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LOCAL WELL-POSEDNESS AND A PRIORI BOUNDS FOR THE MODIFIED BENJAMIN-ONO EQUATION

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Abstract. We prove that the complex-valued modified Benjamin-Ono (mBO) equation is analytically locally well posed if the initial data ϕ belongs to H^s for $s \geq 1/2$ with $\|\phi\|_{L^2}$ sufficiently small, without performing a gauge transformation. The key ingredient is that the logarithmic divergence in the high-low frequency interaction can be overcome by a combination of $X^{s,b}$ structure and smoothing effect structure. We also prove that the real-valued H^∞ solutions to the mBO equation satisfy a priori local-in-time H^s bounds in terms of the H^s size of the initial data for $s > 1/4$.

1. Introduction

In this paper, we study the initial-value problem for the (defocusing) modified Benjamin-Ono equation (also the equation with focusing nonlinearity of the form $-u^2u_x$ can be treated by our method) of the form

$$\begin{aligned} &u_t + \mathcal{H}u_{xx} = u^2u_x,\ (x,t) \in \mathbb{R}^2, \\ &u(x,0) = \phi(x), \end{aligned} \tag{1.1}$$

where $u : \mathbb{R}^2 \to \mathbb{C}$ is a complex-valued function and $\mathcal{H}$ is the Hilbert transform which is defined as follows:

$$\mathcal{H}(u)(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{+\infty} \frac{u(y)}{x-y}dy. \tag{1.2}$$

The equation with quadratic nonlinearity

$$u_t + \mathcal{H}u_{xx} = uu_x \tag{1.3}$$

was derived by Benjamin [2] and Ono [38] as a model for one-dimensional waves in deep water. On the other hand, the cubic nonlinearity is also of

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much interest for long wave models [1, 24]. Since the Hilbert transform maps a real-valued function to a real-valued function, we will refer to (1.1) as the real-valued (complex-valued) mBO if ϕ is real valued (complex valued).

The mBO equation (1.1) has several symmetries. The first one is the scaling invariance: for $\lambda > 0$

$$u(x,t) \to u_\lambda = \frac{1}{\lambda^{1/2}} u(\frac{x}{\lambda}, \frac{t}{\lambda^2}), \quad \phi_\lambda = \frac{1}{\lambda^{1/2}} \phi(\frac{x}{\lambda}). \tag{1.4}$$

Then we see L^2 is the critical space in the sense that $\|\frac{1}{\lambda^{1/2}}\phi(\frac{\cdot}{\lambda})\|_{L^2} = \|\phi\|_{L^2}$. This leads to the constraint $s \geq 0$ on the well posedness for (1.1). There are at least the following three conservation laws preserved under the flow of the real-valued mBO equation (1.1):

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)dx = 0, \tag{1.5}$$

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)^2 dx = 0, \tag{1.6}$$

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2} u\mathcal{H}u_x - \frac{1}{12}|u(x,t)|^4 dx = 0. \tag{1.7}$$

These conservation laws provide a priori bounds on the solution. For instance, we get from (1.6) and (1.7) that the $H^{1/2}$ norm of the solution remains bounded for finite time if the initial data belongs to $H^{1/2}$.

The initial-value problems for (1.1) and for the Benjamin-Ono equation (1.3) have been extensively studied [3, 21, 39, 8, 12, 13, 14, 17, 22, 25, 26, 30, 32, 33, 35, 36, 10, 11]. For instance, the energy method provides local well posedness in H^s for $s > 3/2$ [21]. This result was improved by a combination of the energy method and the dispersive effects. The result in [39] that $s \geq 3/2$ was the first place where such a combination as a consequence of the commutator estimates in [23] appears and was later improved to $s > 5/4$ in [32], and $s > 9/8$ in [22]. Tao [41] obtained global well posedness in H^s for $s \geq 1$ by performing a gauge transformation as for the derivative Schrödinger equation. This result was improved to $s \geq 0$ by Ionescu and Kenig [18], and to $s \geq 1/4$ (local well posedness) by Burq and Planchon [5]. Recently, Molinet and Pilod [34] gave a simplified proof for $s \geq 0$ and obtained unconditional uniqueness for $s > 1/4$.

For the modified Benjamin-Ono equation (1.1), the energy method provides local well posedness in H^s for $s > 3/2$ [21]. This was improved to $s \geq 1$ by Kenig-Koenig [22] by the enhanced energy methods. These results only hold for the real-valued mBO and only obtain the continuity of the solution map. Molinet and Ribaud [36] obtained analytic local well posedness for the complex-valued mBO in H^s for $s > 1/2$ and $B^{1/2}_{2,1}$ with a small L^2 norm, improving the result of Kenig-Ponce-Vega [25] for $s > 1$. The smallness condition of $H^s(s > 1/2)$ results was later removed in [35] by using Tao's gauge transformation [41]. The result for $s = 1/2$ was obtained by Kenig and Takaoka [30] by using frequency dyadically localized gauge transformations. Their result is sharp in the sense that the solution map is not locally uniformly continuous in H^s for $s < 1/2$ (The failure of C^3 smoothness was obtained in [36].). Due to the gauge transform, the latter results without a smallness condition hold only for the real-valued mBO.

In the first part of this paper, we will prove local well posedness for the complex-valued mBO equation in $H^{1/2}$ under an extra condition that the L^2 norm of the initial data is small. Let's recall the smoothing effect method which was used in [25] and [36]. The main difficulty is the loss of derivative in the nonlinear term. To recover one derivative, the following local smoothing effect plays a crucial role:

$$\||\partial_x|u\|_{L^\infty_x L^2_t} \lesssim \|(\partial_t + \mathcal{H}\partial_{xx})u\|_{L^1_x L^2_t}. \tag{1.8}$$

Using this structure, one needs to combine naturally with a maximal function structure $\|u\|_{L^2_x L^\infty_t}$, see [36]. In this paper, we use different structures. We start with the Bourgain space $X^{s,b}$ associated to the mBO equation defined as a closure of the following space:

$$\{f \in \mathcal{S}(\mathbb{R}^2) : \|f\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle\tau - \omega(\xi)\rangle^b \widehat{f}(\xi,\tau)\|_{L^2} < \infty\}.$$

According to the standard methods in [6, 27], a direct perturbative approach in $X^{s,b}$ space reduces to the trilinear estimate

$$\|\partial_x(u^3)\|_{X^{s,b-1}} \lesssim \|u\|^3_{X^{s,b}}, \quad \text{for some } b \in [1/2, 1). \tag{1.9}$$

However, we find that this trilinear estimates fails for any s due to some logarithmic divergences involving the modulation variable (see Proposition 5.2, 5.3 below). The key observation in this paper is that these logarithmic divergences can be removed by using Banach spaces which combine $X^{s,b}$ structure with smoothing effect structure (1.8). The spaces of these structures were first found and used by Ionescu and Kenig [18] to remove some logarithmic divergences. These structures exploit the resonance of the wave interactions (see Proposition 5.1) while the smoothing effect method (1.8) does not. This explains why we can cover the endpoint $s = 1/2$ in [36]. Now we state our first result.

Theorem 1.1. *Let $s \geq 1/2$, $R > 0$, $\phi \in H^s$ with $\|\phi\|_{L^2} \ll 1$ and $\|\phi\|_{H^{1/2}} \leq R$. Then there exists a $T = T(R) > 0$ and a unique solution u to the mBO equation* (1.1) *(or its focusing version) satisfying*

$$u \in F^s(T) \subset C([-T,T] : H^s) \text{ and } \|u\|_{C([-T,T]:L^2)} \ll 1 \tag{1.10}$$

where $F^s(T)$ is defined in Section 2. Moreover, the solution map $\phi \to u$ is uniformly continuous from $\{u_0 \in H^s : \|u_0\|_{H^{1/2}} \leq R, \|u_0\|_{L^2} \ll 1\}$ to $C([-T,T] : H^s)$.

Remark 1.2. For the real-valued mBO, the uniqueness holds in $F^s(T)$, due to the L^2 conservation law. However, for the complex-valued mBO, we can only obtain the uniqueness under the smallness condition.

We will only exploit the dispersive effect. Theorem 1.1 actually holds for general dispersive equations. More precisely, we recall the definition in [15].

Definition 1.3. *Let $\beta > 0$ and $h : \mathbb{R} \to \mathbb{R}$ be smooth on $\mathbb{R} \setminus \{0\}$. h is said to have β-degree dispersive effect at high frequency, denoted by $h \in D_{hi}(\beta)$, if for $|\xi| \gtrsim 1$*

$$|\partial_\xi^k h(\xi)| \sim |\xi|^{\beta-k},\ k = 1, 2; \quad |\partial_\xi^j h(\xi)| \lesssim |\xi|^{\beta-j},\ j \geq 3.$$

The dispersive degree of the equation is given via its dispersive function.

For example, for the KdV equation the dispersion $\xi^3 \in D_{hi}(3)$, and for the Benjamin-Ono equation the dispersion $-|\xi|\xi \in D_{hi}(2)$. Theorem 1.1 holds for the following general cubic dispersive equations:

$$\partial_t u + Lu = \lambda u^2 u_x, \quad u(x,0) = u_0(x), \tag{1.11}$$

where $\mathcal{F}(Lf)(\xi) = i\omega(\xi)\mathcal{F}f(\xi)$ and $\omega \in D_{hi}(2)$. In particular, (1.11) includes the following two additional equations: derivative Schrödinger equation

$$u_t - iu_{xx} = |u|^2 u_x, \quad u(x,0) = u_0(x), \tag{1.12}$$

and modified finite-depth-fluid equation

$$\partial_t u - \mathcal{G}_\delta(\partial_x^2 u) \pm u^2 u_x = 0, \quad u(x,0) = u_0(x),$$

where $\mathcal{G}_\delta f = -i\mathcal{F}^{-1}[\coth(2\pi\delta\xi) - \frac{1}{2\pi\delta\xi}]\mathcal{F}f$, $\delta \gtrsim 1$. The latter equation was studied in [16].

Remark 1.4. The smallness condition in Theorem 1.1 is due to the fact that the mBO equation (1.1) is L^2-critical. Technically, this condition is needed to handle the nonlinear term $(P_{\leq 0}u)^2 u_x$. On the other hand, Theorem 1.1 holds without the smallness condition for the equation

$$u_t + \mathcal{H}u_{xx} = u^2 u_x - (P_{\leq 0}u)^2 u_x.$$

This bad term is removed via a gauge transformation in the previous results. Our result indicates that, for the real-valued mBO, one may also remove the smallness condition by performing a gauge transform as follows:

$$v(x,t) = e^{-(i/2)\int_{-\infty}^{x}(P_{\lesssim 1}u(y,t))^2 dy} P_+ P_{\gg 1} u(x,t),$$

and using the similar methods in [18]. This gauge transform involves only one dyadic piece in [30].

For the real-valued mBO equation, from the conservation laws (1.6), (1.7), and iterating Theorem 1.1, we obtain the following corollary.

Corollary 1.5. *The real-valued mBO equation* (1.1) (*or its focusing version*) *is globally well posed if* ϕ *belongs to* H^s *for* $s \geq 1/2$ *with* $\|\phi\|_{L^2}$ *sufficiently small.*

Remark 1.6. The global well posedness without the smallness condition was proved by Kenig and Takaoka [30]. The novelty of Corollary 1.5 is that the solution map of mBO is analytic and the solution lies in some strong class instead of its gauge transform.

In the second part of this paper, we study the low regularity problem of the real-valued mBO equation (1.1). From the ill-posedness result in [30], we see that for $s < 1/2$ one cannot use the direct Picard iteration method to study mBO (1.1). However, we expect that some well-posedness results hold in the weak sense: only require the continuity of the solution map instead of uniform continuity. More precisely, the following problem is of great interest.

Problem 1.7. *What is the optimal* s *such that the solution map* $S_T^\infty : H^\infty \to C([-T,T] : H^\infty)$ *of the real-valued mBO* (1.1) *can be uniquely extended to a continuous map from* H^s *to* $C([-T,T] : H^s)$ *for some* $T > 0$*?*

The well posedness in the sense of the limit of classical solutions is the most general one. There are some works with the flavor of Problem 1.7, e.g., the Benjamin-Ono equation [18] and KP-I equation [20], dispersion generalized BO equation [15]. To extend the smooth solution map to low regularity space, generally there are two parts of arguments:

- a priori part: for a sequence of smooth initial data $\{\phi_n \in H^\infty\}$ with $\phi_n \to \phi$ in H^s, the smooth solutions u_n with initial data ϕ_n exist on a uniform interval $[-T,T]$, e.g., $T = T(\|\phi\|_{H^s}) > 0$. Moreover, $\{u_n\}$ satisfies, for any $a \geq s$,

$$\|u_n\|_{C([-T,T]:H^a)} \leq C(\|\phi_n\|_{H^a}).$$

- difference part: $\{u_n\}$ is a Cauchy sequence in $C([-T,T]:H^s)$.

The two parts are in the same spirit of the classical energy methods, e.g., Bona-Smith [4]. The second main result of this paper is the following.

Theorem 1.8. *Let $s>1/4$. For any $M>0$, there exist $T,C>0$ such that, for any $\phi\in H^\infty$ satisfying $\|\phi\|_{H^s}\le M$, the solution u to the real-valued mBO equation* (1.1) *satisfies for any $a\ge s$*

$$\|u\|_{C([-T,T]:H^a)}\le C\|\phi\|_{H^a}.$$

Remark 1.9. Theorem 1.8 also holds for some other equations, for example, for the derivative nonlinear Schrödinger equation (1.12).

We discuss now some of the ingredients in the proof of Theorem 1.8. We will use the method of Ionescu, Kenig and Tataru [20] which approaches the problem in a less perturbative way. It can be viewed as a combination of the energy method and the $X^{s,b}$ method.

Since the mBO equation (1.1) is L^2-critical, we need to assume $\|\phi\|_{L^2}\ll 1$ if working in H^s. In order to remove this smallness condition, we work in $\dot H^{s_1}\cap\dot H^{s_2}$. More precisely, we will define $F^{l,s}(T),N^{l,s}(T)$ and the energy space $E^{l,s}(T)$, and show that, if u is a smooth solution of (1.1) on $\mathbb{R}\times[-T,T]$ with $\|u\|_{E^{l,s}(T)}\ll 1$, then

$$\begin{cases} \|u\|_{F^{l,s}(T)}\lesssim\|u\|_{E^{l,s}(T)}+\|\partial_x(u^3)\|_{N^{l,s}(T)};\\ \|\partial_x(u^3)\|_{N^{l,s}(T)}\lesssim\|u\|^3_{F^{l,s}(T)};\\ \|u\|^2_{E^{l,s}(T)}\lesssim\|\phi\|^2_{\dot H^l\cap\dot H^s}+\|u\|^6_{F^{l,s}(T)}.\end{cases}\tag{1.13}$$

The inequalities (1.13) and a simple continuity argument still suffice to control $\|u\|_{F^{l,s}(T)}$, provided that $\|\phi\|_{\dot H^l\cap\dot H^s}\ll 1$ (which can by arranged by rescaling if $l,s>0$). The first inequality in (1.13) is the analogue of the linear estimate. The second inequality in (1.13) is the analogue of the trilinear estimate (1.9). The last inequality in (1.13) is an energy-type estimate.

We explain the strategies in [20] to define the main normed and semi-normed spaces. As was explained before, standard use of $X^{s,b}$ spaces for fixed-point argument will lead to a logarithmic divergence in the key trilinear estimate. But we use $X^{s,b}$-type structures only on small, frequency dependent time intervals. The second step is to define $\|u\|_{E^{l,s}(T)}$ sufficiently large to be able to still prove the linear estimate

$$\|u\|_{F^{l,s}(T)}\lesssim\|u\|_{E^{l,s}(T)}+\|\partial_x(u^3)\|_{N^{l,s}(T)}.$$

Finally, we use an I-method to prove the energy estimate

$$\|u\|^2_{E^{l,s}(T)} \lesssim \|\phi\|^2_{\dot{H}^l\cap \dot{H}^s} + \|u\|^6_{F^{l,s}(T)}.$$

The rest of the paper is organized as follows: In Section 2 we present some notation and Banach function spaces. We summarize some properties of the spaces in Section 3. A symmetric estimate will be given in Section 4 which is used in Section 5 to show a dyadic trilinear estimate. The proof of Theorem 1.1 is given in Section 6. In Section 7 we prove a short time dyadic trilinear estimate and in Section 8 we prove an energy estimate. Finally, in Section 9 we prove Theorem 1.8.

2. Notation and Definitions

For $x, y \in \mathbb{R}^+$, $x \lesssim y$ means that there exists $C > 0$ such that $x \leq Cy$. By $x \sim y$ we mean $x \lesssim y$ and $y \lesssim x$. For $f \in \mathcal{S}'$ we denote by $\widehat{f}$ or $\mathcal{F}(f)$ the Fourier transform of f for both spatial and time variables,

$$\widehat{f}(\xi,\tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(x,t)dxdt.$$

In addition, we use $\mathcal{F}_x$ and $\mathcal{F}_t$ to denote the Fourier transform with respect to space and time variables respectively. Let $\mathbb{Z}_+ = \mathbb{Z}\cap[0,\infty)$. For $k\in\mathbb{Z}$ let $I_k = \{\xi : |\xi| \in [2^{k-1}, 2^{k+1}]\ \}$. For $k \in \mathbb{Z}_+$ let $\widetilde{I}_k = [-2, 2]$ if $k = 0$ and $\widetilde{I}_k = I_k$ if $k \geq 1$.

Let $\eta_0 : \mathbb{R} \to [0, 1]$ denote an even smooth function supported in $[-\frac{8}{5}, \frac{8}{5}]$ and equal to 1 in $[-5/4, 5/4]$. For $k \in \mathbb{Z}$ let $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$, χ_k supported in $\{\xi : |\xi| \in [(5/8)\cdot 2^k, (8/5)\cdot 2^k]\}$, and

$$\chi_{[k_1,k_2]} = \sum_{k=k_1}^{k_2} \chi_k \text{ for any } k_1 \leq k_2 \in \mathbb{Z}.$$

For simplicity of notation, let $\eta_k = \chi_k$ if $k \geq 1$ and $\eta_k \equiv 0$ if $k \leq -1$. Also, for $k_1 \leq k_2 \in \mathbb{Z}$ let

$$\eta_{[k_1,k_2]} = \sum_{k=k_1}^{k_2} \eta_k \text{ and } \eta_{\leq k_2} = \sum_{k=-\infty}^{k_2} \eta_k.$$

Roughly speaking, $\{\chi_k\}_{k\in\mathbb{Z}}$ is the homogeneous decomposition function sequence and $\{\eta_k\}_{k\in\mathbb{Z}_+}$ is the non-homogeneous decomposition function sequence to the frequency space. For $k \in \mathbb{Z}$ let $k_+ = \max(k, 0)$, and let P_k

denote the operators on $L^2(\mathbb{R})$ defined by

$$\widehat{P_k u}(\xi) = \chi_k(\xi)\widehat{u}(\xi).$$

By a slight abuse of notation we also define the operators P_k on $L^2(\mathbb{R}\times\mathbb{R})$ by formulas $\mathcal{F}(P_k u)(\xi,\tau) = \chi_k(\xi)\mathcal{F}(u)(\xi,\tau)$. For $l\in\mathbb{Z}$ let

$$P_{\le l} = \sum_{k\le l} P_k, \quad P_{\ge l} = \sum_{k\ge l} P_k.$$

Let $a_1,a_2,a_3,a_4\in\mathbb{R}$. It will be convenient to define the quantities $a_{max}\ge a_{sub}\ge a_{thd}\ge a_{min}$ to be the maximum, sub-maximum, third-maximum, and minimum of a_1,a_2,a_3,a_4 respectively. We also denote $\mathrm{sub}(a_1,a_2,a_3,a_4)=a_{sub}$ and $\mathrm{thd}(a_1,a_2,a_3,a_4)=a_{thd}$. Usually we use k_i and j_i to denote integers, $N_i=2^{k_i}$ and $L_i=2^{j_i}$ for $i=1,2,3,4$ to denote dyadic numbers. For $a\in\mathbb{R}$ we define $a-$ to be the real number $a-\epsilon$ for some $0<\epsilon\ll 1$. Similarly we also define $a+$.

For $\xi\in\mathbb{R}$ let

$$\omega(\xi) = -\xi|\xi|, \tag{2.1}$$

which is the dispersion relation associated to the linear Benjamin-Ono equation. For $\phi\in L^2(\mathbb{R})$ let $W(t)\phi\in C(\mathbb{R}:L^2)$ denote the solution of the free Benjamin-Ono evolution given by

$$\mathcal{F}_x[W(t)\phi](\xi,t) = e^{it\omega(\xi)}\hat{\phi}(\xi), \tag{2.2}$$

where $\omega(\xi)$ is defined in (2.1). For $k\in\mathbb{Z}_+$ and $j\ge 0$ let $D_{k,j}=\{(\xi,\tau)\in\mathbb{R}\times\mathbb{R}:\xi\in\widetilde{I}_k, \tau-\omega(\xi)\in\widetilde{I}_j\}$. For $k\in\mathbb{Z}_+$ we define the Banach spaces $X_k(\mathbb{R}\times\mathbb{R})$:

$$X_k = \left\{ \begin{array}{l} f\in L^2(\mathbb{R}^2): f \text{ is supported in } \widetilde{I}_k\times\mathbb{R} \text{ and} \\ \|f\|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\|\eta_j(\tau-w(\xi))\cdot f(\xi,\tau)\|_{L^2_{\xi,\tau}} < \infty \end{array} \right\}, \tag{2.3}$$

where

$$\beta_{k,j} = 1 + 2^{(j-2k)/2}. \tag{2.4}$$

Here the spaces X_k are the same as those used by Ionescu and Kenig [18] for $k>0$. X_0 is different, since we don't have the special structures in the low frequency. The use of the weight $\beta_{k,j}$ is important in order for all the trilinear estimates in Section 5 to hold. We see that if k is small then $\beta_{k,j}\approx 2^{j/2}$. This weight is particularly important in controlling the high-low interaction. From the technical level, we know from the K-Z method of Tao [42] that the worst interaction is that the low frequency component has a largest modulation. The weight $\beta_{k,j}$ will strengthen those components. In this paper, we can take any weight of the form $\beta_{k,j}=1+2^{(j-2k)\theta}$ for some $0 < \theta \leq 1/2$, but we choose to use the same weight as in [18]. The logarithmic divergence caused by the other interaction, namely the high frequency component with largest modulation, can be overcome by using a smoothing effect structure.

As in [18], the spaces X_k are not sufficient for our purpose, due to various logarithmic divergences involving the modulation variable (See Proposition 5.2 below). For $k \geq 100$ we also define the Banach spaces $Y_k = Y_k(\mathbb{R}^2)$. For $k \geq 100$ we define

$$Y_k = \left\{ \begin{array}{l} f \in L^2(\mathbb{R}^2) : f \text{ is supported in } \bigcup_{j=0}^{k-1} D_{k,j} \text{ and} \\ \|f\|_{Y_k} := 2^{-k/2} \|\mathcal{F}^{-1}[(\tau - \omega(\xi) + i) f(\xi,\tau)]\|_{L^1_x L^2_t} < \infty \end{array} \right\}. \tag{2.5}$$

Then for $k \in \mathbb{Z}_+$ we define

$$Z_k := X_k \text{ if } k \leq 99 \text{ and } Z_k := X_k + Y_k \text{ if } k \geq 100. \tag{2.6}$$

The spaces Z_k are our basic Banach spaces. For $s \geq 0$ we define the Banach spaces $F^s = F^s(\mathbb{R} \times \mathbb{R})$:

$$F^s = \{u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}) : \|u\|_{F^s}^2 = \sum_{k=0}^{\infty} 2^{2sk} \|\eta_k(\xi) \mathcal{F}(u)\|_{Z_k}^2 < \infty\}, \tag{2.7}$$

and $N^s = N^s(\mathbb{R} \times \mathbb{R})$ which is used to measure the nonlinear term and can be viewed as an analogue of $X^{s,b-1}$

$$N^s = \{u \in \mathcal{S}'(\mathbb{R}^2) : \|u\|_{N^s}^2 = \sum_{k=0}^{\infty} 2^{2sk} \|\eta_k(\xi)(\tau - \omega(\xi) + i)^{-1} \mathcal{F}(u)\|_{Z_k}^2 < \infty\}. \tag{2.8}$$

The spaces F^s and N^s have the same structures in high frequency as those in [18], but have different structures in low frequency. For $T \in (0,1]$ we define the normed spaces

$$\begin{aligned} F^s(T) &= \{f \in C([-T,T] : H^s) : \|f\|_{F^s(T)} = \inf_{\widetilde{f}=f \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{f}\|_{F^s} < \infty\}; \\ N^s(T) &= \{f \in C([-T,T] : H^s) : \|f\|_{N^s(T)} = \inf_{\widetilde{f}=f \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{f}\|_{N^s} < \infty\}. \end{aligned}$$

In order to prove a priori bounds, we will need another set of norms and semi-norms which were first used by Ionescu, Kenig and Tataru in [20] for the KP-I equation. Similar ideas can be found in [31]. For $k \in \mathbb{Z}$ we define

$$B_k = \left\{ \begin{array}{l} f \in L^2(\mathbb{R}^2) : f \text{ is supported in } I_k \times \mathbb{R} \text{ and} \\ \|f\|_{B_k} := \sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - w(\xi)) \cdot f(\xi,\tau)\|_{L^2_{\xi,\tau}} < \infty \end{array} \right\}. \tag{2.9}$$

These l^1-type $X^{s,b}$ structures were first introduced and used in [44, 18, 20].

At frequency 2^k we will use the $X^{s,b}$ structure given by the B_k norm, uniformly on the 2^{-k_+} time scale. For $k \in \mathbb{Z}$ we define the normed spaces

$$F_k = \left\{ \begin{array}{l} f \in L^2(\mathbb{R}^2) : \widehat{f} \text{ is supported in } I_k \times \mathbb{R} \text{ and} \\ \|f\|_{F_k} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[f \cdot \eta_0(2^{k_+}(t-t_k))]\|_{B_k} < \infty \end{array} \right\},$$

$$N_k = \left\{ \begin{array}{l} f \in L^2(\mathbb{R}^2) : \widehat{f} \text{ is supported in } I_k \times \mathbb{R} \text{ and } \|f\|_{N_k} = \\ \sup_{t_k \in \mathbb{R}} \|(\tau - \omega(\xi) + i2^{k_+})^{-1}\mathcal{F}[f \cdot \eta_0(2^{k_+}(t-t_k))]\|_{B_k} < \infty \end{array} \right\}.$$

The bounds we obtain for smooth solutions of the equation (1.1) are on a fixed time interval, while the above function spaces are not. Thus we define a local version of the spaces. Denote $E_k = \{u \in L^2 : \hat{u} \text{ is supported in } I_k\}$. For $T \in (0,1]$ we define the normed spaces

$$F_k(T) = \{f \in C([-T,T] : E_k) : \|f\|_{F_k(T)} = \inf_{\widetilde{f}=f \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{f}\|_{F_k} < \infty\};$$

$$N_k(T) = \{f \in C([-T,T] : E_k) : \|f\|_{N_k(T)} = \inf_{\widetilde{f}=f \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{f}\|_{N_k} < \infty\}.$$

For $l, s \geq 0$ and $T \in (0,1]$, we define the normed spaces

$$F^{l,s}(T) = \left\{ \begin{array}{l} u \in C([-T,T] : H^\infty) : \quad \|u\|_{F^{l,s}}^2 = \\ \sum_{k=-\infty}^{0} 2^{2lk}\|P_k(u)\|_{F_k(T)}^2 + \sum_{k=1}^{\infty} 2^{2sk}\|P_k(u)\|_{F_k(T)}^2 < \infty \end{array} \right\},$$

$$N^{l,s}(T) = \left\{ \begin{array}{l} u \in C([-T,T] : H^\infty) : \quad \|u\|_{N^{l,s}}^2 = \\ \sum_{k=-\infty}^{0} 2^{2lk}\|P_k(u)\|_{N_k(T)}^2 + \sum_{k=1}^{\infty} 2^{2sk}\|P_k(u)\|_{N_k(T)}^2 < \infty \end{array} \right\}.$$

We still need an energy space. For $l, s \geq 0$ and $u \in C([-T,T] : H^\infty)$ we define

$$\|u\|_{E^{l,s}(T)}^2 = \|P_{\leq 0}(u(0))\|_{\dot{H}^l}^2 + \sum_{k\geq 1} \sup_{t_k \in [-T,T]} 2^{2sk}\|P_k(u(t_k))\|_{L^2}^2.$$

3. Properties of the Spaces Z_k

This section is devoted to studying the properties of the spaces Z_k. Using the definitions, if $k \in \mathbb{Z}_+$ and $f_k \in Z_k$ then f_k can be written in the form

$$\left\{ \begin{array}{l} f_k = \sum_{j=0}^{\infty} f_{k,j} + g_k; \\ \sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\|f_{k,j}\|_{L^2} + \|g_k\|_{Y_k} \leq 2\|f_k\|_{Z_k}, \end{array} \right. \tag{3.1}$$

such that $f_{k,j}$ is supported in $D_{k,j}$ and g_k is supported in $\cup_{j=0}^{k-1} D_{k,j}$ (if $k \leq 99$ then $g_k \equiv 0$). We start with the elementary properties.

Lemma 3.1 (Lemma 4.1, [18])**.** *(a) If $m, m' : \mathbb{R} \to \mathbb{C}$, $k \in \mathbb{Z}_+$, and $f_k \in Z_k$ then*

$$\begin{cases} \|m(\xi) f_k(\xi,\tau)\|_{Z_k} \le C\|\mathcal{F}^{-1}(m)\|_{L^1(\mathbb{R})}\|f_k\|_{Z_k}; \\ \|m'(\tau) f_k(\xi,\tau)\|_{Z_k} \le C\|m'\|_{L^\infty(\mathbb{R})}\|f_k\|_{Z_k}. \end{cases} \tag{3.2}$$

(b) If $k \in \mathbb{Z}_+$, $j \ge 0$, and $f_k \in Z_k$, then

$$\|\eta_j(\tau - \omega(\xi)) f_k(\xi,\tau)\|_{X_k} \le C\|f_k\|_{Z_k}. \tag{3.3}$$

(c) If $k \ge 1$, $j \in [0,k]$, and f_k is supported in $I_k \times \mathbb{R}$, then

$$\|\mathcal{F}^{-1}[\eta_{\le j}(\tau - \omega(\xi)) f_k(\xi,\tau)]\|_{L^1_x L^2_t} \le C\|\mathcal{F}^{-1}(f_k)\|_{L^1_x L^2_t}. \tag{3.4}$$

We study now the embedding properties of the spaces Z_k. We will see that the spaces X_k and Y_k are very close.

Lemma 3.2. *Let $k \in \mathbb{Z}_+$, $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ be an interval. Let Y be $L^p_x L^q_{t\in I}$ or $L^q_{t\in I} L^p_x$ for some $1 \le p, q \le \infty$ with the property that*

$$\|e^{-t\mathcal{H}\partial_x^2} f\|_Y \lesssim 2^{ks}\|f\|_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$ with $\widehat{f}$ supported in $\widetilde{I}_k$. Then we have: for any $f_k \in Z_k$

$$\|\mathcal{F}^{-1}(f_k)\|_Y \lesssim 2^{ks}\|f_k\|_{Z_k}. \tag{3.5}$$

Proof. We assume first that $f_k = f_{k,j}$ with $\|f_k\|_{X_k} = 2^{j/2}\beta_{k,j}\|f_{k,j}\|_{L^2}$ and $f_{k,j}$ is supported in $D_{k,j}$ for some $j \ge 0$. Then we have

$$\begin{aligned} \mathcal{F}^{-1}(f_k)(x,t) &= \int f_{k,j}(\xi,\tau) e^{ix\xi} e^{it\tau} d\xi d\tau \\ &= \int_{\widetilde{I}_j} e^{it\tau} \int f_{k,j}(\xi, \tau + \omega(\xi)) e^{ix\xi} e^{it\omega(\xi)} d\xi d\tau. \end{aligned}$$

From the hypothesis on Y, we obtain

$$\begin{aligned} \|\mathcal{F}^{-1}(f_k)(x,t)\|_Y &\lesssim \int \eta_j(\tau) \Big\| e^{it\tau} \int f_{k,j}(\xi, \tau + \omega(\xi)) e^{ix\xi} e^{it\omega(\xi)} d\xi \Big\|_Y d\tau \\ &\lesssim 2^{ks} 2^{j/2} \|f_{k,j}\|_{L^2}, \end{aligned}$$

which completes the proof in this case.

We assume now that $k \ge 100$ and $f_k = g_k \in Y_k$. From the definition g_k can be written in the form

$$\begin{cases} g_k(\xi,\tau) = 2^{k/2}\chi_{[k-1,k+1]}(\xi)(\tau - \omega(\xi) + i)^{-1}\eta_{\le k}(\tau - \omega(\xi))\mathcal{F}_x h(\xi,\tau); \\ \|g_k\|_{Y_k} = C\|h\|_{L^1_x L^2_\tau}. \end{cases} \tag{3.6}$$

It suffices to prove that if

$$f(\xi,\tau)=2^{k/2}\chi_{[k-1,k+1]}(\xi)(\tau-\omega(\xi)+i)^{-1}\eta_{\leq k}(\tau-\omega(\xi))\cdot h(\tau)$$

then

$$\Big\|\int_{\mathbb{R}^2} f(\xi,\tau)e^{ix\xi}e^{it\tau}d\xi d\tau\Big\|_Y \lesssim 2^{ks}\|h\|_{L^2}, \tag{3.7}$$

which follows from the proof of Lemma 4.2 (b) in [18]. □

In order to obtain the more specific embedding properties of the spaces Z_k, we need the estimate for the free Benjamin-Ono equation. We recall the Strichartz estimates, smoothing effects, and maximal function estimates (for the proof, see, e.g., [29, 28] and the references therein)

Lemma 3.3. *Let $k\in\mathbb{Z}_+$ and $I\subset\mathbb{R}$ be an interval with $|I|\lesssim 1$. Then, for all $\phi\in L^2(\mathbb{R})$ with $\widehat{\phi}$ supported in $\widetilde{I}_k$, we have the following.*

(*a*) *Strichartz estimates: if $2\leq q,r\leq\infty$ and $2/q=1/2-1/r$,*

$$\|e^{-t\mathcal{H}\partial_x^2}\phi\|_{L_t^qL_x^r}\leq C\|\phi\|_{L^2(\mathbb{R})}.$$

(*b*) *Smoothing effect:*

$$\|e^{-t\mathcal{H}\partial_x^2}\phi\|_{L_x^\infty L_t^2}\leq C2^{-k/2}\|\phi\|_{L^2(\mathbb{R})}.$$

(*c*) *Maximal function estimate:*

$$\|e^{-t\mathcal{H}\partial_x^2}\phi\|_{L_x^2L_{t\in I}^\infty}\leq C2^{k/2}\|\phi\|_{L^2(\mathbb{R})},\ \|e^{-t\mathcal{H}\partial_x^2}\phi\|_{L_x^4L_t^\infty}\leq C2^{k/4}\|\phi\|_{L^2(\mathbb{R})}.$$

In particular we note the case $(6,6)$ is admissible which we will use in the sequel. From Lemma 3.2, 3.3, we immediately get the following.

Lemma 3.4. *Let $k\in\mathbb{Z}_+$ and $I\subset\mathbb{R}$ be an interval with $|I|\lesssim 1$. Assume (q,r) is admissible and $f_k\in Z_k$. Then*

$$\|\mathcal{F}^{-1}(f_k)\|_{L_t^qL_x^r}\lesssim\|f_k\|_{Z_k},\ \|\mathcal{F}^{-1}(f_k)\|_{L_x^\infty L_t^2}\lesssim 2^{-k/2}\|f_k\|_{Z_k},$$
$$\|\mathcal{F}^{-1}(f_k)\|_{L_x^4L_t^\infty}\lesssim 2^{k/4}\|f_k\|_{Z_k},\ \|\mathcal{F}^{-1}(f_k)\|_{L_x^2L_{t\in I}^\infty}\lesssim 2^{k/2}\|f_k\|_{Z_k}.$$

As a consequence, $F^s\subseteq C(\mathbb{R};H^s)$ for any $s\in\mathbb{R}$.

Now we turn to studying the properties of the space $F^{l,s}$. The definition shows easily that, if $k\in\mathbb{Z}$ and $f_k\in B_k$, then

$$\Big\|\int_{\mathbb{R}}|f_k(\xi,\tau')|d\tau'\Big\|_{L_\xi^2}\lesssim\|f_k\|_{B_k}. \tag{3.8}$$

Moreover, if $k\in\mathbb{Z}$, $l\in\mathbb{Z}_+$, and $f_k\in B_k$, then

$$\sum_{j=l+1}^{\infty}2^{j/2}\Big\|\eta_j(\tau-\omega(\xi))\cdot\int_{\mathbb{R}}|f_k(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\Big\|_{L^2} \tag{3.9}$$
$$+\,2^{l/2}\Big\|\eta_{\le l}(\tau-\omega(\xi))\cdot\int_{\mathbb{R}}|f_k(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\Big\|_{L^2}\lesssim\|f_k\|_{B_k}.$$

In particular, if $k\in\mathbb{Z}$, $l\in\mathbb{Z}_+$, $t_0\in\mathbb{R}$, $f_k\in B_k$, and $\gamma\in\mathcal{S}(\mathbb{R})$, then

$$\|\mathcal{F}[\gamma(2^l(t-t_0))\cdot\mathcal{F}^{-1}(f_k)]\|_{B_k}\lesssim\|f_k\|_{B_k}. \tag{3.10}$$

Indeed, to prove (3.9), first for the second term on the left-hand side of (3.9), we immediately get from the Cauchy-Schwarz inequality and (3.8) that

$$\begin{aligned}&2^{l/2}\Big\|\eta_{\le l}(\tau-\omega(\xi))\cdot\int_{\mathbb{R}}|f_k(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\Big\|_{L^2}\\ &\lesssim\Big\|\int_{\mathbb{R}}|f_k(\xi,\tau')|d\tau'\Big\|_{L^2_\xi}\lesssim\|f_k\|_{B_k}.\end{aligned}$$

For the first term on the left-hand side of (3.9), we decompose $f_k(\xi,\tau')=\sum_{j_1\ge 0}f_{k,j_1}$ where $f_{k,j_1}=f_k(\xi,\tau')\eta_{j_1}(\tau'-\omega(\xi))$, and then we get

$$\begin{aligned}&\sum_{j=l+1}^{\infty}2^{j/2}\left\|\eta_j(\tau-\omega(\xi))\cdot\int_{\mathbb{R}}|f_k(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\right\|_{L^2}\\ &\lesssim\sum_{j=l+1}^{\infty}\sum_{j_1=0}^{\infty}2^{j/2}\left\|\eta_j(\tau-\omega(\xi))\int_{\mathbb{R}}|f_{k,j_1}(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\right\|_{L^2}\\ &=\sum_{j=l+1}^{\infty}\Big(\sum_{j_1>j+5}+\sum_{j_1<j-5}+\sum_{j-5\le j_1\le j+5}\Big)\\ &\qquad 2^{j/2}\left\|\eta_j(\tau-\omega(\xi))\int_{\mathbb{R}}|f_{k,j_1}(\xi,\tau')|\cdot 2^{-l}(1+2^{-l}|\tau-\tau'|)^{-4}d\tau'\right\|_{L^2}\\ &=:I+II+III.\end{aligned}$$

For the contribution of I, we first observe that $|\tau-\tau'|\sim 2^{j_1}$. Then

$$I\lesssim\sum_{j_1\ge l}\sum_{j\le j_1}2^{j}2^{3l}2^{-4j_1}\left\|\int_{\mathbb{R}}|f_{k,j_1}(\xi,\tau')|d\tau'\right\|_{L^2_\xi}\le\|f_k\|_{B_k}.$$

Similarly we can estimate II. For the third term III, using Young's inequality, we then get

$$III \lesssim \sum_{j=l+1}^{\infty} \sum_{|j-j_1|\leq 5} 2^{j/2}\|f_{k,j_1}\|_{L^2} \lesssim \|f_k\|_{B_k}.$$

As in [20], for any $k \in \mathbb{Z}$ we define the set S_k of $k-$*acceptable* time multiplication factors

$$S_k = \{m_k : \mathbb{R} \to \mathbb{R} : \|m_k\|_{S_k} = \sum_{j=0}^{10} 2^{-jk_+}\|\partial^j m_k\|_{L^\infty} < \infty\}. \tag{3.11}$$

For instance, $\eta(2^{k_+}t) \in S_k$ for any η that satisfies $\|\partial_x^k \eta\|_{L^\infty} \leq C$ for $j = 1, 2, \cdots, 10$. Direct estimates using the definitions of $F^{l,s}(T)$ and (3.9) show that for any $s \geq 0$ and $T \in (0, 1]$

$$\begin{cases} \|\sum_{k\in\mathbb{Z}} m_k(t) \cdot R_k(u)\|_{F^{l,s}(T)} \lesssim (\sup_{k\in\mathbb{Z}} \|m_k\|_{S_k}) \cdot \|u\|_{F^{l,s}(T)}; \\ \|\sum_{k\in\mathbb{Z}} m_k(t) \cdot R_k(u)\|_{N^{l,s}(T)} \lesssim (\sup_{k\in\mathbb{Z}} \|m_k\|_{S_k}) \cdot \|u\|_{N^{l,s}(T)}; \\ \|\sum_{k\in\mathbb{Z}} m_k(t) \cdot R_k(u)\|_{E^{l,s}(T)} \lesssim (\sup_{k\in\mathbb{Z}} \|m_k\|_{S_k}) \cdot \|u\|_{E^{l,s}(T)}. \end{cases} \tag{3.12}$$

Actually, for instance we show the first inequality in (3.12). In view of the definition, it suffices to prove that, if $u_k \in F_k$, then

$$\|m_k(t)u_k\|_{F_k} \lesssim \|u_k\|_{F_k}\|m_k\|_{S_k}, \quad \forall\, k \in \mathbb{Z}.$$

From (3.10) we see that we only need to prove that

$$\left|\mathcal{F}\left[m_k(\cdot)\eta_0(2^{k_+}(\cdot - t_k))\right]\right| \lesssim 2^{-k_+}(1 + 2^{-k_+}|\tau|)^{-4}\|m_k\|_{S_k},$$

which follows from partial integration and the definition of S_k.

4. A Symmetric Estimate

In this section we prove a symmetric estimate which will be used to prove a trilinear estimate. For $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ and $\omega : \mathbb{R} \to \mathbb{R}$ as in (2.1) let

$$\Omega(\xi_1, \xi_2, \xi_3) = \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) - \omega(\xi_1 + \xi_2 + \xi_3). \tag{4.1}$$

This is the resonance function that plays a crucial role in the trilinear estimate of the $X^{s,b}$-type space. See [42] for a perspective discussion. For compactly supported functions $f, g, h, u \in L^2(\mathbb{R} \times \mathbb{R})$ let

$$\begin{aligned} J(f, g, h, u) = \int_{\mathbb{R}^6} & f(\xi_1, \mu_1)g(\xi_2, \mu_2)h(\xi_3, \mu_3) \\ & u(\xi_1 + \xi_2 + \xi_3, \mu_1 + \mu_2 + \mu_3 + \Omega(\xi_1, \xi_2, \xi_3))d\xi_1 d\xi_2 d\xi_3 d\mu_1 d\mu_2 d\mu_3. \end{aligned}$$

Lemma 4.1. *Assume $k_i \in \mathbb{Z}$, $j_i \in \mathbb{Z}_+$, and $f_{k_i,j_i} \in L^2(\mathbb{R}^2)$ are nonnegative functions supported in $I_{k_i} \times \cup_{l=0}^{j_i} \widetilde{I}_l$, $1 \le i \le 4$, $k_1 \le k_2 \le k_3 \le k_4$. For simplicity we write $J = J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})$. Then*

(*a*) *For any $k_1 \le k_2 \le k_3 \le k_4$ and $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$,*

$$J \lesssim 2^{(j_{min}+j_{thd})/2} 2^{(k_{min}+k_{thd})/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}.$$

(*b*) *If $k_2 \le k_3 - 5$ and $j_2 \neq j_{max}$,*

$$J \lesssim 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} 2^{-k_{max}/2} 2^{k_{min}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}; \tag{4.2}$$

if $k_2 \le k_3 - 5$ and $j_2 = j_{max}$,

$$J \lesssim 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} 2^{-k_{max}/2} 2^{k_{thd}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}. \tag{4.3}$$

(*c*) *For any $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ and $j_1, j_2, j_3, j_4 \in \mathbb{Z}_+$,*

$$J \lesssim 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}. \tag{4.4}$$

(*d*) *If $k_{min} \le k_{max} - 10$, then*

$$J \lesssim 2^{(j_1+j_2+j_3+j_4)/2} 2^{-k_{max}} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}. \tag{4.5}$$

Proof. Let $A_{k_i}(\xi) = [\int_{\mathbb{R}} |f_{k_i,j_i}(\xi,\mu)|^2 d\mu]^{1/2}$, $i = 1,2,3,4$. Using the Cauchy-Schwartz inequality and the support properties of the functions f_{k_i,j_i},

$$\begin{aligned}
&|J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})| \\
\le& C 2^{(j_{min}+j_{thd})/2} \int_{\mathbb{R}^3} A_{k_1}(\xi_1) A_{k_2}(\xi_1) A_{k_3}(\xi_1) A_{k_4}(\xi_1+\xi_2+\xi_3) d\xi_1 d\xi_2 d\xi_3 \\
\le& C 2^{(k_{min}+k_{thd})/2} 2^{(j_{min}+j_{thd})/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2},
\end{aligned}$$

which is part (a), as desired.

For part (b), in view of the support properties of the functions, it is easy to see that $J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4}) \equiv 0$ unless

$$k_4 \le k_3 + 5. \tag{4.6}$$

Simple changes of variables in the integration and the observation that the function ω is odd show that

$$|J(f,g,h,u)| = |J(g,f,h,u)| = |J(f,h,g,u)| = |J(\widetilde{f},g,u,h)|,$$

where $\widetilde{f}(\xi,\mu) = f(-\xi,-\mu)$. We assume first that $j_2 \neq j_{max}$. Then we have several cases: if $j_4 = j_{max}$, then we will prove that, if $g_i : \mathbb{R} \to \mathbb{R}_+$ are L^2 functions supported in I_{k_i}, $i = 1,2,3$, and $g : \mathbb{R}^2 \to \mathbb{R}_+$ is an L^2 function supported in $I_{k_4} \times \widetilde{I}_{j_4}$, then

$$\begin{aligned}&\int_{\mathbb{R}^3} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1+\xi_2+\xi_3,\Omega(\xi_1,\xi_2,\xi_3))d\xi_1 d\xi_2 d\xi_3\\ \lesssim& 2^{-k_{max}/2}2^{k_{min}/2}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}.\end{aligned}\tag{4.7}$$

This suffices for (4.2).

To prove (4.7), we first observe that since $k_2 \leq k_3 - 5$ then $|\xi_3+\xi_2| \sim |\xi_3|$. By a change of variables $\xi_1' = \xi_1$, $\xi_2' = \xi_2$, $\xi_3' = \xi_3+\xi_2$, we get that the left side of (4.7) is bounded by

$$\begin{aligned}&\int_{|\xi_1|\sim 2^{k_1},|\xi_2|\sim 2^{k_2},|\xi_3|\sim 2^{k_3}} g_1(\xi_1)g_2(\xi_2)\\ &\cdot g_3(\xi_3-\xi_2)g(\xi_1+\xi_3,\Omega(\xi_1,\xi_2,\xi_3-\xi_2))d\xi_1 d\xi_2 d\xi_3.\end{aligned}\tag{4.8}$$

Note that in the integration area we have

$$\left|\partial_{\xi_2}\left[\Omega(\xi_1,\xi_2,\xi_3-\xi_2)\right]\right| = |\omega'(\xi_2)-\omega'(\xi_3-\xi_2)| \sim 2^{k_3},$$

where we use the facts that $\omega'(\xi) = |\xi|$ and $k_2 \leq k_3 - 5$. By a change of variables $\mu_2 = \Omega(\xi_1,\xi_2,\xi_3-\xi_2)$, we get that (4.8) is bounded by

$$\begin{aligned}&2^{-k_3/2}\int_{|\xi_1|\sim 2^{k_1}} g_1(\xi_1)\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}d\xi_1\\ \lesssim& 2^{-k_{max}/2}2^{k_{min}/2}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g_1\|_{L^2}\|g\|_{L^2}.\end{aligned}\tag{4.9}$$

If $j_3 = j_{max}$, this case is identical to the case $j_4 = j_{max}$ in view of (4.6). If $j_1 = j_{max}$ it suffices to prove that if $g_i : \mathbb{R} \to \mathbb{R}_+$ are L^2 functions supported in I_{k_i}, $i = 2,3,4$, and $g : \mathbb{R}^2 \to \mathbb{R}_+$ is an L^2 function supported in $I_{k_1} \times \widetilde{I}_{j_1}$, then

$$\begin{aligned}&\int_{\mathbb{R}^3} g_2(\xi_2)g_3(\xi_3)g_4(\xi_4)g(\xi_2+\xi_3+\xi_4,\Omega(\xi_2,\xi_3,\xi_4))d\xi_2 d\xi_3 d\xi_4\\ \lesssim& 2^{-k_{max}/2}2^{k_{min}/2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g_4\|_{L^2}\|g\|_{L^2}.\end{aligned}\tag{4.10}$$

Indeed, by a change of variables $\xi_2'=\xi_2,\xi_3'=\xi_3,\xi_4'=\xi_2+\xi_3+\xi_4$ and noting that in the area $|\xi_2'|\sim 2^{k_2},|\xi_3'|\sim 2^{k_3},|\xi_4'|\sim 2^{k_1}$,

$$\left|\partial_{\xi_2'}\left[\Omega(\xi_2',\xi_3',\xi_4'-\xi_2'-\xi_3')\right]\right|=|\omega'(\xi_2')-\omega'(\xi_4'-\xi_2'-\xi_3')|\sim 2^{k_3},$$

we get from the Cauchy-Schwarz inequality that

$$\begin{aligned}&\int_{\mathbb{R}^3} g_2(\xi_2)g_3(\xi_3)g_4(\xi_4)g(\xi_2+\xi_3+\xi_4,\Omega(\xi_2,\xi_3,\xi_4))d\xi_2 d\xi_3 d\xi_4\\ \lesssim&\int_{|\xi_2'|\sim 2^{k_2},|\xi_3'|\sim 2^{k_3},|\xi_4'|\sim 2^{k_1}} g_2(\xi_2')g_3(\xi_3')\\ &\quad\cdot g_4(\xi_4'-\xi_2'-\xi_3')g(\xi_4',\Omega(\xi_2',\xi_3',\xi_4'-\xi_2'-\xi_3'))d\xi_2'd\xi_3'd\xi_4'\\ \lesssim&2^{-k_3/2}\int_{|\xi_3'|\sim 2^{k_3},|\xi_4'|\sim 2^{k_1}} g_3(\xi_3')\|g_2(\xi_2')g_4(\xi_4'-\xi_2'-\xi_3')\|_{L^2_{\xi_2'}}\|g(\xi_4',\cdot)\|_{L^2_{\xi_2'}}d\xi_3'd\xi_4'\\ \lesssim&2^{-k_{max}/2}2^{k_{min}/2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g_4\|_{L^2}\|g\|_{L^2}. \end{aligned}\tag{4.11}$$

We assume now that $j_2=j_{max}$. This case is identical to the case $j_1=j_{max}$. We note that we actually prove that if $k_2\le k_3-5$ then

$$J\le C2^{(j_1+j_2+j_3+j_4)/2}2^{-j_{sub}/2}2^{-k_{max}/2}2^{k_{min}/2}\prod_{i=1}^4\|f_{k_i,j_i}\|_{L^2}. \tag{4.12}$$

Therefore, we have completed the proof for part (b).

For part (c), setting $f^\sharp_{k_i,j_i}=f_{k_i,j_i}(\xi,\tau-\omega(\xi))$, $i=1,2,3$, we then get

$$\begin{aligned}&|J(f_{k_1,j_1},f_{k_2,j_2},f_{k_3,j_3},f_{k_4,j_4})|=|\int f^\sharp_{k_1,j_1}*f^\sharp_{k_2,j_2}*f^\sharp_{k_3,j_3}\cdot f^\sharp_{k_4,j_4}|\\ \lesssim&\|f^\sharp_{k_1,j_1}*f^\sharp_{k_2,j_2}*f^\sharp_{k_3,j_3}\|_{L^2}\|f^\sharp_{k_4,j_4}|\|_{L^2}\lesssim\prod_{i=1}^3\|\mathcal{F}^{-1}(f^\sharp_{k_i,j_i})\|_{L^6}\|f_{k_4,j_4}|\|_{L^2}.\end{aligned}$$

On the other hand, from

$$\begin{aligned}\mathcal{F}^{-1}(f^\sharp_{k_1,j_1})&=\int f_{k_i,j_i}(\xi,\tau-\omega(\xi))e^{ix\xi}e^{it\tau}d\xi d\tau\\ &=\int f_{k_i,j_i}(\xi,\tau)e^{ix\xi}e^{it\omega(\xi)}e^{it\tau}d\xi d\tau,\end{aligned}$$

it then follows from Lemma 3.3 (a) that

$$\|\mathcal{F}^{-1}(f^\sharp_{k_1,j_1})\|_{L^6}\lesssim\int_{\mathbb{R}}\left\|\int_{\mathbb{R}} f_{k_i,j_i}(\xi,\tau)e^{ix\xi}e^{it\omega(\xi)}d\xi\right\|_{L^6}d\tau\lesssim 2^{j_i/2}\|f_{k_i,j_i}\|_{L^2}.$$

Thus part (c) follows from the symmetry.

For part (d), we only need to consider the worst cases $\xi_1\cdot\xi_2<0$ and $k_2\leq k_3-5$. Indeed in the other cases we get (4.5) from the fact that $|\Omega(\xi_1,\xi_2,\xi_3)|\gtrsim 2^{k_2+k_3}$, which implies that $j_{max}\geq k_2+k_3-20$ by checking the support properties. Thus (d) follows from (b) and (c) in these cases. We assume now $\xi_1\cdot\xi_2<0$ and $k_2\leq k_3-5$. If $j_4=j_{max}$, it suffices to prove that if g_i is an L^2 nonnegative function supported in I_{k_i}, $i=1,2,3$, and g is an L^2 nonnegative function supported in $I_{k_4}\times\widetilde{I}_{j_4}$, then

$$\begin{aligned}&\int_{\mathbb{R}^3\cap\{\xi_1\cdot\xi_2<0\}} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1+\xi_2+\xi_3,\Omega(\xi_1,\xi_2,\xi_3))d\xi_1d\xi_2d\xi_3\\ &\lesssim 2^{j_4/2}2^{-k_3}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}.\end{aligned}\tag{4.13}$$

By localizing $|\xi_1+\xi_2|\sim 2^l$ for $l\in\mathbb{Z}$, we get that the right-hand side of (4.13) is bounded by

$$\sum_l\int_{\mathbb{R}^3}\chi_l(\xi_1+\xi_2)g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1+\xi_2+\xi_3,\Omega(\xi_1,\xi_2,\xi_3))d\xi_1d\xi_2d\xi_3.\tag{4.14}$$

From the support properties of the functions g_i, g and the fact that in the integration area $|\Omega(\xi_1,\xi_2,\xi_3)|=(\xi_1+\xi_2)(\xi_1+\xi_3)\sim 2^{l+k_3}$, we get that

$$j_{max}\geq l+k_3-20.\tag{4.15}$$

By changing the variables of integration $\xi_1'=\xi_1+\xi_2$, $\xi_2'=\xi_2$, $\xi_3'=\xi_1+\xi_3$, we obtain that (4.14) is bounded by

$$\begin{aligned}&\sum_l\int_{|\xi_1'|\sim 2^l,|\xi_2'|\sim 2^{k_2},|\xi_3'|\sim 2^{k_3}}\chi_l(\xi_1')g_1(\xi_1'-\xi_2')g_2(\xi_2')g_3(\xi_2'+\xi_3'-\xi_1')\\ &\quad g(\xi_2'+\xi_3',\Omega(\xi_1'-\xi_2',\xi_2',\xi_2'+\xi_3'-\xi_1'))d\xi_1'd\xi_2'd\xi_3'.\end{aligned}\tag{4.16}$$

Since in the integration area

$$\left|\partial_{\xi_1'}[\Omega(\xi_1'-\xi_2',\xi_2',\xi_2'+\xi_3'-\xi_1')]\right|\sim 2^{k_3},\tag{4.17}$$

we then get from (4.17) that (4.16) is bounded by

$$\begin{aligned}&\sum_l\int_{|\xi_1'|\sim 2^l}\|g_1\|_2\|g_3\|_2\|g_2(\xi_2')g(\xi_2'+\xi_3',\Omega(\xi_1'-\xi_2',\xi_2',\xi_2'+\xi_3'-\xi_1'))\|_{L^2_{\xi_2',\xi_3'}}d\xi_1'\\ &\quad\lesssim\sum_l 2^{l/2}2^{-k_3/2}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}\\ &\quad\lesssim 2^{j_{max}/2}2^{-k_3}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2},\end{aligned}$$

where we used (4.15) in the last inequality.

From symmetry we know the case $j_3 = j_{max}$ is identical to the case $j_4 = j_{max}$, and the case $j_1 = j_{max}$ is identical to the case $j_2 = j_{max}$, thus it reduces to proving the case $j_2 = j_{max}$. It suffices to prove that if g_i is an L^2 nonnegative function supported in I_{k_i}, $i = 1, 3, 4$, and g is an L^2 nonnegative function supported in $I_{k_2} \times \widetilde{I}_{j_2}$, then

$$\begin{aligned}&\int_{\mathbb{R}^3 \cap \{\xi_1 \cdot \xi_2 < 0\}} g_1(\xi_1) g_3(\xi_3) g_4(\xi_4) g(\xi_1 + \xi_3 + \xi_4, \Omega(\xi_1, \xi_3, \xi_4)) d\xi_1 d\xi_3 d\xi_4 \\ &\lesssim 2^{j_2/2} 2^{-k_3} \|g_1\|_{L^2} \|g_4\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2}. \end{aligned} \tag{4.18}$$

As in the case $j_4 = j_{max}$, we get that the right-hand side of (4.18) is bounded by

$$\sum_l \int_{\mathbb{R}^3} \chi_l(\xi_3 + \xi_4) g_1(\xi_1) g_4(\xi_4) g_3(\xi_3) g(\xi_1 + \xi_4 + \xi_3, \Omega(\xi_1, \xi_3, \xi_4)) d\xi_1 d\xi_4 d\xi_3. \tag{4.19}$$

From the support properties of the functions g_i, g, and

$$|\Omega(\xi_1, \xi_2, \xi_3)| = |(\xi_1 + \xi_4)(\xi_4 + \xi_3)| \sim 2^{l+k_3},$$

we get that $j_{max} \geq l + k_3 - 20$. By changing the variables of integration $\xi_1' = \xi_1 + \xi_3$, $\xi_3' = \xi_3 + \xi_4$, $\xi_4' = \xi_1 + \xi_3 + \xi_4$, we obtain that (4.19) is bounded by

$$\begin{aligned}&\sum_l \int_{|\xi_3'| \sim 2^l, |\xi_4'| \sim 2^{k_2}, |\xi_1'| \sim 2^{k_3}} \chi_l(\xi_3') g_1(\xi_4' - \xi_3') g_3(\xi_1' + \xi_3' - \xi_4') g_4(\xi_4' - \xi_1') \\ &\quad g(\xi_4', \Omega(\xi_4' - \xi_3', \xi_1' + \xi_3' - \xi_4', \xi_4' - \xi_1')) d\xi_1' d\xi_3' d\xi_4'. \end{aligned} \tag{4.20}$$

Since, in the integration area,

$$\left| \partial_{\xi_3'} [\Omega(\xi_4' - \xi_3', \xi_1' + \xi_3' - \xi_4', \xi_4' - \xi_1')] \right| \sim 2^{k_3}, \tag{4.21}$$

we then get from (4.21) that (4.20) is bounded by

$$\begin{aligned}&\sum_l \int_{|\xi_3'| \sim 2^l} \chi_l(\xi_3') \|g_1\|_{L^2} \|g_3\|_{L^2} \\ &\quad \|g_4(\xi_4' - \xi_1') g(\xi_4', \Omega(\xi_4' - \xi_3', \xi_1' + \xi_3' - \xi_4', \xi_4' - \xi_1'))\|_{L^2_{\xi_1', \xi_4'}} d\xi_3' \\ &\lesssim \sum_l 2^{l/2} 2^{-k_3/2} \|g_1\|_{L^2} \|g_3\|_{L^2} \|g_4\|_{L^2} \|g\|_{L^2} \\ &\lesssim 2^{j_{max}/2} 2^{-k_3} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2}. \end{aligned} \tag{4.22}$$

Thus we complete the proof of (d). □

Remark 4.2. From the proof, we see that Lemma 4.1 also holds if $k_i \in \mathbb{Z}_+$, with I_{k_i} replaced by $\widetilde{I}_{k_i}$. It also holds for general dispersive functions $\omega(\xi) \in D_{hi}(2)$. The arguments work for general functions $\omega \in D_{hi}(\beta)$ for some $\beta > 1$.

5. Trilinear Estimates

This section is devoted to proving some dyadic trilinear estimates, using the symmetric estimates obtained in the last section.

Proposition 5.1. *Let $k_1, k_2, k_3, k_4 \in \mathbb{Z}_+$. Assume $f_{k_i} \in Z_{k_i}$ with $\mathcal{F}^{-1}(f_{k_i})$ compactly supported (in time) in J_0, $|J_0| \lesssim 1$, $i = 1, 2, 3$. Denote*

$$N(k_1, k_2, k_3, k_4) = 2^{k_4} \|\eta_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}}.$$

Then we have the following.

(*a*) *If $k_3 \geq 110$, $|k_4 - k_3| \leq 5$, $0 \leq k_1, k_2 \leq k_3 - 10$, $|k_1 - k_2| \leq 10$, then*

$$N(k_1, k_2, k_3, k_4) \lesssim 2^{(k_1+k_2)/2} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}. \tag{5.1}$$

(*b*) *If $k_3 \geq 110$, $|k_4 - k_3| \leq 5$, $0 \leq k_1, k_2 \leq k_3 - 10$, $k_2 \geq 10$, $k_1 \leq k_2 - 5$, then*

$$N(k_1, k_2, k_3, k_4) \lesssim 2^{(k_1+k_2)/4} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}. \tag{5.2}$$

(*c*) *If $k_3 \geq 110$, $|k_4 - k_3| \leq 5$, $k_3 - 10 \leq k_2 \leq k_3$, $0 \leq k_1 \leq k_2 - 10$, then*

$$N(k_1, k_2, k_3, k_4) \lesssim 2^{(k_1+k_2)/4} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}. \tag{5.3}$$

(*d*) *If $k_3 \geq 110$, $|k_4 - k_3| \leq 5$, $k_3 - 30 \leq k_1$, $k_2 \leq k_3$, then*

$$N(k_1, k_2, k_3, k_4) \lesssim 2^{k_3/2} \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}. \tag{5.4}$$

(*e*) *If $k_1 \geq 110$, $|k_1 - k_2| \leq 5$, $0 \leq k_3 \leq k_1 + 10$, $0 \leq k_4 \leq k_1 - 10$, then*

$$N(k_1, k_2, k_3, k_4) \lesssim k_1^4 \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}. \tag{5.5}$$

(*f*) *If* $0 \leq k_1, k_2, k_3, k_4 \leq 120$, *then*

$$N(k_1, k_2, k_3, k_4) \lesssim \prod_{i=1}^{3} \|f_{k_i}\|_{Z_{k_i}}. \tag{5.6}$$

Proof. First we prove (a). We divide it into three parts, according to the modulation.

$$\begin{aligned}
&2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
\leq&2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
&+2^{k_4}\|\chi_{k_4}(\xi)\eta_{[k_4,2k_4]}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
&+2^{k_4}\|\chi_{k_4}(\xi)\eta_{\geq 2k_4+1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
=&I+II+III.
\end{aligned}$$

We consider first the contribution of I. Using the Y_k norm, we then get from Lemma 3.1 (a), (c) and Lemma 3.4 that

$$\begin{aligned}
I\leq&2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Y_{k_4}}\\
\lesssim&2^{k_4/2}\|\mathcal{F}^{-1}[f_{k_1}*f_{k_2}*f_{k_3}]\|_{L_x^1L_t^2}\\
\lesssim&2^{k_4/2}\|\mathcal{F}^{-1}(f_{k_3})\|_{L_x^\infty L_t^2}\|\mathcal{F}^{-1}(f_{k_2})\|_{L_x^2L_{t\in I_0}^\infty}\|\mathcal{F}^{-1}(f_{k_1})\|_{L_x^2L_{t\in I_0}^\infty}\\
\lesssim&2^{(k_1+k_2)/2}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}},
\end{aligned}$$

which is (5.1) as desired.

For the contribution of II, we use the X_k norm. We then get from Lemma 3.4 that

$$\begin{aligned}
II\leq&2^{k_4}\|\chi_{k_4}(\xi)\eta_{[k_4,2k_4]}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\
\leq&\sum_{k_4\leq j\leq 2k_4}2^{k_4}2^{-j/2}\|1_{D_{k_4,j}}(\xi,\tau)f_{k_1}*f_{k_2}*f_{k_3}\|_{L^2}\\
\leq&\sum_{k_4\leq j\leq 2k_4}2^{k_4}2^{-j/2}\|\mathcal{F}^{-1}(f_{k_3})\|_{L_x^\infty L_t^2}\|\mathcal{F}^{-1}(f_{k_1})\|_{L_x^4L_t^\infty}\|\mathcal{F}^{-1}(f_{k_2})\|_{L_x^4L_t^\infty}\\
\lesssim&2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}},
\end{aligned}\tag{5.7}$$

which is acceptable.

Finally, we consider the contribution of III. For $j_i \geq 0$, $i = 1, 2, 3$, let $f_{k_i,j_i}(\xi,\tau) = f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$. Using the X_k norm, we get

$$III \leq \sum_{j_4\geq 2k_4+1}\sum_{j_1,j_2,j_3\geq 0} \|1_{D_{k_4,j_4}}(\xi,\tau) f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}\|_{L^2}. \tag{5.8}$$

Since, in the area $\{|\xi_i| \in \widetilde{I}_{k_i}, i = 1, 2, 3\}$, we have $|\Omega(\xi_1,\xi_2,\xi_3)| \ll 2^{2k_4}$. By checking the support properties of f_{k_i,j_i}, we get $|j_{max} - j_{sub}| \leq 5$. We consider only the worst case $|j_4 - j_3| \leq 5$, since the other cases are better. It follows from Lemma 4.1 and Lemma 3.1 (b) that

$$\begin{aligned}
III \lesssim & \sum_{j_3\geq 2k_4+1}\sum_{j_1,j_2\geq 0} 2^{(j_1+j_2)/2} 2^{(k_1+k_2)/2} \|f_{k_1,j_1}\|_{L^2}\|f_{k_2,j_2}\|_{L^2}\|f_{k_3,j_3}\|_{L^2}\\
\lesssim & \sum_{j_3\geq 2k_4+1} 2^{j_3/4} 2^{k_3-j_3} 2^{(k_1+k_2)/2} 2^{j_3-k_3} \|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}\|f_{k_3,j_3}\|_{L^2}\\
\lesssim & \sum_{j_3\geq 2k_4+1} 2^{k_3-\frac{3}{4}j_3} 2^{(k_1+k_2)/2} \|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}\|f_{k_3}\|_{Z_{k_3}}\\
\lesssim & 2^{(k_1+k_2)/4} \|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}\|f_{k_3}\|_{Z_{k_3}}.
\end{aligned} \tag{5.9}$$

Therefore, we have completed the proof of part (a).

Next we prove (b). We first observe that in this case we have

$$|\Omega(\xi_1,\xi_2,\xi_3)| \sim 2^{k_3+k_2}, \tag{5.10}$$

which follows from the fact that $\xi_1+\xi_2+\xi_3$ has the same sign as ξ_3, $k_1 \leq k_2 - 10$ and in the area $\{\xi_i \in \widetilde{I}_{k_i} : i = 1, 2, 3\}$, $|\omega(\xi_3) - \omega(\xi_1+\xi_2+\xi_3)| \sim 2^{k_3+k_2}$. Dividing it into three parts as before, we obtain

$$\begin{aligned}
& 2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}}\\
\leq & 2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}}\\
& + 2^{k_4}\|\chi_{k_4}(\xi)\eta_{[k_4,2k_4]}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}}\\
& + 2^{k_4}\|\chi_{k_4}(\xi)\eta_{\geq 2k_4+1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_4}}\\
= & I + II + III.
\end{aligned}$$

For the last two terms II, III, we can use the same argument as for II, III in the proof of part (a). We consider now the first term I.

$$\begin{aligned}
I \leq & 2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}^h\|_{Z_{k_4}}\\
& + 2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1} f_{k_1} * f_{k_2} * f_{k_3}^l\|_{Z_{k_4}}\\
:= & I_1 + I_2,
\end{aligned}$$

where $f_{k_3}^h = f_{k_3}(\xi,\tau)\eta_{\geq k+k_2-10}(\tau-\omega(\xi))$, $f_{k_3}^l = f_{k_3}(\xi,\tau)\eta_{\leq k+k_2-9}(\tau-\omega(\xi))$.

For the contribution of I_1, we observe first that from the support of $f_{k_3}^h$ and the definition of Y_k, one easily gets that $\|f_{k_3}^h\|_{X_{k_3}}\lesssim\|f_{k_3}\|_{Z_{k_3}}$. Thus, from the definition of Y_k, and from Hölder's inequality, Lemma 3.1 (a), (c) and Lemma 3.4, we get

$$\begin{aligned}
I_1&\lesssim 2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}^h\|_{Y_{k_4}}\\
&\lesssim 2^{k_4/2}\|\mathcal{F}^{-1}[f_{k_1}*f_{k_2}*f_{k_3}^h]\|_{L_x^1L_t^2}\\
&\lesssim 2^{k_4/2}\|\mathcal{F}^{-1}(f_{k_3}^h)\|_{L_x^2L_t^2}\|\mathcal{F}^{-1}(f_{k_1})\|_{L_x^4L_t^\infty}\|\mathcal{F}^{-1}(f_{k_2})\|_{L_x^4L_t^\infty}\\
&\lesssim 2^{k_4/2}2^{(k_1+k_2)/4}\|f_{k_3}^h\|_{L^2}\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}.
\end{aligned}$$

Then from the fact that

$$2^{k_4/2}\|f_{k_3}^h\|_{L^2}\lesssim\sum_{j\geq k_4+k_2-10}2^{k_4/2}\|f_{k_3}^h\eta_j(\tau-\omega(\xi))\|_{L^2}\lesssim\|f_{k_3}^h\|_{X_{k_3}}\lesssim\|f_{k_3}\|_{Z_{k_3}}$$

we conclude the proof for I_1.

We consider now the contribution of I_2. Let $f_{k_i,j_i}(\xi,\tau)=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i\geq 0$, $i=1,2,3$. Using the X_k norm, we get

$$I_2\leq\sum_{j_4\leq k_4-1}\sum_{j_3\leq k_4+k_2-9}\sum_{j_1,j_2\geq 0}2^{k_4}2^{-j_4/2}\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}.$$

By checking the support properties of the functions f_{k_i,j_i} $(i=1,2,3)$ and from (5.10), we get that $1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\equiv 0$ unless

$$\begin{cases} j_1,j_2\geq k_3+k_2-10, |j_1-j_2|\leq 5;\ \text{or}\\ |j_1-k_3-k_2|\leq 5, j_2\leq k_3+k_2-10;\ \text{or switch } j_1,j_2.\end{cases}$$

For the first case, it follows from Lemma 4.1 (b) and Lemma 3.1 (b) that

$$\begin{aligned}
I_2&\lesssim\sum_{j_3,j_4\leq 2k_4}\sum_{j_1,j_2\geq 0}2^{k_4}2^{-j_4/2}2^{(j_1+j_2+j_3+j_4)/2}2^{-j_1/2}2^{-k_3/2}2^{k_1/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_2\\
&\lesssim\sum_{j_1,j_2\geq 0}k_4^2 2^{j_2/2}2^{k_3/2}2^{k_1/2}\prod_{i=1}^{2}\|f_{k_i,j_i}\|_2\|f_{k_3}\|_{Z_{k_3}}\\
&\lesssim\sum_{j_1,j_2\geq 0}k_4^2 2^{j_2/2}2^{k_3/2}2^{k_1/2}2^{k_1+k_2-j_1-j_2}\prod_{i=1}^{2}(2^{j_i-k_i}\|f_{k_i,j_i}\|_2)\|f_{k_3}\|_{Z_{k_3}}
\end{aligned}$$

$$\lesssim 2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}. \tag{5.11}$$

The second case follows from Lemma 4.1 (b). Therefore, we have completed the proof of part (b).

Next we prove part (c). We first observe that this case corresponds to an integration in the area $\{|\xi_i|\in I_{k_i},\ i=1,2,3\}\cap\{|\xi_1+\xi_2+\xi_3|\in I_{k_4}\}$, where we have $|\Omega(\xi_1,\xi_2,\xi_3)|\sim 2^{2k_3}$. Let $f_{k_i,j_i}(\xi,\tau)=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i\geq 0$ and $i=1,2,3$. Using the X_k norm, we get

$$\begin{aligned}&2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}} \\ \lesssim&\sum_{j_1,j_2,j_3,j_4\geq 0}2^{k_4}2^{-j_4/2}(1+2^{(j_4-2k_4)/2})\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}.\end{aligned} \tag{5.12}$$

From the support properties of the functions f_{k_i,j_i}, $i=1,2,3$, it is easy to see that $1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\equiv 0$ unless

$$\begin{cases} j_{max},j_{sub}\geq 2k_3-10, |j_{max}-j_{sub}|\leq 5;\ \text{or}\\ |j_{max}-2k_3|\leq 5, j_{sub}\leq 2k_3-10.\end{cases}$$

Case 1. $j_{max},j_{sub}\geq 2k_3-10, |j_{max}-j_{sub}|\leq 5$.

By Lemma 4.1 (a) we get that the right-hand side of (5.12) is bounded by

$$\sum_{j_i\geq 0}2^{k_4}2^{(j_1+j_2+j_3)/2}(1+2^{(j_4-2k_4)/2})2^{-(j_{sub}+j_{max})/2}2^{(k_1+k_2)/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_2. \tag{5.13}$$

It suffices to consider the worst case $j_3,j_4=j_{max},j_{sub}$. We get from Lemma 3.1 (b) that (5.13) is bounded by

$$\sum_{j_3\geq 2k_3-10}2^{k_4}2^{-\frac{3}{4}j_3}2^{(k_1+k_2)/2}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}\lesssim 2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}.$$

Case 2. $|j_{max}-2k_3|\leq 5, j_{sub}\leq 2k_3-10$.

By Lemma 4.1 (c) we get that the right-hand side of (5.12) is bounded by

$$\sum_{j_1,j_2,j_3,j_4\geq 0}2^{k_4}2^{(j_1+j_2+j_3)/2}2^{-j_{max}/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_2\lesssim 2^{(k_1+k_2)/4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}},$$

where we used Lemma 3.1 (b). Thus, we have completed the proof of part (c).

Next we prove part (d). First we divide it into two parts.

$$\begin{aligned}&2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ \lesssim&2^{k_4}\|\chi_{k_4}(\xi)\eta_{\leq 2k_4+20}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ &+2^{k_4}\|\chi_{k_4}(\xi)\eta_{\geq 2k_4+21}(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ :=&I+II.\end{aligned}$$

For the first term I, using the X_k norm and Lemma 3.4, we then get

$$\begin{aligned}I\lesssim&2^{k_4}\sum_{j_4\geq 0}^{2k_4}2^{-j_4/2}\|1_{D_{k_4,j_4}}f_{k_1}*f_{k_2}*f_{k_3}\|_{L^2}\\ \lesssim&2^{k_4}\prod_{i=1}^{3}\|\mathcal{F}^{-1}(f_{k_i})\|_{L^6}\lesssim 2^{k_4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}.\end{aligned}$$

We consider now the contribution of the second term II. Let $f_{k_i,j_i}(\xi,\tau)=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i\geq 0$, $i=1,2,3$. Using the X_k norm, we get

$$II\lesssim\sum_{j_4\geq 2k_4+20}\sum_{j_1,j_2,j_3\geq 0}\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}.$$

Since in the area $\{|\xi_i|\in I_{k_i},\ i=1,2,3\}$ we have $|\Omega(\xi_1,\xi_2,\xi_3)|\lesssim 2^{2k_3}$, by checking the support properties of the functions f_{k_i,j_i}, $i=1,2,3$, we get $|j_{max}-j_{sub}|\leq 5$ and $j_{sub}\geq 2k_3+10$. From symmetry, we assume $j_3,j_4=j_{max},j_{sub}$, then we get

$$\begin{aligned}II\lesssim&\sum_{j_4\geq 2k_4+20}\sum_{j_1,j_2,j_3\geq 0}2^{(j_1+j_2)/2}2^{k_3}2^{k_3-j_3}2^{j_3-k_3}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_2\\ \lesssim&\sum_{j_3\geq 2k_4+20}2^{2k_3}2^{-j_3/2}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}\lesssim 2^{k_4}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}.\end{aligned}$$

Therefore we have completed the proof of part (d).

Next we prove part (e). We divide the argument into two cases. Let $f_{k_i,j_i}(\xi,\tau)=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i\geq 0$, $i=1,2,3$. Using the X_k norm, we then get

$$\begin{aligned}&2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ \lesssim&\sum_{j_i\geq 0}2^{k_4}2^{-j_4/2}(1+2^{(j_4-2k_4)/2})\|1_{D_{k_4,j_4}}(\xi,\tau)f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3}\|_{L^2}.\end{aligned}\tag{5.14}$$

Case 1. $\max(j_1,j_2,j_3,j_4)\leq 2k_1+20$.

It follows from Lemma 4.1 (d) that (5.14) is bounded by

$$\sum_{j_i\geq 0} 2^{k_4}(1+2^{(j_4-2k_4)/2})2^{(j_1+j_2+j_3)/2}2^{-k_1}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}\lesssim k_1^4\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}},$$

where we used Lemma 3.1 (b).

Case 2. $\max(j_1,j_2,j_3,j_4)\geq 2k_1+20$.

By checking the support properties, we get $|j_{max}-j_{sub}|\leq 5$. We consider only the worst case $j_1,j_4=j_{max},j_{sub}$. It follows from Lemma 4.1 (a) and Lemma 3.1 (b) that the right side of (5.14) is bounded by

$$\sum_{j_i\geq 0} 2^{k_4}2^{-j_4/2}(1+2^{(j_4-2k_4)/2})2^{(j_2+j_3)/2}2^{k_4}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}\lesssim k_1^4\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}}.$$

Therefore, we have completed the proof of part (e).

Finally we prove part (f). Let $f_{k_i,j_i}(\xi,\tau)=f_{k_i}(\xi,\tau)\eta_{j_i}(\tau-\omega(\xi))$, $j_i\geq 0$, $i=1,2,3$. Using the X_k norm, Lemma 4.1 (a) and Lemma 3.1 (b), we then get

$$\begin{aligned}&2^{k_4}\|\chi_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ \lesssim&\sum_{j_1,j_2,j_3,j_4\geq 0}2^{(j_{min}+j_{thd})/2}2^{(k_{min}+k_{thd})/2}\prod_{i=1}^{3}\|f_{k_i,j_i}\|_{L^2}\\ \lesssim&2^{(k_{min}+k_{thd})/2}\prod_{i=1}^{3}\|f_{k_i}\|_{Z_{k_i}},\end{aligned}$$

since for the case $j_{max}\geq 200$ we have $|j_{max}-j_{sub}|\leq 5$ by checking the support properties of the functions f_{k_i,j_i}, $i=1,2,3$. $\square$

Finally we present two counterexamples. The first one shows a logarithmic divergence if we only use X_k, which is the reason for us applying the Y_k structure. The other one shows why we use an l^1-type $X^{s,b}$ structure. See also the similar phenomenon in [19] for the complex-valued Benjamin-Ono equation.

Proposition 5.2. *Assume $k\geq 200$. Then there exist $f_1\in X_1$, $f_k\in X_k$ such that*

$$2^k\|\eta_k(\xi)(\tau-\omega(\xi)+i)^{-1}f_1*f_1*f_k\|_{X_k}\gtrsim k\|f_1\|_{X_1}\|f_1\|_{X_1}\|f_k\|_{X_k}.$$

Proof. From the proof of Proposition 5.1, we easily see that the worst interaction comes from the case that the largest frequency component has a largest modulation. So we construct this case explicitly. Let $I = [1/2, 1]$, and take

$$f_1(\xi,\tau) = \chi_I(\xi)\eta_1(\tau-\omega(\xi)),\ \ f_k(\xi,\tau) = \chi_{I_k}(\xi)\eta_k(\tau-\omega(\xi)).$$

From the definition, we easily get $\|f_1\|_{X_1} \sim 1$, $\|f_k\|_{X_k} \sim 2^{3k/2}$ and

$$2^k\|\eta_k(\xi)(\tau-\omega(\xi)+i)^{-1}f_1 * f_1 * f_k\|_{X_k} \gtrsim 2^k\sum_{j=0}^{k/2} 2^{-j/2}\|1_{D_{k,j}}\cdot f_1 * f_1 * f_k\|_{L^2_{\xi,\tau}}.$$

On the other hand, we have for $j \le k/2$

$$\begin{aligned}
&1_{D_{k,j}}(\xi,\tau)\cdot f_1 * f_1 * f_k\\
&=\int f_1(\xi_1,\tau_1)f_2(\xi_2,\tau_2)f_k(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&=\int \chi_I(\xi_1)\chi_I(\xi_2)\eta_1(\tau_1)\eta_1(\tau_2)\chi_{I_k}(\xi-\xi_1-\xi_2)\\
&\quad\cdot\eta_k(\tau-\tau_1-\tau_2-\omega(\xi_1)-\omega(\xi_2)-\omega(\xi-\xi_1-\xi_2))d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&\gtrsim \chi_{[\frac{2^{10}-1}{2^{10}}2^k,\frac{2^{10}+1}{2^{10}}2^k]}(\xi)\eta_j(\tau-\omega(\xi)).
\end{aligned}$$

Therefore, we get

$$2^k\sum_{j=0}^{k/2} 2^{-j/2}\|1_{D_{k,j}}\cdot f_1 * f_1 * f_k\|_{L^2_{\xi,\tau}} \gtrsim k2^{3k/2}, \tag{5.15}$$

which completes the proof of the proposition. □

Proposition 5.3. *For any $s \in \mathbb{R}$, there does not exist $b \in \mathbb{R}$ such that*

$$\|\partial_x(uvw)\|_{X^{s,b-1}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}\|w\|_{X^{s,b}}. \tag{5.16}$$

Proof. It is easy to see that the counterexample in the proof of Proposition 5.2 shows that (5.16) fails for $b = 1/2$ with a $k^{1/2}$ divergence in (5.15). We assume now $b \neq 1/2$. By using Plancherel's equality, we get that (5.16) is equivalent to

$$\begin{aligned}
&\Big\|\frac{\langle\xi\rangle^s\xi}{\langle\tau-\omega(\xi)\rangle^{1-b}}\int\frac{u(\xi_1,\tau_1)}{\langle\xi_1\rangle^s\langle\tau_1-\omega(\xi_1)\rangle^b}\frac{v(\xi_2,\tau_2)}{\langle\xi_2\rangle^s\langle\tau_2-\omega(\xi_2)\rangle^b}\\
&\quad\cdot\frac{w(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{\langle\xi-\xi_1-\xi_2\rangle^s\langle\tau-\tau_1-\tau_2-\omega(\xi-\xi_1-\xi_2)\rangle^b}d\tau_1 d\tau_2 d\xi_1 d\xi_2\Big\|_{L^2_{\xi,\tau}}\\
&\lesssim \|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.
\end{aligned}\tag{5.17}$$

Fix any dyadic number $N \gg 1$. Let

$$A = \{1/2 \le \xi \le 10,\ |\tau| \le 1\} \text{ and } B = \{N/2 \le \xi \le 2N,\ |\tau| \le 2^{10}\}.$$

Take $u(\xi,\tau) = v(\xi,\tau) = \chi_A(\xi,\tau-\omega(\xi))$, $w(\xi,\tau) = \chi_B(\xi,\tau-\omega(\xi))$. We easily see that $\|u\|_{L_2} = \|v\|_{L_2} \sim 1$ and $\|w\|_{L^2} \sim N^{1/2}$. Denote $f(\xi,\tau) = u * v * w(\xi,\tau+\omega(\xi))$. Then we have

$$\begin{aligned} f(\xi,\tau) &= \int u(\xi_1,\tau_1)v(\xi_2,\tau_2)w(\xi-\xi_1-\xi_2,\tau+\omega(\xi)-\tau_1-\tau_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\ &= \int \chi_{\le 2^{10}}(\tau-\tau_1-\tau_2+\omega(\xi)-\omega(\xi-\xi_1-\xi_2)-\omega(\xi_1)-\omega(\xi_2)) \\ &\quad \chi_A(\xi_1,\tau_1)\chi_A(\xi_2,\tau_2)\chi_{[N/2,2N]}(\xi-\xi_1-\xi_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\ &= \int \chi_{\le 2^{10}}(\tau-\tau_1-\tau_2+2(\xi_1+\xi_2)\xi+(\xi_1-\xi_2)^2-\omega(\xi_1)-\omega(\xi_2)) \\ &\quad \chi_A(\xi_1,\tau_1)\chi_A(\xi_2,\tau_2)\chi_{[N/2,2N]}(\xi-\xi_1-\xi_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2. \end{aligned}$$

Therefore, fixing $M \gg 1$, we get, for any $(\xi,\tau) \in [(M-1)N/M,(M+1)N/M] \times [-8N,-4N]$, that $\tau = -C_0\xi$ for some $2 \le C_0 \le 9$ and

$$f(\xi,\tau) \gtrsim \int \chi_A(\xi_1,\tau_1)\chi_A(\xi_2,\tau_2)\chi_{|\xi_1+\xi_2-C_0|\lesssim N^{-1}} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \gtrsim N^{-1}.$$

Thus we see that the left-hand side of (5.17) is larger than N^b, while the right-hand side is $N^{1/2}$, which implies $b < 1/2$.

Similarly, by taking $B' = \{N/2 \le \xi \le 2N,\ N \le |\tau| \le N\}$ as before, we obtain that $b > 1/2$. Therefore we have completed the proof of the proposition. □

6. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1 by using the standard fixed-point machinery. From Duhamel's principle, we get that the equation (1.1) is equivalent to the following integral equation:

$$u = W(t)\phi + \int_0^t W(t-t')(\partial_x(u^3)(t'))dt'. \tag{6.1}$$

We will mainly work on the following truncated version:

$$u = \psi(t)W(t)\phi + \psi(t)\int_0^t W(t-t')(\partial_x[(\psi(t')u)^3](t'))dt', \tag{6.2}$$

where $\psi(t) = \eta_0(t)$ is a smooth cut-off function. Then we easily see that, if u is a solution to (6.2) on $\mathbb{R}$, then u solves (6.1) on $t \in [-1, 1]$. Our first lemma is on the estimate for the linear solution.

Lemma 6.1. *If $s \geq 0$ and $\phi \in H^s$, then*

$$\|\psi(t) \cdot (W(t)\phi)\|_{F^s} \leq C\|\phi\|_{H^s}. \tag{6.3}$$

Proof. A direct computation shows that

$$\mathcal{F}[\psi(t) \cdot (W(t)\phi)](\xi, \tau) = \widehat{\phi}(\xi)\widehat{\psi}(\tau - \omega(\xi)).$$

In view of the definition, it suffices to prove that if $k \in \mathbb{Z}_+$ then

$$\|\eta_k(\xi)\widehat{\phi}(\xi)\widehat{\psi}(\tau - \omega(\xi))\|_{Z_k} \leq C\|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2}. \tag{6.4}$$

Indeed, from the definition we have

$$\begin{aligned}
&\|\eta_k(\xi)\widehat{\phi}(\xi)\widehat{\psi}(\tau - \omega(\xi))\|_{Z_k} \leq \|\eta_k(\xi)\widehat{\phi}(\xi)\widehat{\psi}(\tau - \omega(\xi))\|_{X_k} \\
&\leq C\sum_{j=0}^{\infty} 2^j \|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2}\|\eta_j(\tau)\widehat{\psi}(\tau)\|_{L^2} \leq C\|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2},
\end{aligned}$$

which is (6.4) as desired. □

The next lemma is on the estimate for the retarded linear term. These estimates were also used in [18]. The only difference is that here we don't have special structure for the low frequency.

Lemma 6.2. *If $l, s \geq 0$ and $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$, then*

$$\left\|\psi(t) \cdot \int_0^t W(t - s)(u(s))ds\right\|_{F^s} \leq C\|u\|_{N^s}. \tag{6.5}$$

Proof. A straightforward computation shows that

$$\begin{aligned}
&\mathcal{F}\left[\psi(t) \cdot \int_0^t W(t - s)(u(s))ds\right](\xi, \tau) \\
&= c\int_{\mathbb{R}} \mathcal{F}(u)(\xi, \tau')\frac{\widehat{\psi}(\tau - \tau') - \widehat{\psi}(\tau - \omega(\xi))}{\tau' - \omega(\xi)}d\tau'.
\end{aligned}$$

For $k \in \mathbb{Z}_+$ let $f_k(\xi, \tau') = \mathcal{F}(u)(\xi, \tau')\chi_k(\xi)(\tau' - \omega(\xi) + i)^{-1}$. For $f_k \in Z_k$ let

$$T(f_k)(\xi, \tau) = \int_{\mathbb{R}} f_k(\xi, \tau')\frac{\widehat{\psi}(\tau - \tau') - \widehat{\psi}(\tau - \omega(\xi))}{\tau' - \omega(\xi)}(\tau' - \omega(\xi) + i)d\tau'.$$

In view of the definitions, it suffices to prove that

$$\|T\|_{Z_k \to Z_k} \leq C \text{ uniformly in } k \in Z_+,$$

which follows from the proof of Lemma 5.2 in [18]. □

We prove a trilinear estimate in the following proposition which is an important component for using fixed-point arguments.

Proposition 6.3. *Let $s \geq 1/2$. Then*

$$\|\partial_x(\psi(t)^3 uvw)\|_{N^s} \lesssim \|u\|_{F^s}\|v\|_{F^{\frac12}}\|w\|_{F^{\frac12}} + \|u\|_{F^{\frac12}}\|v\|_{F^s}\|w\|_{F^{\frac12}} + \|u\|_{F^{\frac12}}\|v\|_{F^{\frac12}}\|w\|_{F^s}. \tag{6.6}$$

Proof. For the simplicity of notation, we write $u = \psi(t)u$, $v = \psi(t)v$ and $w = \psi(t)w$. In view of the definition, we get

$$\|\partial_x(uvw)\|_{N^s}^2 = \sum_{k_4=0}^{\infty} 2^{2sk_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}\mathcal{F}(\partial_x(uvw))\|_{Z_k}^2.$$

Setting $f_{k_1} = \eta_{k_1}(\xi)\mathcal{F}(u)$, $f_{k_2} = \eta_{k_2}(\xi)\mathcal{F}(v)$, $f_{k_3} = \eta_{k_3}(\xi)\mathcal{F}(w)$, we then get

$$\begin{aligned}&2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}\mathcal{F}(uvw)\|_{Z_{k_4}}\\ &\lesssim \sum_{k_1,k_2,k_3\in\mathbb{Z}_+} 2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}.\end{aligned}$$

From symmetry it suffices to bound

$$\sum_{0\leq k_1\leq k_2\leq k_3} 2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}.$$

Dividing the summation into several parts, we get

$$\begin{aligned}&\sum_{k_1\leq k_2\leq k_3} 2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\\ &\leq \sum_{j=1}^{6}\sum_{(k_1,k_2,k_3,k_4)\in A_j} 2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}},\end{aligned}\tag{6.7}$$

where we denote

$$\begin{aligned}&A_1 = \{0\leq k_1\leq k_2\leq k_3-10, k_3\geq 110, |k_4-k_3|\leq 5, |k_1-k_2|\leq 10\};\\ &A_2 = \{0\leq k_1\leq k_2\leq k_3-10, k_3\geq 110, |k_4-k_3|\leq 5, k_1\leq k_2-5\};\\ &A_3 = \{0\leq k_1\leq k_2\leq k_3, k_2\geq k_3-10\geq k_1+9, k_3\geq 110, |k_4-k_3|\leq 5\};\\ &A_4 = \{0\leq k_1\leq k_2\leq k_3, k_1\geq k_3-30, k_3\geq 110, |k_4-k_3|\leq 5\};\\ &A_5 = \{0\leq k_1\leq k_2\leq k_3, k_4\leq k_3-10, k_3\geq 110, |k_2-k_3|\leq 5\};\\ &A_6 = \{0\leq k_1\leq k_2\leq k_3, \max(k_3,k_4)\leq 120\}.\end{aligned}$$

Noting that $\mathcal{F}^{-1}(f_{k_i})(x,t)$ is supported in $\mathbb{R}\times I$ with $|I|\lesssim 1$, we will apply Proposition 5.1 obtained in the last section to bound the six terms in (6.7). For example, for the first term, from Proposition 5.1, we have

$$\begin{aligned}&\big\|2^{sk_4}\sum_{k_i\in A_1}2^{k_4}\|\eta_{k_4}(\xi)(\tau-\omega(\xi)+i)^{-1}f_{k_1}*f_{k_2}*f_{k_3}\|_{Z_{k_4}}\big\|_{l^2_{k_4}}\\ &\leq C\big\|2^{sk_4}\sum_{k_i\in A_1}2^{(k_1+k_2)/2}\|f_{k_1}\|_{Z_{k_1}}\|f_{k_2}\|_{Z_{k_2}}\|f_{k_3}\|_{Z_{k_3}}\big\|_{l^2_{k_4}}\\ &\leq \|u\|_{F^{1/2}}\|v\|_{F^{1/2}}\|w\|_{F^s}.\end{aligned}$$

We can handle the other terms in similar ways. Therefore we have completed the proof of the proposition. □

Now we prove Theorem 1.1.

Step 1: existence.

To begin with, we renormalize the data a bit via scaling. By the scaling (1.4), we see that if $s\geq 1/2$ then $\|\phi_\lambda\|_{L^2}=\|\phi\|_{L^2}$ and $\|\phi_\lambda\|_{\dot H^s}=\lambda^{-s}\|\phi\|_{\dot H^s}$. From the assumption $\|\phi\|_{L^2}\ll 1$ and by taking λ sufficiently large, we can first restrict ourselves to considering (1.1) with data ϕ satisfying

$$\|\phi\|_{H^s}=r\ll 1. \tag{6.8}$$

This indicates the reason why we assume that $\|\phi\|_{L^2}\ll 1$.

Define the operator

$$\Phi_\phi(u)=\psi(t)W(t)\phi+\psi(t)\int_0^t W(t-t')(\partial_x((\psi(t')u)^3)(t'))dt',$$

and we will prove that $\Phi_\phi(\cdot)$ is a contraction mapping from $\mathcal{B}=\{w\in F^s:\|w\|_{F^s}\leq 2cr\}$ into itself. From Lemma 6.1, 6.2 and Proposition 6.3 we get that, if $w\in\mathcal{B}$, then

$$\begin{aligned}\|\Phi_\phi(w)\|_{F^s}\leq& c\|\phi\|_{H^s}+\|\partial_x(\psi(t)^3w^3(\cdot,t))\|_{N^s}\\ \leq& cr+c\|w\|_{F^s}^3\leq cr+c(2cr)^3\leq 2cr,\end{aligned}\tag{6.9}$$

provided that r satisfies $8c^3r^2\leq 1/2$. Similarly, for $w,h\in\mathcal{B}$,

$$\begin{aligned}\|\Phi_\phi(w)-\Phi_\phi(h)\|_{F^s}\leq& c\left\|L\partial_x(\psi^3(\tau)(w^3(\tau)-h^3(\tau)))\right\|_{F^s}\\ \leq& c(\|w\|_{F^s}^2+\|h\|_{F^s}^2)\|w-h\|_{F^s}\\ \leq& 8c^3r^2\|w-h\|_{F^s}\leq\frac12\|w-h\|_{F^s}.\end{aligned}\tag{6.10}$$

Thus $\Phi_\phi(\cdot)$ is a contraction. Therefore, there exists a unique $u \in \mathcal{B}$ such that

$$u = \psi(t)W(t)\phi + \psi(t)\int_0^t W(t-t')(\partial_x[(\psi(t')u)^3](t'))dt'.$$

Hence u solves the integral equation (6.1) in the time interval $[-1,1]$. By inverting the scaling, we obtain a solution v to mBO (1.1) defined on $[-T,T]$ for some $T = T(\|\phi\|_{H^s}) > 0$. Moreover, $v \in F^s(T)$.

Step 2: uniqueness.

We will follow the arguments in [37] to prove the uniqueness [1]. We only need to prove the uniqueness for $s = 1/2$. Assume $u_1, u_2 \in F^{1/2}(T)$ are both solutions to the mBO equation (1.1) with the same initial data $\phi \in \{\phi \in H^{1/2} : \|\phi\|_{L^2} \ll 1\}$ and $\|u_1\|_{C([-T,T]:L^2)}, \|u_2\|_{C([-T,T]:L^2)} \ll 1$. We need to prove $u_1(t) = u_2(t)$ for $t \in [-T,T]$. Since $u_1, u_2 \in C([-T,T] : H^{1/2})$, we define $E = \{t \in [-T,T] : u_1(t) = u_2(t) \text{ in } H^{1/2}\}$. Obviously, $E \neq \varnothing$ since $0 \in E$. It suffices to prove that E is both an open and closed subset of $[-T,T]$. It follows immediately from $u_1, u_2 \in C([-T,T] : H^{1/2})$ that E is closed. It remains to prove E is open.

Since $\|u_1\|_{C([-T,T]:L^2)}, \|u_2\|_{C([-T,T]:L^2)} \ll 1$, it suffices to prove that 0 is an inner point of E (that the arbitrary point is an inner point follows in the same way). Namely, we need to prove that there is a $\delta > 0$ such that $(-\delta, \delta) \subset E$. We need the following lemma.

Lemma 6.4. *For any $u \in F^{1/2}$ satisfying $u(x,0) = 0$,*

$$\lim_{T\to 0+} \|u\|_{F^{1/2}(T)} = 0. \tag{6.11}$$

Proof. Denote by $X_T^{s,b}$ the restriction of $X^{s,b}$ to $[-T,T]$. First we prove that

$$\lim_{T\to 0+} \|u\|_{X_T^{1/2,1}} = 0 \tag{6.12}$$

holds for all $u \in X^{1/2,1}$ with $u(x,0) = 0$. Indeed, for $\epsilon > 0$, choose a Schwartz function v such that $\|u - v\|_{X^{1/2,1}} \le \epsilon/4$, thus $\|v(0)\|_{H^{1/2}} \le \epsilon/4$. Then

$$\|u\|_{X_T^{1/2,1}} \le \|u-v\|_{X^{1/2,1}} + \|v - W(t)v(0)\|_{X_T^{1/2,1}} + \|v(0)\|_{H^{1/2}}.$$

[1] The author thanks Nobu Kishimoto for pointing out the reference for the uniqueness.

It suffices to prove $\|v - W(t)v(0)\|_{X_T^{1/2,1}} \leq \epsilon/4$ for some small $T > 0$. From the fact that

$$v - W(t)v(0) = \int_0^t \partial_s(v(x,s) - W(s)v(0))ds$$

we get

$$\begin{aligned}
\|v - W(t)v(0)\|_{X_T^{1/2,1}} \leq \left\|\eta_0(t/T)\int_0^t \partial_s(v(x,s) - W(s)v(0))ds\right\|_{X^{1/2,1}} \\
\lesssim \left\|\|\langle\xi\rangle^{1/2}e^{-it\omega(\xi)}\eta_0(t/T)\int_0^t \partial_s(\mathcal{F}_x v(\xi,s) - e^{is\omega(\xi)}\mathcal{F}(v_0)(\xi))ds\|_{H_t^1}\right\|_{L_\xi^2}.
\end{aligned}$$

From $\|f(t)\|_{H^1}\lesssim\|f\|_{L^2}+\|f'\|_{L^2}$, we can easily get that $\|v-W(t)v(0)\|_{X_T^{1/2,1}} \leq \epsilon/4$ for some small $T > 0$.

For $u \in F^{1/2}$, taking $v = \mathcal{F}^{-1}\eta_{\leq N}(\xi)\eta_{\leq N}(\tau - \omega(\xi))\hat{u}(\xi,\tau)$, then $\|u - v\|_{F^{1/2}} \leq \epsilon/4$ for large N. Fixing N, we get that $\|v\|_{F^{1/2}}\lesssim\|v\|_{X^{1/2,1}}$. Therefore, from

$$\|u\|_{F^{1/2}(T)} \leq \|u - v\|_{F^{1/2}} + \|v - W(t)v(0)\|_{X_T^{1/2,1}} + \|v(0)\|_{H^{1/2}}$$

we immediately get $\lim_{T\to 0+}\|u\|_{F^{1/2}(T)} = 0$. □

By the scaling (1.4), we may assume $\|\phi\|_{H^{1/2}} \ll 1$. Let $u = u_1 - u_2$. Then u satisfies

$$u(t) = \int_0^t W(t-s)\partial_x(u_1^3(s) - u_2^3(s))ds.$$

Then from the linear and trilinear estimates we get for any $0 < \delta < T$

$$\begin{aligned}
\|u\|_{F^{1/2}(\delta)}&\lesssim\|\partial_x(u_1^3 - u_2^3)\|_{N^{1/2}(\delta)} \\
&\lesssim\|u\|_{F^{1/2}(\delta)}\prod_{j=1}^{2}(\|u_j - W(t)\phi\|_{F^{1/2}(\delta)} + \|\phi\|_{H^{1/2}}).
\end{aligned}$$

From Lemma 6.4 we know that by taking $\delta > 0$ sufficiently small we have

$$\prod_{j=1}^{2}(\|u_j - W(t)\phi\|_{F^{1/2}(\delta)} + \|\phi\|_{H^{1/2}}) \ll 1,$$

which implies $u = 0$ for $t \in (-\delta, \delta)$, thus $(-\delta, \delta) \subset E$. We have completed the proof of uniqueness.

Step 3: uniform continuity.

For any $\phi_1, \phi_2 \in \{u_0 \in H^{1/2} : \|u_0\|_{H^{1/2}} \le R, \|u_0\|_{L^2} \ll 1\}$, let u_1, u_2 be the corresponding solutions. Using the scaling (1.4), we obtain two solutions $\tilde{u}_1, \tilde{u}_2 \in F^{1/2}(1)$ with $\|\tilde{u}_1(0)\|_{H^{1/2}}, \|\tilde{u}_2(0)\|_{H^{1/2}} \ll 1$. By the existence in Step 1 and the uniqueness in Step 2, we get

$$\|\tilde{u}_1\|_{F^{1/2}(1)}, \|\tilde{u}_2\|_{F^{1/2}(1)} \ll 1.$$

Then uniform continuity follows immediately. Therefore, we have completed the proof of Theorem 1.1.

7. Short-time Trilinear Estimates

This section is devoted to proving some dyadic trilinear estimates in the spaces $F_k,\ N_k$. For $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ let

$$\widetilde{D}_{k,j} = \{(\xi, \tau) : \xi \in I_k, |\tau - \omega(\xi)| \le 2^j\}. \tag{7.1}$$

Proposition 7.1. *Let $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. Assume $u_{k_1} \in F_{k_1}, v_{k_2} \in F_{k_2}, w_{k_3} \in F_{k_3}$.*

(a) If $k_4 \ge 20$, $|k_3 - k_4| \le 5$, $k_1 \le k_2 \le k_3 - 10$, then

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \lesssim \min(2^{k_1/2}, \langle k_2\rangle)\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}. \tag{7.2}$$

(b) If $k_4 \ge 20$, $|k_3 - k_4| \le 5$, $k_3 - 10 \le k_2 \le k_3$ and $k_1 \le k_2 - 20$, then

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \lesssim \min(2^{k_1/2}, 1)\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}.$$

(c) If $k_4 \ge 20$, $|k_3 - k_4| \le 5$, $k_3 - 10 \le k_2 \le k_3$ and $k_2 - 30 \le k_1 \le k_2$, then

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \lesssim 2^{k_3/2}\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}.$$

(d) If $k_3 \ge 20$, $|k_3 - k_2| \le 5$, $k_2 - 10 \le k_1 \le k_2$ and $k_4 \le k_1 - 30$, then

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \lesssim \min(2^{k_4}, 1)|k_2|\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}. \tag{7.3}$$

(e) If $k_3 \ge 20$, $|k_3 - k_2| \le 5$, and $k_1, k_4 \le k_2 - 10$, then

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \lesssim \min(2^{k_1/2}, 1)|k_2|\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}.$$

(f) If $k_1, k_2, k_3, k_4 \le 200$, then

$$\|P_{k_4}\partial_x(u_{k_1}v_{k_2}w_{k_3})\|_{N_{k_4}} \lesssim 2^{k_{min}/2}2^{k_{thd}/2}\|u_{k_1}\|_{F_{k_1}}\|v_{k_2}\|_{F_{k_2}}\|w_{k_3}\|_{F_{k_3}}. \tag{7.4}$$

Proof. First we prove (a). Using the definitions and (3.10), we get that the left-hand side of (7.2) is dominated by

$$\begin{aligned} C\sup_{t_k\in\mathbb{R}} \|&(\tau - \omega(\xi) + i2^{k_4})^{-1} \cdot 2^{k_4} 1_{I_k}(\xi) \cdot \mathcal{F}[u_{k_1}\eta_0(2^{k_4-2}(t - t_k))] \\ &* \mathcal{F}[v_{k_2}\eta_0(2^{k_4-2}(t - t_k))] * \mathcal{F}[w_{k_3}\eta_0(2^{k_4-2}(t - t_k))]\|_{B_k}. \end{aligned} \tag{7.5}$$

It suffices to prove that if $j_i \geq k_4$ and $f_{k_i,j_i} : \mathbb{R}^2 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$ for $i = 1, 2, 3$, then

$$2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}$$
$$\lesssim \min(2^{k_1/2}, \langle k_2 \rangle) 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2} 2^{j_2/2} \|f_{k_2,j_2}\|_{L^2} 2^{j_3/2} \|f_{k_3,j_3}\|_{L^2}. \quad (7.6)$$

We assume first (7.6). Let $f_{k_1} = \mathcal{F}[u_{k_1} \cdot \eta_0(2^{k_4-2}(t - t_k))]$, $f_{k_2} = \mathcal{F}[v_{k_2} \cdot \eta_0(2^{k_4-2}(t-t_k))]$ and $f_{k_3} = \mathcal{F}[w_{k_3} \cdot \eta_0(2^{k_4-2}(t-t_k))]$. Then from the definition of B_k we get that (7.5) is dominated by

$$\sup_{t_k \in \mathbb{R}} 2^{k_4} \sum_{j_4=0}^{\infty} 2^{j_4/2} \sum_{j_1,j_2,j_3 \geq k_4} \|(2^{j_4} + i2^{k_4})^{-1} 1_{\dot{D}_{k_4,j_4}} \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}\|_{L^2}, \quad (7.7)$$

where $f_{k_i,j_i} = f_{k_i}(\xi, \tau)\eta_{j_i}(\tau - \omega(\xi))$ for $j_i > k_4$ and $f_{k_i,k_4} = f_{k_i}(\xi, \tau)\eta_{\leq k_4}(\tau - \omega(\xi))$, $i = 1, 2, 3$. For the summation on the terms $j_4 < k_4$ in (7.7), we get from the fact $1_{D_{k_4,j_4}} \leq 1_{\widetilde{D}_{k_4,j_4}}$ that

$$\sup_{t_k \in \mathbb{R}} 2^{k_4} \sum_{j_4 < k_4} 2^{j_4/2} \sum_{j_1,j_2,j_3 \geq k_4} \|(2^{j_4} + i2^{k_4})^{-1} 1_{\dot{D}_{k_4,j_4}} \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}\|_{L^2}$$
$$\lesssim \sup_{t_k \in \mathbb{R}} \sum_{j_1,j_2,j_3 \geq k_4} 2^{k_4/2} \|1_{\widetilde{D}_{k_4,k_4}} \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}\|_{L^2}. \quad (7.8)$$

From the fact that f_{k_i,j_i} is supported in $\widetilde{D}_{k_i,j_i}$ for $i = 1, 2, 3$ and using (7.6), we get that

$$\sup_{t_k \in \mathbb{R}} \sum_{j_1,j_2,j_3 \geq k_4} 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}\|_{L^2}$$
$$\lesssim \sup_{t_k \in \mathbb{R}} \min(2^{k_1/2}, \langle k_2 \rangle) \sum_{j_1,j_2,j_3 \geq k_4} 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2} 2^{j_2/2} \|f_{k_2,j_2}\|_{L^2} 2^{j_3/2} \|f_{k_3,j_3}\|_{L^2}.$$

Thus using (3.9) and (3.10) we obtain (7.2), as desired.

To prove (7.6), we consider first the case $|k_1 - k_2| \leq 5$. If $k_2 \geq 0$, it follows from Lemma 4.1 (b) and Remark 4.2 that

$$2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}$$
$$\lesssim 2^{k_4} \sum_{j_4 \geq k_4 + k_2} 2^{-j_4/2} 2^{(j_1+j_2+j_3)/2} 2^{-k_4/2} 2^{k_1/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}$$

$$+ 2^{k_4} \sum_{k_4 \leq j_4 \leq k_4+k_2} 2^{-j_4/2} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-k_4} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}$$

$$\lesssim (1+k_2) 2^{(j_1+j_2+j_3)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}, \tag{7.9}$$

which is (7.6) as desired. If $k_2 < 0$, it follows similarly from Lemma 4.1 (a) and Remark 4.2.

We assume now $k_1 < k_2 - 5$. If $k_2 < 0$, then arguing as above we get (7.6) as desired. If $k_2 > 0$, then (7.9) also holds in this case. On the other hand, by checking the support properties of the function f_{k_i,j_i}, $i = 1, 2, 3$, we get that $1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \equiv 0$ unless $j_{max} \geq k_4 + k_2 - 20$. For the summation on the terms $j_4 > k_4 + k_2 - 30$ in (7.6), we have

$$2^{k_4} \sum_{j_4 \geq k_4+k_2-30} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}$$

$$\lesssim 2^{k_4} \sum_{j_4 \geq k_4+k_2-30} 2^{-j_4/2} 2^{(j_1+j_2+j_3)/2} 2^{-j_3/2} 2^{k_1/2} 2^{k_2/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}$$

$$\lesssim 2^{k_1/2} 2^{(j_1+j_2+j_3)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \tag{7.10}$$

For the summation on the terms $j_4 < k_4 + k_2 - 30$, we have $j_4 \leq j_{med}$. Thus we can sum over using Lemma 4.1 (a).

Next we prove (b). As in the proof of part (a), it suffices to prove that if $j_i \geq k_4$, and $f_{k_i,j_i} : \mathbb{R}^2 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$, $i = 1, 2, 3$, then

$$2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}$$

$$\lesssim \min(2^{k_1/2}, 1) 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2,j_2}\|_{L^2} \cdot 2^{j_3/2} \|f_{k_3,j_3}\|_{L^2}. \tag{7.11}$$

Since $|\Omega(\xi_1, \xi_2, \xi_3)| \sim 2^{2k_4}$ in the area $\{|\xi_i| \in \widetilde{I}_{k_i}, i = 1, 2, 3\} \cap \{|\xi_1 + \xi_2 + \xi_3| \in I_{k_4}\}$, then we get $1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \equiv 0$ unless $j_{max} \geq 2k_4 - 30$. By Lemma 4.1 (a) we get that the left-hand side of (7.11) is bounded by

$$2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} 2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{max}/2} 2^{-j_{sub}/2} 2^{k_1/2} 2^{k_2/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}. \tag{7.12}$$

Then we get the bound (7.11) by considering either $j_4 = j_{max}$ or $j_4 \neq j_{max}$.

Next we prove (c). As in the proof of part (a), using (3.9) and (3.10), it suffices to prove that, if $j_i \geq k_4$, and $f_{k_i,j_i} : \mathbb{R}^2 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$, $i = 1,2,3$, then

$$2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}$$
$$\leq C 2^{k_4/2} \cdot 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2,j_2}\|_{L^2} \cdot 2^{j_3/2} \|f_{k_3,j_3}\|_{L^2},$$

which follows immediately from Lemma 4.1 (c).

Next we prove part (d). Let $\gamma : \mathbb{R} \to [0,1]$ denote a smooth function supported in $[-1,1]$ with the property that $\sum_{n\in\mathbb{Z}} \gamma^3(x-n) \equiv 1, \quad x \in \mathbb{R}$. Using the definitions, the left-hand side of (7.3) is dominated by

$$C \sup_{t_k \in \mathbb{R}} \Big\|(\tau - \omega(\xi) + i2^{k_{4+}})^{-1} \cdot 2^{k_4} 1_{I_{k_4}}(\xi) \cdot \sum_{|m| \leq C 2^{k_2 - k_{4+}}}$$
$$\mathcal{F}[u_{k_1} \eta_0(2^{k_{4+}}(t-t_k))\gamma(2^{k_2}(t-t_k)-m)] *$$
$$\mathcal{F}[v_{k_2} \eta_0(2^{k_{4+}}(t-t_k))\gamma(2^{k_2}(t-t_k)-m)] *$$
$$\mathcal{F}[w_{k_3} \eta_0(2^{k_{4+}}(t-t_k))\gamma(2^{k_2}(t-t_k)-m)]\Big\|_{B_k}.$$

In view of the definitions, (3.9) and (3.10), it suffices to prove that, if $j_i \geq k_2$, and $f_{k_i,j_i} : \mathbb{R}^3 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$, $i = 1,2,3$, then

$$2^{k_4} 2^{k_2 - k_{4+}} \sum_{j_4 \geq k_{4+}} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}$$
$$\lesssim \min(2^{k_4},1)|k_2| 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2,j_2}\|_{L^2} \cdot 2^{j_3/2} \|f_{k_3,j_3}\|_{L^2}. \tag{7.13}$$

From the same argument as in Proposition 7.1, we get $j_{max} \geq 2k_2 - 30$. Then (7.13) follows from Lemma 4.1.

Next we prove part (e). As in the proof of part (d), it suffices to prove that, if $j_1, j_2, j_3 \geq k_2$, and $f_{k_i,j_i} : \mathbb{R}^3 \to \mathbb{R}_+$ are supported in $\widetilde{D}_{k_i,j_i}$, $i = 1,2,3$, then

$$2^{k_4} 2^{k_2 - k_{4+}} \sum_{j_4 \geq k_{4+}} 2^{-j_4/2} \|1_{\widetilde{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})\|_{L^2}$$
$$\lesssim \min(2^{k_1/2},1)|k_2| \cdot 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2} \cdot 2^{j_2/2} \|f_{k_2,j_2}\|_{L^2} \cdot 2^{j_3/2} \|f_{k_3,j_3}\|_{L^2} \tag{7.14}$$

In proving (7.14) we may assume $j_4 \leq 10k_2$ in the summation of (7.14), otherwise we use Lemma 4.1 (a). Using Lemma 4.1 (a) for $k_1 \leq 0$, else using

Lemma 4.1 (b), we get

$$2^{k_4}2^{k_2-k_{4+}}\sum_{j_4\geq k_{4+}}2^{-j_4/2}\|1_{\widetilde{D}_{k_4,j_4}}\cdot(f_{k_1,j_1}*f_{k_2,j_2}*f_{k_3,j_3})\|_{L^2}$$
$$\lesssim\min(2^{k_1/2},1)|k_2|2^{j_1/2}\|f_{k_1,j_1}\|_{L^2}2^{j_2/2}\|f_{k_2,j_2}\|_{L^2}2^{j_3/2}\|f_{k_3,j_3}\|_{L^2}.$$

Finally we prove part (f). This follows immediately from the definitions, Lemma 4.1 (a), Remark 4.2 and (3.9) and (3.10). □

8. Energy Estimates

In this section we prove an energy estimate by using the I-method [9], following some ideas in [31]. For the difference equation of two modified Benjamin-Ono equations, we don't know how to prove a similar energy estimate due to the lack of symmetry. That's why we can only solve the a-priori part.

Proposition 8.1. *Assume that $T\in(0,1]$ and $u\in C([-T,T]:H^\infty)$ is a real-valued solution of the initial-value problem*

$$\begin{cases} u_t+\mathcal{H}u_{xx}=u^2u_x,\ (x,t)\in\mathbb{R}\times(-T,T);\\ u(x,0)=\phi(x).\end{cases}\tag{8.1}$$

Then, for $0\leq l<1/4$ and $s>1/4$, there exists $\delta_0>0$ such that if $\|u\|_{E^{l,s}(T)}\leq\delta_0$ then we have

$$\|u\|^2_{E^{l,s}(T)}\lesssim\|\phi\|^2_{\dot H^l\cap\dot H^s}+\|u\|^4_{F^{\frac14-,\frac14+}(T)}\cdot\|u\|^2_{F^{l,s}(T)}.\tag{8.2}$$

The following definition was first introduced in [31].

Definition 8.2. *Let $s\in\mathbb{R}$ and $\epsilon>0$. Then S^s_ϵ is the class of spherical symmetric symbols with the following properties:*

(*i*) *symbol regularity:*

$$|\partial^\alpha a(\xi)|\lesssim a(\xi)(1+\xi^2)^{-\alpha/2}$$

(*ii*) *decay at infinity, for $|\xi|\gg1$:*

$$s\leq\frac{\log a(\xi)}{\log(1+\xi^2)}\leq s+\epsilon,\quad s-\epsilon\leq\frac{d\log a(\xi)}{d\log(1+\xi^2)}\leq s+\epsilon.$$

Assume $u\in C([-T,T]:H^\infty)$ solves (8.1) and $a\in S^s_\epsilon$. Denote $A(D)=\mathcal{F}^{-1}a(\xi)\mathcal{F}$. We first set

$$E_0(u)=(A(D)u,u)=\int_{\xi_1+\xi_2=0}a(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2).$$

Using the equation (8.1) and noting that $a(\xi)$ is even while $\omega(\xi)$ is odd, we then easily get that

$$\begin{aligned}&\frac{d}{dt}E_0(u) = R_4(u)\\&= -\frac{1}{6}\int_{\Gamma_4} i[\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)]\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{u}(\xi_4),\end{aligned}$$

where for $k \in \mathbb{N}$, we denote $\Gamma_k = \{\xi_1 + \xi_2 + \ldots + \xi_k = 0\}$. Following the idea of the I-method, we define a multi-linear correction term to achieve a cancellation

$$E_1(u) = \int_{\Gamma_4} b_4(\xi_1, \xi_2, \xi_3, \xi_4)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{u}(\xi_4),$$

where b_4 will be determined soon. Again using the equation (1.1), we get

$$\begin{aligned}&\frac{d}{dt}E_1(u) = R_6(u)\\&+ \int_{\Gamma_4} ib_4(\xi_1, \xi_2, \xi_3, \xi_4)[\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4)]\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{u}(\xi_4),\end{aligned}$$

where $R_6(u) = C\int_{\Gamma_6} b_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6)\prod_{j=1}^{6}\widehat{u}(\xi_j)$. To achieve the cancellation of the quadrilinear form we define b_4 on Γ_4 by

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = C\frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4)}.$$

Thus we get $\frac{d}{dt}(E_0(u) + E_1(u)) = R_6(u)$.

Proposition 8.3. *Assume that $a \in S^s_\epsilon$. Then for each dyadic $\lambda \leq \alpha \leq \mu$ there is an extension of b_4 from the diagonal set $\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4 : |\xi_1| \sim \lambda, |\xi_2| \sim \alpha, |\xi_3|, |\xi_4| \sim \mu\}$ to the dyadic set $\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}_4 : |\xi_1| \sim \lambda, |\xi_2| \sim \alpha, |\xi_3|, |\xi_4| \sim \mu\}$ which satisfies*

$$|b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim a(\mu)\mu^{-1} \tag{8.3}$$

and

$$\sum_{j=1}^{4} |\partial_j b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim a(\alpha)\mu^{-1} + a(\mu)\mu^{-2}. \tag{8.4}$$

Proof. From symmetry we may assume $|\xi_1| \leq |\xi_2| \leq |\xi_3| \leq |\xi_4|$ and $\xi_3 > 0, \xi_4 < 0$. We first consider the case that $\xi_1\xi_2 > 0$, say $\xi_1, \xi_2 > 0$. Then $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = -2(\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3)$. Thus in Γ_4 we have

$$Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1\xi_2 + (\xi_2 + \xi_1)\xi_3} + \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3}.$$

Using $\xi_1+\xi_2+\xi_3+\xi_4=0$ we get

$$\frac{\xi_3 a(\xi_3)+\xi_4 a(\xi_4)}{\xi_1\xi_2+\xi_2\xi_3+\xi_1\xi_3}=\frac{\xi_1\xi_2(\xi_3 a(\xi_3)+\xi_4 a(\xi_4))}{(\xi_1\xi_2+\xi_2\xi_3+\xi_1\xi_3)(\xi_3(\xi_3+\xi_4))}-\frac{\xi_3 a(\xi_3)+\xi_4 a(\xi_4)}{\xi_3(\xi_3+\xi_4)}.$$

Therefore, we extend b_4 by setting

$$\begin{aligned}Cb_4(\xi_1,\xi_2,\xi_3,\xi_4)&=\frac{\xi_1 a(\xi_1)+\xi_2 a(\xi_2)}{\xi_1\xi_2+\xi_2\xi_3+\xi_1\xi_3}\\&+\frac{\xi_1\xi_2}{\xi_1\xi_2+\xi_2\xi_3+\xi_1\xi_3}\frac{\xi_3 a(\xi_3)+\xi_4 a(\xi_4)}{\xi_3(\xi_3+\xi_4)}-\frac{\xi_3 a(\xi_3)+\xi_4 a(\xi_4)}{\xi_3(\xi_3+\xi_4)}.\end{aligned}\tag{8.5}$$

It is easy to see from the properties of $a(\xi)$ that $|b_4(\xi_1,\xi_2,\xi_3,\xi_4)|\lesssim a(\mu)\mu^{-1}$. It remains to check the derivatives. We only consider $|\partial_1 b_4|$, since the others can be handled in similar ways. For $|\partial_1 b_4|$ it suffices to consider the first term on the right-hand side of (8.5). Direct computations show that

$$\begin{aligned}&\partial_{\xi_1}\left(\frac{\xi_1 a(\xi_1)+\xi_2 a(\xi_2)}{\xi_1\xi_2+(\xi_2+\xi_1)\xi_3}\right)\\&=\frac{[a(\xi_1)-a(\xi_2)]\xi_2\xi_3-\xi_2^2 a(\xi_2)}{(\xi_1\xi_2+(\xi_2+\xi_1)\xi_3)^2}+\frac{a'(\xi_1)\xi_1}{\xi_1\xi_2+(\xi_2+\xi_1)\xi_3},\end{aligned}$$

which satisfies (8.4) as desired.

We consider now $\xi_1\xi_2<0$, say $\xi_1<0,\xi_2>0$. Thus we get $\omega(\xi_1)+\omega(\xi_2)+\omega(\xi_3)+\omega(\xi_4)=\xi_1^2-\xi_2^2-\xi_3^2+\xi_4^2=(\xi_1+\xi_2)(\xi_1+\xi_3)$. We will extend b_4 in the following cases.

(a) $\lambda\ll\mu$, $\alpha\le\mu$. Then the extension of b_4 is defined using the formula

$$b_4(\xi_1,\xi_2,\xi_3,\xi_4)=\frac{\xi_1 a(\xi_1)+\xi_2 a(\xi_2)}{(\xi_1+\xi_2)(\xi_1+\xi_3)}-\frac{\xi_3 a(\xi_3)+\xi_4 a(\xi_4)}{(\xi_3+\xi_4)(\xi_1+\xi_3)}.$$

Since $\lambda\ll\mu$, we see that $|\xi_1+\xi_3|\sim\mu$. By using the properties of $a(\xi)$ we see (8.3) and (8.4) are satisfied as desired.

(b) $\lambda\sim\mu$. Then the extension of b_4 is defined using the formula

$$b_4(\xi_1,\xi_2,\xi_3,\xi_4)=\frac{\xi_1 a(\xi_1)+\xi_2 a(\xi_2)+\xi_3 a(\xi_3)-(\xi_1+\xi_2+\xi_3)a(\xi_1+\xi_2+\xi_3)}{(\xi_1+\xi_2)(\xi_1+\xi_3)}.$$

To check the properties, setting $q(\xi_1,\xi_2)=\frac{\xi_1 a(\xi_1)+\xi_2 a(\xi_2)}{\xi_1+\xi_2}$, we then get that

$$b_4(\xi_1,\xi_2,\xi_3,\xi_4)=\frac{q(\xi_1,\xi_2)-q(\xi_1-(\xi_1+\xi_3),\xi_2+(\xi_1+\xi_3))}{\xi_1+\xi_3},$$

from which we easily verify (8.3) and (8.4). □

In view of the definition, for Proposition 8.1 we are mainly concerned with the control of the energy in high frequency. From the definition we see that if $a \in S^s_\epsilon$ then $a(\xi)\eta_{\geq 1}(\xi) \in S^s_\epsilon$. Let $A^s_\epsilon = \{a(\xi)\eta_{\geq 1}(\xi) : a \in S^s_\epsilon\}$.

Proposition 8.4. *Assume $a \in A^s_\epsilon$ and $s - \epsilon \geq 0$; then we have*

$$|E_1(u)| \lesssim \|u\|^2_{\dot{H}^{1/4-} \cap \dot{H}^{1/4+}} E_0(u).$$

Proof. Using the definition, we get

$$|E_1(u)| \leq \int_{\Gamma_4} |b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \cdot |\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{u}(\xi_4)|.$$

From symmetries, we may assume that $|\xi_1| \leq |\xi_2| \leq |\xi_3| \leq |\xi_4|$. Localizing $|\xi_j| \sim N_j$ for N_j a dyadic number, we may assume $N_3 \sim N_4 \gtrsim 1$. Then it follows from Proposition 8.3 that

$$\begin{aligned}
|E_1(u)| &\lesssim \sum_{N_j} \int_{\Gamma_4, |\xi_j| \sim N_j} {N_4}^{-1} |\widehat{u}(\xi_1)\widehat{u}(\xi_2) a(N_4)^{1/2} \widehat{u}(\xi_3) a(N_4)^{1/2} \widehat{u}(\xi_4)| \\
&\lesssim \sup_{N_4 \gtrsim 1} \sum_{N_1 \leq N_2 \leq N_4} N_4^{-1/2+\epsilon} {N_1}^{1/2} \|R_{N_1} u\|_2 \|R_{N_2} u\|_2 E_0(u) \\
&\lesssim \|u\|^2_{\dot{H}^{1/4-} \cap \dot{H}^{1/4+}} E_0(u).
\end{aligned}$$

Therefore we have completed the proof of the proposition. □

Proposition 8.5. *Assume $a \in A^s_\epsilon$, $s - \epsilon \geq 0$ and $T \in (0, 1]$. Then*

$$\left| \int_{-T}^{T} R_6(u) dt \right| \lesssim \|u\|^4_{F^{1/4-,1/4+}(T)} \|u\|^2_{F^{l,s}(T)}.$$

Proof. We first fix extension $\widetilde{u} \in C_0(\mathbb{R} : H^\infty)$ of u such that $\|R_k(\widetilde{u})\|_{F_k} \leq 2\|R_k(u)\|_{F_k(T)}$, $k \in \mathbb{Z}$. It suffices to prove that

$$\left| \int_{0}^{T} R_6(\widetilde{u}) dt \right| \lesssim \|\widetilde{u}\|^4_{F^{1/4-,1/4+}} \|\widetilde{u}\|^2_{F^{l,s}}. \tag{8.6}$$

For simplicity of the notation we still write $\widetilde{u} = u$. From symmetry, we get

$$\begin{aligned}
CR_6(u) = \int_{\Gamma_6} &[b_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6) \\
&- b_4(-\xi_4, -\xi_5, -\xi_6, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6)] \prod_{j=1}^{6} \widehat{u}(\xi_j).
\end{aligned}$$

Localizing $|\xi_j| \sim N_j = 2^{k_j}$ and using symmetry, we may assume $N_1 \leq N_2 \leq N_3$, $N_4 \leq N_5 \leq N_6$ and $\max(N_j) \sim \mathrm{sub}(N_j) \gtrsim 1$ where $\max(N_j)$ and sub(N_j) are the maximum and second-maximum of $N_j, j = 1, 2, \dots, 6$. Let $u_k = R_k(u)$ and $\xi_{456} = \xi_4 + \xi_5 + \xi_6$. Thus

$$\Big|\int_0^T R_6(u)dt\Big| \lesssim \sum_{N_j} |\int_0^T \int_{\Gamma_6} [b_4(\xi_1, \xi_2, \xi_3, \xi_{456})(\xi_{456}) - b_4(-\xi_4, -\xi_5, -\xi_6, \xi_{456})(\xi_{456})] \prod_{j=1}^{6} \widehat{u_{k_j}}(\xi_j)dt|. \quad (8.7)$$

Let $\gamma : \mathbb{R} \to [0, 1]$ denote a positive smooth function supported in $[-1, 1]$ with the property that $\sum_{n\in\mathbb{Z}} \gamma^6(x - n) \equiv 1, \quad x \in \mathbb{R}$. We will bound (8.6) in several cases.

Case 1. $N_3 \lesssim N_5, N_6$ and $N_5 \sim N_6 \gtrsim 1$. Then we get that the right-hand side of (8.7) is bounded by

$$\sum_{N_j} \sum_{|n|\lesssim 2^{k_6}} |\int_{\mathbb{R}} \int_{\Gamma_6} [b_4(\xi_1, \xi_2, \xi_3, \xi_{456}) - b_4(-\xi_4, -\xi_5, -\xi_6, \xi_{456})] \xi_{456}[\gamma(2^{k_6}t - n)1_{[0,T]}(t)\widehat{u_{k_1}}(\xi_1)] \prod_{j=2}^{6} [\gamma(2^{k_6}t - n)\widehat{u_{k_j}}(\xi_j)]dt|. \quad (8.8)$$

We observe first that

$$|A| = |\{n : \gamma(2^{k_6}t - n)1_{[0,T]}(t) \neq \gamma(2^{k_6}t - n), 0, \ \forall\, t \in \mathbb{R}\}| \leq 4.$$

Let $f_{k_j}(\xi, \tau) = \mathcal{F}[\gamma(2^{k_6}t - n)R_k u], \ j = 1, 2, \dots, 6$. Using Proposition 8.3 and Plancherel's theorem, we get that the summation for $n \in A^c$ of (8.8) is bounded by

$$\sum_{N_j} \sum_{|n|\lesssim 2^{k_6}, n\in A^c} (a(N_3)N_3^{-1} + a(N_6)N_6^{-1})N_3 \Big|\int_{\Gamma_6(\mathbb{R}^2)} \prod_{j=1}^{6} f_{k_j}(\xi_j, \tau_j)\Big|. \quad (8.9)$$

Using Hölder's inequality and the embedding properties of B_k, we get

$$\begin{aligned}\Big|\int_{\Gamma_6} \prod_{j=1}^{6} f_{k_j}(\xi_j, \tau_j)\Big| &\lesssim \prod_{j=1}^{4} \|\mathcal{F}^{-1}(f_{k_j})\|_{L_x^4 L_t^\infty} \|\mathcal{F}^{-1}(f_{k_5})\|_{L_x^\infty L_t^2} \|\mathcal{F}^{-1}(f_{k_6})\|_{L_x^\infty L_t^2} \\ &\lesssim N_5^{-1} \prod_{j=1}^{4} N_j^{1/4} \prod_{j=1}^{6} \|f_{k_j}\|_{B_k}.\end{aligned}$$

Then we can bound (8.9) by

$$\sum_{N_j}(a(N_3)N_3^{-1}+a(N_6)N_6^{-1})N_3\prod_{j=1}^{4}N_j^{1/4}\prod_{j=1}^{6}\|f_{k_j}\|_{F_k}\lesssim\|u\|^4_{F^{1/4-,1/4+}}\|u\|^2_{F^{l,s}},$$

which is (8.6) as desired.

For the summation of $n\in A$, we observe that, if $I\subset\mathbb{R}$ is an interval, $k\in\mathbb{Z}$, $f_k\in B_k$, and $f_k^I=\mathcal{F}(1_I(t)\cdot\mathcal{F}^{-1}(f_k))$, then

$$\sup_{j\in\mathbb{Z}_+}2^{j/2}\|\eta_j(\tau-\omega(\xi))\cdot f_k^I\|_{L^2}\lesssim\|f_k\|_{B_k}.$$

Let $f_{k_i,j_i}=\eta_{j_i}(\tau-\omega(\xi))\mathcal{F}[\gamma(2^{k_6}t-n)R_ku]$, $i=1,2,\ldots,5$, and $f_{k_6,j_6}=\eta_{j_6}(\tau-\omega(\xi))\mathcal{F}[\gamma(2^{k_6}t-n)1_{[0,T]}(t)R_ku]$. If $j_6\geq 100k_6$, then by checking the support properties we get $\int_{\Gamma_6(\mathbb{R}^2)}\prod_{i=1}^6 f_{k_i,j_i}(\xi_j,\tau_j)\equiv 0$ unless $|j_{max}-j_{sub}|\leq 10$ and $j_{max}\geq 100k_6$, where j_{max} and j_{sub} are the maximum and sub-maximum of $j_1,j_2,\ldots,j_6$. By using the Cauchy-Schwarz inequality, we get that

$$\sum_{j_i}\Big|\int_{\Gamma_6(\mathbb{R}^2)}\prod_{i=1}^{6}f_{k_i,j_i}(\xi_j,\tau_j)\Big|\lesssim 2^{(k_1+\cdots+k_6)/2}2^{(j_1+\cdots+j_6)/2}2^{-(j_{max})}\prod_{i=1}^{6}\|f_{k_i,j_i}\|_{L^2}$$
$$\lesssim N_5^{-1}\prod_{j=1}^{4}N_j^{1/4}\prod_{j=1}^{6}\|f_{k_j}\|_{F_k},$$

which is acceptable. If $j_6\leq 100k_6$, then we argue as before for $n\in A^c$, then

$$\sum_{j_i}\Big|\int_{\Gamma_6(\mathbb{R}^2)}\prod_{i=1}^{6}f_{k_i,j_i}(\xi_j,\tau_j)\Big|\lesssim k_6N_5^{-1}\prod_{j=1}^{4}N_j^{1/4}\prod_{j=1}^{6}\|f_{k_j}\|_{F_k},$$

which combined with (8.9) gives (8.6).

Case 2. $N_6\lesssim N_2,N_3$ and $N_2\sim N_3\gtrsim 1$. From symmetry, this case is identical to Case 1. We omit the details.

Case 3. $N_2,N_5\ll N_3,N_6$ and $N_6\sim N_3\gtrsim 1$. From the proof of Proposition 8.3, we get that

$$|b_4(\xi_1,\xi_2,\xi_3,\xi_{456})-b_4(-\xi_4,-\xi_5,-\xi_6,\xi_{456})|\cdot|\xi_{456}|\lesssim a(N_3)+a(N_6)N_6^{-1}.$$

Then following the same argument as in Case 1, we obtain (8.6) as desired. □

Lemma 8.6 (Lemma 5.5, [31]). *There is a sequence $\{\beta_\lambda\}$ with the following properties:*

(a) $\lambda^{2s}\|P_\lambda(u_0)\|_{L^2}^2 \le \beta_\lambda \|u_0\|_{H^s}^2$,
(b) $\sum \beta_\lambda \lesssim 1$,
(c) β_λ *is slowly varying in the sense that*

$$|\log_2 \beta_\lambda - \log_2 \beta_\mu| \le \frac{\epsilon}{2}|\log_2 \lambda - \log_2 \mu|.$$

Proof of Proposition 8.1. In view of the definition, we get

$$\|u\|_{E^{l,s}(T)}^2 \lesssim \|P_{\le 0}u_0\|_{\dot{H}^l}^2 + \sum_{k\ge 1}\sup_{t\in[-T,T]} 2^{2ks}\|P_k(u(t))\|_{L^2}^2.$$

We will prove that, if $k \ge 1$, then

$$\sup_{t\in[-T,T]} 2^{2ks}\|P_k(u(t))\|_{L^2}^2 \lesssim \beta_k(\|P_{\ge 1}u_0\|_{H^s}^2 + \|u\|_{F^{\frac14-,\frac14+}(T)}^4 \cdot \|u\|_{F^{l,s}(T)}^2), \tag{8.10}$$

which suffices to prove Proposition 8.1 in view of Lemma 8.6 (b). In order to prove (8.10) for some fixed k_0 we define the sequence

$$a_k = 2^{2ks}\max(1, \beta_{k_0}^{-1}2^{-\epsilon|k-k_0|}).$$

Using the slowly varying condition (iii), we then get

$$\begin{aligned}\sum_{k\ge 1} a_k\|P_k(u_0)\|_{L^2}^2 &\lesssim \sum_k 2^{2ks}\|P_k(u_0)\|_{L^2}^2 + 2^{-\epsilon|k-k_0|/2}2^{2ks}\beta_k^{-1}\|P_k(u_0)\|_{L^2}^2\\ &\lesssim \|P_{\ge 1}(u_0)\|_{H^s}^2.\end{aligned}$$

We may assume that $\beta_0 = 1$. Then we see that $\max(|\beta_k|, |\beta_k^{-1}|) \le 2^{k\epsilon/2}$. Correspondingly we find a function $a(\xi) \in S_\epsilon^s$ so that $a(\xi) \sim a_k$, $|\xi| \sim 2^k$. Thus we apply Proposition 8.4, 8.5 for $a(\xi)\eta_{\ge 1}(\xi)$, then we get

$$\sup_{t\in[-T,T]} |E_0(u(t) + E_1(u(t))| \le |E_0(u_0) + E_1(u_0)| + \left|\int_{-T}^T R_6(u)dt\right|,$$

from which we see that

$$\sup_{t\in[-T,T]} |E_0(u(t)| \le |E_0(u_0)| + \|u\|_{F^{1/4-,1/4+}(T)}^4\|u\|_{F^{l,s}(T)}^2.$$

Therefore, we get

$$\sum_{k\ge 1} a_k\|P_k(u(t))\|_{L^2}^2 \lesssim \|P_{\ge 1}u_0\|_{H^s}^2 + \|u\|_{F^{1/4-,1/4+}(T)}^4\|u\|_{F^{l,s}(T)}^2,$$

which at $k = k_0$ gives (8.10) as desired. □

9. Proof of Theorem 1.8

This section is devoted to proving Theorem 1.8. The main ingredients are energy estimates and short-time trilinear estimates.

Proposition 9.1. *Let $l, s \geq 0$, $T \in (0,1]$, and $u \in F^{l,s}(T)$; then*

$$\sup_{t\in[-T,T]} \|u(t)\|_{\dot{H}^l\cap\dot{H}^s} \lesssim \|u\|_{F^{l,s}(T)}.$$

Proof. In view of the definitions, it suffices to prove that, if $k \in \mathbb{Z}$, $t_k \in [-1,1]$, and $\widetilde{u}_k \in F_k$ then $\|\mathcal{F}[\widetilde{u}_k(t_k)]\|_{L^2_\xi} \lesssim \|\mathcal{F}[\widetilde{u}_k \cdot \eta_0(2^{k_+}(t-t_k))]\|_{B_k}$. Let $f_k = \mathcal{F}[\widetilde{u}_k \cdot \eta_0(2^{k_+}(t-t_k))]$, so $\mathcal{F}[\widetilde{u}_k(t_k)](\xi) = c\int_{\mathbb{R}} f_k(\xi,\tau)e^{it_k\tau}d\tau$. From the definition of B_k, we get that

$$\|\mathcal{F}[\widetilde{u}_k(t_k)]\|_{L^2_\xi} \lesssim \left\|\int_{\mathbb{R}} |f_k(\xi,\tau)|d\tau\right\|_{L^2_\xi} \lesssim \|f_k\|_{B_k},$$

which completes the proof of the proposition. □

Proposition 9.2. *Assume $T \in (0,1]$, $u, v \in C([-T,T] : H^\infty)$ and*

$$u_t + \mathcal{H}u_{xx} = v \ \text{on } \mathbb{R}^2 \times (-T,T). \tag{9.1}$$

Then for any $l, s \geq 0$,

$$\|u\|_{F^{l,s}(T)} \lesssim \|u\|_{E^{l,s}(T)} + \|v\|_{N^{l,s}(T)}. \tag{9.2}$$

Proof. In view of the definitions, we see that the square of the right-hand side of (9.2) is equivalent to

$$\begin{aligned}&\sum_{k\leq 0}\left(2^{2lk}\|P_k(u(0))\|_{L^2}^2 + 2^{2lk}\|P_k(v)\|_{N_k(T)}^2\right)\\ &+\sum_{k\geq 1}\Big(\sup_{t_k\in[-T,T]} 2^{2sk}\|P_k(u(t_k))\|_{L^2}^2 + 2^{2sk}\|P_k(v)\|_{N_k(T)}^2\Big).\end{aligned} \tag{9.3}$$

Thus, from the definitions, it suffices to prove that, if $k \in \mathbb{Z}$ and $u, v \in C([-T,T] : H^\infty)$ solve (9.1), then

$$\begin{aligned}&\|P_k(u)\|_{F_k(T)} \lesssim \|P_k(u(0))\|_{L^2} + \|P_k(v)\|_{N_k(T)} \text{ if } k \leq 0;\\ &\|P_k(u)\|_{F_k(T)} \lesssim \sup_{t_k\in[-T,T]} \|P_k(u(t_k))\|_{L^2} + \|P_k(v)\|_{N_k(T)} \text{ if } k \geq 1.\end{aligned} \tag{9.4}$$

Let $\widetilde{v}$ denote an extension of $R_k(v)$ such that $\|\widetilde{v}\|_{N_k} \leq C\|v\|_{N_k(T)}$. Using (3.12), we may assume that $\widetilde{v}$ is supported in $\mathbb{R}\times[-T-2^{-k_+-10}, T+2^{-k_+-10}]$, $k \in \mathbb{Z}$. Indeed, let $\beta(t)$ be a smooth function such that

$$\beta(t) = 1, \text{ if } t \geq 1; \quad \beta(t) = 0, \text{ if } t \leq 0.$$

Thus $\beta(2^{k_++10}(t+T+2^{-k_+-10}))$, $\beta(-2^{k_++10}(t-T-2^{-k_+-10}))\in S_k$. Then we see that $\beta(2^{k_++10}(t+T+2^{-k_+-10}))\beta(-2^{k_++10}(t-T-2^{-k_+-10}))$ is supported in $[-T-2^{-k_+-10}, T+2^{-k_+-10}]$, and equal to 1 in $[-T,T]$. For $t\geq T$ we define

$$\widetilde{u}(t)=\eta_0(2^{k_++5}(t-T))[W(t-T)R_k(u(T))+\int_T^t W(t-s)(R_k(\widetilde{v}(s)))ds].$$

For $t\leq -T$ we define

$$\widetilde{u}(t)=\eta_0(2^{k_++5}(t+T))[W(t+T)R_k(u(-T))+\int_{-T}^t W(t-s)(R_k(\widetilde{v}(s)))ds].$$

For $t\in[-T,T]$ we define $\widetilde{u}(t)=u(t)$. It is clear that $\widetilde{u}$ is an extension of u. Also, using (3.12), we get

$$\|u\|_{F_k(T)}\lesssim \sup_{t_k\in[-T,T]}\|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k}. \tag{9.5}$$

Indeed, to prove (9.5), it suffices to prove that

$$\sup_{t_k\in\mathbb{R}}\|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k}\lesssim \sup_{t_k\in[-T,T]}\|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k}.$$

For $t_k>T$, since $\widetilde{u}$ is supported in $[-T-2^{-k_+-5}, T+2^{-k_+-5}]$, it is easy to see that $\widetilde{u}\eta_0(2^{k_+}(t-t_k))=\widetilde{u}\eta_0(2^{k_+}(t-T))\eta_0(2^{k_+}(t-t_k))$. Thus, we get from (3.10) that

$$\sup_{t_k>T}\|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k}\lesssim \sup_{t_k\in[-T,T]}\|\mathcal{F}[\widetilde{u}\cdot\eta_0(2^{k_+}(t-t_k))]\|_{B_k}.$$

Using the same method for $t_k<-T$, we obtain (9.5) as desired.

It remains to prove (9.4). In view of the definitions, (9.5) and (3.10), it suffices to prove that if $k\in\mathbb{Z}$, $\phi_k\in L^2$ with $\widehat{\phi_k}$ supported in I_k, and $v_k\in N_k$ then

$$\|\mathcal{F}[u_k\cdot\eta_0(2^{k_+}t)]\|_{B_k}\lesssim\|\phi_k\|_{L^2}+\|(\tau-\omega(\xi)+i2^{k_+})^{-1}\cdot\mathcal{F}(v_k)\|_{B_k},$$

where $u_k(t)=W(t)(\phi_k)+\int_0^t W(t-s)(v_k(s))ds$. We have

$$\mathcal{F}[u_k\cdot\eta_0(2^{k_+}t)](\xi,\tau)=\widehat{\phi_k}(\xi)\cdot 2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau-\omega(\xi)))$$
$$+C\int_{\mathbb{R}}\mathcal{F}(v_k)(\xi,\tau')\cdot\frac{2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau-\tau'))-2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau-\omega(\xi)))}{\tau'-\omega(\xi)}d\tau'.$$

We observe now that

$$\left|\frac{2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau-\tau'))-2^{-k_+}\widehat{\eta_0}(2^{-k_+}(\tau-\omega(\xi)))}{\tau'-\omega(\xi)}\cdot(\tau'-\omega(\xi)+i2^{k_+})\right|$$

$$\lesssim 2^{-k_+}(1+2^{-k_+}|\tau-\tau'|)^{-4}+2^{-k_+}(1+2^{-k_+}|\tau-\omega(\xi)|)^{-4}.$$

Using (3.8) and (3.9), we complete the proof of the proposition. □

Proposition 9.3. *Let* $0\le l\le 1/4$ *and* $s\ge 1/4$. *Then*

$$\begin{aligned}\|\partial_x(uvw)\|_{N^{l,s}(T)}\lesssim&\|u\|_{F^{l,s}(T)}\|v\|_{F^{1/4,1/4}(T)}\|w\|_{F^{1/4,1/4}(T)}\\&+\|u\|_{F^{1/4,1/4}(T)}\|v\|_{F^{l,s}(T)}\|w\|_{F^{1/4,1/4}(T)}\\&+\|u\|_{F^{1/4,1/4}(T)}\|v\|_{F^{1/4,1/4}(T)}\|w\|_{F^{l,s}(T)}.\end{aligned}\tag{9.6}$$

Proof. Since $R_kR_j=0$ if $k\neq j$, we can fix extensions $\widetilde{u},\widetilde{v},\widetilde{w}$ of u,v,w such that $\|R_k(\widetilde{u})\|_{F_k}\le 2\|R_k(u)\|_{F_k(T)}$, $\|R_k(\widetilde{v})\|_{F_k}\le 2\|R_k(v)\|_{F_k(T)}$ and $\|R_k(\widetilde{w})\|_{F_k}\le 2\|R_k(w)\|_{F_k(T)}$ for any $k\in\mathbb{Z}$. In view of the definition, we get

$$\|\partial_x(\widetilde{u}\widetilde{v}\widetilde{w})\|^2_{N^{l,s}}=\sum_{k_4=-\infty}^{-1}2^{2lk_4}\|R_{k_4}(\partial_x(\widetilde{u}\widetilde{v}\widetilde{w}))\|^2_{N_{k_4}}+\sum_{k_4=0}^{\infty}2^{2sk_4}\|R_{k_4}(\partial_x(\widetilde{u}\widetilde{v}\widetilde{w}))\|^2_{N_{k_4}}.$$

Let $\widetilde{u}_k=R_k(\widetilde{u})$, $\widetilde{v}_k=R_k(\widetilde{v})$ and $\widetilde{w}_k=R_k(\widetilde{u})$. Then we get

$$\|R_{k_4}(\partial_x(\widetilde{u}\widetilde{v}\widetilde{w}))\|_{N_{k_4}}\lesssim\sum_{k_1,k_2,k_3\in\mathbb{Z}}\|R_{k_4}(\partial_x(\widetilde{u}_{k_1}\widetilde{v}_{k_2}\widetilde{w}_{k_3}))\|_{N_{k_4}}.$$

From symmetry we may assume $k_1\le k_2\le k_3$. Dividing the summation into several parts, we get

$$\sum_{k_1\le k_2\le k_3}\|R_{k_4}(\partial_x(\widetilde{u}_{k_1}\widetilde{v}_{k_2}\widetilde{w}_{k_3}))\|_{N_{k_4}}\le\sum_{j=1}^{6}\sum_{\{k_i\}\in A_j}\|R_{k_4}(\partial_x(\widetilde{u}_{k_1}\widetilde{v}_{k_2}\widetilde{w}_{k_3}))\|_{N_{k_4}},\tag{9.7}$$

where A_j, $j=1,2,\cdots,6$, are as in the proof of Proposition 6.3. We will apply Proposition 7.1 obtained in Section 7 to bound the six terms in (9.7). For example, for the first term, from Proposition 7.1, we have

$$\begin{aligned}&\Big\|2^{sk_4}\sum_{\{k_i\}\in A_1}\|R_{k_4}(\partial_x(\widetilde{u}_{k_1}\widetilde{v}_{k_2}\widetilde{w}_{k_3}))\|_{N_{k_4}}\Big\|_{l^2_{k_4}}\\\le&C\Big\|2^{sk_4}\sum_{k_i\in A_1}\min(2^{k_1/2},|k_2|+1)\|\widetilde{u}_{k_1}\|_{F_{k_1}}\|\widetilde{v}_{k_2}\|_{F_{k_2}}\|\widetilde{w}_{k_3}\|_{F_{k_3}}\Big\|_{l^2_{k_4}}\\\le&\|\widetilde{u}\|_{F^{1/4,1/4}}\|\widetilde{v}\|_{F^{1/4,1/4}}\|\widetilde{w}\|_{F^{l,s}}.\end{aligned}$$

For the other terms we can handle them in similar ways. Therefore we have completed the proof of the proposition. □

We prove now Theorem 1.8. Fix $0 < l < 1/4$ and $s > 1/4$. By the scaling (1.4) we may assume that

$$\|\phi\|_{\dot H^l \cap \dot H^s} \leq \delta_0/M, \tag{9.8}$$

where $M \gg 1$ and δ_0 is given as in Proposition 8.1. For any $T' \in [0,1]$, we denote $X(T') = \|u\|_{E^{l,s}(T')} + \|\partial_x(u^3)\|_{N^{l,s}(T')}$. We assume first that $X(T')$ is continuous and satisfies

$$\lim_{T'\to 0} X(T') \lesssim \|\phi\|_{\dot H^l \cap \dot H^s}. \tag{9.9}$$

We will prove that

$$X(T') \leq 2c\delta_0/M, \qquad \text{for any } T' \in [0,1]. \tag{9.10}$$

By using a bootstrap (see e.g. [43]), we may assume that $X(T') \leq 3c\delta_0/M$ for any $T' \in [0,1]$. It follows from Proposition 9.2, 9.3 that for any $T' \in [0,1]$ we have

$$\begin{cases} \|u\|_{F^{l,s}(T')} \lesssim \|u\|_{E^{l,s}(T')} + \|\partial_x(u^3)\|_{N^{l,s}(T')}; \\ \|\partial_x(u^3)\|_{N^{l,s}(T')} \lesssim \|u\|^3_{F^{l,s}(T')}; \\ \|u\|^2_{E^{l,s}(T')} \lesssim \|\phi\|^2_{\dot H^l \cap \dot H^s} + \|u\|^6_{F^{l,s}(T')}. \end{cases} \tag{9.11}$$

Thus we get

$$X(T')^2 \lesssim \|\phi\|^2_{\dot H^l \cap \dot H^s} + X(T')^6,$$

from which, (9.8) and the assumption $X(T') \leq 3c\delta_0/M$, we obtain (9.10) as desired. Then, using (9.11), (9.10) and Proposition 9.1, we have

$$\|u\|_{C([-1,1];\dot H^l \cap \dot H^s)} \lesssim \|\phi\|_{\dot H^l \cap \dot H^s}.$$

For general ϕ and the L^2 norm of the solution, we just use the scaling (1.4) and the L^2 conservation law (1.6).

It remains to prove that $X(T)$ is continuous and (9.9). Obviously, for $u \in C([-T,T] : H^\infty)$ the first component $T' \to \|u\|_{E^{l,s}(T')}$ is increasing and continuous on $[-T,T]$ and

$$\lim_{T'\to 0} \|u\|_{E^{l,s}(T')} \lesssim \|\phi\|_{\dot H^l \cap \dot H^s}.$$

For the second component it follows from a similar argument as in the proof of Lemma 4.2 in [20]. We omit the details.

Remark 9.4. From the proof we see that we actually prove a stronger result than that stated in Theorem 1.8.

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