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POTENTIAL WELL AND EXACT BOUNDARY CONTROLLABILITY FOR SEMILINEAR WAVE EQUATIONS

Yi Zhou[1] and Ning-An Lai
School of Mathematical Sciences, Fudan University
Shanghai 200433, P. R. China

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Abstract. In this paper, we consider the exact boundary controllability for cubic focusing semilinear wave equations in $1 \le n \le 3$ space dimensions. When the initial data and the final data are in the so-called potential well, we find that the sufficient condition for the global existence is also sufficient to ensure the exact boundary controllability of the problem. Moreover, in one space dimension, the control time can be that of the linear wave equation.

1. Introduction and main results

The problem of controllability is clearly of significant practical interest. There are an extremely large number of publications on these topics. Some classical references are Lions [7] and Russell [10], and also see Zhang [14] for recent updates.

In this paper, we continue to study the exact boundary controllability problem for semilinear wave equations. In Zhou et al [19], we studied the global exact boundary controllability for semilinear wave equations with a defocusing nonlinearity. The aim of this paper is to study the focusing case. The problem can be described as follows: Let Ω be a bounded open subset of R^n (n= 1, 2 or 3), with a smooth boundary Γ. The equation under consideration takes the form

$$\Box u = u^3, \quad x \in \Omega, \quad 0 < t < T, \tag{1.1}$$

where $\Box = \partial_t^2 - \Delta$ is the wave operator, with $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ being the Laplacian.

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[1]Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education of China, P.R. China

Consider the initial data

$$u(0,x) = f_0(x), \quad u_t(0,x) = f_1(x), \quad x \in \Omega, \tag{1.2}$$

and the final data

$$u(T,x) = g_0(x), \quad u_t(T,x) = g_1(x), \quad x \in \Omega. \tag{1.3}$$

Assume the boundary Γ consists of two disjoint parts Γ_0 and Γ_1; we impose the following boundary conditions:

$$u(t,x) = 0, \quad x \in \Gamma_0, 0 < t < T, \tag{1.4}$$

$$u(t,x) = h(t,x), \quad x \in \Gamma_1, \quad 0 < t < T. \tag{1.5}$$

Then the problem of exact boundary controllability for the equation (1.1) is stated as follows: Given $T > 0$, is it possible to find a corresponding boundary control function $h(t,x)$ driving the equation (1.1) with the initial state (f_0, f_1) to the desired state (g_0, g_1) at time T?

Without any restriction on the initial and final data, the exact boundary control problem under consideration will be impossible. This is because the nonlinearity of the equation is of focusing type, so blow up in finite time can happen even for regular data. Thus, it is desirable to inspect which data lead to global existence and which data lead to blow up in finite time. For that purpose, we first consider the initial-boundary-value problem for (1.1) with initial condition (1.2) with the homogeneous Dirichlet boundary condition:

$$u(t,x) = 0, \quad x \in \Gamma, \quad t > 0. \tag{1.6}$$

For this problem, if $(f_0, f_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, then the initial-boundary-value problem is locally well posed in the space $C\big([0,T_*], H^2(\Omega)\big) \cap C^1\big([0,T_*], H^1(\Omega)\big)$ for small T_* and the energy $E(u(t))$ is conserved

$$\frac{dE(u(t))}{dt} = 0, \ t \in [0, T_*], \tag{1.7}$$

where

$$E(u(t)) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)} + J(u(t)), \tag{1.8}$$

and $\frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)}$ is the kinetic energy,

$$J(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{4}\|u\|^4_{L^4(\Omega)} \tag{1.9}$$

is the potential energy.

In this case, the problem of determining under what conditions on the initial data there will be global existence and under what conditions on the initial data there will be finite time blow up is known at least when the initial data is in the so called potential well. This kind of result is due to D. H. Sattinger [11] with many followers and recent developments (see [8], [12], [13], [6], [5]). To describe the potential well, we define the height of the potential well

$$d_0 = \inf_{v \neq 0} \{J(v) : I(v) = 0\}, \tag{1.10}$$

where

$$I(v) = \|\nabla v\|^2_{L^2(\Omega)} - \|v\|^4_{L^4(\Omega)}. \tag{1.11}$$

We will prove that $d_0 > 0$ in Section 2. We say initial data is in the potential well if

$$E(u(0)) < d_0. \tag{1.12}$$

In this case, it is now a classical result that if $I(f_0) \geq 0$, then a solution to the initial-boundary-value problem exists globally, while if $I(f_0) < 0$, then the solution blows up in finite time. It follows from the above discussion that if the initial data and final data are both in the potential well, then a necessary condition for the exact boundary controllability is

$$I(f_0) \geq 0, I(g_0) \geq 0. \tag{1.13}$$

We point out that, generally speaking, there exists a big difference between controllability problems and pure PDEs problems, see the discussions in [15].

To state our main theorem, we need to make the following assumption on the boundary $\partial\Omega$: $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and Γ_0, Γ_1 are nonempty. Furthermore there exists an x_0 such that

$$\begin{aligned} &(x - x_0) \cdot \nu(x) < 0, \ \forall x \in \Gamma_0, \\ &(x - x_0) \cdot \nu(x) \geq 0, \ \forall x \in \Gamma_1, \end{aligned} \tag{1.14}$$

where ν is the outward normal to Γ_0 and Γ_1. This means Γ_0 is concave and Γ_1 convex. Without loss of generality, we assume that $x_0 = 0$.

Since u is not necessarily zero on Γ_1, we modify $I(u)$ for

$$K(u) = \|\nabla u\|^2_{L^2(\Omega)} - \|u\|^4_{L^4(\Omega)} - 4 \int_{\Gamma_1} x \cdot \nu u^4 ds, \tag{1.15}$$

where ν is the outward unit normal to the boundary Γ_1, and ds denotes the induced Lebesgue measure on Γ_1. Then the corresponding height of the potential well is

$$d = \inf_{v \neq 0} \{J(v) : K(v) = 0\}. \tag{1.16}$$

Precisely, we will prove the following theorem.

Theorem 1.1. *Assume that* $\partial\Omega = \Gamma_0 \cup \Gamma_1$ *satisfies the condition* (1.14). *Suppose that* $f_0, g_0 \in W = \{f \in H^2(\Omega) : f = 0 \text{ on } \Gamma_0\}$ *and* $f_1, g_1 \in H_0^1(\Omega)$, *such that*

$$\tfrac{1}{2}\|f_1\|^2_{L^2(\Omega)} + J(f_0) < d, \quad K(f_0) \geq 0,$$
$$\tfrac{1}{2}\|g_1\|^2_{L^2(\Omega)} + J(g_0) < d, \quad K(g_0) \geq 0.$$

Then there exists a sufficiently large positive constant T_0 *depending only on* n, *the geometry of* Ω *and its boundary* Γ_0, Γ_1 *and a boundary control function* h, *such that the cubic focusing semi-linear wave equation* (1.1) *with the initial state* (1.2) *and the boundary conditions* (1.4), (1.5) *admits a unique solution* $u \in C\big([0,T], H^2(\Omega)\big) \cap C^1\big([0,T], H^1(\Omega)\big)$ *on the domain* $(0,T) \times \Omega$ *which satisfies the desired state* (1.3), *provided that* $T > T_0$.

In one space dimension, the time T_0 can be taken to be the same as that of the linear wave equation, see the discussions in Section 3. For the proof of Theorem 1.1, we apply a dissipative boundary feedback to stabilizing the system and then use the constructive method introduced by Zhou and Lei [18].

2. The potential well

In this section, we consider the minimizing problem (1.10). This problem is classical. However, for the convenience of the reader, we write down the proofs.

Lemma 2.1. *Suppose that* v *is not identically zero and* $I(v) \leq 0$. *Then*

$$\|\nabla v\|_{L^2(\Omega)} \geq \frac{1}{C_*^2}, \tag{2.1}$$

where C_* *is the Sobolev constant in the Sobolev inequality*

$$\|v\|_{L^4(\Omega)} \leq C_* \|\nabla v\|_{L^2(\Omega)}. \tag{2.2}$$

On the other hand, if $I(v) \geq 0$, *then*

$$J(v) \geq \tfrac{1}{4}\|\nabla v\|^2_{L^2(\Omega)}. \tag{2.3}$$

Proof. By the Sovolev embedding theorem and the fact that $I(v) \leq 0$ and we obtain

$$\|\nabla v\|_{L^2(\Omega)}^2 \leq \|v\|_{L^4(\Omega)}^4 \leq C_*^4 \|\nabla v\|_{L^2(\Omega)}^4, \tag{2.4}$$

then (2.1) follows.

For (2.3) one just has to recall the expressions for $I(v)$ and $J(v)$. □

We now prove that $d_0 > 0$. From the definition (1.10), (1.11) and Sobolev embedding theory, we have

$$||\nabla v||_{L^2(\Omega)}^2 = ||v||_{L^4(\Omega)}^4 \leq C_*^4 ||\nabla v||_{L^2(\Omega)}^4. \tag{2.5}$$

This implies

$$||\nabla v||_{L^2(\Omega)} \geq \frac{1}{C_*^2} > 0. \tag{2.6}$$

Then the desired result follows from (2.3) and (2.6). Obviously, d_0 is a constant depending only on the geometry of Ω.

Remark 2.2. Since $K(v) \leq 0$, by the Sobolev embedding theorem and trace theorem we have

$$\|\nabla v\|_{L^2(\Omega)}^2 \leq \|v\|_{L^4(\Omega)}^4 + 4 \int_{\Gamma_1} x \cdot \nu v^4 ds \leq C_*^4 \|\nabla v\|_{L^2(\Omega)}^4 + C \|\nabla v\|_{L^2(\Omega)}^4, \tag{2.7}$$

so if we replace $I(v)$ by $K(v)$ and d_0 by d in the proof above, then the results still hold with another different positive constant $\overline{C}_*$.

3. ONE-DIMENSIONAL SPACE CASE

In this section, we consider the one-dimensional space case

$$u_{tt} - u_{xx} = u^3, \quad 0 < x < L, \quad 0 < t < T, \tag{3.1}$$

with the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = h(t), \tag{3.2}$$

and initial conditions

$$u(x,0) = f_0(x), \quad u_t(0,x) = f_1(x). \tag{3.3}$$

We want to find a control function h that drives the solution to the desired state at time T:

$$u(x,T) = g_0(x), \quad u_t(x,T) = g_1(x). \tag{3.4}$$

Our main result can be stated as follows.

Theorem 3.1. *Suppose that* $f_0, g_0 \in H_0^1(0,L)$, $f_1, g_1 \in L^2(0,L)$, *such that*

$$\tfrac{1}{2}\|f_1\|^2_{L^2(0,L)} + J(f_0) < d_0, \quad I(f_0) \geq 0, \tag{3.5}$$

$$\tfrac{1}{2}\|g_1\|^2_{L^2(0,L)} + J(g_0) < d_0, \quad I(g_0) \geq 0. \tag{3.6}$$

Then there exists a boundary control function $h \in H^1([0,T])$, *such that the cubic focusing semi-linear wave equation* (3.1) *with the initial state* (3.3) *and the boundary conditions* (3.2) *admits a unique solution on the domain* $[0,L] \times (0,T)$ *which satisfies the desired state* (3.4), *provided that the time* $T > 2L$.

To prove Theorem 3.1, we use a constructive method of Li [17]. We first prove the following lemma.

Lemma 3.2. *Under the assumptions of Theorem* 3.1, *the initial-boundary-value problem*

$$u(x,0) = f_0(x), \quad u_t(0,x) = f_1(x), \tag{3.7}$$

$$u(0,t) = 0, \tag{3.8}$$

for equation (3.1), *admits a unique solution on the dependence domain* $D = \{0 < x < L-t, 0 < t < L\}$. *Moreover, there is a positive constant* c *depending only on* d_0 *such that*

$$\int_0^L (u_t - u_x)^2(t, L-t)dt \leq c^2. \tag{3.9}$$

To prove this lemma, we only need to consider the initial-boundary-value problem

$$\begin{cases} w_{tt} - w_{xx} = w^3, \ 0 < x < L, t > 0, \\ w(0,t) = 0, w(L,t) = 0, \ t > 0, \\ w(x,0) = f_0(x), w_t(x,0) = f_1(x), \ 0 < x < L. \end{cases} \tag{3.10}$$

We shall prove this problem has a global solution provided that $\frac{1}{2}\|f_1\|^2_{L^2(0,L)} + J(f_0) < d_0$ and $I(f_0) \geq 0$. Then by finite propagation speed, the restriction of w to D is just the solution u in Lemma 3.2. To prove that w exists globally, we first use local existence results to conclude that there is a small time T_* such that there exists a unique solution $w \in C\big([0,T_*], H_0^1(0,L)\big) \cap C^1\big([0,T_*], L^2(0,L)\big)$. Moreover, the energy is conserved:

$$E(w(t)) = \tfrac{1}{2}\|w_t(t)\|^2_{L^2(0,L)} + J(w(t)) = \tfrac{1}{2}\|f_1\|^2_{L^2(0,L)} + J(f_0). \tag{3.11}$$

First we claim that

$$I(w(t)) \geq 0 \quad 0 \leq t \leq T_*. \tag{3.12}$$

We shall argue by contradiction. Suppose that (3.12) is not true, then there exists $0 < t_1 \leq T_*$ such that $I(w(t_1)) < 0$. Let

$$t_* = \inf\{t \in [0, t_1) : I(w(t)) < 0\}. \tag{3.13}$$

Then by continuity $I(w(t_*)) = 0$ and there exists a sequence t_n such that $t_n > t_*, t_n \to t_*$ and $I(w(t_n)) < 0$. By Lemma 2.1, we have

$$\|\nabla w(t_n)\|_{L^2(0,L)} \geq \frac{1}{C_*^2}, \tag{3.14}$$

which implies

$$\|\nabla w(t_*)\|_{L^2(0,L)} \geq \frac{1}{C_*^2} > 0, \tag{3.15}$$

so $w(t_*) \neq 0$. But from the energy inequality (3.11), we have

$$E(w(t_*)) = \tfrac{1}{2}\|w_t(t_*)\|^2_{L^2(0,L)} + J(w(t_*)) \leq E(0) < d_0, \tag{3.16}$$

which implies $J(w(t_*)) < d_0$, a contradiction. Therefore, we proved that on the domain of existence, we always have $I(w(t)) \geq 0$, and by Lemma 2.1,

$$J(w(t)) \geq \tfrac{1}{4}\|\nabla w(t)\|^2_{L^2(0,L)}. \tag{3.17}$$

Thus, by the energy equality

$$\tfrac{1}{2}\|w_t(t)\|^2_{L^2(0,L)} + \tfrac{1}{4}\|\nabla w(t)\|^2_{H^1(0,L)} < d_0. \tag{3.18}$$

It follows that the solution exists globally. Finally (3.9) follows by an energy integration on D. Similarly, the backward problem

$$\begin{cases} v_{tt} - v_{xx} = v^3, \quad 0 < x < t + L - T, \quad T - L < t < T, \\ v(0,t) = 0, \ T - L \leq t \leq T, \\ v(x,T) = g_0(x), v_t(x,T) = g_1(x) \end{cases} \tag{3.19}$$

admits a global solution on the domain $D' = \{0 < x < t + L - T, T - L < t < T\}$. Moreover, there is a positive constant c' depending only on d_0 such that

$$\int_0^L (v_t + v_x)^2(t, t + L - T)dt \leq c'^2. \tag{3.20}$$

Having obtained solutions on D and D', we change the role of t and x and solve the Cauchy-Goursat problem

$$u_{xx} - u_{tt} + u^3 = 0, \tag{3.21}$$

where we prescribe the boundary conditions

$$\begin{cases} u = w(t) \text{ on } x = L - t, \\ u = v(t) \text{ on } x = t + L - T, \end{cases} \tag{3.22}$$

and let

$$u = 0 \quad \text{for} \quad L \le t \le T - L, \; x = 0. \tag{3.23}$$

However, the nonlinearity becomes defocusing and by standard energy methods there exists a unique global solution! Set

$$h = u(L, t), \quad \text{for} \quad 0 \le t \le T, \tag{3.24}$$

then we finish the proof of Theorem 3.1.

In a similar way, we can consider the two-sided control problem

$$u_{tt} - u_{xx} = u^3, \quad 0 < x < L, \quad 0 < t < T, \tag{3.25}$$

with the boundary conditions

$$u(0, t) = h_1(t), \quad u(L, t) = h_2(t), \tag{3.26}$$

and initial conditions

$$u(x, 0) = f_0(x), \quad u_t(0, x) = f_1(x). \tag{3.27}$$

We want to find two control functions h_1, h_2 that drive the solution to the desired state at time T:

$$u(x, T) = g_0(x), \quad u_t(x, T) = g_1(x). \tag{3.28}$$

Our main result can be stated as follows.

Theorem 3.3. *Suppose that $f_0, g_0 \in H_0^1(0, L)$, $f_1, g_1 \in L^2(0, L)$, such that*

$$\tfrac{1}{2}\|f_1\|^2_{L^2(0,L)} + J(f_0) < d_0, \quad I(f_0) \ge 0, \tag{3.29}$$

$$\tfrac{1}{2}\|g_1\|^2_{L^2(0,L)} + J(g_0) < d_0, \quad I(g_0) \ge 0 \tag{3.30}$$

then there exist two boundary control functions $h_1, h_2 \in H^1([0, T])$, such that the cubic focusing semi-linear wave equation (3.25) with the initial state (3.27) and the boundary conditions (3.26) admits a unique solution on the domain $[0, L] \times (0, T)$ which satisfies the desired state (3.28), provided that the time $T > L$.

4. Global existence and exponentially dissipative energy estimates for the focusing cubic semi-linear wave equation

Consider the mixed initial-boundary-value problem:

$$\begin{cases} \partial_{tt}u - \Delta u = u^3, \ t \geq 0, x \in \Omega, \\ u(0) = f_0, u_t(0) = f_1, \ x \in \Omega, \\ \partial_t u + \partial_\nu u = 0, \ t \geq 0, x \in \Gamma_1, \\ u = 0, \ t \geq 0, x \in \Gamma_0, \end{cases} \tag{4.1}$$

where $(f_0, f_1) \in W \times H_0^1(\Omega)$.

First we use local existence results to conclude that there is a small time T_* such that there exists a unique solution

$$u \in C\big([0, T_*], H^2(\Omega)\big) \cap C^1\big([0, T_*], H^1(\Omega)\big);$$

next we will prove the global existence and exponential dissipative energy estimates provided that $\frac{1}{2}\|f_1\|^2_{L^2\Omega} + J(f_0) < d$ and $K(f_0) \geq 0$. We shall need the following inequalities:

$$\exists \alpha > 0, \ \int_\Omega u^2 dx \leq \alpha^2 \int_\Omega |\nabla u|^2 dx, \ \forall u \in V, \tag{4.2}$$

$$\exists \beta > 0, \ \int_{\Gamma_1} u^2 dx \leq \beta^2 \int_\Omega |\nabla u|^2 dx, \ \forall u \in V, \tag{4.3}$$

and

$$\exists \gamma > 0, \ \int_{\Gamma_1} u_t^2 dx \leq \gamma^2 \int_\Omega |\nabla u_t|^2 dx, \ \forall u \in V, \tag{4.4}$$

where $V = \{u \in C\big([0, T_*], H^2(\Omega)\big) \cap C^1\big([0, T_*], H^1(\Omega)\big) : u = 0 \ on \ \Gamma_0\}$ is a subspace of the Sobolev space $H^2(\Omega)$ for each $t \in [0, T_*]$.

If $u = u(t, x)$ is a solution of system (4.1), then we define its energy by

$$E(u(t)) = \frac{1}{2} \int_\Omega (u_t^2 + |\nabla u|^2 - \frac{1}{2} u^4) dx, \ \forall t \geq 0, \tag{4.5}$$

and the energy decreases with time as we have

$$\frac{1}{2} \frac{d}{dt} \big(\|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{2}\|u\|^4_{L^4(\Omega)}\big) + \|u_t\|^2_{L^2(\Gamma_1)} = 0, \tag{4.6}$$

which is obtained by taking the L^2 inner product of the equation in system (4.1) with u_t and using the boundary condition. And so

$$E(u(t)) = \tfrac{1}{2}\|u_t\|^2_{L^2(\Omega)} + J(u(t)) \leq E(0) = \tfrac{1}{2}\|f_1\|^2_{L^2(\Omega)} + J(f_0). \tag{4.7}$$

As above we claim that

$$K(u(t)) \geq 0,\ \ 0 \leq t \leq T_*. \tag{4.8}$$

We shall prove this by contradiction too. Suppose that (4.8) is not true, then there exists $0 < t_1 \leq T_*$ such that $K(u(t_1)) < 0$. Let

$$t_* = \inf\{t \in [0, t_1) : K(u(t)) < 0\}. \tag{4.9}$$

Then by continuity $K(u(t_*)) = 0$ and there exists a sequence t_n such that $t_n > t_*, t_n \to t_*$ and $K(u(t_n)) < 0$. By 2.1, we have

$$\|\nabla u(t_n)\|_{L^2(\Omega)} \geq \frac{1}{C_*^2}, \tag{4.10}$$

which implies

$$\|\nabla u(t_*)\|_{L^2(\Omega)} \geq \frac{1}{C_*^2} > 0, \tag{4.11}$$

so $u(t_*) \neq 0$. But by the energy inequality (4.7), we have

$$E(u(t_*)) = \tfrac{1}{2}\|u_t(t_*)\|^2_{L^2(\Omega)} + J(u(t_*)) \leq E(0) < d, \tag{4.12}$$

which implies

$$J(u(t_*)) < d, \tag{4.13}$$

but this is in contradiction with the definition of d. So we have proved that, on the domain of existence, we always have $K(u(t)) \geq 0$ and, by Lemma 2.1,

$$J(u) \geq \tfrac{1}{4}\|\nabla u\|^2_{L^2(\Omega)}, \tag{4.14}$$

then by the energy inequality we have

$$\tfrac{1}{2}\|u_t\|^2_{L^2(\Omega)} + \tfrac{1}{4}\|\nabla u\|^2_{L^2(\Omega)} < d. \tag{4.15}$$

Applying ∂_t to system (4.1) and taking the L^2 inner product of the resulting equation with $\partial_{tt}u$, we get

$$\tfrac{1}{2}\partial_t u_{tt}^2 - \nabla \cdot (\nabla u_t u_{tt}) + \tfrac{1}{2}\partial_t|\nabla u_t|^2 = 3u^2 u_t u_{tt}. \tag{4.16}$$

Integrating this equality and using the boundary conditions $u_{tt} + \partial_\nu u_t = 0, x \in \Gamma_1$ we obtain the second-order energy inequality

$$\frac{1}{2}\frac{d}{dt}\big(\|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)}\big) + \|u_{tt}\|^2_{L^2(\Gamma_1)} = \int_\Omega 3u^2 u_t u_{tt} dx. \tag{4.17}$$

By the Hölder inequality and Sobolev embedding theorem, we deduce

$$\frac{d}{dt}\big(\|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)}\big) \leq C\|\nabla u\|^2_{L^2(\Omega)}\|\nabla u_t\|_{L^2(\Omega)}\|u_{tt}\|_{L^2(\Omega)}$$

$$\leq C\|\nabla u\|^2_{L^2(\Omega)}\big(\|u_{tt}\|^2_{L^2(\Omega)}+\|\nabla u_t\|^2_{L^2(\Omega)}\big),$$

which implies

$$\ln\frac{\|u_{tt}(t)\|^2_{L^2(\Omega)}+\|\nabla u_t(t)\|^2_{L^2(\Omega)}}{\|u_{tt}(0)\|^2_{L^2(\Omega)}+\|\nabla u_t(0)\|^2_{L^2(\Omega)}}\leq C\int_0^t\|\nabla u(s)\|^2_{L^2(\Omega)}ds. \tag{4.18}$$

Since $u_{tt}(0)=\Delta u(0)+f_0^3, u_t(0)=f_1$, (4.18) means

$$\begin{aligned}&\|u_{tt}(t)\|^2_{L^2(\Omega)}+\|\nabla u_t(t)\|^2_{L^2(\Omega)}\\ \leq& C\big(\|f_0\|^2_{H^2(\Omega)}+\|f_0\|^6_{H^1(\Omega)}+\|f_1\|^2_{H^1(\Omega)}\big)e^{C\int_0^t\|\nabla u(s)\|^2_{L^2(\Omega)}ds},\end{aligned} \tag{4.19}$$

where C is a positive constant which may change from one place to another.

In order to prove global existence, we need the following lemma.

Lemma 4.1. *Suppose that Ω is a bounded domain with smooth boundary $\partial\Omega=\Gamma_0\cup\Gamma_1$ and Γ_0,Γ_1 are nonempty. Consider the boundary value problem*

$$\begin{cases}-\Delta v=f,\ x\in\Omega,\\ \dfrac{\partial v}{\partial\nu}=g,\ x\in\Gamma_1,\\ v=0,\ x\in\Gamma_0.\end{cases} \tag{4.20}$$

Then, for any given $f\in L^2(\Omega)$ and $g\in H^{\frac12}(\partial\Omega)$, system (4.20) admits a solution $v\in H^2(\Omega)$ satisfying

$$\|v\|_{H^2(\Omega)}\leq C\big(\|f\|_{L^2(\Omega)}+\|g\|_{H^{\frac12}(\partial\Omega)}+\|v\|_{L^2(\Omega)}\big). \tag{4.21}$$

Sketch of the proof: Since the boundary $\partial\Omega$ is smooth, there exists for each point $x_0\in\partial\Omega$, a ball $B(x_0)$ and a one-to-one smooth mapping Φ from B onto an open set $D\subset R^n$ such that $\Phi(B\cap\Omega)\subset R^n_+=\{x\in R^n: x_n>0\}, \Phi(B\cap\partial\Omega)\subset\partial R^n_+$. By a partition of unity, the problem reduces to the upper plane. To illustrate the main idea of the proof, we consider the following system for simplicity:

$$\begin{cases}-\Delta\widetilde{v}=\widetilde{f},\ x\in R^n_+,\\ \dfrac{\partial\widetilde{v}}{\partial x_n}=\widetilde{g},\ x_1>0,\ x_n=0,\\ \widetilde{v}=0,\ x_1<0,\ x_n=0.\end{cases} \tag{4.22}$$

According to the Lax-Milgram theorems this system has a weak solution $\widetilde{v} \in H^1(R^n_+)$. Formally, $\frac{\partial \widetilde{v}}{\partial x_n}$ satisfies the system below:

$$\begin{cases} -\Delta \widetilde{w} = \dfrac{\partial \widetilde{f}}{\partial x_n},\ x \in R^n_+, \\ \dfrac{\partial \widetilde{w}}{\partial x_n} = -\Delta_{n-1}\widetilde{v} - \widetilde{f} = -\widetilde{f},\ x_1 < 0,\ x_n = 0, \\ \widetilde{w} = \widetilde{g},\ x_1 > 0,\ x_n = 0. \end{cases} \tag{4.23}$$

Again, by Lax-Milgram theorems, there exists a unique solution $\widetilde{w} \in H^1(R^n_+)$. Approximating $\frac{\partial \widetilde{v}}{\partial x_n}$ by difference quotients, we can prove that it indeed equals $\widetilde{w}$. Thus, we get $\frac{\partial \widetilde{v}}{\partial x_n} \in H^1(R^n_+)$. By a similar argument, we can conclude that the tangential derivatives of $\widetilde{v}$ belong to $H^1(R^n_+)$, and we finish our proof.

If we regard system (4.1) as the boundary-value problem of elliptic equations:

$$\begin{cases} -\Delta u = -u_{tt} + u^3,\ x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = -u_t,\ x \in \Gamma_1, \\ u = 0,\ x \in \Gamma_0, \end{cases} \tag{4.24}$$

then using inequality (4.21), together with Sobolev embedding theory and trace theory, we obtain

$$\begin{aligned} \|u\|^2_{H^2(\Omega)} &\le C\big(\|u_{tt} + u^3\|^2_{L^2(\Omega)} + \|-u_t\|^2_{H^{\frac{1}{2}}(\partial\Omega)} + \|u\|^2_{L^2(\Omega)}\big) \\ &\le C\big(\|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u\|^6_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\big). \end{aligned} \tag{4.25}$$

Then, combining (4.19) and (4.25), we get prime estimates of $\|u(t)\|_{H^2(\Omega)}$. The boundedness of $\|u(t)\|_{H^1(\Omega)}$ can be easily obtained from (4.15), and we have proved the global existence.

Next we will show the exponentially dissipative energy estimates; our main result is as follows.

Theorem 4.2. *Let u be a solution of system* (4.1) *and suppose $\partial\Omega = \Gamma_0 \cup \Gamma_1$ satisfies the condition* (1.14)*; then there exist constants $C, \omega > 0$ depending only on the domain Ω and the dimension n, such that for initial data $(f_0, f_1) \in W \times H^1_0(\Omega)$ which satisfies $K(f_0) \ge 0$, the energy of the solution of system* (4.1) *satisfies the inequality*

$$E(u(t)) \le Ce^{-\omega t}E(0),\ \forall t \ge 0. \tag{4.26}$$

Proof. For the one-dimensional space case we have obtained the result in Section 3; in the following we just need to finish the proof when the dimension n equals 2 or 3. From $K(f_0) \geq 0$, we easily get

$$K(u(t)) \geq 0, \ \forall t \geq 0, \tag{4.27}$$

which implies $\|u\|^4_{L^4(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)}$, and furthermore we have

$$\begin{aligned}&\tfrac{1}{4}(\|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})\\ &\leq E(u(t)) = \tfrac{1}{2}(\|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|u\|^4_{L^4(\Omega)})\\ &\leq \tfrac{1}{2}(\|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}).\end{aligned} \tag{4.28}$$

We get the energy estimate of Morawetz type by taking the L^2 inner product of the equation in system (4.1) with $x \cdot \nabla u$:

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (x\cdot\nabla u)u_t dx - \int_\Omega x\cdot\nabla u_t u_t dx - \int_\Omega \Delta u x\cdot\nabla u dx - \int_\Omega u^3 x\cdot\nabla u dx\\ =&\frac{d}{dt}\int_\Omega (x\cdot\nabla u)u_t dx - \frac{1}{2}\int_\Omega x\cdot(u_t^2)dx - \int_\Omega \nabla_k(\nabla_k u x\cdot\nabla u)\\ &+\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega x\cdot\nabla(|\nabla u|^2)dx - \frac{1}{4}\int_\Omega x\cdot\nabla(u^4)dx = 0.\end{aligned} \tag{4.29}$$

By integration by parts and using the boundary condition, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (x\cdot\nabla u)u_t dx - \frac{1}{2}\int_{\Gamma_0} x\cdot\nu(\partial_\nu u)^2 ds - \int_{\Gamma_1} x\cdot\nu u_t^2 ds\\ &+\frac{3}{2}\|u_t\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{4}\int_{\Gamma_1} x\cdot\nu u^4 ds + \frac{3}{4}\|u\|^4_{L^4(\Omega)} = 0.\end{aligned} \tag{4.30}$$

By taking the L^2 inner product of the equation in system (4.1) with u, we get

$$\begin{aligned}\int_\Omega (u_t^2 - |\nabla u|^2)dx &= \frac{d}{dt}\int_\Omega u u_t dx - \int_\Omega \big(u(\Delta u + u^3) + |\nabla u|^2\big)dx\\ &= \frac{d}{dt}\int_\Omega u u_t dx + \int_{\Gamma_1} u u_t ds - \|u\|^4_{L^4(\Omega)},\end{aligned} \tag{4.31}$$

where the boundary condition is used.

Combining (4.30) and (4.31), we obtain

$$E(u(t)) = \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2 - \frac{1}{2}u^4)dx$$

$$
\begin{aligned}
&= -\frac{d}{dt}\int_\Omega u_t(u + x\cdot\nabla u)dx + \frac{1}{2}\int_{\Gamma_0} x\cdot\nu(\partial_\nu u)^2 ds \\
&+ \int_{\Gamma_1} x\cdot\nu u_t^2 ds + \frac{1}{4}\int_{\Gamma_1} x\cdot\nu u^4 ds - \int_{\Gamma_1} u u_t ds \qquad (4.32)\\
&\le -\frac{d}{dt}\int_\Omega u_t(u + x\cdot\nabla u)dx + \int_{\Gamma_1} x\cdot\nu u_t^2 ds + \frac{1}{16}\|\nabla u\|^2_{L^2(\Omega)} - \int_{\Gamma_1} u u_t ds.
\end{aligned}
$$

In order to deal with the boundary terms such as $\int_{\Gamma_1} x\cdot\nu u_t^2 ds$ and prove the exponentially decay we introduce the functional as in [4]

$$E_\varepsilon\big(u(t)\big) = E\big(u(t)\big) + \varepsilon P\big(u(t)\big) \qquad (4.33)$$

with

$$P\big(u(t)\big) = \int_\Omega u_t(t,x)[2x\cdot\nabla u + 2u(t,x)]dx,\ \ t\ge 0, \qquad (4.34)$$

and $\varepsilon > 0$ a fixed constant.

Setting $m = \sup_{x\in\Omega}|x|$, by (4.2) and (4.28) we easily get

$$
\begin{aligned}
&\varepsilon^{-1}|E_\varepsilon\big(u(t)\big) - E\big(u(t)\big)| = |\int_\Omega u_t(2x\cdot\nabla u + 2u)| \\
\le&\|u_t\|_{L^2(\Omega)}(2m\|\nabla u\|_{L^2(\Omega)} + 2\|u\|_{L^2(\Omega)}) \\
\le&(2m + 2\alpha)\|u_t\|_{L^2(\Omega)}|\nabla u\|_{L^2(\Omega)} \le C_1 E\big(u(t)\big), \qquad (4.35)
\end{aligned}
$$

which implies

$$|E_\varepsilon\big(u(t)\big) - E\big(u(t)\big)| \le C_1\varepsilon E\big(u(t)\big),\ \forall t\ge 0,\ \forall\varepsilon > 0, \qquad (4.36)$$

where C_1 is a constant depending only on the dimension n and the domain Ω. Then we can choose $0 < \varepsilon < \frac{1}{C_1}$ such that

$$(1 - C_1\varepsilon)E(u(t)) \le E_\varepsilon(u(t)) \le (1 + C_1\varepsilon)E(u(t)); \qquad (4.37)$$

that is, $E(u(t)) \sim E_\varepsilon(u(t))$. Differentiating (4.33), and combining (4.6), we obtain

$$E'_\varepsilon\big(u(t)\big) = E'\big(u(t)\big) + \varepsilon P'\big(u(t)\big), \qquad (4.38)$$

with

$$E'\big(u(t)\big) = -\|u_t\|^2_{L^2(\Gamma_1)}, \qquad (4.39)$$

$$P'\big(u(t)\big) = \int_\Omega (2u_{tt}x\cdot\nabla u + 2u_{tt}u + 2u_t x\cdot\nabla u_t + 2u_t^2)dx, \qquad (4.40)$$

for all $t \geq 0$. Applying the divergence theorem and using the equation and inequality (4.3), we obtain

$$\begin{aligned} 2\int_\Omega u_{tt} u dx =& 2\int_\Omega (\Delta u + u^3) u dx \\ =& 2\int_\Omega \nabla \cdot (\nabla u u) dx - 2\|\nabla u\|^2_{L^2(\Omega)} + 2\int_\Omega u^4 dx \\ =& -2\int_{\Gamma_1} u u_t ds - 2\|\nabla u\|^2_{L^2(\Omega)} + 2\int_\Omega u^4 dx \\ \leq& \frac{1}{8\beta^2}\int_{\Gamma_1} u^2 ds + 8\beta^2 \int_{\Gamma_1} u_t^2 ds - 2\|\nabla u\|^2_{L^2(\Omega)} + 2\int_\Omega u^4 dx \\ \leq& -\frac{15}{8}\|\nabla u\|^2_{L^2(\Omega)} + 4\beta^2 \int_{\Gamma_1} u_t^2 ds + 2\int_\Omega u^4 dx, \end{aligned} \tag{4.41}$$

$$\begin{aligned} & 2\int_\Omega u_{tt} x\cdot \nabla u dx = 2\int_\Omega (\Delta u + u^3) x \cdot \nabla u dx \\ =& 2\int_\Omega \nabla_k (\nabla_k u x \cdot \nabla u) dx - 2\|\nabla u\|^2_{L^2(\Omega)} - \int_\Omega x\cdot \nabla(|\nabla u|^2) dx + \tfrac{1}{2}\int_\Omega x \cdot \nabla (u^4) dx \\ =& 2\int_{\Gamma_0} x\cdot \nu (\partial_\nu u)^2 ds + 2\int_{\Gamma_1} x\cdot\nu u_t^2 ds - 2\|\nabla u\|^2_{L^2(\Omega)} - \int_{\Gamma_0} x\cdot \nu (\partial_\nu u)^2 ds \\ & - \int_{\Gamma_1} x \cdot \nu u_t^2 ds + 3\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{\Gamma_1} x\cdot \nu u^4 ds - \frac{3}{2}\int_\Omega u^4 dx \\ =& \int_{\Gamma_0} x \cdot \nu (\partial_\nu u)^2 ds + \int_{\Gamma_1} x\cdot \nu u_t^2 ds \\ & + \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{\Gamma_1} x\cdot\nu u^4 ds - \frac{3}{2}\int_\Omega u^4 dx \\ \leq& \int_{\Gamma_1} x\cdot\nu u_t^2 ds + \|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{8}\|\nabla u\|^2_{L^2(\Omega)} - \frac{3}{2}\int_\Omega u^4 dx \\ =& \int_{\Gamma_1} x\cdot \nu u_t^2 ds + \frac{9}{8}\|\nabla u\|^2_{L^2(\Omega)} - \frac{3}{2}\int_\Omega u^4 dx, \end{aligned} \tag{4.42}$$

noting that $K(u(t)) \geq 0, t \geq 0$ and $x \cdot \nu < 0$ on Γ_0.

For the third term we have

$$2\int_\Omega u_t x\cdot \nabla u_t dx = \int_\Omega \nabla \cdot (x u_t^2) dx - 3\|u_t\|^2_{L^2(\Omega)} = \int_{\Gamma_1} x \cdot \nu u_t^2 ds - 3\|u_t\|^2_{L^2(\Omega)}. \tag{4.43}$$

From (4.38)-(4.43) we conclude that

$$\begin{aligned} &E'_\varepsilon\big(u(t)\big) \le -\|u_t\|^2_{L^2(\Gamma_1)} + \varepsilon[8\beta^2 \int_{\Gamma_1} u_t^2 ds + 2\int_{\Gamma_1} x\cdot \nu u_t^2 ds \\ &- \|u_t\|^2_{L^2(\Omega)} - \frac{3}{4}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^4_{L^4(\Omega)}] \\ &\le (-1+8\beta^2\varepsilon + 2m\varepsilon)\|u_t\|^2_{L^2(\Gamma_1)} + \varepsilon(-\|u_t\|^2_{L^2(\Omega)} \\ &- \frac{3}{4}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4}\|u\|^4_{L^4(\Omega)} + \frac{1}{4}\|\nabla u\|^2_{L^2(\Omega)}) \\ &\le -\varepsilon E\big(u(t)\big), \ \forall t \ge 0, \end{aligned} \tag{4.44}$$

if we choose $\varepsilon \le \frac{1}{8\beta^2+2m}$.

Let $C_2 = \min\{\frac{1}{C_1}, \frac{1}{8\beta^2+2m}\}$. If we fix $0 < \varepsilon < C_2$, then both (4.37) and (4.44) hold, and by direct computation we get

$$E(u(t)) \le \frac{1+C_1\varepsilon}{1-C_1\varepsilon} e^{-\frac{\varepsilon}{1+C_1\varepsilon}t} E(0), \tag{4.45}$$

so we have completed the proof of Theorem 4.2 with $C = \frac{1+C_1\varepsilon}{1-C_1\varepsilon}$ and $\omega = \frac{\varepsilon}{1+C_1\varepsilon}$. □

Define the second-order energy by

$$\overline{E}\big(u(t)\big) = \frac{1}{2}(\|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)}). \tag{4.46}$$

Thanks to Theorem 4.2, we can also prove the exponentially dissipative estimates of the second-order energy; that is, we have the following.

Theorem 4.3. *Let u be a solution of system* (4.1) *and suppose $\partial\Omega = \Gamma_0 \cup \Gamma_1$ satisfies the condition* (1.14)*; then there exists a constant $\delta > 0$ and a constant $C_3 > 0$ depending only on the domain Ω, the dimension n, $E(0)$, ω and δ, such that for the initial data $(f_0, f_1) \in W \times H_0^1(\Omega)$ which satisfies $K(f_0) \ge 0$, the second-order energy of the solution of system* (4.1) *satisfies the inequality*

$$\overline{E}(u(t)) \le C_3 e^{-\delta t}\overline{E}(u(0)), \ \forall t \ge 0, \tag{4.47}$$

Proof. Applying ∂_t to system (4.1) and taking the L^2 inner product of the resulting equations with u_{tt}, then using the boundary condition $u_{tt} + \partial_\nu u_t = 0, x \in \Gamma_1$ and the Sobolev embedding theorem, we obtain the second-order energy estimates:

$$\frac{1}{2}\frac{d}{dt}(\|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)}) + \|u_{tt}\|^2_{L^2(\Gamma_1)}$$

$$
\begin{aligned}
&= 3\int_\Omega u^2 u_t u_{tt} dx \le 3\|u\|^2_{L^6(\Omega)}\|u_t\|_{L^6(\Omega)}\|u_{tt}\|_{L^2(\Omega)} \\
&\le C_4\|\nabla u\|^2_{L^2(\Omega)}\|\nabla u_t\|_{L^2(\Omega)}\|u_{tt}\|_{L^2(\Omega)} \le C_5 e^{-\omega t}\overline{E}\big(u(t)\big), \qquad (4.48)
\end{aligned}
$$

where C_5 is a positive constant depending on the domain Ω, the dimension n and $E(0)$. This inequality means that there exists a positive constant $C_6 > 0$ depending on the domain Ω, the dimension n and $E(0)$ such that the inequality $\overline{E}\big(u(t)\big) \le C_6\overline{E}(u(0))$ holds for all $t \ge 0$.

Applying ∂_t to system (4.1) and taking the L^2 inner product of the resulting equations with $x\cdot\nabla u_t$, then using the boundary condition $u_{tt} + \partial_\nu u_t = 0, x \in \Gamma_1$ and the Sobolev embedding theorem, we get second-order Morawetz's energy estimates:

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega x\cdot\nabla u_t u_{tt} - \int_{\Gamma_1} x\cdot\nu u_{tt}^2 ds + \frac{3}{2}\|u_{tt}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla u_t\|^2_{L^2(\Omega)} \\
&= \int_{\Gamma_0} x\cdot\nu\partial_\nu u_t^2 ds + 3\int_\Omega u^2 u_t x\cdot\nabla u_t dx \le C_7 e^{-\omega t}\overline{E}\big(u(t)\big), \qquad (4.49)
\end{aligned}
$$

where C_7 is a positive constant depending on the domain Ω, the dimension n and $E(0)$.

Applying ∂_t to system (4.1) and taking the L^2 inner product of the resulting equation with u_t, then using the boundary condition $u_{tt} + \partial_\nu u_t = 0, x \in \Gamma_1$ and the Sobolev embedding theorem, we get the following estimates:

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t u_{tt} dx - \|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)} + \int_{\Gamma_1} u_t u_{tt} ds \\
&= 3\int_\Omega u^2 u_t^2 dx \le 3\|u\|^2_{L^4(\Omega)}\|u_t\|^2_{L^4(\Omega)} \le C_8 e^{-\omega t}\overline{E}\big(u(t)\big), \qquad (4.50)
\end{aligned}
$$

where C_8 is a positive constant depending on the domain Ω, the dimension n and $E(0)$.

The combination of (4.49) and (4.50) shows that

$$
\begin{aligned}
\overline{E}\big(u(t)\big) =& \frac{1}{2}(\|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)}) \\
\le& -\frac{d}{dt}\int_\Omega u_{tt}(x\cdot\nabla u_t + u_t)dx \\
&+ \int_{\Gamma_1} x\cdot\nu u_{tt}^2 ds - \int_{\Gamma_1} u_t u_{tt} ds + C_9 e^{-\omega t}\overline{E}\big(u(t)\big) \\
\le& -\frac{d}{dt}\int_\Omega u_{tt}(x\cdot\nabla u_t + u_t)dx + m\|u_{tt}\|^2_{L^2(\Gamma_1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4\gamma^2}\int_{\Gamma_1} u_t^2 ds + 2\gamma^2\int_{\Gamma_1} u_{tt}^2 ds + C_9 e^{-\omega t}\overline{E}\big(u(t)\big) \\
\le &-\frac{d}{dt}\int_\Omega u_{tt}(x\cdot\nabla u_t + u_t)dx + \frac{1}{4}\|\nabla u_t\|^2_{L^2(\Omega)} \\
&+(2\gamma^2+m)\|u_{tt}\|^2_{L^2(\Gamma_1)} + C_9 e^{-\omega t}\overline{E}\big(u(t)\big), \qquad (4.51)
\end{aligned}
$$

where C_9 is a positive constant depending on the domain Ω, the dimension n and $E(0)$. This inequality implies

$$
\begin{aligned}
\frac{1}{2}\overline{E}\big(u(t)\big) \le &-\frac{d}{dt}\int_\Omega u_{tt}(x\cdot\nabla u_t + u_t)dx \\
&-(2\gamma^2+m)\frac{d}{dt}\overline{E}\big(u(t)\big) + C_9 e^{-\omega t}\overline{E}\big(u(t)\big). \qquad (4.52)
\end{aligned}
$$

On the other hand, a straightforward calculation shows

$$
\begin{aligned}
&\Big|\int_\Omega u_{tt}(x\cdot\nabla u_t + u_t)dx\Big| \\
\le&\frac{1}{2}\|u_{tt}\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega (x\cdot\nabla u_t + u_t)^2 dx \le C_{10}\overline{E}\big(u(t)\big), \qquad (4.53)
\end{aligned}
$$

where C_{10} is a positive constant depending only on the domain Ω and the dimension n. From (4.48), for any $0 < S < T$ we have

$$
\overline{E}\big(u(T)\big) - \overline{E}\big(u(S)\big) \le \int_S^T C_5 e^{-\omega t}\overline{E}\big(u(t)\big)dt. \qquad (4.54)
$$

Then (4.52)-(4.54) implies that, for any $0 \le S \le T$,

$$
\int_S^T \overline{E}(t)dt \le C_{11}\overline{E}(S) + \int_S^T C_{12}e^{-\omega t}\overline{E}(t)dt, \qquad (4.55)
$$

where C_{11}, C_{12} are positive constants depending on the domain Ω, the dimension n and $E(0)$.

Let $T \to \infty$; by the inequality (4.55), we have

$$
\begin{aligned}
\int_S^\infty \overline{E}(t)dt &\le C_{11}\overline{E}(S) + \int_S^\infty C_{12}e^{-\omega t}C_6\overline{E}(u(0))dt \\
&\le C_{11}\overline{E}(S) + C_{13}e^{-\omega S}\overline{E}(0), \qquad (4.56)
\end{aligned}
$$

where C_{13} is a positive constant depending on the domain Ω, the dimension n, $E(0)$ and ω.

Obviously, there exists a positive constant $\delta > 0$ such that $\delta < \omega$ and $\frac{1}{\delta} > C_{11}$. Therefore, (4.56) implies

$$\int_S^\infty \overline{E}(t)dt \leq \frac{1}{\delta}\overline{E}(S) + C_{13}e^{-\omega S}\overline{E}(0). \tag{4.57}$$

Let $\overline{M}(s) = e^{\delta S}\int_S^\infty \overline{E}(t)dt$; then from (4.57) we conclude that

$$\overline{M}'(S) \leq C_{14}e^{(\delta-\omega)S}, \tag{4.58}$$

with $C_{14} > 0$ depending on the domain Ω, the dimension n, $E(0)$, ω and δ. This implies

$$\int_S^{S+1} \overline{E}(t)dt \leq \int_S^\infty \overline{E}(t)dt \leq C_{15}e^{-\delta S}\overline{E}(0), \tag{4.59}$$

with $C_{15} > 0$ depending on the domain Ω, the dimension n, $E(0)$, ω and δ. Here we use the inequality (4.57) when $S = 0$.

From (4.48), we have $\frac{d}{dt}\overline{E}(t) \leq C_{16}e^{-\omega t}\overline{E}(0)$, with C_{16} depending on the domain Ω, the dimension n and $E(0)$, which means that, for any $0 \leq \theta \leq 1$,

$$\overline{E}(S+1) - \overline{E}(S+\theta) \leq C_{17}e^{-\omega S}\overline{E}(0), \tag{4.60}$$

where $C_{17} > 0$ depends on the domain Ω, the dimension n, $E(0)$ and ω.

Combining (4.59) and (4.60), we obtain

$$\overline{E}(S+1) \leq C_{18}e^{-\delta S}\overline{E}(0),\ \forall S \geq 0, \tag{4.61}$$

where $C_{18} > 0$ depends on the domain Ω, the dimension n, $E(0)$, ω and δ. This inequality means there exists a positive constant C_3 depending only on $\delta, \|f_0\|_{H^1(\Omega)}, \|f_1\|_{L^2(\Omega)}$ such that

$$\overline{E}(u(t)) \leq C_3e^{-\delta t}\overline{E}(0),\ \forall t \geq 0, \tag{4.62}$$

and we have completed the proof of Theorem 4.3. □

Now let us take a look at the inverted initial-boundary-value problem:

$$\begin{cases} \partial_{tt}u - \Delta u = u^3,\ t \geq 0, x \in \Omega, \\ u(T) = g_0, u_t(T) = g_1,\ x \in \Omega, \\ -\partial_t u + \partial_\nu u = 0,\ 0 \leq t \leq T, x \in \Gamma_1, \\ u = 0,\ 0 \leq t \leq T, x \in \Gamma_0. \end{cases} \tag{4.63}$$

We make a change of variables $t \to T - t$ and then system (4.63) converts into the problem (4.1). According to Theorem 4.2 and Theorem 4.3, we have the following theorems for system (4.63).

Theorem 4.4. *Let u be a solution of system* (4.63) *and suppose $\partial\Omega = \Gamma_0 \cup \Gamma_1$ satisfies the condition* (1.14)*; then there exist constants $C_{19}, \omega > 0$ depending only on the domain Ω and the dimension n, such that for the initial data $(g_0, g_1) \in W \times H_0^1(\Omega)$ satisfying $K(g_0) \geq 0$, the energy of the solution of system* (4.63) *satisfies the inequality*

$$E(u(t)) \leq C_{19} e^{-\omega t} E(0), \ \forall t \geq 0. \tag{4.64}$$

Theorem 4.5. *Let u be a solution of system* (4.63) *and suppose $\partial\Omega = \Gamma_0 \cup \Gamma_1$ satisfies the condition* (1.14)*; then there exist constants $\delta > 0$ and $C_{20} > 0$ which depend only on the domain Ω, the dimension n, ω, $\delta, \|g_0\|_{H^1(\Omega)}$ and $\|g_1\|_{L^2(\Omega)}$ such that for the initial data $(g_0, g_1) \in W \times H_0^1(\Omega)$ satisfying $K(g_0) \geq 0$, the second-order energy of the solution of system* (4.63) *satisfies the inequality*

$$\|u_{tt}\|^2_{L^2(\Omega)} + \|\nabla u_t\|^2_{L^2(\Omega)} \leq C_{20} e^{-\delta t} \overline{E}(0), \ \forall 0 \leq t \leq T. \tag{4.65}$$

5. PROOF OF THEOREM 1.1

In this section, we will give the proof of Theorem 1.1, which is completely parallel to Zhou et al [19]; for the convenience of the reader, we sketch the proof as follows.

We divide the proof into several steps.

Step 1. By Lemma 4.1 and in view of Theorem 4.2, 4.3, 4.4 and 4.5, for any T_1 and T_2 satisfying $0 < T_1, T_2 < T$, system (4.1) admits a unique solution u_1 on the domain $[0, T_1] \times \Omega$ and there exists a unique solution u_2 to system (4.63) on the domain $[T - T_2, T] \times \Omega$. In addition, there exists a positive constant C_{21} depending on the domain Ω, the dimension n, $\delta, \|f_0\|_{H^1(\Omega)}, \|f_1\|_{L^2(\Omega)}$ such that

$$\begin{aligned}
&\|u_{1tt}(T_1,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u_{1t}(T_1,\cdot)\|^2_{L^2(\Omega)} + \|u_{1t}(T_1,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u_1(T_1,\cdot)\|^2_{L^2(\Omega)} \\
&\leq C_{21}(\Omega, n, \delta, \|f_0\|_{H^1(\Omega)}, \|f_1\|_{L^2(\Omega)}) e^{-\delta T_1} (\|u_{1tt}(0)\|^2_{L^2(\Omega)} \\
&+ \|\nabla u_{1t}(0)\|^2_{L^2(\Omega)} + \|u_{1t}(0)\|^2_{L^2(\Omega)} + \|\nabla u_1(0)\|^2_{L^2(\Omega)}),
\end{aligned} \tag{5.1}$$

and a positive constant C_{22} depending on the domain Ω, the dimension n, $\delta, \|g_0\|_{H^1(\Omega)}, \|g_1\|_{L^2(\Omega)}$ such that

$$\begin{aligned}
&\|u_{2tt}(T-T_2,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u_{2t}(T-T_2,\cdot)\|^2_{L^2(\Omega)} \\
&+ \|u_{2t}(T-T_2,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u_2(T-T_2,\cdot)\|^2_{L^2(\Omega)} \\
&\leq C_{22}(\Omega, n, \delta, \|g_0\|_{H^1(\Omega)}, \|g_1\|_{L^2(\Omega)}) e^{-\delta T_2} (\|u_{2tt}(T)\|^2_{L^2(\Omega)}
\end{aligned}$$

$$+\|\nabla u_{2t}(T)\|^2_{L^2(\Omega)} + \|u_{2t}(T)\|^2_{L^2(\Omega)} + \|\nabla u_2(T)\|^2_{L^2(\Omega)}). \tag{5.2}$$

Take $T > T_1 + T_2$.

To best illustrate our proof, let $T_1 = T_2 = \frac{T}{4}$. Then it suffices to study the exact boundary controllability problem

$$\begin{cases} \partial_{tt} u - \Delta u = u^3, \ \frac{1}{4}T \le t \le \frac{3}{4}T, x \in \Omega, \\ u(\frac{1}{4}T) = u_1, u_t(\frac{1}{4}T) = u_{1t}, \ x \in \Omega, \\ u(\frac{3}{4}T) = u_2, u_t(\frac{3}{4}T) = u_{2t}, \ x \in \Omega. \end{cases} \tag{5.3}$$

Define

$$\begin{aligned} D^1_\Lambda(\psi) &= \sup_{\frac{1}{4}T \le t \le \frac{3}{4}T} \Big(\|\partial_t \psi(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla \psi\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}}, \\ D^2_\Lambda(\psi) &= \sup_{\frac{1}{4}T \le t \le \frac{3}{4}T} \Big(\|\partial_{tt} \psi(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla \psi_t\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}}, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \Lambda^\epsilon_\eta = \Big\{ &\psi : [\tfrac{1}{4}T, \tfrac{3}{4}T] \times \Omega \to \mathbb{R} : \psi(\tfrac{1}{4}T, \cdot) = u_1(\tfrac{1}{4}T, \cdot), \\ &\psi_t(\tfrac{1}{4}T, \cdot) = u_{1t}(\tfrac{1}{4}T, \cdot), \quad \psi(\tfrac{3}{4}T, \cdot) = u_2(\tfrac{3}{4}T, \cdot), \\ &\psi_t(\tfrac{3}{4}T, \cdot) = u_{2t}(\tfrac{3}{4}T, \cdot), \quad D^1_\Lambda(\psi) \le \epsilon, \ D^2_\Lambda(\psi) \le C\eta \Big\}, \end{aligned} \tag{5.5}$$

where $\epsilon > 0$ is small enough, $C > 0$ is a constant large enough and

$$\eta = \|f_0\|_{H^2(\Omega)} + \|f_1\|_{H^1(\Omega)} + \|g_0\|_{H^2(\Omega)} + \|g_1\|_{H^1(\Omega)}.$$

Obviously, Λ^ϵ_η is a closed subset of $C\big([\frac{1}{4}T, \frac{3}{4}T], H^1(\Omega)\big)$.

For any $\phi \in \Lambda^\epsilon_\eta$, we define a map Π: $\phi \to v + w$, where v satisfies the mixed initial-boundary-value problem:

$$\begin{cases} \partial_{tt} v - \Delta v = \phi^3, \ \frac{1}{4}T \le t \le \frac{3}{4}T, x \in \Omega, \\ v = 0, \partial_t v = 0, \ t = \frac{1}{4}T, x \in \Omega, \\ v = 0, \ \frac{1}{4}T \le t \le \frac{3}{4}T, x \in \partial\Omega, \end{cases} \tag{5.6}$$

and w is defined via the exact boundary controllability problem:

$$\begin{cases} \partial_{tt} w - \Delta w = 0, \ \frac{1}{4}T \le t \le \frac{3}{4}T, x \in \Omega, \\ w = u_1, \partial_t w = u_{1t}, \ t = \frac{1}{4}T, x \in \Omega, \\ w = u_2 - v, \partial_t w = u_{2t} - v_t, \ t = \frac{3}{4}T, x \in \Omega. \end{cases} \tag{5.7}$$

Step 2. For system (5.6), we have the energy estimate

$$\frac{1}{2}\frac{d}{dt}\Big(\|v_t\|^2_{L^2(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}\Big)=\int_\Omega \phi^3 v_t dx\le C\|\nabla\phi\|^3_{L^2(\Omega)}\|v_t\|_{L^2(\Omega)},$$
$$\frac{1}{2}\frac{d}{dt}\Big(\|v_{tt}\|^2_{L^2(\Omega)}+\|\nabla v_t\|^2_{L^2(\Omega)}\Big)=\int_\Omega 3\phi^2\phi_t v_{tt}dx$$
$$\le C\|\nabla\phi\|^2_{L^2(\Omega)}\|\nabla\phi_t\|_{L^2(\Omega)}\|v_{tt}\|_{L^2(\Omega)}; \tag{5.8}$$

there is no boundary term since $\partial_t^l v(t,x)\equiv 0$ for $l=0,1,2$ while $x\in\partial\Omega$. This means

$$D^1_\Lambda(v)\le CT\epsilon^3,$$
$$D^2_\Lambda(v)\le \|\partial_{tt}v(\frac{T}{4},\cdot)\|_{L^2(\Omega)}+CT\epsilon^2\eta \tag{5.9}$$
$$\le \|\phi^3(\frac{T}{4})\|_{L^2(\Omega)}+CT\epsilon^2\eta\le\|\nabla\phi(\frac{T}{4})\|^3_{L^2(\Omega)}+CT\epsilon^2\eta\le\epsilon^3+CT\epsilon^2\eta.$$

So we can choose ϵ small enough and T depending on ϵ such that

$$D^1_\Lambda(v)\le\tfrac{1}{2}\epsilon,\qquad D^2_\Lambda(v)\le C\eta. \tag{5.10}$$

Step 3. Now we shall prove that w is well defined, using the constructive method introduced by Zhou and Lei [18].

We define the series $\varphi^{(i)}$ and $\psi^{(i)}$ as follows:

Let $\varphi^{(1)}$ be the solution of the initial-boundary-value problem

$$\begin{cases}\Box\varphi^{(1)}=0,\ \frac{1}{4}T\le t\le\frac{3}{4}T, x\in\Omega,\\ \varphi^{(1)}_t+\varphi^{(1)}_\nu=0,\ \frac{1}{4}T\le t\le\frac{3}{4}T, x\in\Gamma_1,\\ \varphi^{(1)}=0,\ \frac{1}{4}T\le t\le\frac{3}{4}T, x\in\Gamma_0,\\ \varphi^{(1)}=u_1,\partial_t\varphi^{(1)}=u_{1t},\ t=\frac{1}{4}T, x\in\Omega,\end{cases} \tag{5.11}$$

and let $\psi^{(1)}$ be the solution of the inverted initial-boundary-value problem

$$\begin{cases}\Box\psi^{(1)}=0,\ \frac{1}{4}T\le t\le\frac{3}{4}T, x\in\Omega,\\ -\psi^{(1)}_t+\psi^{(1)}_\nu=0,\ \frac{1}{4}T\le t\le\frac{3}{4}T, x\in\Gamma_1,\\ \psi^{(1)}=0,\ \frac{1}{4}T\le t\le\frac{3}{4}T, x\in\Gamma_0,\\ \psi^{(1)}=u_2-v,\partial_t\psi^{(1)}=u_{2t}-v_t,\ t=\frac{3}{4}T, x\in\Omega.\end{cases} \tag{5.12}$$

For $j \geq 2, \varphi^{(j)}$ is defined inductively as the solution of the initial-boundary-value problem

$$\begin{cases} \Box \varphi^{(j)} = 0, \ \frac{1}{4}T \leq t \leq \frac{3}{4}T, x \in \Omega, \\ \varphi_t^{(j)} + \varphi_\nu^{(j)} = 0, \ \frac{1}{4}T \leq t \leq \frac{3}{4}T, x \in \Gamma_1, \\ \varphi^{(j)} = 0, \ \frac{1}{4}T \leq t \leq \frac{3}{4}T, x \in \Gamma_0, \\ \varphi^{(j)} = \psi^{(j-1)}, \partial_t \varphi^{(j)} = \psi_t^{(j-1)}, \ t = \frac{1}{4}T, x \in \Omega, \end{cases} \tag{5.13}$$

and $\psi^{(j)}$ is defined as the solution of the inverted initial-boundary-value problem

$$\begin{cases} \Box \psi^{(j)} = 0, \ \frac{1}{4}T \leq t \leq \frac{3}{4}T, x \in \Omega, \\ -\psi_t^{(j)} + \psi_\nu^{(j)} = 0, \ \frac{1}{4}T \leq t \leq \frac{3}{4}T, x \in \Gamma_1, \\ \psi^{(j)} = 0, \ \frac{1}{4}T \leq t \leq \frac{3}{4}T, x \in \Gamma_0, \\ \psi^{(j)} = \varphi^{(j-1)}, \partial_t \psi^{(j)} = \varphi_t^{(j-1)}, \ t = \frac{3}{4}T, x \in \Omega. \end{cases} \tag{5.14}$$

Let

$$w^m = \sum_{j=1}^{m} (-1)^{j-1} (\varphi^{(j)} + \psi^{(j)}). \tag{5.15}$$

First of all, we observe that for any $m \geq 1$

$$\Box w^m = 0, \quad \frac{1}{4}T \leq t \leq \frac{3}{4}T, \quad x \in \Omega, \tag{5.16}$$

and

$$\begin{cases} w^m(\frac{1}{4}T, x) = u_1(\frac{1}{4}T, x) + (-1)^{m-1}\psi^{(m)}(\frac{1}{4}T, x), \ x \in \Omega, \\ w_t^m(\frac{1}{4}T, x) = u_{1t}(\frac{1}{4}T, x) + (-1)^{m-1}\psi_t^{(m)}(\frac{1}{4}T, x), \ x \in \Omega, \\ w^m(\frac{3}{4}T, x) = (u_2 - v)(\frac{3}{4}T, x) + (-1)^{m-1}\varphi^{(m)}(\frac{3}{4}T, x), \ x \in \Omega, \\ w_t^m(\frac{3}{4}T, x) = (u_{2t} - v_t)(\frac{3}{4}T, x) + (-1)^{m-1}\varphi_t^{(m)}(\frac{3}{4}T, x), \ x \in \Omega. \end{cases} \tag{5.17}$$

To show that the sequence $\{w^m\}$ defined by (5.15) is convergent, let's take a look at the mixed initial-boundary-value problem

$$\begin{cases} \Box u = 0, \quad t \geq 0, x \in \Omega, \\ u(0) = f, u_t(0) = g, \ x \in \Omega, \\ u_t + u_\nu = 0, \ t \geq 0, x \in \Gamma_1, \\ u = 0, \ t \geq 0, x \in \Gamma_0. \end{cases} \tag{5.18}$$

Theorem 5.1. *Assume $\partial\Omega = \Gamma_0 \cup \Gamma_1$ satisfies the condition (1.14). If $(f, g) \in W \times H_0^1(\Omega)$, then there exist a global solution $u(t, x)$ to system (5.18) and constants $\sigma > 0$, $C_{23} > 0$ depending on the domain Ω, the dimension n, $\sigma, \|f\|_{H^1(\Omega)}, \|g\|_{L^2(\Omega)}$ such that for any $t \geq 0$*

$$\|u_{tt}(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u_t(t,\cdot)\|^2_{L^2(\Omega)} + \|u_t(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u(t,\cdot)\|^2_{L^2(\Omega)}$$

$$\leq C_{23}(\Omega, n, \sigma, \|f\|_{H^1(\Omega)}, \|g\|_{L^2(\Omega)})e^{-\sigma t}(\|u_{tt}(0)\|^2_{L^2(\Omega)} + \|\nabla u_t(0)\|^2_{L^2(\Omega)} + \|u_t(0)\|^2_{L^2(\Omega)} + \|\nabla u(0)\|^2_{L^2(\Omega)}). \tag{5.19}$$

The only difference between system (4.1) and (5.18) is that the latter is a homogeneous problem, and the corresponding exponentially dissipative energy estimates stated in Theorem 5.1 can be obtained by the same way as that used in Theorem 4.2 and Theorem 4.3. The proof may be easier since system (5.18) has no cubic focusing term, and we omit the details here.

By the inequality (5.19), we know there exists a positive constant C_{24} such that

$$\begin{aligned}
&\|\varphi^{(j)}_{tt}(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\varphi^{(j)}_t(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&\quad + \|\varphi^{(j)}_t(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\varphi^{(j)}(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&\leq C_{24}(\Omega, n, \sigma, \|f_0\|_{H^1(\Omega)}, \|f_1\|_{L^2(\Omega)}, \|g_0\|_{H^1(\Omega)}, \|g_1\|_{L^2(\Omega)})e^{-\frac{\sigma T}{2}} \\
&\big(\|\psi^{(j-1)}_{tt}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\psi^{(j-1)}_t(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&+ \|\psi^{(j-1)}_t(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\psi^{(j-1)}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)}\big),
\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}
&\|\psi^{(j)}_{tt}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\psi^{(j)}_t(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&\quad + \|\psi^{(j)}_t(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\psi^{(j)}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&\leq C_{24}(\Omega, n, \sigma, \|f_0\|_{H^1(\Omega)}, \|f_1\|_{L^2(\Omega)}, \|g_0\|_{H^1(\Omega)}, \|g_1\|_{L^2(\Omega)})e^{-\frac{\sigma T}{2}} \\
&\quad\big(\|\varphi^{(j-1)}_{tt}(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\varphi^{(j-1)}_t(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&\quad + \|\varphi^{(j-1)}_t(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\varphi^{(j-1)}(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)}\big).
\end{aligned} \tag{5.21}$$

Combining (5.20) and (5.21), we arrive at

$$\begin{aligned}
&\|\psi^{(m)}_{tt}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\psi^{(m)}_t(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\psi^{(m)}_t(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&+ \|\nabla\psi^{(m)}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\varphi^{(m)}_{tt}(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\varphi^{(m)}_t(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&+ \|\varphi^{(m)}_t(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla\varphi^{(m)}(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&\leq \big(C_{24}e^{-\frac{\sigma T}{2}}\big)^m\Big(\|u_{1tt}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u_{1t}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|u_{1t}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&+ \|\nabla u_1(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|(u_2-v)_{tt}(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla(u_{2t}-v_t)(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&+ \|(u_{2t}-v_t)(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla(u_2-v)(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)}\Big).
\end{aligned} \tag{5.22}$$

Take a large enough positive constant T depending only on $\|f_0\|_{H^1(\Omega)}$, $\|f_1\|_{L^2(\Omega)}$, $\|g_0\|_{H^1(\Omega)}$ and $\|g_1\|_{L^2(\Omega)}$ such that $C_{24}e^{\frac{-\sigma T}{2}} < \frac{1}{2}$. Then by the inequality (5.22) and (4.25) we deduce the conclusion that when $m \to \infty$, $w^m(\frac{1}{4}T,.) \to u_1(\frac{1}{4}T,.)$, $w^m(\frac{3}{4}T,.) \to \big(u_2 - v\big)(\frac{3}{4}T,.)$ in $H^2(\Omega)$ and $w_t^m(\frac{1}{4}T,.) \to u_{1t}(\frac{1}{4}T,.)$, $w_t^m(\frac{3}{4}T,.) \to \big(u_{2t} - v_t\big)(\frac{3}{4}T,.)$ in $H^1(\Omega)$. What's more, w^m is a Cauchy sequence in $\bigcap_{j=0}^2 C^j([\frac{1}{4}T, \frac{3}{4}T], H^{2-j}(\Omega))$. Denote $w^m \to w$. Hence w satisfies system (5.7). For any $\frac{1}{4}T \le t \le \frac{3}{4}T$, it follows that

$$\begin{aligned}
&\|w_t(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla w(t,\cdot)\|^2_{L^2(\Omega)} \\
\le&\big(C_{24}e^{-\frac{\sigma T}{2}}\big)^m\Big(\|u_{1t}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u_1(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&+ \|(u_{2t} - v_t)(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla(u_2 - v)(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)}\Big), \\
&\|w_{tt}(t,\cdot)\|^2_{L^2(\Omega)} + \|\nabla w_t(t,\cdot)\|^2_{L^2(\Omega)} \\
\le&\big(C_{24}e^{-\frac{\sigma T}{2}}\big)^m\Big(\|u_{1tt}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla u_{1t}(\tfrac{1}{4}T,\cdot)\|^2_{L^2(\Omega)} \\
&+ \|(u_{2tt} - v_{tt})(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)} + \|\nabla(u_{2t} - v_t)(\tfrac{3}{4}T,\cdot)\|^2_{L^2(\Omega)}\Big),
\end{aligned} \tag{5.23}$$

which implies that $D^1_\Lambda(w) \le \frac{1}{2}\epsilon$, $D^2_\Lambda(w) \le C\eta$ by using the inequality (5.1) and (5.2). Therefore,

$$D^1_\Lambda(v + w) \le D^1_\Lambda(v) + D^1_\Lambda(w) \le \epsilon,$$

$$D^2_\Lambda(v + w) \le D^2_\Lambda(v) + D^2_\Lambda(w) \le C\eta.$$

So $v + w \in \Lambda^\epsilon_\eta$, which says the map $\Pi : \Lambda^\epsilon_\eta \to \Lambda^\epsilon_\eta$.

Step 4. In the end, we prove that Π is a strict contraction. For any $\phi_1, \phi_2 \in \Lambda_\eta$, define $\Pi\phi_1 = v_1 + w_1$, $\Pi\phi_2 = v_2 + w_2$, $\overline{\phi} = \phi_1 - \phi_2$, $\overline{v} = v_1 - v_2$, $\overline{w} = w_1 - w_2$. Hence $\overline{v}$ solves the initial-boundary-value problem

$$\begin{cases} \Box\overline{v} = \phi_1^3 - \phi_2^3,\ \frac{1}{4}T \le t \le \frac{3}{4}T, x \in \Omega, \\ \overline{v} = 0, \partial_t\overline{v} = 0,\ t = \frac{1}{4}T, x \in \Omega, \\ \overline{v} = 0,\ \frac{1}{4}T \le t \le \frac{3}{4}T, x \in \partial\Omega, \end{cases} \tag{5.24}$$

and $\overline{w}$ solves the exact boundary controllability problem

$$\begin{cases} \Box\overline{w} = 0,\ \frac{1}{4}T \le t \le \frac{3}{4}T, x \in \Omega, \\ \overline{w} = 0, \partial_t\overline{w} = 0,\ t = \frac{1}{4}T, x \in \Omega, \\ \overline{w} = -\overline{v}, \partial_t\overline{w} = -\overline{v}_t,\ t = \frac{3}{4}T, x \in \Omega. \end{cases} \tag{5.25}$$

Similarly we have

$$D^1_\Lambda(\overline{w}) \leq \left(C_{24}e^{-\frac{\sigma T}{2}}\right)^m D^1_\Lambda(\overline{v}), \qquad D^2_\Lambda(\overline{w}) \leq \left(C_{24}e^{-\frac{\sigma T}{2}}\right)^m D^2_\Lambda(\overline{v}),$$
$$D^1_\Lambda(\overline{v}) \leq C\epsilon^2 T D^1_\Lambda(\overline{\phi}), \qquad D^2_\Lambda(\overline{v}) \leq C\epsilon^3 + C\epsilon^2 T\eta.$$

Choosing ϵ small enough and T depending on ϵ large enough, we get

$$D^1_\Lambda(\Pi\phi_1 - \Pi\phi_2) \leq D^1_\Lambda(\overline{v}) + D^1_\Lambda(\overline{w}) \leq \tfrac{1}{2}D^1_\Lambda(\overline{\phi}),$$
$$D^2_\Lambda(\Pi\phi_1) \leq D^2_\Lambda(v_1) + D^2_\Lambda(w_1) \leq C\eta,$$
$$D^2_\Lambda(\Pi\phi_2) \leq D^2_\Lambda(v_2) + D^2_\Lambda(w_2) \leq C\eta.$$

Therefore, Π is a strict contraction from Λ^ϵ_η to Λ^ϵ_η.

By the standard contraction mapping theorem, there exists a fixed point $u_0 \in \Lambda_\eta$ such that $\Pi u_0 = u_0$. It follows that u_0 solves system (5.3).

In order to prove Theorem 1.1, we set $h(t,x) = u_0(t,x)$ for $(t,x) \in [\frac{1}{4}T, \frac{3}{4}T] \times \partial\Omega$, set $h(t,x) = u_1(t,x)$ for $(t,x) \in [0, \frac{1}{4}T] \times \partial\Omega$ and set $h(t,x) = u_2(t,x)$ for $(t,x) \in [\frac{3}{4}T, T] \times \partial\Omega$. Therefore h is the desired boundary control function. Furthermore, the existence of the solution on the time interval [0,T] to the initial-boundary-value problem (1.1), (1.2) with the mixed boundary condition (1.4), (1.5) is apparent from the proof above. Also we can obtain the uniqueness of the solution to the initial-boundary-value problem (1.1), (1.2), (1.4) and (1.5) by doing energy estimates and applying Gronwall's inequality. Then the proof of Theorem 1.1 is completed.

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