathbb{R}^2} |x|^2v(t,x) \, dx = 4M \left(1 - \frac{M}{8\pi}\right)t. \tag{4.16}
$$

Proof. For $R > 1$, let $\varphi_R \in C_0^\infty(\mathbb{R}^2)$ be the same cutoff function as in the proof of Proposition 4.3. Take $\Phi_R = |x|^2\varphi_R$ and $t$ as $\varphi$ and $T$ in (4.13), respectively:

$$
\int_{\mathbb{R}^2} \Phi_R(x)v(t,x) \, dx = \int_0^t \int_{\mathbb{R}^2} \Delta \Phi_R(x)v(s,x) \, dx \, ds
+ \int_0^t \int_{\mathbb{R}^2} v(s,x)(\nabla N \ast v)(s,x) \cdot \nabla \Phi_R(x) \, dx \, ds. \tag{4.17}
$$

We note

$$
\int_{\mathbb{R}^2} v(t,x) \, dx = M
$$

by (4.11), and $\|\Delta \Phi_R\|_\infty \leq C$ by the definition of $\Phi_R$, where $C$ is independent of $R$. Then

$$
\int_0^t \int_{\mathbb{R}^2} |v(s,x)\Delta \Phi_R(x)| \, dx \, ds \leq CMt.
$$

We observe that the second term on the right-hand side of (4.17) may be rewritten as

$$
I_R(s) := \int_{\mathbb{R}^2} v(s,x)(\nabla N \ast v)(s,x) \cdot \nabla \Phi_R(x) \, dx
= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} v(s,x)v(s,y) \frac{x-y}{|x-y|^2} \cdot \nabla \Phi_R(x) \, dy \, dx
= -\frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} v(s,x)v(s,y) \frac{(x-y) \cdot (\nabla \Phi_R(x) - \nabla \Phi_R(y))}{|x-y|^2} \, dy \, dx.
$$

Since

$$
|\nabla \Phi_R(x) - \nabla \Phi_R(y)| \leq \|\nabla^2 \Phi_R\|_\infty |x-y| \leq C|x-y|,
$$

where $C$ is a constant independent of $R$, we have

$$
|I_R(s)| \leq C \left(\int_{\mathbb{R}^2} v(s,x) \, dx\right)^2 = CM^2,
$$

and hence

$$
\int_{\mathbb{R}^2} \Phi_R(x)v(t,x) \, dx \leq C(M + M^2)t.
$$
Letting \( R \to \infty \), by Fatou’s lemma we obtain
\[
\int_{\mathbb{R}^2} |x|^2 v(t, x) \, dx \leq C(M + M^2) t.
\]
Noting that as \( R \to \infty \),
\[
\Delta \Phi_R(x) \to 4, \quad \frac{(x - y) \cdot (\nabla \Phi_R(x) - \nabla \Phi_R(y))}{|x - y|^2} \to 2,
\]
by the Lebesgue convergence theorem we deduce
\[
\int_0^t \int_{\mathbb{R}^2} v(s, x) \Delta \Phi_R(x) \, dx \, ds \to 4 \int_0^t \int_{\mathbb{R}^2} v(s, x) \, dx \, ds = 4Mt,
\]
\[
\int_0^t I_R(s) \, ds \to -\frac{1}{2\pi} \int_0^t \left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} v(s, x) v(s, y) \, dy \, dx \right) \, ds = -\frac{M^2}{2\pi} t,
\]
and hence (4.16) by letting \( R \to \infty \) in (4.17).

For a nonnegative weak solution \( v \) of (1.5)–(1.6) with initial data \( \hat{M}\delta_0(x) \), we show that if \( 0 < \hat{M} < 8\pi \) then \( v = U_{\hat{M}} \), where \( U_{\hat{M}} \) is the radially symmetric self-similar solution of (1.5) with
\[
\int_{\mathbb{R}^2} U_{\hat{M}}(t, x) \, dx = \hat{M}
\]
mentioned in Section 2. In order to show \( v = U_{\hat{M}} \), we need a result by Gallagher-Gallay-Lions [15] in which they proved the uniqueness of weak solutions of the vorticity equation in \( \mathbb{R}^2 \) with a Dirac measure by applying rearrangement techniques. To mention this result (Proposition 4.5 below), we introduce the notion of rearrangements.

Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a measurable function satisfying
\[
||f||_\theta := |\{x \in \mathbb{R}^d : |f(x)| > \theta\}| < \infty \quad \text{for any } \theta > 0,
\]
where \( |A| \) is the Lebesgue measure of a Lebesgue-measurable set \( A \) in \( \mathbb{R}^d \). The distribution function \( \mu_f \) of \( f \) is defined by \( \mu_f(\theta) = ||f||_\theta \) \( (\theta \geq 0) \), and the decreasing rearrangement \( f^* \) of \( f \) by
\[
f^*(s) = \inf\{\theta \geq 0 : \mu_f(\theta) \leq s\} \quad (s \geq 0).
\]
The function \( f^*(x) \), called the symmetric rearrangement or the Schwarz symmetrization of \( f \), is defined by \( f^*(x) = f^*(c_d|x|^d) \), where \( c_d \) is the volume of the unit ball in \( \mathbb{R}^d \).

Some basic properties about rearrangements are as follows (see [1, 27, 30, 41] for example):

(i) \( f^* \) is nonincreasing and right-continuous on \([0, \infty)\).
(ii) If \( f \) is continuous on \( \mathbb{R}^d \), then \( f^* \) and \( f^\sharp \) are continuous on \([0, \infty)\) and \( \mathbb{R}^d \), respectively.

(iii) If \( f : \mathbb{R}^d \to [0, \infty) \) is radially symmetric and nonincreasing with respect to \(|x|\), then \( f = f^\sharp \).

(iv) For every Borel-measurable function \( \Phi : \mathbb{R} \to [0, \infty) \),
\[
\int_{\mathbb{R}^d} \Phi(|f(x)|) \, dx = \int_{\mathbb{R}^d} \Phi(f^\sharp(x)) \, dx = \int_0^\infty \Phi(f^*(s)) \, ds.
\]

(v) Let \( f, g : \mathbb{R}^d \to \mathbb{R} \) be integrable on \( \mathbb{R}^d \). If \( \int_0^s f^*(\sigma) \, d\sigma \leq \int_0^s g^*(\sigma) \, d\sigma \) for all \( s > 0 \), then
\[
\int_{\mathbb{R}^d} \Phi(|f(x)|) \, dx \leq \int_{\mathbb{R}^d} \Phi(|g(x)|) \, dx
\]
for all convex functions \( \Phi : [0, \infty) \to [0, \infty) \) with \( \Phi(0) = 0 \).

(vi) (Contraction property) Let \( 1 \leq p \leq \infty \). For \( f, g \in L^p(\mathbb{R}^d) \),
\[
\|f^* - g^*\|_{L^p([0, \infty))} \leq \|f - g\|_{L^p(\mathbb{R}^d)}, \quad \|f^\sharp - g^\sharp\|_{L^p(\mathbb{R}^d)} \leq \|f - g\|_{L^p(\mathbb{R}^d)}.
\]

Let \( T > 0 \). For a measurable function \( f : (0, T) \times \mathbb{R}^d \to \mathbb{R} \), we denote by \( f^* \) the decreasing rearrangement of \( f \) with respect to the space variable \( x \in \mathbb{R}^d \); that is, \( f^*(t, s) = f(t)^*(s) \) for \( t \in (0, T) \) and \( s \geq 0 \), where \( f(t)^* \) is the decreasing rearrangement of \( f(t) \). Define the Schwarz symmetrization \( f^\sharp \) of \( f \) with respect to the space variable by \( f^\sharp(t, x) = f^*(t, c_d|x|^d) = f(t)^*(c_d|x|^d) \).

**Proposition 4.5** (Proposition 4.2, [15]). Let \( f, g : \mathbb{R}^d \to [0, +\infty) \) be continuous and integrable functions satisfying

(i) \( \int_0^s f^*(\sigma) \, d\sigma \leq \int_0^s g^*(\sigma) \, d\sigma \) for all \( s > 0 \),

(ii) \( g \) is radially symmetric and nonincreasing with respect to \(|x|\),

(iii) \( \int_{\mathbb{R}^d} f(x) \, dx = \int_{\mathbb{R}^d} g(x) \, dx \),

(iv) \( \int_{\mathbb{R}^d} \sigma f(x) \, dx = \int_{\mathbb{R}^d} |x|^d g(x) \, dx < \infty \).

Then \( f = g \).

For the nonnegative mild solution \( u \) of (1.5)–(1.6) with nonnegative initial data \( u_0 \in L^1 \) and the radially symmetric self-similar solution \( U^\sharp_M \) with \( M = \int_{\mathbb{R}^2} u_0 \, dx \), the following comparison between \( \int_0^s u^*(t, \sigma) \, d\sigma \) and \( \int_0^s U^\sharp_M(t, \sigma) \, d\sigma \) was obtained in [32].

**Proposition 4.6.** Assume \( M := \int_{\mathbb{R}^2} u_0 \, dx < 8\pi \) for the nonnegative initial data \( u_0 \in L^1 \). Then for the nonnegative mild solution \( u \) of (1.5)–(1.6) on \([0, T)\), it holds that for each \( 0 < t < T \),
\[
\int_0^s u^*(t, \sigma) \, d\sigma \leq \int_0^s U^\sharp_M(t, \sigma) \, d\sigma \quad \text{for all} \quad s > 0.
\]
Applying Proposition 4.6, we have the following.

**Proposition 4.7.** Let \( v \) be a nonnegative weak solution of (1.5)–(1.6) with initial data \( \hat{M}\delta_0(x) \) and assume \( 0 < \hat{M} < 8\pi \). Then for each \( t > 0 \),

\[
\int_0^s v^*(t, \sigma) \, d\sigma \leq \int_0^s U^*_M(t, \sigma) \, d\sigma \quad \text{for all } s > 0.
\]  

(4.18)

**Proof.** Take an arbitrary number \( \tau > 0 \) and fix it. Define \( w \) on \([0, \infty) \times \mathbb{R}^2\) by \( w(t, x) = v(t + \tau, x) \). Then \( w \in C([0, \infty); L^1 \cap L^{4/3}) \), and we see that \( w \) satisfies the following: For any \( \varphi \in C_0^\infty([0, \infty) \times \mathbb{R}^2) \),

\[
0 = \int_{\mathbb{R}^2} \varphi(0, x)v(\tau, x) \, dx + \int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + \Delta \varphi) w \, dx \, dt \quad \text{(4.19)}
\]

\[+ \int_0^\infty \int_{\mathbb{R}^2} \nabla \varphi \cdot f \, dx \, dt,\]

where \( f = w(\nabla N \ast w) \). By (2.2) and (i) and (ii) of Definition 4.1, \( f \in C([0, \infty); L^1) \). We claim that for all \( t > 0 \),

\[
w(t) = e^{t\Delta}v(\tau) - \int_0^t \nabla \cdot e^{(t-s)\Delta} f(s) \, ds.
\]  

(4.20)

To prove this claim, we define \( \tilde{w} \) on \([0, \infty) \times \mathbb{R}^2\) by the right-hand side of (4.20). We observe that \( \tilde{w} \) satisfies (4.19) replacing \( w \) by \( \tilde{w} \). In fact, for \( \{f_n\}_{n=1}^\infty \subset C_0^\infty([0, \infty) \times \mathbb{R}^2) \) satisfying

\[
\max_{0 \leq t \leq T} \|f_n(t) - f(t)\|_1 \to 0 \quad \text{as } n \to \infty \quad \text{for all } T > 0,
\]

define \( \tilde{w}_n \) on \([0, \infty) \times \mathbb{R}^2\) by

\[
\tilde{w}_n(t) = e^{t\Delta}v(\tau) - \int_0^t \nabla \cdot e^{(t-s)\Delta} f_n(s) \, ds.
\]

Then \( \tilde{w}_n \in C([0, \infty); L^1 \cap L^{4/3}) \cap C^\infty((0, \infty) \times \mathbb{R}^2) \), and applying \( L^p-L^1 \) estimates for \( e^{t\Delta} \) yields that for all \( 1 \leq p < 2 \),

\[
\max_{0 \leq t \leq T} \|\tilde{w}_n(t) - \tilde{w}(t)\|_p \to 0 \quad \text{as } n \to \infty \quad \text{for all } T > 0.
\]

Since \( \tilde{w}_n \) satisfies (4.19) replacing \( w \) and \( f \) by \( \tilde{w}_n \) and \( f_n \), respectively (see Chapter 4 of [17] for example), \( \tilde{w} \) satisfies (4.19) replacing \( w \) by \( \tilde{w} \) by letting \( n \to \infty \). Hence, by the uniqueness of weak solutions for the heat equation (see Theorem 4.4.2 of [17] for example), we conclude \( w = \tilde{w} \) and hence (4.20).
From (4.20) it follows that \( w \) is a nonnegative mild solution of (1.5)–(1.6) with initial data \( v(\tau) \). Since \( \int_{\mathbb{R}^2} v(\tau) \, dx = \hat{M} < 8\pi \), applying Proposition 4.6 yields that for each \( t > 0 \),
\[
\int_0^s v^*(t + \tau, \sigma) \, d\sigma = \int_0^s w^*(t, \sigma) \, d\sigma \leq \int_0^s U_{\hat{M}}^*(t, \sigma) \, d\sigma \quad \text{for all} \quad s > 0.
\]
We observe \( \|v^*(t + \tau) - v^*(t)\|_1 \to 0 \) as \( \tau \to 0 \) by the contraction property of the decreasing rearrangement, and hence, letting \( \tau \to 0 \) in (4.21), we conclude (4.18).

By Proposition 4.5, we have the following result on uniqueness.

**Theorem 4.1.** Let \( v \) be a nonnegative weak solution of (1.5)–(1.6) with initial data \( \hat{M}\delta_0(x) \). If \( 0 < \hat{M} < 8\pi \), then \( v = U_{\hat{M}} \).

**Proof.** To prove this theorem, we apply Proposition 4.5 with \( f = v(t) \) and \( g = U_{\hat{M}}(t) \). For each \( t > 0 \), the function \( x \mapsto U_{\hat{M}}(t, x) \) is radially symmetric and nonincreasing with respect to \(|x|\). Proposition 4.7 and (i) of Proposition 4.3 imply (i) and (iii) of Proposition 4.5, respectively. To prove (iv) of Proposition 4.5, we claim
\[
\int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(t, x) \, dx = 4\hat{M}\left(1 - \frac{\hat{M}}{8\pi}\right) t \quad \text{for} \quad t > 0.
\]
In fact, by Proposition 2.2, we see that for any \( 0 < \varepsilon < t \),
\[
\int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(t, x) \, dx = \int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(\varepsilon, x) \, dx + 4\hat{M}\left(1 - \frac{\hat{M}}{8\pi}\right)(t - \varepsilon).
\]
From this relation, (4.22) is deduced because
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(\varepsilon, x) \, dx = 0
\]
by virtue of (2.3). Hence it follows from Proposition 4.4 and (4.22) that
\[
\int_{\mathbb{R}^2} |x|^2 v(t, x) \, dx = \int_{\mathbb{R}^2} |x|^2 U_{\hat{M}}(t, x) \, dx.
\]
Therefore, Proposition 4.5 ensures \( v(t) = U_{\hat{M}}(t) \) for each \( t > 0 \).

**Remark 4.1.** To prove uniqueness by our method requires nonnegativity for weak solutions. For small \( M \in \mathbb{R} \), the uniqueness of weak solutions without nonnegativity is obtained by the same method as in [18, 24].
4.2. **Convergence to a radially symmetric self-similar solution.** We come back to study the convergence of \( u_{\lambda j} \) to \( U \) as \( j \to \infty \), where \( u_{\lambda j} \) and \( U \) are the same as before.

**Proposition 4.8.** Let \( 1 \leq p \leq \infty \). Then \( u_{\lambda j}(t) \to U(t) \) in \( L^p \) as \( j \to \infty \) for all \( t > 0 \).

**Proof.** For fixed \( t > 0 \), \( u_{\lambda j}(t,x) \to U(t,x) \) as \( j \to \infty \) for all \( x \in \mathbb{R}^2 \), and \( \|u_{\lambda j}(t)\|_1 = \hat{M} = \|U(t)\|_1 \) by (i) of Proposition 4.3. Hence,

\[
\lim_{j \to \infty} \|u_{\lambda j}(t) - U(t)\|_1 = 0.
\]

Let \( 1 < p < \infty \). Then

\[
\|u_{\lambda j}(t) - U(t)\|_p \leq \|u_{\lambda j}(t) - U(t)\|_1^{1-1/p}\|u_{\lambda j}(t) - U(t)\|_1^{1/p},
\]

from which together with \( t\|u_{\lambda j}(t) - U(t)\|_\infty \leq C(\hat{M}) \) by (4.3) and (4.4) it follows that

\[
\lim_{j \to \infty} \|u_{\lambda j}(t) - U(t)\|_p = 0.
\]

We consider the case \( p = \infty \). Recall the following interpolation inequalities (for example, see Theorem 9.3 of [14]): Let \( 2 < p < \infty \). Then there is a positive constant \( C \), depending only on \( p \), such that for any \( f \in W^{1,p}(\mathbb{R}^2) \),

\[
\|f\|_\infty \leq C\|
abla f\|_{p}^{2/p}\|f\|_{p}^{1-2/p}.
\]

Applying this inequality yields

\[
\|u_{\lambda j}(t) - U(t)\|_\infty \leq C\|\nabla(u_{\lambda j}(t) - U(t))\|_{p}^{2/p}\|u_{\lambda j}(t) - U(t)\|_{p}^{1-2/p}.
\]

Since \( t^{1-1/p}\|\nabla(u_{\lambda j}(t) - U(t))\|_{p} \leq C(\hat{M},p) \) by (4.3) and (4.4), we deduce

\[
\lim_{j \to \infty} \|u_{\lambda j}(t) - U(t)\|_\infty = 0.
\]

Therefore, we establish the proof of this proposition. \( \square \)

We are now in a position to mention our main result.

**Theorem 4.2.** Assume \( \hat{M} = \int_{\mathbb{R}^2} u_0 \, dx < 8\pi \) for the nonnegative initial data \( u_0 \in L^1 \). Then for the nonnegative global mild solution \( u \) of (1.5)--(1.6), it holds that for all \( 1 \leq p \leq \infty \),

\[
\lim_{t \to \infty} t^{1-1/p}\|u(t) - U_{\hat{M}}(t)\|_p = 0.
\] (4.23)
Proof. By Proposition 4.2, $U$ is a nonnegative weak solution of (1.5)–(1.6) with initial data $\hat{M}\delta_0$, and by Theorem 4.1, $U = \hat{U}$. Hence, by Proposition 4.8,

$$\lim_{j \to \infty} \|u_{\lambda_j}(t) - \hat{U}(t)\|_p = 0$$

(4.24)

for all $t > 0$, where $1 \leq p \leq \infty$. Since for any sequence $\{\lambda_j\}$ satisfying $\lambda_j \not\to \infty (j \not\to \infty)$ there exists a subsequence of $\{\lambda_j\}$ for which (4.24) is satisfied, we deduce that for all $t > 0$ and for all $1 \leq p \leq \infty$,

$$\lim_{\lambda \to \infty} \|u_\lambda(t) - \hat{U}(t)\|_p = 0.$$ 

Putting $t = 1$ and then taking $\lambda = t^{1/2}$ yields that

$$\lim_{t \to \infty} t^{1-1/p}\|u(t, \cdot) - \frac{1}{t} \hat{U}(1, \frac{1}{\sqrt{t}})\|_p = 0,$$

which completes the proof of Theorem 4.2 because

$$t^{-1} \hat{U}(1, xt^{-1/2}) = \hat{U}(t, x).$$

References

[34] T. Nagai and T. Ogawa, Global existence of solutions to a parabolic-elliptic system of drift-diffusion type in $\mathbb{R}^2$, submitted.
[38] Y. Naito and T. Senba, Bounded and unbounded oscillating solutions to a parabolic-elliptic system in two dimensional solutions, preprint.