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HOMOGENIZATION OF A COUPLED PROBLEM FOR SOUND PROPAGATION IN POROUS MEDIA

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Abstract. In this paper we study the acoustic properties of a microstructured material such as glass, wool, or foam. In our model, the solid matrix is governed by linear elasticity and the surrounding fluid obeys the Stokes equations. The microstructure is assumed to be periodic at some small scale ε and the viscosity coefficient of the fluid is assumed to be of order ε^2. We consider the time-harmonic regime forced by vibrations applied on a part of the boundary. We use the two-scale convergence theory to prove the convergence of the displacements to the solution of a homogeneous problem as the size of the microstructure shrinks to 0.

1. Introduction

Phonic insulating properties of porous media, such as glass, wool, or foam, are currently used for industrial applications, assemblies of such materials being currently used in the context of noise reduction for aeroplane cabins or cars. However, the acoustic properties of such materials are difficult to deduce accurately from those of each of its constitutive elements (air, glass, etc.) because of their complex microstructure. For practical engineering applications in the context of poroelasticity, some simplified models are used.

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For example, the Biot–Allard model [4] is heuristically derived from experiments at low and high frequencies: some of the parameters have to be fitted from experiments, while attempts for deriving rigorous models can be found in [6] for instance.

Mathematically speaking, deriving macroscopic properties of a microstructured medium from those of its components pertains to the homogenization theory. In this paper, we use the two-scale homogenization theory [2, 17, 19] to derive the homogenized macroscopic equations governing the propagation of sound in a porous medium. In particular, the coefficients of this macroscopic equation are evaluated by solving cell problems.

Let us set more precisely the problem we are interested in: the connected domain Ω representing the porous medium splits into a solid part Ω^s and a fluid part Ω^f. We study the acoustic properties of the structure through its response to a boundary harmonic forcing. Namely, we consider that the displacement (of both the fluid and the solid) and the pressure in the fluid have the time-harmonic form

$$\begin{aligned} &\boldsymbol{U}(x,t) = \boldsymbol{u}_f(x)e^{i\omega t} \quad \text{on } \Omega^f, \\ &\boldsymbol{U}(x,t) = \boldsymbol{u}_s(x)e^{i\omega t} \quad \text{on } \Omega^s, \quad \text{and} \quad P(x,t) = p(x)e^{i\omega t}. \end{aligned}$$

As proposed in [18], the behavior of the fluid is described by $(\boldsymbol{u}_f, p)$ a (complex) displacement/pressure field satisfying the incompressible Stokes equations, written in time-harmonic regime:

$$\left\{ \begin{array}{rcll} -\rho_f\omega^2\boldsymbol{u}_f - i\omega\eta\Delta\boldsymbol{u}_f + \nabla p &=& \boldsymbol{f}^f, & \text{in } \Omega^f, \\ \nabla\cdot\boldsymbol{u}_f &=& 0, & \text{in } \Omega^f, \end{array} \right.$$

where η and $\rho_f \geq 0$ stand for the fluid's viscosity and density respectively and $\boldsymbol{f}^f$ represents the force density applied to the fluid. Let us emphasize that $\boldsymbol{u}_f$ denotes a displacement field and not a velocity field as is usually the case. The Stokes equations inherently admit the characteristic length

$$l_f \sim \sqrt{\frac{\eta}{\omega\rho_f}},$$

which is typically of the order of magnitude of $10^{-5}m$ for air at $\omega = 10$ kHz.

In the solid domain, we assume that the displacement field $\boldsymbol{u}_s$ satisfies the linear elasticity equations:

$$-\rho_s\omega^2\boldsymbol{u}_s - (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}_s - \mu\Delta\boldsymbol{u}_s \quad = \quad \boldsymbol{f}^s, \quad \text{in } \Omega^s,$$

where λ and μ are the two Lamé coefficients of the material and ρ_s its density $(\lambda, \mu \geq 0,\ \rho_s \geq 0)$. Similarly, $\boldsymbol{f}^s$ represents the force density applied to the solid, and this equation admits a characteristic length

$$l_s \sim \sqrt{\frac{\max(\lambda,\mu)}{\omega^2\rho_s}} \sim 10^{-2}m$$

for glass, still at $\omega = 10$ kHz.

Furthermore, the boundary of the domain is decomposed in $\partial\Omega = \Gamma_D \cup \Gamma_N$. The time-harmonic forcing is modeled by a Dirichlet boundary condition $\boldsymbol{g}$ only applied on Γ_D, and we assume free boundary conditions on Γ_N. On the interface $\partial\Omega^f \cap \partial\Omega^s$ between the solid and the fluid, we assume continuity of the displacement and stress equilibrium.

The microscopic complexity of the material is represented by a structure varying at a small length scale that we call ε. For instance, in glass wool, glass fibers of thickness 10^{-5} m are common while the wool is typically a few centimeters thick. To emphasize the fact that the fluid, here the air, possesses a characteristic length compatible with ε, we rescale the viscosity η as

$$\eta = \nu\varepsilon^2.$$

Eventually, neglecting other external forces (e.g., weight), we rewrite all the equations into the following system:

$$\left\{\begin{array}{l}
\left[\begin{array}{rcll}
-\rho_f\omega^2\boldsymbol{u}_f - \nabla\cdot\boldsymbol{\sigma}^f_\varepsilon(\boldsymbol{u}_f,p) &=& 0 & \text{in } \Omega^f,\\
\nabla\cdot\boldsymbol{u}_f &=& 0 & \text{in } \Omega^f,\\
\boldsymbol{u}_f &=& \boldsymbol{g} & \text{on } \Gamma_D\cap\partial\Omega^f,\\
\boldsymbol{\sigma}^f_\varepsilon(\boldsymbol{u}_f,p)\cdot\boldsymbol{n}_f &=& 0 & \text{on } \Gamma_N\cap\partial\Omega^f,
\end{array}\right.\\
\\
\left[\begin{array}{rcll}
-\rho_s\omega^2\boldsymbol{u}_s - \nabla\cdot\boldsymbol{\sigma}^s(\boldsymbol{u}_s) &=& 0 & \text{in } \Omega^s,\\
\boldsymbol{u}_s &=& \boldsymbol{g} & \text{on } \Gamma_D\cap\partial\Omega^s,\\
\boldsymbol{\sigma}^s(\boldsymbol{u}_s)\cdot\boldsymbol{n}_s &=& 0 & \text{on } \Gamma_N\cap\partial\Omega^s,
\end{array}\right.\\
\\
\left[\begin{array}{rcll}
\boldsymbol{u}_s - \boldsymbol{u}_f &=& 0 & \text{on } \partial\Omega^s\cap\partial\Omega^f,\\
\boldsymbol{\sigma}^s(\boldsymbol{u}_s)\cdot\boldsymbol{n}_s + \boldsymbol{\sigma}^f_\varepsilon(\boldsymbol{u}_f,p)\cdot\boldsymbol{n}_f &=& 0 & \text{on } \partial\Omega^s\cap\partial\Omega^f,
\end{array}\right.
\end{array}\right. \tag{1.1}$$

where $\boldsymbol{n}_s$ (respectively $\boldsymbol{n}_f$) denotes the exterior normal to Ω_s (respectively Ω_f) and where the stress tensors $\boldsymbol{\sigma}^s$ and $\boldsymbol{\sigma}^f_\varepsilon$ are given by

$$\begin{aligned}
\boldsymbol{\sigma}^f_\varepsilon(\boldsymbol{u}_f,p) &:= \varepsilon^2 i\omega\nu\left(\nabla\boldsymbol{u}_f + {}^t\nabla\boldsymbol{u}_f\right) - p\mathbf{Id},\\
\boldsymbol{\sigma}^s(\boldsymbol{u}_s) &:= \lambda(\nabla\cdot u_s)\mathbf{Id} + \mu\left(\nabla\boldsymbol{u}_s + {}^t\nabla\boldsymbol{u}_s\right),
\end{aligned}$$

and where $\mathbf{Id}$ stands for the 3×3 identity matrix.

Remark 1.1. The system (1.1) is of mixed type. The equations in the solid domain are of Helmholtz type since they are the time-harmonic version of the linear elastic wave equation, whereas, in the fluid, the imaginary coefficient is reminiscent of the parabolic nature of the Stokes equations. In particular, due to the Helmholtz form of the equations in the solid domain, the underlying operator is not coercive. This operator may also admit some so-called resonant frequencies where its resolvent is singular. In our study, we avoid this (discrete) set of frequencies.

In order to obtain a homogenized system, we have to pass to the limit as ε tends to 0. Let us underline that the small parameter ε not only appears in the fluid equation but also in the geometry of the structure, as we shall see in the next section. The relevance of the limiting process and its mathematical difficulties result precisely from the interaction between them. We refer the reader to [5], [7], or [21] for the derivation of the limiting system using an asymptotic expansion in the case of a simpler system. The homogenization of similar fluid-structure systems, though not in time-harmonic regime, have also been considered in [10, 11, 14, 15] using weak-convergence arguments or asymptotic expansions.

In this work, we use instead the two-scale convergence method introduced by G. Allaire and G. Nguetseng [2, 19]. Let us note that a very similar coupled problem, where the time-dependent fluid-structure problem is considered, has been also treated by T. Clopeau *et al.* in [12]. The present paper is very close in spirit, but the time-harmonic regime brings several specific difficulties. Our approach is therefore complementary to [12], and in view of the applications in particular, the aims are clearly different.

2. Setting of the problem and main results

We consider that the physical domain Ω satisfies the following assumption.

($\mathbf{H_1}$) *Ω is a bounded, Lipschitz, and connected open set.*

In order to describe the microstructure, we introduce two subsets of $\mathbf{R}^3$, $\mathcal{S}$ and $\mathcal{F}$, that are assumed to satisfy

($\mathbf{H_2}$) *$\mathcal{S}$ and $\mathcal{F}$ are two disjoint open sets with integer periodicity*

$$\mathcal{S} = \mathcal{S} + k, \quad \mathcal{F} = \mathcal{F} + k, \quad \forall\, k \in \mathbf{Z}^3,$$

and are such that $\mathbf{R}^3 = \overline{\mathcal{S}} \cup \overline{\mathcal{F}}$. The interface $\mathcal{I} = \partial\mathcal{S} = \partial\mathcal{F}$ is assumed to be smooth. Moreover, $\mathcal{S}$ is locally connected in the sense that for any $Q = (0,a) \times (0,b) \times (0,c)$, where (a,b,c) is any permutation of $(2,1,1)$, the set $Q \cap \mathcal{S}$ is connected.

Now, for $\varepsilon > 0$, we define the fluid and solid subdomains,

$$\Omega_\varepsilon^f := (\varepsilon\mathcal{F}) \cap \Omega, \qquad \Omega_\varepsilon^s := (\varepsilon\mathcal{S}) \cap \Omega.$$

We study the fluid-structure system (1.1) posed in these domains. To emphasize the dependency in ε, we will use the notation $\boldsymbol{u}_\varepsilon$ for the velocity (both in the fluid and solid domains) and p_ε for the pressure in the fluid domain. External vibrations are applied as boundary conditions posed on Γ_D, which is a part of the boundary that satisfies assumption $(\mathbf{H}_3)$.

(H$_3$) *Γ_D is a nonempty Lipschitz open subset of $\partial\Omega$.*

We further assume that the fluid domain Ω_ε^f satisfies $(\mathbf{H}_4)$.

(H$_4$) *For $\varepsilon > 0$ small enough, the fluid domain Ω_ε^f is connected and its boundary intersects Γ_D on a set of positive two-dimensional measure.*

We consider that the boundary condition $\boldsymbol{g}$ admits a lifting $\boldsymbol{h} \in H^1(\Omega, \mathbf{C}^3)$ and we translate the unknown displacement $\boldsymbol{u} = \boldsymbol{h} + \boldsymbol{u}_\varepsilon$, where $\boldsymbol{u}_\varepsilon$ belongs to

$$H_D(\Omega) := \{\boldsymbol{w} \in H^1(\Omega, \mathbf{C}^3) : \boldsymbol{w} = 0 \text{ on } \Gamma_D\}.$$

We are led to look for solutions of the variational problem: find $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in H_D(\Omega) \times L^2(\Omega_\varepsilon^f, \mathbf{C})$ such that for every $(\boldsymbol{\psi}, q) \in H_D(\Omega) \times L^2(\Omega_\varepsilon^f, \mathbf{C})$, we have

$$-\omega^2\rho_s \int_{\Omega_\varepsilon^s} \boldsymbol{u}_\varepsilon \cdot \overline{\boldsymbol{\psi}} + \int_{\Omega_\varepsilon^s} \left\{2\mu \boldsymbol{G}(\boldsymbol{u}_\varepsilon) : \overline{\boldsymbol{G}(\boldsymbol{\psi})} + \lambda \nabla \cdot \boldsymbol{u}_\varepsilon \, \overline{\nabla \cdot \boldsymbol{\psi}}\right\} \tag{2.1}$$

$$-\omega^2\rho_f \int_{\Omega_\varepsilon^f} \boldsymbol{u}_\varepsilon \cdot \overline{\boldsymbol{\psi}} + i2\omega\nu\varepsilon^2 \int_{\Omega_\varepsilon^f} \boldsymbol{G}(\boldsymbol{u}_\varepsilon) : \overline{\boldsymbol{G}(\boldsymbol{\psi})} - \int_{\Omega_\varepsilon^f} p_\varepsilon \overline{\nabla \cdot \boldsymbol{\psi}} = F_\varepsilon(\boldsymbol{h}, \boldsymbol{\psi}),$$

$$\int_{\Omega_\varepsilon^f} \overline{q}\, \nabla \cdot \boldsymbol{u}_\varepsilon = 0, \tag{2.2}$$

where to lighten the notation, we have introduced the infinitesimal strain tensor

$$\boldsymbol{G}(\boldsymbol{w}) := \frac{\nabla \boldsymbol{w} + {}^t\nabla \boldsymbol{w}}{2}. \tag{2.3}$$

The right-hand side of (2.1) is defined by

$$F_\varepsilon(\boldsymbol{h}, \boldsymbol{\psi}) := \int_\Omega \omega^2(\rho_s \mathbf{1}_{\Omega_\varepsilon^s} + \rho_f \mathbf{1}_{\Omega_\varepsilon^f}) \boldsymbol{h} \cdot \overline{\boldsymbol{\psi}}$$

$$-\int_{\Omega_\varepsilon^s} \left\{2\mu \boldsymbol{G}(\boldsymbol{h}) : \overline{\boldsymbol{G}(\boldsymbol{\psi})} + \lambda \nabla \cdot \boldsymbol{h}\, \overline{\nabla \cdot \boldsymbol{\psi}}\right\} - 2i\omega\nu\varepsilon^2 \int_{\Omega_\varepsilon^f} \boldsymbol{G}(\boldsymbol{h}) : \overline{\boldsymbol{G}(\boldsymbol{\psi})}, \tag{2.4}$$

with $\mathbf{1}_{\mathcal{O}}$ the characteristic function of $\mathcal{O}$.

We establish the following result showing that this problem is uniformly well-posed as ε goes to 0.

Theorem 2.1. *Assuming that assumptions* $(\mathbf{H}_1)$–$(\mathbf{H}_4)$ *hold,*

a) *there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and any* $\boldsymbol{h} \in H^1(\Omega, \mathbf{C}^3)$ *the variational problem* (2.1)–(2.2) *admits a unique solution, i.e.,* $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in H_D(\Omega) \times L^2(\Omega_\varepsilon^f, \mathbf{C})$;

b) *moreover, there exists* $C > 0$ *independent of* ε *such that the following estimate holds:*

$$\|\boldsymbol{u}_\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{G}(\boldsymbol{u}_\varepsilon)\|^2_{L^2(\Omega_\varepsilon^s)} + \varepsilon^2 \|\boldsymbol{G}(\boldsymbol{u}_\varepsilon)\|^2_{L^2(\Omega_\varepsilon^f)} + \|p_\varepsilon\|^2_{L^2(\Omega_\varepsilon^f)} \ \leq \ C\|\boldsymbol{h}\|^2_{H^1(\Omega)}. \tag{2.5}$$

Part a) of this result is established in Section 4; part b) is proved in Section 5.

Let us note that the classical method to obtain the energy-like estimate (2.5) consists in using the test function $\boldsymbol{\psi} = \boldsymbol{u}_\varepsilon$ in (2.1). Unfortunately, here we would obtain on the left-hand side:

$$-\omega^2 \int_\Omega (\rho_s \mathbf{1}_{\Omega_\varepsilon^s} + \rho_f \mathbf{1}_{\Omega_\varepsilon^f}) |\boldsymbol{u}_\varepsilon|^2 + \int_{\Omega_\varepsilon^s} \left\{ 2\mu |\boldsymbol{G}(\boldsymbol{u}_\varepsilon)|^2 + \lambda |\nabla \cdot \boldsymbol{u}_\varepsilon|^2 \right\} + i 2\omega\nu\varepsilon^2 \int_{\Omega_\varepsilon^f} |\boldsymbol{G}(\boldsymbol{u}_\varepsilon)|^2.$$

The two terms composing the real part of this quantity have opposite signs which prevents us from obtaining directly the desired *a priori* estimate. Although this Helmholtz-type situation is usual in the time-harmonic regime, our situation is not classical since we furthermore need an estimate which is uniform in ε. It turns out that we obtain part *b)* of the theorem as a byproduct of the homogenization process as ε goes to 0, in the spirit of [8].

The main subject of the paper is the study of the homogenization process for the problem (2.1)–(2.2). We use the tools of the two-scale convergence theory as presented in the seminal paper of Allaire [2]. Let us first recall basic definitions and properties.

Definition 2.2. *Let* $Y = \mathbf{R}^3/\mathbf{Z}^3$ *be the unit three-dimensional torus. We say that a family of mappings* $\{\boldsymbol{w}_\varepsilon\}_{\varepsilon>0} \subset L^2(\Omega)$ *two-scale converges to* $\boldsymbol{w}_0 \in L^2(\Omega \times Y)$ *as* $\varepsilon \to 0$ *and we write* $\boldsymbol{w}_\varepsilon \twoheadrightarrow \boldsymbol{w}_0$ *in* $\Omega \times Y$ *if*

$$\int_\Omega \boldsymbol{w}_\varepsilon(x) \cdot \boldsymbol{\varphi}\left(x, x/\varepsilon\right) \, dx \ \xrightarrow{\varepsilon \downarrow 0} \ \int_{\Omega \times Y} \boldsymbol{w}_0(x,y) \cdot \boldsymbol{\varphi}(x,y) \, dx \, dy$$

for every $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \times \overline{Y})$.

We also use a two-scale convergence localized in the solid part of the domain.

Definition 2.3. *Defining $S = \mathcal{S}/\mathbf{Z}^3$, we say that the family $\{\boldsymbol{w}_\varepsilon\}_{\varepsilon>0}$ with $\boldsymbol{w}_\varepsilon \in L^2(\Omega_\varepsilon^s)$ two-scale converges in the solid domain to $\boldsymbol{w}_0 \in L^2(\Omega \times S)$ as $\varepsilon \to 0$ and we write $\boldsymbol{w}_\varepsilon \twoheadrightarrow \boldsymbol{w}_0$ in $\Omega \times S$ if*

$$\int_{\Omega_\varepsilon^s} \boldsymbol{w}_\varepsilon(x) \cdot \boldsymbol{\varphi}\left(x, x/\varepsilon\right)\, dx \;\xrightarrow{\varepsilon\downarrow 0}\; \int_{\Omega\times S} \boldsymbol{w}_0(x,y)\cdot\boldsymbol{\varphi}(x,y)\, dx\, dy$$

for every $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \times \overline{S})$.

Notice that, extending $\boldsymbol{w}_\varepsilon$ and $\boldsymbol{w}_0$ by 0 in Ω_ε^f and $\Omega \times F$ respectively, the last definition amounts to two-scale convergence in $\Omega \times Y$. We also use the similar notion of restricted two-scale convergence in the fluid part.

Definition 2.4. *Defining $F = \mathcal{F}/\mathbf{Z}^3$, we say that the family $\{\boldsymbol{w}_\varepsilon\}_{\varepsilon>0}$ with $\boldsymbol{w}_\varepsilon \in L^2(\Omega_\varepsilon^f)$ two-scale converges in the solid domain to $\boldsymbol{w}_0 \in L^2(\Omega \times F)$ as $\varepsilon \to 0$ and we write $\boldsymbol{w}_\varepsilon \twoheadrightarrow \boldsymbol{w}_0$ in $\Omega \times F$ if*

$$\int_{\Omega_\varepsilon^f} \boldsymbol{w}_\varepsilon(x) \cdot \boldsymbol{\varphi}\left(x, x/\varepsilon\right)\, dx \;\xrightarrow{\varepsilon\downarrow 0}\; \int_{\Omega\times F} \boldsymbol{w}_0(x,y)\cdot\boldsymbol{\varphi}(x,y)\, dx\, dy$$

for every $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \times \overline{F})$.

Let us finally define the space of rigid movements $\mathcal{R}$ by

$$\mathcal{R} := \left\{ \boldsymbol{w} \in H^1(S, \mathbf{C}^3),\ \|\boldsymbol{G}(\boldsymbol{w})\|_{L^2(S)} = 0 \right\}.$$

Remark 2.5. In the preceding definitions, the sets Y, S, and F are defined as *periodic* cells. In particular, functions in $H^1(Y)$, $H^1(S)$, $H^1(F)$ or in $\mathcal{D}(Y)$, $\mathcal{D}(\overline{S})$, $\mathcal{D}(\overline{F})$ are meant with periodic boundary conditions. For example, a function $f \in H^1(Y)$ admits a periodic lifting $\tilde{f} \in H^1_{loc}(\mathbf{R}^3)$.

In Section 3 below we also briefly use the unit cells $\tilde{Y} := [0,1)^3$, $\tilde{S} := \tilde{Y} \cap \mathcal{S}$, and $\tilde{F} := \tilde{Y} \cap \mathcal{F}$, which in turn are not periodic.

Using this setting, we are able to identify an homogenized problem associated to the limits of the solutions of (2.1)–(2.2) as ε goes to 0. We have used in the following the notation $\boldsymbol{G}_y(\boldsymbol{w})$ for the strain tensor (2.3) where the derivatives are taken with respect to the y variable, and where the variable $\boldsymbol{w}$ is defined inside the solid part of the unit cell $S \subset Y$.

Theorem 2.6. *Let $\boldsymbol{h}$ in $H^1(\Omega, \mathbf{C}^3)$ and $\{(\boldsymbol{u}_\varepsilon, p_\varepsilon)\}_{\varepsilon>0}$ be the family of solutions of* (2.1)–(2.2) *with right-hand side $F_\varepsilon(\boldsymbol{h}, \cdot)$ given by* (2.4)*. Then, under assumptions* $(\mathbf{H}_1)$–$(\mathbf{H}_4)$*, there exists a* (*possibly empty*) *countable set $N \subset \boldsymbol{R}$ of the form $N := \{\pm\omega_i,\ i \geq 1\}$, where $(\omega_i) \subset \boldsymbol{R}_+$ is either finite or converging to infinity, such that if $\omega \notin N$, the two following properties hold.*

a) *There exist* $\boldsymbol{u} \in H^1(\Omega, \mathbf{C}^3)$, $\boldsymbol{u}_1 \in L^2(\Omega, H^1(S, \mathbf{C}^3)/\mathcal{R})$, $\boldsymbol{v} \in L^2(\Omega, H_0^1(F, \mathbf{C}^3))$, *and* $p \in H^1(\Omega, \mathbf{C})$ *such that*

$$\boldsymbol{u}_\varepsilon \twoheadrightarrow \boldsymbol{u} + \boldsymbol{v} \quad \text{in } \Omega \times Y, \tag{2.6}$$

$$\varepsilon \boldsymbol{G}(\boldsymbol{u}_\varepsilon) \twoheadrightarrow \boldsymbol{G}_y(\boldsymbol{v}) \quad \text{in } d\Omega \times F, \quad \mathbf{1}_{\Omega_\varepsilon^s} \boldsymbol{G}(\boldsymbol{u}_\varepsilon) \twoheadrightarrow \boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{G}_y(\boldsymbol{u}_1) \quad \text{in } \Omega \times S, \tag{2.7}$$

$$p_\varepsilon \;\twoheadrightarrow\; p \quad \text{in } \Omega \times F. \tag{2.8}$$

b) *Moreover, the limit* $(\boldsymbol{u}, p, \boldsymbol{u}_1, \boldsymbol{v})$ *solves the homogenized problem* (2.9)–(2.13) *described below.*

Remark 2.7. The countable set N can be considered as the set of the forbidden frequencies of the coupled fluid-structure problem. The values of these frequencies cannot be explicitly determined in general (however, a numerical approximation could be used to compute them in practice), as they are defined by using the spectral theory for compact operators. They physically correspond to the resonant frequencies of the porous material.

Remark 2.8. Let us emphasize that the two-scale limit of the problem splits into two parts: the first one $(\boldsymbol{u}, p)$ corresponds to the macroscopic part and only depends on the macroscopic variable $x \in \Omega$, while the second one $(\boldsymbol{u}_1, \boldsymbol{v})$ describes the displacement at small scale in the solid and fluid domains respectively. More precisely, the function $\boldsymbol{u}_1$ represents the second term in the asymptotic expansion of the displacement $\boldsymbol{u}_\varepsilon$: $\boldsymbol{u}_\varepsilon(x) = \boldsymbol{u}(x) + \boldsymbol{v}(x, x/\varepsilon) + \varepsilon \boldsymbol{u}_1(x, x/\varepsilon)$. This intuition is confirmed in Theorem 2.12.

Before stating the limit problem, let us introduce further notation. For $U \subset Y$ and any function $\phi \in L^1_{loc}(\Omega \times U)$, we write

$$\langle \phi \rangle_U \;:=\; \int_U \phi(x, y)\, dy.$$

The quantities $|S| = \langle 1 \rangle_S$ and $|F| = \langle 1 \rangle_F$ denote the relative volumes of S and F and $\rho := |S|\rho_s + |F|\rho_f = \langle \rho_s \mathbf{1}_S + \rho_f \mathbf{1}_F \rangle_Y$ denotes the homogenized density at $x \in \Omega$.

We establish in the sequel that the limit $(\boldsymbol{u}, p, \boldsymbol{u}_1, \boldsymbol{v})$ of Theorem 2.6 (a) solves the following homogenized variational problem composed of one macroscopic problem (2.9)–(2.10) and two uncoupled cell problems (2.11)–(2.12) and (2.13).

The macroscopic problem reads as follows: for every $(\boldsymbol{\psi}, q) \in H_D(\Omega) \times L^2(\Omega, \mathbf{C})$, we have

$$-\omega^2\rho\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{\psi}}-\omega^2\rho_f\int_\Omega \langle\boldsymbol{v}\rangle_F\cdot\overline{\boldsymbol{\psi}}+|S|\int_\Omega\left\{2\mu\boldsymbol{G}(\boldsymbol{u}):\overline{\boldsymbol{G}(\boldsymbol{\psi})}+\lambda\nabla\cdot\boldsymbol{u}\,\overline{\nabla\cdot\boldsymbol{\psi}}\right\}$$
$$+\int_\Omega\left\{2\mu\langle\boldsymbol{G}_y(\boldsymbol{u}_1)\rangle_S:\overline{\boldsymbol{G}(\boldsymbol{\psi})}+\lambda\langle\nabla_y\cdot\boldsymbol{u}_1\rangle_S\,\overline{\nabla\cdot\boldsymbol{\psi}}\right\}-|F|\int_\Omega p\overline{\nabla\cdot\boldsymbol{\psi}}$$
$$=\ \omega^2\rho\int_\Omega\boldsymbol{h}\cdot\overline{\boldsymbol{\psi}}-|S|\int_\Omega\left\{2\mu\boldsymbol{G}(\boldsymbol{h}):\overline{\boldsymbol{G}(\boldsymbol{\psi})}+\lambda\nabla\cdot\boldsymbol{h}\,\overline{\nabla\cdot\boldsymbol{\psi}}\right\},\quad(2.9)$$

$$\int_\Omega\overline{q}\left(|F|\nabla\cdot\boldsymbol{u}+\nabla\cdot\langle\boldsymbol{v}\rangle_F-\langle\nabla_y\cdot\boldsymbol{u}_1\rangle_S\right)\ =\ 0.\quad(2.10)$$

In this formulation, the new unknowns $\boldsymbol{v}\in L^2(\Omega,H_0^1(F,\mathbf{C}^3))$ and $\boldsymbol{u}_1\in L^2(\Omega,H^1(S,\mathbf{C}^3)/\mathcal{R})$ are uniquely determined by $\boldsymbol{u}$ and p as solutions of cell problems.

In the fluid domain, the fast displacement $\boldsymbol{v}(x,\cdot)$ is solution of a Stokes problem in F with data depending on $\boldsymbol{u}(x)$ and $\nabla p(x)$: there exists $p_1\in L^2(\Omega\times F)$, with $\int_F p_1=0$, such that for almost every $x\in\Omega$ and for every $(\boldsymbol{\varphi},q)\in H_0^1(F,\mathbf{C}^3)\times L^2(F,\mathbf{C})$, we have

$$-\,\omega^2\rho_f\int_F\boldsymbol{v}\cdot\overline{\boldsymbol{\varphi}}+2i\omega\nu\int_F\boldsymbol{G}_y(\boldsymbol{v}):\overline{\boldsymbol{G}_y(\boldsymbol{\varphi})}-\int_F p_1\overline{\nabla_y\cdot\boldsymbol{\varphi}}\quad(2.11)$$
$$=\left(\omega^2\rho_f(\boldsymbol{u}+\boldsymbol{h})-\nabla_x p\right)\cdot\int_F\overline{\boldsymbol{\varphi}},$$

$$\int_F\overline{q}\,\nabla_y\cdot\boldsymbol{v}=0.\quad(2.12)$$

In the solid domain, the displacement $\boldsymbol{u}_1(x,\cdot)$ is a solution of an elasticity problem in S: for almost every $x\in\Omega$ and every $\boldsymbol{\varphi}\in H^1(S,\mathbf{C}^3)$, we have

$$\int_S\left\{2\mu\boldsymbol{G}_y(\boldsymbol{u}_1):\overline{\boldsymbol{G}_y(\boldsymbol{\varphi})}+\lambda\nabla_y\cdot\boldsymbol{u}_1\,\overline{\nabla_y\cdot\boldsymbol{\varphi}}\right\}\quad(2.13)$$
$$=-\int_S\left\{2\mu\boldsymbol{G}(\boldsymbol{u}+\boldsymbol{h}):\overline{\boldsymbol{G}_y(\boldsymbol{\varphi})}+\lambda\nabla\cdot(\boldsymbol{u}+\boldsymbol{h})\,\overline{\nabla_y\cdot\boldsymbol{\varphi}}\right\}-p\int_S\overline{\nabla_y\cdot\boldsymbol{\varphi}}.$$

Remark 2.9. Before going further, we remark that the linearity of these three coupled problems yields a decoupling procedure. Both the cell problems are initially solved for generic right-hand sides; the unknown functions $\boldsymbol{v}$ and $\boldsymbol{u}_1$ are then linked to $\boldsymbol{u}$ and p by linear relations involving the microstructure properties (in particular its geometry). Eventually, the macroscopic problem is solved, where $\boldsymbol{v}$ and $\boldsymbol{u}_1$ are expressed in terms of $\boldsymbol{u}$ and p.

As we shall see (Remark 4.2), the second term in (2.11) is still of dissipative nature.

The following theorem specifies the well-posedness character of the problem (2.9)–(2.13). The proof can be found in Section 4.

Theorem 2.10. *Under assumptions* $(\mathbf{H}_1)$–$(\mathbf{H}_4)$*, the three following properties hold.*

a) *The fluid cell problem* (2.11)–(2.12) *is well-posed; i.e., for any* $\boldsymbol{e} \in \mathbf{C}^3$*, there exists a unique couple* $(\boldsymbol{v}, p_1) \in H_0^1(F, \mathbf{C}^3) \times L^2(F, \mathbf{C})/\mathbf{R}$ *such that for every* $(\boldsymbol{\varphi}, q) \in H_0^1(F, \mathbf{C}^3) \times L^2(F, \mathbf{C})$*, we have*

$$-\omega^2 \rho_f \int_F \boldsymbol{v} \cdot \overline{\boldsymbol{\varphi}} + 2i\omega\nu \int_F \boldsymbol{G}_y(\boldsymbol{v}) : \overline{\boldsymbol{G}_y(\boldsymbol{\varphi})} - \int_F p_1 \overline{\nabla_y \cdot \boldsymbol{\varphi}} = \int_S \boldsymbol{e} \cdot \overline{\boldsymbol{\varphi}}, \tag{2.14}$$

$$\int_F \overline{q} \nabla_y \cdot \boldsymbol{v} = 0. \tag{2.15}$$

Moreover, $\boldsymbol{v}$ *and* p_1 *are smooth on* $\overline{F}$*.*

b) *The solid cell problem* (2.13) *is well-posed; i.e., for any* $\boldsymbol{E} \in \mathbf{C}^9$*, there exists a unique* $\boldsymbol{u}_1 \in H^1(S, \mathbf{C}^3)/\mathcal{R}$ *such that for any* $\boldsymbol{\varphi} \in H^1(S, \mathbf{C}^3)$*, we have*

$$\int_S \left\{ 2\mu \boldsymbol{G}_y(\boldsymbol{u}_1) : \overline{\boldsymbol{G}_y(\boldsymbol{\varphi})} + \lambda \nabla_y \cdot \boldsymbol{u}_1 \, \overline{\nabla_y \cdot \boldsymbol{\varphi}} \right\} = \int_S \boldsymbol{E} \cdot \overline{\boldsymbol{G}(\boldsymbol{\varphi})}. \tag{2.16}$$

Moreover, this solution $\boldsymbol{u}_1$ *is smooth on* $\overline{S}$*.*

c) *Let* N *be the (at most countable) set of frequencies introduced in Theorem* 2.6*, and assume* $\omega \notin N$*. For every* $\boldsymbol{h} \in H^1(\Omega, \mathbf{C}^3)$*, there exists a unique solution* $\boldsymbol{u} \in H_D(\Omega)$*,* $p \in H^1(\Omega, \mathbf{C})$*,* $\boldsymbol{u}_1 \in L^2(\Omega, H^1(S, \mathbf{C}^3)/\mathcal{R})$*,* $\boldsymbol{v} \in L^2(\Omega, H_0^1(F))$*,* $p_1 \in L^2(\Omega \times F)$ *of the variational problem* (2.9)–(2.13)*. Moreover, there exists a constant* $C(\omega)$*, such that this solution satisfies the following estimate:*

$$\begin{aligned} &\|\boldsymbol{u}\|^2_{H^1(\Omega)} + \|p\|^2_{H^1(\Omega)} + \|\nabla_y \boldsymbol{u}_1\|^2_{L^2(\Omega\times S)} \\ &+ \|\nabla_y \boldsymbol{v}\|^2_{L^2(\Omega\times F)} + \|\boldsymbol{v}\|_{L^2(\Omega\times F)} + \|p_1\|^2_{L^2(\Omega\times F)} \;\le\; C(\omega)\|\boldsymbol{h}\|^2_{H^1(\Omega)}. \end{aligned} \tag{2.17}$$

Remark 2.11. Unsurprisingly, the homogenized problem (2.9)–(2.13) turns out to be the harmonic version of the homogenized problem found in [12], although there is no right-hand side here and our problem is posed in a bounded domain with boundary conditions.

Finally, we improve the two-scale convergences of Theorem 2.6 by establishing the following corrector result. This result indicates that, at leading order, no oscillations have been averaged out by the two-scale weak convergence process, so we can claim that the homogenized problem (2.9)–(2.13) contains the physically relevant information.

Theorem 2.12. *With the hypothesis and notation of Theorem* 2.6, *we have furthermore*

$$\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_{L^2(\Omega_\varepsilon^s)} \overset{\varepsilon\downarrow 0}{\longrightarrow} 0, \qquad \|\boldsymbol{u}_\varepsilon(x) - \boldsymbol{u}(x) - \boldsymbol{v}\,(x, x/\varepsilon)\,\|_{L^2(\Omega_\varepsilon^f)} \overset{\varepsilon\downarrow 0}{\longrightarrow} 0,$$

$$\|\boldsymbol{G}(\boldsymbol{u}_\varepsilon)(x) - \boldsymbol{G}(\boldsymbol{u})(x) - \boldsymbol{G}_y(\boldsymbol{u}_1)\,(x, x/\varepsilon)\,\|_{L^2(\Omega_\varepsilon^s)} \overset{\varepsilon\downarrow 0}{\longrightarrow} 0,$$

$$\|\varepsilon\boldsymbol{G}(\boldsymbol{u}_\varepsilon)(x) - \boldsymbol{G}_y(\boldsymbol{v})\,(x, x/\varepsilon)\,\|_{L^2(\Omega_\varepsilon^f)} \overset{\varepsilon\downarrow 0}{\longrightarrow} 0.$$

The different theorems have been stated in a natural order: first we establish the uniform bound of Theorem 2.1 b), then deduce two-scale weak compactness of bounded sequences in $L^2(\Omega)$ to obtain Theorem 2.6, and finally use the problem satisfied by the two-scale limits to obtain the corrector result Theorem 2.12. This scheme is misleading. Indeed, the proof of these results are interdependent. It turns out that the fundamental bound of part b) of Theorem 2.1 will be obtained at the end of the homogenization process. Roughly speaking, we start by proving that if these bounds were true, then the conclusions of Theorems 2.6 and 2.12 would hold. These weaker results are then used to prove part b) of Theorem 2.1 by contradiction. So the uniform bound was indeed true and we can enter the loop and conclude. To our knowledge, this kind of proof, which circumvents the lack of coercivity of the variational problem, has been initiated in [8] for the homogenization of a Helmholtz problem.

Another difficulty arises from the shape of the domain. Since we consider a general bounded Lipschitz domain Ω, its boundary may intersect $\mathcal{I}$ in a way such that, for example, there is no uniformly bounded family of extension operators $\{T_\varepsilon^s : H^1(\Omega_\varepsilon^s) \to H^1(\Omega)\}_{\varepsilon>0}$. To overcome this difficulty, we use truncation arguments and develop a specific treatment in the neighborhood of $\partial\Omega$ to establish compactness and strong convergence results up to the boundary.

Unsurprisingly, the homogenized problem (2.9)–(2.13) turns out to be the the harmonic version of the homogenized problem found in [12]. Pay attention however, to the fact that there is no forcing term here and our problem is posed in a bounded domain with natural boundary conditions. As in [12], there is a strong connection with the Biot–Allard model, which is classically used in poroelasticity.

The remainder of the paper is organized as follows. Technical lemmas are first stated in Section 3. Section 4 is devoted to the proof of existence and uniqueness of a solution to problem (2.1)–(2.2) for a fixed and small enough ε (part a) of Theorem 2.1). Then, we establish the well-posedness of the cell problems (parts b) and c) of Proposition 2.10) and eventually we establish that the complete homogenized problem admits at most one solution (uniqueness for part c) of Proposition 2.10). Finally, in Section 5, we prove the rest of the results along the lines described above.

3. Technical lemmas

In this paper, we use different types of Poincaré and Korn inequalities, depending on the domain and the nature of the boundary condition (if there is one). In this direction, the main tool we need is stated in the following lemma.

Lemma 3.1. *There exists $C > 0$ and $\nu_0 > 0$ such that for $\varepsilon \in (0, \nu_0)$ and any $\boldsymbol{\psi} \in H^1(\Omega, \mathbf{C}^3)$, if $\boldsymbol{\psi} \equiv 0$ on Ω_ε^s, or $\boldsymbol{\psi} \equiv 0$ on Ω_ε^f, then we have the estimate*

$$\int_\Omega |\boldsymbol{\psi}|^2 \;\le\; C\varepsilon^2 \int_\Omega |\boldsymbol{G}(\boldsymbol{\psi})|^2. \tag{3.1}$$

Proof. Let us first consider a single cell. Let $\tilde{Y} := [0,1)^3$ be a unit cell considered without periodicity and let $\tilde{S} := \tilde{Y} \cap \mathcal{S}$ and $\tilde{F} := \tilde{Y} \cap \mathcal{F}$ be the corresponding solid and fluid domains. By the Korn and Poincaré inequalities, there exists a constant $C > 0$ such that if $\boldsymbol{\psi} \in H^1(\tilde{Y})$ is such that either $u \equiv 0$ on S or $u \equiv 0$ on F, then

$$\int_{\tilde{Y}} |\boldsymbol{\psi}|^2 \;\le\; C \int_{\tilde{Y}} |\boldsymbol{G}(\boldsymbol{\psi})|^2.$$

Now, for $\varepsilon > 0$, let us define

$$Q_\varepsilon := \bigcup \Big\{ \varepsilon(\tilde{Y} + k) : k \in \mathbf{Z}^3,\ B(\varepsilon(k + (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})), \varepsilon) \subset \Omega \Big\}.$$

Applying the (rescaled) preceding inequality on cubes $\varepsilon(\tilde{Y}+k)$ and summing we obtain the following partial result.

Lemma 3.2. *Let $\boldsymbol{\psi} \in H^1(\Omega_\varepsilon, \mathbf{C}^3)$ such that $\boldsymbol{\psi} \equiv 0$ on Ω_ε^s, or $\boldsymbol{\psi} \equiv 0$ on Ω_ε^f. We have*

$$\int_{Q_\varepsilon} |\boldsymbol{\psi}|^2 \;\le\; C\varepsilon^2 \int_\Omega |\boldsymbol{G}(\boldsymbol{\psi})|^2.$$

Now we want to improve this estimate by establishing that it is valid up to the boundary. For this, we use the following result.

Lemma 3.3. *There exists $C > 0$ and $\nu_0 > 0$ such that for all $\boldsymbol{\psi} \in H^1(\Omega, \mathbf{C}^3)$ and any $\nu \in (0, \nu_0)$, we have*

$$\int_{\mathcal{N}(\nu)} |\boldsymbol{\psi}|^2 \;\le\; C\Big(\int_{\mathcal{N}(2\nu)\setminus\mathcal{N}(\nu)} |\boldsymbol{\psi}|^2 + \nu^2 \int_{\mathcal{N}(2\nu)} |\boldsymbol{G}(\boldsymbol{\psi})|^2 \Big),$$

with the notation $\mathcal{N}(\nu) := \{x \in \Omega : d(x, \partial\Omega) < \nu\}$, for $\nu > 0$.

Since $\Omega = \mathcal{N}(2\varepsilon) \cup Q_\varepsilon$ and $\mathcal{N}(2\varepsilon) \setminus \mathcal{N}(\varepsilon) \subset Q_\varepsilon$, the estimate (3.1) on the whole domain Ω follows from Lemma 3.2 and Lemma 3.3. □

It remains thus to prove Lemma 3.3. Using the fact that Ω is a bounded Lipschitz open set, Lemma 3.3 is a direct consequence of the following Lemma and a (finite) covering argument.

Lemma 3.4. *Let $D = B(0, 2r) \subset \mathbf{R}^2$ be an open disc and f be a Lipschitz function on D. For any $\delta > 0$ and $0 < \rho < 2r$, let us define the domain*

$$E_{\rho,\delta} = \big\{ x = (\tilde{x}, x_3) \in \mathbf{R}^3 : |\tilde{x}| < \rho,\ x_3 \in (f(\tilde{x}), f(\tilde{x}) + \delta) \big\}.$$

For every $\delta_0 > 0$, there exists $C > 0$ such that, for any $\boldsymbol{\psi} \in H^1(E_{2r,2\delta_0}, \mathbf{R}^3)$ and any $0 < \delta \le \delta_0$, we have

$$\|\boldsymbol{\psi}\|^2_{L^2(E_{r,\delta})} \;\le\; C\left(\delta^2 \|\boldsymbol{G}(\boldsymbol{\psi})\|^2_{L^2(E_{2r,2\delta})} + \|\boldsymbol{\psi}\|^2_{L^2(E_{2r,2\delta}\setminus E_{2r,\delta})} \right).$$

Proof. Let $\boldsymbol{\psi} \in H^1(E_{2r,2\delta_0}, \mathbf{R}^3)$ and $0 < \delta \le \delta_0$. For $i = 1, 2, 3$ and almost every $x = (\tilde{x}, x_3) \in E_{2r,\delta}$, we have

$$|\boldsymbol{\psi}_i|^2(x) \;=\; |\boldsymbol{\psi}_i|^2(\tilde{x}, x_3 + \delta) - 2\int_{x_3}^{x_3+\delta} (\boldsymbol{\psi}_i \cdot \partial_3 \boldsymbol{\psi}_i)(\tilde{x}, z)\, dz.$$

Integrating on $x_3 \in (f(\tilde{x}), f(\tilde{x}) + \delta)$, we obtain, using Fubini's theorem, for almost every $\tilde{x} \in D$,

$$\begin{aligned} \int_{f(\tilde{x})}^{f(\tilde{x})+\delta} |\boldsymbol{\psi}_i|^2(\tilde{x}, x_3)\, dx_3 = &\int_{f(\tilde{x})+\delta}^{f(\tilde{x})+2\delta} |\boldsymbol{\psi}_i|^2(\tilde{x}, x_3)\, dx_3 \\ &- 2\int_{f(\tilde{x})}^{f(\tilde{x})+2\delta} \varphi_\delta(z - f(\tilde{x}))\, (\boldsymbol{\psi}_i \cdot \partial_3 \boldsymbol{\psi}_i)(\tilde{x}, z)\, dz, \end{aligned} \tag{3.2}$$

where φ_δ is the hat function defined by $\varphi_\delta(t) = t$ on $[0, \delta]$ and $\varphi_\delta(t) = 2\delta - t$ on $[\delta, 2\delta]$.

Let us first consider the case $i = 3$ and integrate (3.2) on $\tilde{x} \in D$. Using successively the Cauchy–Schwarz and Young inequalities we obtain

$$\|\boldsymbol{\psi}_3\|^2_{L^2(E_{2r,\delta})} \le \|\boldsymbol{\psi}_3\|^2_{L^2(E_{2r,2\delta}\setminus E_{2r,\delta})} + 2\delta \|\partial_3 \boldsymbol{\psi}_3\|_{L^2(E_{2r,2\delta})} \|\boldsymbol{\psi}_3\|_{L^2(E_{2r,2\delta})}$$

$$\leq \|\boldsymbol{\psi}_3\|^2_{L^2(E_{2r,2\delta}\setminus E_{2r,\delta})} + 2\delta^2\|\partial_3\boldsymbol{\psi}_3\|^2_{L^2(E_{2r,2\delta})} + \frac{1}{2}\|\boldsymbol{\psi}_3\|^2_{L^2(E_{2r,2\delta})}.$$

Taking into account the identity $\boldsymbol{G}_{3,3}(\boldsymbol{\psi}) = \partial_3\boldsymbol{\psi}_3$, this yields

$$\|\boldsymbol{\psi}_3\|^2_{L^2(E_{2r,\delta})} \leq 3\|\boldsymbol{\psi}_3\|^2_{L^2(E_{2r,2\delta}\setminus E_{2r,\delta})} + 4\delta^2\|\boldsymbol{G}_{3,3}(\boldsymbol{\psi})\|^2_{L^2(E_{2r,2\delta})}. \tag{3.3}$$

Next, we consider the case $i = 1$ or $i = 2$. Let us introduce the Lipschitz-continuous function $\chi \in C(\overline{D})$ defined by $\chi(\tilde{x}) = 1$ if $|\tilde{x}| \leq r$ and $\chi(\tilde{x}) = 2 - |\tilde{x}|/r$ if $r < |\tilde{x}| \leq 2r$. Multiplying (3.2) by $\chi^2(\tilde{x})$, integrating on D, and using the identity, $\partial_3\boldsymbol{\psi}_i = 2\boldsymbol{G}_{i,3}(\boldsymbol{\psi}) - \partial_i\boldsymbol{\psi}_3$, we obtain

$$\begin{aligned}
&\int_{E_{2r,\delta}} \chi^2(\tilde{x})|\boldsymbol{\psi}_i|^2(x)\,dx \\
&= \int_{E_{2r,2\delta}\setminus E_{2r,\delta}} \chi^2(\tilde{x})|\boldsymbol{\psi}_i|^2(x)\,dx - 2\int_{E_{2r,2\delta}} \chi^2(\tilde{x})\varphi_\delta(x_3 - f(\tilde{x}))\boldsymbol{\psi}_i \cdot \boldsymbol{G}_{i,3}(\boldsymbol{\psi})\,dx \\
&+ 2\int_{E_{2r,2\delta}} \chi^2(\tilde{x})\varphi_\delta(x_3 - f(\tilde{x}))\,\boldsymbol{\psi}_i \cdot \partial_i\boldsymbol{\psi}_3\,dx.
\end{aligned} \tag{3.4}$$

Treating the second term on the right-hand side as above we see that it is bounded by

$$4\delta^2\|\boldsymbol{G}_{i,3}\|^2_{L^2(E_{2r,2\delta})} + \frac{1}{4}\int_{E_{2r,2\delta}} \chi^2(\tilde{x})|\boldsymbol{\psi}_i|^2(x)\,dx.$$

For the last term, we integrate by parts. Since $\chi^2(\tilde{x})\varphi_\delta(x_3 - f(\tilde{x})) = 0$ for $(\tilde{x}, x_3) \in \partial E_{2r,2\delta}$, the boundary terms vanish and we get

$$\begin{aligned}
2\Big|&\int_{E_{2r,2\delta}} \chi^2(\tilde{x})\varphi_\delta(x_3 - f(\tilde{x}))\,\boldsymbol{\psi}_i \cdot \partial_i\boldsymbol{\psi}_3\,dx\Big| \\
&\leq 2\int_{E_{2r,2\delta}} \chi^2(\tilde{x})\varphi_\delta(x_3 - f(\tilde{x}))|\partial_i\boldsymbol{\psi}_i||\boldsymbol{\psi}_3|\,dx \\
&\quad + 2\Big(\frac{2\delta}{r} + L\Big)\int_{E_{2r,2\delta}} \chi(\tilde{x})|\boldsymbol{\psi}_i||\boldsymbol{\psi}_3|\,dx,
\end{aligned}$$

where L is the Lipschitz constant of f. Now we notice that $G_{i,i}(\boldsymbol{\psi}) = 2\partial_i\boldsymbol{\psi}_i$ and we use the Cauchy–Schwarz and Young inequalities as before to get

$$\begin{aligned}
2\Big|&\int_{E_{2r,2\delta}} \chi^2(\tilde{x})\varphi_\delta(x_3 - f(\tilde{x}))\,\boldsymbol{\psi}_i \cdot \partial_i\boldsymbol{\psi}_3\,dx\Big| \\
&\leq \Big[\frac{1}{2} + 4\Big(\frac{2\delta_0}{r} + L\Big)^2\Big]\int_{E_{2r,2\delta}} |\boldsymbol{\psi}_3|^2(x)\,dx
\end{aligned}$$

$$+\frac{\delta^2}{2}\|\boldsymbol{G}_{i,i}\|^2_{L^2(E_{2r,2\delta})}+\frac{1}{4}\int_{E_{2r,2\delta}}\chi^2(\tilde{x})|\boldsymbol{\psi}_i|^2(x)\,dx.$$

Collecting the estimates, we conclude that (3.4) leads to

$$\begin{aligned}\|\boldsymbol{\psi}_i\|^2_{L^2(E_{r,\delta})}\le{}& 3\|\boldsymbol{\psi}_i\|^2_{L^2(E_{2r,2\delta}\setminus E_{2r,\delta})}\\&+\delta^2\Big(8\|\boldsymbol{G}_{i,3}(\boldsymbol{\psi})\|^2_{L^2(E_{2r,2\delta})}+\|\boldsymbol{G}_{i,i}(\boldsymbol{\psi})\|^2_{L^2(E_{2r,2\delta})}\Big)\\&+\Big[1+8\Big(\frac{2\delta_0}{r}+L\Big)^2\Big]\|\boldsymbol{\psi}_3\|^2_{L^2(E_{2r,2\delta})},\end{aligned}\tag{3.5}$$

where the last term has been cut in two pieces, namely on $E_{2r,2\delta}\setminus E_{2r,\delta}$ and on $E_{2r,\delta}$. Finally, (3.3) and (3.5) imply the lemma. □

In the sequel, we use the following Korn inequality (without boundary conditions) in the solid subdomain with a constant independent of ε. This result is proved by A. O. Oleinik *et al.* in [20].

Lemma 3.5. *There exists $C>0$ such that for any $\varepsilon>0$ with $Q^s_\varepsilon\neq\emptyset$ and any $\boldsymbol{\psi}\in H^1(\Omega^s_\varepsilon,\mathbf{C}^3)$,*

$$\int_{Q_\varepsilon\cap\Omega^s_\varepsilon}|\nabla\boldsymbol{\psi}|^2\le C\Big(\int_{Q_\varepsilon\cap\Omega^s_\varepsilon}|\boldsymbol{G}(\boldsymbol{\psi})|^2+\int_{Q_\varepsilon\cap\Omega^s_\varepsilon}|\boldsymbol{\psi}|^2\Big).$$

We also need an extension lemma which is due to E. Acerbi *et al.* (Theorem 2.1 in [1]).

Lemma 3.6. *There exists $\varepsilon_0>0$, $\xi>0$, and $C>0$ and a family of linear extension operators $\{T^s_\varepsilon : H^1(\Omega^s_\varepsilon)\to H^1(\Omega)\}_{\varepsilon\in(0,\varepsilon_0)}$ such that for $\varphi\in H^1(\Omega^s_\varepsilon)$, we have the identity $T^s_\varepsilon(\varphi)=\varphi$ on Ω^s_ε and the estimates*

$$\|T^s_\varepsilon\varphi\|^2_{L^2(\Omega\setminus\mathcal{N}(\xi\varepsilon))}\le C\|\varphi\|^2_{L^2(\Omega^s_\varepsilon)},\quad \|\nabla[T^s_\varepsilon\varphi]\|^2_{L^2(\Omega\setminus\mathcal{N}(\xi\varepsilon))}\le C\|\nabla\varphi\|^2_{L^2(\Omega^s_\varepsilon)}.$$

Finally, we use an interior version of the Rellich theorem.

Lemma 3.7. *Let $\xi>0$ and let $\{\varphi_\varepsilon\}_{\varepsilon>0}\subset H^1(\Omega)$ be such that*

$$\{\|\varphi_\varepsilon\|_{H^1(\Omega\setminus\mathcal{N}(\xi\varepsilon))}\}_{\varepsilon>0}$$

is bounded. Then there exists $\varphi_0\in H^1(\Omega)$ and a sequence $(\varepsilon_n)_{n\ge0}$ converging to 0, such that $\|\varphi_{\varepsilon_n}-\varphi_0\|_{L^2(\Omega\setminus\mathcal{N}(\xi\varepsilon_n))}\longrightarrow 0$.

Proof. Since Ω is Lipschitz, there exists $\varepsilon_0>0$ and a family of bi-Lipschitz mappings $\{\boldsymbol{\psi}_\varepsilon:\Omega\setminus\mathcal{N}(\xi\varepsilon)\to\Omega\}_{\varepsilon\in(0,\varepsilon_0)}$ such that

$$\{\|\nabla\boldsymbol{\psi}_\varepsilon\|_\infty\}_{\varepsilon\in(0,\varepsilon_0)},\quad\{\|\nabla\boldsymbol{\psi}^{-1}_\varepsilon\|_\infty\}_{\varepsilon\in(0,\varepsilon_0)}$$

are uniformly bounded and $\boldsymbol{\psi}_\varepsilon \equiv \mathrm{Id}$ on $\Omega \setminus \mathcal{N}(2\xi\varepsilon)$. We consider the family $\{\tilde{\varphi}_\varepsilon := \varphi_\varepsilon \circ \boldsymbol{\psi}_{\boldsymbol{\varepsilon}}\}_{\varepsilon\in(0,\varepsilon_0)}$. By the chain rule, this family is bounded in $H^1(\Omega)$, so we can apply the Rellich theorem. There exists $\varphi_0 \in H^1(\Omega)$ and $(\varepsilon_n)_{n\geq 0}$ converging to 0, such that $(\varphi_{\varepsilon_n} \circ \boldsymbol{\psi}_{\varepsilon_n})_{n\geq 0}$ converges to φ_0 in $L^2(\Omega)$. Using the identity $\boldsymbol{\psi}_{\varepsilon_n} \equiv \mathrm{Id}$ on $\Omega \setminus \mathcal{N}(2\xi\varepsilon_n)$, we have

$$\begin{aligned} &\|\varphi_{\varepsilon_n} - \varphi_0\|^2_{L^2(\Omega\setminus\mathcal{N}(\xi\varepsilon_n))} \\ &= \|\tilde{\varphi}_{\varepsilon_n} - \varphi_0\|^2_{L^2(\Omega\setminus\mathcal{N}(2\xi\varepsilon_n))} + \|\varphi_{\varepsilon_n} - \varphi_0\|^2_{L^2(\mathcal{N}(2\xi\varepsilon_n)\setminus\mathcal{N}(\xi\varepsilon_n))}\,. \end{aligned} \tag{3.6}$$

The first term tends to 0. Using the change of variables $x = \boldsymbol{\psi}_{\varepsilon_n}(y)$ and the bounds on $\nabla\boldsymbol{\psi}_\varepsilon$, we see that the remaining term is bounded by

$$C\Big(\int_{\boldsymbol{\psi}^{-1}_{\varepsilon_n}(\mathcal{N}(2\xi\varepsilon_n))} |\tilde{\varphi}_{\varepsilon_n}(y)|^2\,dy + \int_{\mathcal{N}(2\xi\varepsilon_n)} |\varphi_0(x)|^2\,dx\Big).$$

Using the compactness of $(\tilde{\varphi}_{\varepsilon_n})_{n\geq 0}$ in $L^2(\Omega)$ and the fact that the volume of the integration domain goes to 0, we conclude that the left-hand side of (3.6) converges to 0. □

4. Well-posedness results

In this section we establish the existence and uniqueness of the solution of (2.1)–(2.2) (part a) of Theorem 2.1), and then we obtain the well-posedness of the problems (2.14)–(2.15) and (2.16) (parts a) and b) of Proposition 2.10). Finally, we will establish the uniqueness part of Proposition 2.10 (part c)) concerning the homogenized problem.

4.1. **Proof of Theorem 2.1.a: Existence and uniqueness of the solution of problem (2.1)–(2.2):** We proceed in two steps. As explained earlier, the problem (2.1)–(2.2) is of Helmholtz type, and we use Fredholm theory to study its well-posedness. The pressure field will be obtained by using De Rham's theorem.

Let$(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ be a solution of (2.1)–(2.2). Then $\boldsymbol{u}_\varepsilon$ also solves the following problem:

$$a_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{\psi}) = F(\boldsymbol{\psi}), \quad \forall\, \boldsymbol{\psi} \in H_f := \big\{\boldsymbol{\psi} \in H_D(\Omega) : \nabla\cdot\boldsymbol{\psi} = 0 \text{ on } \Omega^f_\varepsilon\big\}, \tag{4.1}$$

where $F(\boldsymbol{\psi}) := F_\varepsilon(\boldsymbol{h}, \boldsymbol{\psi})$ defines a continuous linear form on $H_D(\Omega)$ and where $a_\varepsilon := b_\varepsilon + c_\varepsilon$ is the sum of the continuous sesquilinear forms on $H_D(\Omega) \times H_D(\Omega)$ defined by

$$b_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{\psi}) = 2\mu \int_{\Omega_s} \boldsymbol{G}(\boldsymbol{u}_\varepsilon) : \overline{\boldsymbol{G}(\boldsymbol{\psi})} + \lambda \int_{\Omega_s} \nabla\cdot\boldsymbol{u}_\varepsilon\,\overline{\nabla\cdot\boldsymbol{\psi}} \tag{4.2}$$

$$+ 2i\omega\eta\varepsilon^2 \int_{\Omega_f} \boldsymbol{G}(\boldsymbol{u}_\varepsilon) : \overline{\boldsymbol{G}(\boldsymbol{\psi})},$$

$$c_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{\psi}) = -\omega^2 \rho_s \int_{\Omega_s} \boldsymbol{u}_\varepsilon \cdot \overline{\boldsymbol{\psi}} - \omega^2 \rho_f \int_{\Omega_f} \boldsymbol{u}_\varepsilon \cdot \overline{\boldsymbol{\psi}}. \tag{4.3}$$

Let us also introduce the continuous linear operators $A_\varepsilon, B_\varepsilon, C_\varepsilon : H_f \to (H_f)'$ naturally defined by

$$\langle A_\varepsilon \boldsymbol{w}, \boldsymbol{\psi} \rangle := a_\varepsilon(\boldsymbol{w}, \boldsymbol{\psi}), \quad \langle B_\varepsilon \boldsymbol{w}, \boldsymbol{\psi} \rangle := b_\varepsilon(\boldsymbol{w}, \boldsymbol{\psi}), \quad \langle C_\varepsilon \boldsymbol{w}, \boldsymbol{\psi} \rangle := c_\varepsilon(\boldsymbol{w}, \boldsymbol{\psi}),$$

for $\boldsymbol{w}, \boldsymbol{\psi} \in H_f$.

Step 1: existence and uniqueness of the solutions of (4.1). First, the compact embedding from $[H^1(\Omega)]^3$ into $[L^2(\Omega)]^3$ gives the compactness of C_ε. Next, we claim that B_ε is invertible by using Lax–Milgram theorem. Indeed, thanks to the symmetry of the tensor $\boldsymbol{G}$, we have for all $\boldsymbol{w} \in H_f$

$$b_\varepsilon(\boldsymbol{w}, \boldsymbol{w}) = 2\mu \|\boldsymbol{G}(\boldsymbol{w})\|^2_{L^2(\Omega^s_\varepsilon)} + \lambda \|\nabla \cdot \boldsymbol{w}\|^2_{L^2(\Omega^s_\varepsilon)} + 2i\omega\nu\varepsilon^2 \|\boldsymbol{G}(\boldsymbol{w})\|^2_{L^2(\Omega^f_\varepsilon)},$$

from which we get

$$|b_\varepsilon(\boldsymbol{w}, \boldsymbol{w})| \geq \delta(\varepsilon) \|\boldsymbol{G}(\boldsymbol{w})\|^2_{L^2(\Omega)},$$

with $\delta(\varepsilon) = \min(2\mu, 2\omega\nu\varepsilon^2) > 0$. The sesquilinear form b_ε is then coercive on H_f by the classical Korn inequality and B_ε is invertible by the Lax–Milgram theorem.

From Fredholm theory, since A_ε is a compact perturbation of C_ε, it is sufficient to show that A_ε is one-to-one in order to establish that it is invertible. For $\boldsymbol{w} \in \ker A_\varepsilon$, we have immediately the identity

$$2\mu \|\boldsymbol{G}(\boldsymbol{w})\|^2_{L^2(\Omega^s_\varepsilon)} + \lambda \|\nabla \cdot \boldsymbol{w}\|^2_{L^2(\Omega^s_\varepsilon)} + 2i\omega\nu\varepsilon^2 \|\boldsymbol{G}(\boldsymbol{w})\|^2_{L^2(\Omega^f_\varepsilon)} - \omega^2 \rho_s \|\boldsymbol{w}\|^2_{L^2(\Omega^s_\varepsilon)} - \omega^2 \rho_f \|\boldsymbol{w}\|^2_{L^2(\Omega^f_\varepsilon)} = 0.$$

Taking the imaginary part yields $\|\boldsymbol{G}(\boldsymbol{w})\|^2_{L^2(\Omega^f_\varepsilon)} = 0$. Then, using the boundary condition $\boldsymbol{w} = 0$ on Γ_D (and assumption ($\mathbf{H}_4$)), we get by the Korn inequality, $\boldsymbol{w} = 0$ on Ω^f_ε. The previous identity then reduces to

$$2\mu \|\boldsymbol{G}(\boldsymbol{w})\|^2_{L^2(\Omega^s_\varepsilon)} + \lambda \|\nabla \cdot \boldsymbol{w}\|^2_{L^2(\Omega^s_\varepsilon)} - \omega^2 \rho_s \|\boldsymbol{w}\|^2_{L^2(\Omega^s_\varepsilon)} = 0,$$

and from (3.1) we get

$$\left(\frac{2\mu}{C\varepsilon^2} - \omega^2 \rho_s \right) \|\boldsymbol{w}\|^2_{L^2(\Omega^s_\varepsilon)} \leq 0,$$

which implies $\boldsymbol{w} = 0$ on Ω_ε^s, for ε small enough. Therefore, A_ε is one-to-one for ε small enough and, thus, invertible.

Step 2: existence and uniqueness of the pressure field. From Step 1, we conclude that there exists a unique $\boldsymbol{u}_\varepsilon \in H_f$ such that (4.1) holds. Equivalently, there exists a unique continuous linear form

$$L \in H_f^\perp := \left\{T \in H'_D(\Omega) : \langle T, \boldsymbol{\psi}\rangle = 0, \, \forall \boldsymbol{\psi} \in H_f\right\},$$

such that $a(\boldsymbol{u}_\varepsilon, \overline{\boldsymbol{\psi}}) + \langle L, \overline{\boldsymbol{\psi}}\rangle = F(\boldsymbol{\psi})$ for every $\boldsymbol{\psi} \in H_D(\Omega)$. We now establish that L can be represented by a unique pressure field.

Let H be the subspace of $H'_D(\Omega)$ defined by

$$H := \left\{\boldsymbol{g}_{q_\varepsilon} \in H'_D(\Omega) : \left\langle \boldsymbol{g}_{q_\varepsilon}, \boldsymbol{\psi}\right\rangle := -\int_{\Omega_\varepsilon^f} q_\varepsilon(x)\, \overline{\nabla \cdot \boldsymbol{\psi}(x)}\, dx, \; q_\varepsilon \in L^2(\Omega_\varepsilon^f)\right\}. \tag{4.4}$$

We have to show that $H = H_f^\perp$. Clearly, $H \subset H_f^\perp$ is a closed subspace of $H'_D(\Omega)$. It remains to prove that $H^\perp \subset H_f$. Let $\boldsymbol{\psi} \in H^\perp$; for every function $\boldsymbol{g}$ in H, we have $\langle \boldsymbol{g}, \boldsymbol{\psi}\rangle = 0$, or equivalently for every $q_\varepsilon \in L^2(\Omega_\varepsilon^f)$, $\int_{\Omega_\varepsilon^f} q_\varepsilon(x) \nabla \cdot \boldsymbol{\psi}(x)\, dx = 0$. Thus $\nabla \cdot \boldsymbol{\psi} = 0$ on Ω_ε^f, and $\boldsymbol{\psi} \in H_f$, which establishes that $H = H_f^\perp$. In particular, there exists a pressure field $q_\varepsilon \in L^2(\Omega_\varepsilon^f)$ such that $L = \boldsymbol{g}_{q_\varepsilon}$.

Finally, we claim that this pressure field is unique. Indeed, assume by linearity that $q_\varepsilon \in L^2(\Omega_\varepsilon^f)$ is such that $\int_{\Omega_\varepsilon^f} q_\varepsilon \overline{\nabla \cdot \boldsymbol{\psi}} = 0$ for any $\boldsymbol{\psi} \in H_0^1(\Omega)$. Following [12] (notice that this expansion is different from the one proposed in [13, 22]), we set

$$\tilde{q}_\varepsilon := \begin{cases} q_\varepsilon - \dfrac{1}{|\Omega|} \displaystyle\int_{\Omega_\varepsilon^f} q_\varepsilon & \text{on } \Omega_\varepsilon^f, \\ -\dfrac{1}{|\Omega|} \displaystyle\int_{\Omega_\varepsilon^f} q_\varepsilon & \text{on } \Omega_\varepsilon^s, \end{cases} \tag{4.5}$$

so that, for any $\boldsymbol{\psi} \in H_0^1(\Omega)$ we have

$$\int_\Omega \tilde{q}_\varepsilon \overline{\nabla \cdot \boldsymbol{\psi}} = \int_{\Omega_\varepsilon^f} q_\varepsilon \overline{\nabla \cdot \boldsymbol{\psi}} = 0.$$

By construction, we also have $\int_\Omega \tilde{q}_\varepsilon = 0$, so we can choose $\boldsymbol{\psi} \in H_0^1(\Omega)$ such that $\nabla \cdot \boldsymbol{\psi} = \tilde{q}_\varepsilon$ in Ω. We then have $0 = \int_\Omega \tilde{q}_\varepsilon \nabla \cdot \overline{\boldsymbol{\psi}} = \int_\Omega |\tilde{q}_\varepsilon|^2$ and $\tilde{q}_\varepsilon$ vanishes. This easily yields $q_\varepsilon = 0$ almost everywhere in Ω_ε^f, which proves the uniqueness of the pressure. $\square$

4.2. **Proof of parts a) and b) of Proposition 2.10: well-posedness of the cell problems.** Both variational problems (2.16) and (2.14)–(2.15) are well-known, and their well-posedness and regularity results are established for instance in [3] and in [9]. In particular, since $p \in H^1(\Omega)$ (see Step 1 of Section 5) we have

$$\|p_1\|_{L^2(\Omega,H^1(F))} + \|\boldsymbol{v}\|_{L^2(\Omega,H^1_0(F))} \leq C\left(\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)}\right). \tag{4.6}$$

4.3. **Uniqueness result for the homogenized problem and definition of N.** We only prove a uniqueness result of the solution of (2.9)–(2.13). Existence and bounds will be obtained in Section 5 as by-products of the two-scale convergences.

By linearity, we consider $(\boldsymbol{u}, p) \in H_D \times H^1(\Omega)$ and $\boldsymbol{u}_1 \in L^2(\Omega, H^1(S))$, $\boldsymbol{v} \in L^2(\Omega, H^1_0(F))$ such that (2.9)–(2.13) holds with $\boldsymbol{h} = 0$ and prove that $(\boldsymbol{u}, p, \boldsymbol{v}, p_1, \boldsymbol{u}_1) = 0$.

Step 1: $\boldsymbol{v} = 0$. Summing (2.9) with test function $\boldsymbol{\psi} = \boldsymbol{u}$, (2.11) with test function $\boldsymbol{\varphi} = \boldsymbol{v}$, and (2.13) with test function $\boldsymbol{\varphi} = \boldsymbol{u}_1$, and using (2.10), we get the energy identity

$$\begin{aligned} &-\omega^2\rho_s|S|\|\boldsymbol{u}\|^2_{L^2(\Omega)} - \omega^2\rho_f\|\boldsymbol{u}+\boldsymbol{v}\|^2_{L^2(\Omega\times F)} + 2i\omega\nu\|\boldsymbol{G}_y(\boldsymbol{v})\|^2_{L^2(\Omega\times F)} \\ &\quad + 2\mu\|\boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{G}_y(\boldsymbol{u}_1)\|^2_{L^2(\Omega\times S)} + \lambda\|\nabla\cdot\boldsymbol{u} + \nabla_y\cdot\boldsymbol{u}_1\|^2_{L^2(\Omega\times S)} = 0. \end{aligned} \tag{4.7}$$

Taking the imaginary part, we get $\|\boldsymbol{G}_y(\boldsymbol{v})\|_{L^2(\Omega\times F)} = 0$, and since $\boldsymbol{v}(x,\cdot) \in H^1_0(F, \mathbf{C}^3)$ for almost every $x \in \Omega$, we deduce from the classical Korn inequality that $\boldsymbol{v}$ vanishes almost everywhere.

Step 2: $p_1 = 0$. Using (2.11), we get a relationship between the two macroscopic variables $\boldsymbol{u}$ and p and the microscopic variable p_1. Namely, for every $\boldsymbol{\varphi} \in L^2(\Omega, H^1_0(F))$, we have

$$\int_\Omega\int_F \left(\omega^2\rho_f\boldsymbol{u} - \nabla p - \nabla_y p_1\right)\cdot\overline{\boldsymbol{\varphi}} \;=\; 0, \tag{4.8}$$

which implies that $\nabla_y p_1$ depends only on x. More precisely, there exists a function $a(x)$ such that $p_1(x,y) = (\omega^2\rho_f\boldsymbol{u} - \nabla p)(x)\cdot y + a(x)$. By the y-periodicity of p_1, $\omega^2\rho_f\boldsymbol{u} - \nabla p = 0$, and from $\int_F p_1 = 0$, we get $a(x) = 0$. Finally, p_1 vanishes almost everywhere in $\Omega \times F$.

Step 3: $\boldsymbol{u} = 0$ *and* $\boldsymbol{u}_1 = 0$. We set

$$K := \left\{ (\boldsymbol{\psi}, \boldsymbol{\varphi}) \in H_D(\Omega) \times L^2(\Omega, H^1(S)/\mathcal{R}),\ |F| \nabla \cdot \boldsymbol{\psi} = \langle \nabla_y \cdot \boldsymbol{\varphi} \rangle_S \right\}.$$

In order to show that $\boldsymbol{u} = \boldsymbol{u}_1 = 0$, we need the following proposition, whose proof is postponed to the end of the section.

Proposition 4.1. *There exists a (possibly empty) discrete closed set $N \subset \mathbb{R}$ such that if $\omega \notin N$, and $(\boldsymbol{u}, \boldsymbol{u}_1) \in K$ satisfies*

$$\begin{aligned} -\omega^2 \rho \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{\psi}} + 2\mu \int_\Omega \int_S (\boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{G}_y(\boldsymbol{u}_1)) : \overline{(\boldsymbol{G}(\boldsymbol{\psi}) + \boldsymbol{G}_y(\boldsymbol{\varphi}))} \\ + \lambda \int_\Omega \int_S (\nabla \cdot \boldsymbol{u} + \nabla_y \cdot \boldsymbol{u}_1)\, \overline{(\nabla \cdot \boldsymbol{\psi} + \nabla_y \cdot \boldsymbol{\varphi})} = 0, \end{aligned} \tag{4.9}$$

for all $(\boldsymbol{\psi}, \boldsymbol{\varphi}) \in K$, then $(\boldsymbol{u}, \boldsymbol{u}_1) = (0, 0)$.

Assuming this result for the time being, we see by summing the variational formulations (2.9)–(2.13), that $(\boldsymbol{u}, \boldsymbol{u}_1)$ solves (4.9) for every $(\boldsymbol{\psi}, \boldsymbol{\psi}_1) \in K$, and therefore, for every $\omega \notin N$, we obtain that $(\boldsymbol{u}, \boldsymbol{u}_1) = (0, 0)$.

Step 4: $p = 0$. Finally, we deduce from (4.8) that p is constant almost everywhere in Ω. Using (2.13) with any test function $\boldsymbol{\varphi}$ such that

$$\int_F \nabla_y \cdot \boldsymbol{\varphi} \neq 0,$$

we obtain $p = 0$. This proves the uniqueness of the solution $(\boldsymbol{u}, p, \boldsymbol{v}, p_1, \boldsymbol{u}_1)$ of (2.9)–(2.13). $\square$

Proof of Proposition 4.1. To complete the proof above, we need to prove Proposition 4.1. We first notice that if $(\boldsymbol{u}, \boldsymbol{u}_1)$ satisfies (4.9), then one has

$$\int_\Omega \int_S (\boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{G}_y(\boldsymbol{u}_1)) : \overline{\boldsymbol{G}_y(\boldsymbol{\varphi}')} = 0, \tag{4.10}$$

for all $\boldsymbol{\varphi}' \in L^2(\Omega, H^1(S)/\mathcal{R})$ which satisfies $\langle \nabla_y \cdot \boldsymbol{\varphi}' \rangle_S = 0$. Clearly, the solution $\boldsymbol{u}_1$ of (4.10) is unique and depends linearly on $\boldsymbol{u}$.

The problem (4.9) is therefore equivalent to

$$D(\boldsymbol{u}, \boldsymbol{\psi}) = \omega^2 Q(\boldsymbol{u}, \boldsymbol{\psi}),\ \forall \boldsymbol{\psi} \in H_D(\Omega), \tag{4.11}$$

where the symmetric positive sesquilinear forms D and Q are given by

$$\begin{aligned} D(\boldsymbol{u}, \boldsymbol{\psi}) = 2\mu \int_\Omega \int_S (\boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{G}_y(\boldsymbol{u}_1)) : \overline{(\boldsymbol{G}(\boldsymbol{\psi}) + \boldsymbol{G}_y(\boldsymbol{\psi}_1))} \\ + \lambda \int_\Omega \int_S (\nabla \cdot \boldsymbol{u} + \nabla_y \cdot \boldsymbol{u}_1)\, \overline{(\nabla \cdot \boldsymbol{\psi} + \nabla_y \cdot \boldsymbol{\psi}_1)}, \end{aligned}$$

with $\boldsymbol{u}_1$ and $\boldsymbol{\psi}_1$ solving (4.10) for $\boldsymbol{u}$ and $\boldsymbol{\psi}$ respectively, and

$$Q(\boldsymbol{u},\boldsymbol{\psi}) = \rho \int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{\psi}}.$$

The result follows from classical spectral theory, provided D is coercive on $H_D(\Omega)$. This is a direct consequence of the coercivity of the sesquilinear form d defined for $(P,\boldsymbol{u}_1),(Q,\boldsymbol{\varphi}) \in \mathcal{M}_3(\mathbf{C}) \times H^1(S)/\mathcal{R}$ by

$$d\big((P,\boldsymbol{u}_1),(Q,\boldsymbol{\varphi})\big) := 2\mu \int_S (P+\boldsymbol{G}_y(\boldsymbol{u}_1)) : \overline{(Q+\boldsymbol{G}_y(\boldsymbol{\varphi}))} \\ + \lambda \int_S (\mathrm{Tr}(P)+\nabla_y\cdot\boldsymbol{u}_1)\ \overline{(\mathrm{Tr}(Q)+\nabla_y\cdot\boldsymbol{\varphi})}.$$

Because of the positivity of λ and μ, it is sufficient to prove the following inequality: there exists $\alpha > 0$ such that for every $(M,\boldsymbol{u}_1) \in \mathcal{M}_3(\mathbf{C}) \times H^1(S)/\mathcal{R}$, we have

$$\alpha\left(|M|^2 + \|\boldsymbol{G}_y(\boldsymbol{u}_1)\|^2_{L^2(S)}\right) \ \le\ \int_S |M+\boldsymbol{G}_y(\boldsymbol{u}_1)|^2 . \tag{4.12}$$

Let us assume for the sake of contradiction that $(M^n,\boldsymbol{u}_1^n)_{n\ge 0} \subset \mathcal{M}_3(\mathbf{C}) \times H^1(S)/\mathcal{R}$ is a sequence such that

$$|M^n|^2 + \|\boldsymbol{G}_y(\boldsymbol{u}_1^n)\|^2_{L^2(S)} = 1 \tag{4.13}$$

and $\|M^n+\boldsymbol{G}_y(\boldsymbol{u}_1^n)\|_{L^2(S)}$ tends to 0. Then, up to extraction, $(M^n)_{n\ge 0}$ strongly converges to a matrix M^∞. Moreover, by the classical Korn inequality on S, $(\|2\boldsymbol{u}_1^n + M^n y\|_{H^1(S)})_{n\ge 0}$ tends to 0. Then, $(\boldsymbol{u}_1^n)_{n\ge 0}$ strongly converges in $H^1(S)/\mathcal{R}$ towards a locally affine mapping $\boldsymbol{u}_1^\infty$. Furthermore, since $\boldsymbol{u}_1^n$ is periodic on Y, $\boldsymbol{u}_1^\infty$ is also periodic and as $\mathcal{S}$ is connected by assumption $(\mathbf{H}_2)$, we deduce that $\boldsymbol{u}_1^\infty = 0$, and then $M^\infty = 0$. The contradiction is obtained after passing to the limit in (4.13).

Remark 4.2. Notice that the imaginary term in (4.7) is the reminiscence of the dissipative character of the homogenized problem (2.9)–(2.13).

5. Periodic homogenization of problem (2.1)–(2.2)

In this section, we investigate the convergence of problem (2.1)–(2.2) by using the theory of homogenization. We proceed in three steps. Let $\boldsymbol{h} \in H^1(\Omega,\mathbf{C}^3)$ and let $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in H_D(\Omega) \times L^2(\Omega_\varepsilon^f,\mathbf{C})$ be the corresponding solution of (2.1)–(2.2) with right-hand side $F_\varepsilon(\boldsymbol{h},\cdot)$ for all $\varepsilon > 0$. We first assume that the family $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ satisfies the additional condition

$$\|\boldsymbol{u}_\varepsilon\|^2_{L^2(\Omega)} + \|\boldsymbol{G}(\boldsymbol{u}_\varepsilon)\|^2_{L^2(\Omega^s_\varepsilon)} + \varepsilon^2\|\boldsymbol{G}(\boldsymbol{u}_\varepsilon)\|^2_{L^2(\Omega^f_\varepsilon)} \;\le\; C_0, \qquad (5.1)$$

for some constant $C_0 > 0$ that does not depend on ε.

We prove in Step 1 that under this additional hypothesis, the conclusion of Theorem 2.6 holds; i.e., the family $\{(\boldsymbol{u}_\varepsilon, p_\varepsilon)\}_{\varepsilon>0}$ admits suitable two-scale limits $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u}_1$, and p satisfying (2.9)–(2.13).

Then in Step 2, with the same hypotheses we prove that the strong convergence results of Theorem 2.12 also hold.

Finally, in Step 3, using Steps 1 and 2, we prove by contradiction part b) of Theorem 2.1. This justifies *a posteriori* the bound (5.1) and thus establishes Theorem 2.6 and Theorem 2.12. We also deduce part c) of Proposition 2.10: the uniqueness has been established in Section 4.3 and existence and bounds follow from Theorem 2.6.

Step 1: $\{(\boldsymbol{u}_\varepsilon, p_\varepsilon)\}_{\varepsilon>0}$ *admits suitable two-scale limits* $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{u}_1$, *and* p *satisfying* (2.9)–(2.13) *(proof of Theorem* 2.6*)*. The *a priori* estimate for the pressure field is not easy since the pressure is only defined on Ω^f_ε, which depends on ε. We therefore extend p_ε to the whole of Ω using $\tilde{p}_\varepsilon$ defined from p_ε by the construction (4.5).

Since $\int_\Omega \tilde{p}_\varepsilon = 0$, there exists $\boldsymbol{\psi}_\varepsilon \in H^1_0(\Omega)$ such that $\nabla\cdot\boldsymbol{\psi}_\varepsilon = \tilde{p}_\varepsilon$ with the estimate

$$\|\boldsymbol{\psi}_\varepsilon\|_{H^1(\Omega)} \;\le\; C\|\tilde{p}_\varepsilon\|_{L^2(\Omega)} \;\le\; C\|p_\varepsilon\|_{L^2(\Omega^f_\varepsilon)}. \qquad (5.2)$$

The variational formulation (2.1) and the hypothesis (5.1) give for any test function $\boldsymbol{\psi} \in H^1_0(\Omega)$

$$\Big|\int_\Omega \tilde{p}_\varepsilon \overline{\nabla\cdot\boldsymbol{\psi}}\Big| = \Big|\int_{\Omega^f_\varepsilon} p_\varepsilon \overline{\nabla\cdot\boldsymbol{\psi}}\Big| \le\; C\;\|\boldsymbol{\psi}\|_{H^1(\Omega)}.$$

Then, using the test function $\boldsymbol{\psi} = \boldsymbol{\psi}_\varepsilon$, we get

$$\int_\Omega |\tilde{p}_\varepsilon|^2 \;\le\; C,$$

from (5.2). This gives, from the definition (4.5) of $\tilde{p}_\varepsilon$, the estimation

$$\int_{\Omega^f_\varepsilon} |p_\varepsilon|^2 \;\le\; C.$$

Thus, up to a subsequence, both $\{\boldsymbol{u}_\varepsilon\}_\varepsilon$ and $\{\widetilde{p_\varepsilon}\}_\varepsilon$ two-scale converge respectively to $\boldsymbol{u}_0$ and p_0. The results (2.6)–(2.7) are well-known (see for example [2] and [16]).

In order to establish that the limit p_0 only depends on the macroscopic variable x, we consider the formulation (2.1) multiplied by ε with a test function in $L^2(\Omega, H_0^1(F))$. Passing to the limit, this leads to $\nabla_y p_0 = 0$ on $\Omega \times F$. So p_0 depends only on x, and we will write $p(x) = p_0(x, y)$.

Now, each of the variational formulations (2.9)–(2.13) is the limit as ε goes to zero of (2.1)–(2.2) with a good choice of the scaling and of the test function space. This type of result is very classical and can be found for example in [2] or [12]. More precisely, the variational formulation (2.11) is the limit of (2.1) with $\boldsymbol{\psi} \in L^2(\Omega, H_0^1(F))$ such that $\nabla_y \cdot \boldsymbol{\psi} = 0$. The variational formulation (2.12) is the limit of $\varepsilon\times$(2.2) with $q \in L^2(\Omega)$, and the variational formulation (2.13) is the limit of $\varepsilon\times$(2.1) with $\boldsymbol{\psi} \in L^2(\Omega, H^1(S))$. Finally, the macroscopic variational formulations are both the limit of (2.1)–(2.2) with macroscopic test functions.

We provide some details of the proof of (2.10), which is not a classical result. Since $\Omega_\varepsilon^f = \Omega \backslash \Omega_\varepsilon^s$, we get from (2.2) that

$$\int_\Omega \overline{q}(\nabla \cdot \boldsymbol{u}_\varepsilon) - \int_{\Omega_\varepsilon^s} \overline{q}(\nabla \cdot \boldsymbol{u}_\varepsilon) \; = \; 0, \quad \forall\, q \in L^2(\Omega).$$

Now, we assume that the test function $q \in L^2(\Omega)$ is smooth and compactly supported in Ω, and we integrate by parts in the first integral. Passing to the limit $\varepsilon \to 0$ and using the two-scale convergences $\boldsymbol{u}_\varepsilon \twoheadrightarrow \boldsymbol{u} + \boldsymbol{v}$ in $\Omega \times F$ and $\boldsymbol{u}_\varepsilon \twoheadrightarrow \boldsymbol{u}$, $\nabla \boldsymbol{u}_\varepsilon \twoheadrightarrow \nabla_x \boldsymbol{u} + \nabla_y \boldsymbol{u}_1$ in $\Omega \times S$, we get

$$-\int_\Omega \int_F \nabla \overline{q} \cdot (\boldsymbol{u} + \boldsymbol{v}) - \int_\Omega \int_S \nabla \overline{q} \cdot \boldsymbol{u} - \int_\Omega \int_S \overline{q}(\nabla_x \cdot \boldsymbol{u} + \nabla_y \cdot \boldsymbol{u}_1) = 0.$$

Eventually, integrating by parts on Ω and using the identity $|Y| - |S| = |F|$, we obtain that (2.10) holds for smooth test functions. We conclude by a density argument.

Finally, we consider for $j = 1, 2, 3$, a test function $\boldsymbol{\varphi}_j \in H_0^1(F)$ such that $\nabla_y \cdot \boldsymbol{\varphi}_j = 0$ and $\int_F \boldsymbol{\varphi}_j = \boldsymbol{e}_j$. Using these test functions in (2.11), we easily deduce that $\nabla p \in L^2(\Omega)$, so that $p \in H^1(\Omega)$. Moreover, taking the scalar product with $\partial_j p$ and integrating on Ω, we obtain the estimate

$$\|\nabla p\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\boldsymbol{v}\|_{L^2(\Omega\times F)} + \|\boldsymbol{G}_y(\boldsymbol{v})\|_{L^2(\Omega\times F)} + \|\boldsymbol{u}\|_{L^2(\Omega)} \right). \qquad (5.3)$$

Step 2: Strong convergence results (*proof of Theorem* 2.12). We now turn to proving strong convergence results. We proceed as in the proof of Theorem 2.6. in [2]. Let us define the quantity

$$q_\varepsilon := -\omega^2\rho_s \int_{\Omega_\varepsilon^s} |\boldsymbol{u}_\varepsilon - \boldsymbol{u}|^2\, dx - \omega^2\rho_f \int_{\Omega_\varepsilon^f} |\boldsymbol{u}_\varepsilon - \boldsymbol{u} - \boldsymbol{v}\,(x, x/\varepsilon)\,|^2\, dx$$
$$+ \int_{\Omega_\varepsilon^s} \{2\mu|\boldsymbol{G}(\boldsymbol{u}_\varepsilon - \boldsymbol{u}) - \boldsymbol{G}_y(\boldsymbol{u}_1)(x, x/\varepsilon)|^2 + \lambda|\nabla\cdot(\boldsymbol{u}_\varepsilon - \boldsymbol{u}) - \nabla_y \cdot \boldsymbol{u}_1\,\big(x, x/\varepsilon)|^2\big\}\, dx$$
$$+ i\frac{\omega\nu}{2} \int_{\Omega_\varepsilon^f} |\varepsilon \boldsymbol{G}(\boldsymbol{u}_\varepsilon) - \boldsymbol{G}_y(\boldsymbol{v})\,(x, x/\varepsilon)\,|^2\, dx. \quad (5.4)$$

We claim that

$$q_\varepsilon \overset{\varepsilon\downarrow 0}{\longrightarrow} 0. \quad (5.5)$$

Proof of Claim (5.5)**.** Expanding the squares, we are led to consider sequences of Hermitian scalar products $\int_\Omega w_\varepsilon^1(x)\overline{w_\varepsilon^2}(x)\, dx$, where the families $\{w_\varepsilon^1\}_\varepsilon$ and $\{w_\varepsilon^2\}_\varepsilon$ are bounded in $L^2(\Omega)$ and two-scale converge to some limits w_0^1 and w_0^2 in $L^2(\Omega \times Y)$. It is well-known (see [2]) that if one of the family, say $\{w_\varepsilon^1\}_\varepsilon$, has the form $w_\varepsilon^1(x) = \varphi(x)\psi(x/\varepsilon)$, with $\varphi, \psi \in L^2(\Omega) \times C(Y)$, then we do have

$$\int_\Omega w_\varepsilon^1(x)\overline{w_\varepsilon^2}(x)\, dx \overset{\varepsilon\downarrow 0}{\longrightarrow} \int_{\Omega\times Y} w_0^1(x,y)\overline{w_0^2}(x,y)\, dx\, dy.$$

Now, since $\boldsymbol{v}$ solves the cells problem (2.14)–(2.15) with right-hand side in $L^2(\Omega)$, this function may be written in the form

$$\boldsymbol{v}(x,y) = \sum_{i=1}^{3} \theta_i(x)\boldsymbol{v}_i(y), \quad (5.6)$$

where, for $i = 1, 2, 3$, $\theta_i \in L^2(\Omega)$ and $\boldsymbol{v}_i$ is the smooth solution of (2.14)–(2.15) with right-hand side $\boldsymbol{e} = \boldsymbol{e}_i$, where the $\boldsymbol{e}_i$ are the three basis functions. A similar decomposition also holds for $\boldsymbol{u}_1$, so we can pass to the limit in all the products of (5.4) involving $\boldsymbol{u}$, $\boldsymbol{u}_1$, $\boldsymbol{v}$, or their gradients. The remainder is formed by the bilinear terms in $\boldsymbol{u}_\varepsilon$. Collecting these terms and using (2.1), we see that their sum is $a(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) = F_\varepsilon(\boldsymbol{h}, \boldsymbol{u}_\varepsilon)$, and we can also easily pass to the limit in this remainder. Finally, we compute

$$q_\varepsilon \overset{\varepsilon\downarrow 0}{\longrightarrow} \omega^2\rho \int_\Omega \boldsymbol{h}\cdot\overline{\boldsymbol{u}} + \omega^2\rho_f \int_{\Omega\times F} \boldsymbol{h}\cdot\overline{\boldsymbol{v}} \quad (5.7)$$
$$- \int_{\Omega\times S} \Big\{2\mu \boldsymbol{G}(\boldsymbol{h}) : (\overline{\boldsymbol{G}_x(\boldsymbol{u})} + \overline{\boldsymbol{G}_y(\boldsymbol{u}_1)}) + \lambda\nabla\cdot\boldsymbol{h}\,(\overline{\nabla\cdot\boldsymbol{u}} + \overline{\nabla_y\cdot\boldsymbol{u}_1})\Big\}$$
$$+ \omega^2\rho_s|S| \int_\Omega |\boldsymbol{u}|^2 + \omega^2\rho_f \int_{\Omega\times F} |\boldsymbol{u}+\boldsymbol{v}|^2$$

$$- \int_{\Omega\times S} \{2\mu|\boldsymbol{G}_x(\boldsymbol{u}) + \boldsymbol{G}_y(\boldsymbol{u}_1)|^2 + \lambda|\nabla\cdot\boldsymbol{u} + \nabla_y\cdot\boldsymbol{u}_1|^2\} - i\frac{\omega\nu}{2}\int_{\Omega\times F} |\boldsymbol{G}(\boldsymbol{v})|^2.$$

Now, let us apply the variational formulations (2.9) with $\boldsymbol{\psi} = \boldsymbol{u}$, (2.11) with $\boldsymbol{\varphi} = \boldsymbol{v}$, and (2.13) with $\boldsymbol{\varphi} = \boldsymbol{u}_1$. Summing and taking into account (2.10), we conclude that the right-hand side of (5.7) vanishes, so (5.5) holds. □

Now, in order to obtain the convergences stated in Theorem 2.12, it is sufficient to prove that the zero-order terms of (5.4) converge to 0; that is,

$$\int_\Omega |\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{u}}_\varepsilon|^2(x)\,dx \xrightarrow{\varepsilon\downarrow 0} 0, \tag{5.8}$$

where we have set

$$\tilde{\boldsymbol{u}}_\varepsilon(x) := \begin{cases} \boldsymbol{u}(x) & \text{if } x\in\Omega_\varepsilon^s, \\ \boldsymbol{u}(x) + \boldsymbol{v}\,(x, x/\varepsilon) & \text{if } x\in\Omega_\varepsilon^f. \end{cases}$$

The remainder of Step 2 is dedicated to the proof of (5.8). We proceed in three steps. First we prove strong compactness in the "interior" solid domain $\Omega_\varepsilon^s \setminus \mathcal{N}(\xi\varepsilon)$, where we recall the notation

$$\mathcal{N}(\nu) := \{x\in\Omega : d(x,\partial\Omega) < \nu\}, \qquad \text{for } \nu > 0.$$

Then we consider the "interior" fluid domain, and finally we establish compactness up to the boundary of Ω.

First, we claim that

$$\int_{\Omega_\varepsilon^s\setminus\mathcal{N}(\xi\varepsilon)} |\boldsymbol{u}_\varepsilon - \boldsymbol{u}|^2 \xrightarrow{\varepsilon\downarrow 0} 0, \tag{5.9}$$

where $\xi > 0$ is the constant in Lemma 3.6.

Proof of (5.9)**.** Using the notation of Lemma 3.6, we consider the extension $T_\varepsilon^s[\boldsymbol{u}_\varepsilon]$. From the bound (5.1) and the estimates of Lemma 3.6, the family $\{\|T_\varepsilon^s[\boldsymbol{u}_\varepsilon]\|_{H^1(\Omega\setminus\mathcal{N}(\xi\varepsilon))}\}_{\varepsilon>0}$ is bounded. Applying Lemma 3.7, we see that there exists $\tilde{\boldsymbol{u}}_0$ such that up to extraction $\{\|T_\varepsilon^s[\boldsymbol{u}_\varepsilon] - \tilde{\boldsymbol{u}}_0\|_{L^2(\Omega\setminus\mathcal{N}(\xi\varepsilon))}\} \to 0$. In particular, $T_\varepsilon^s[\boldsymbol{u}_\varepsilon]$ weakly converges to $\tilde{\boldsymbol{u}}_0$. We already know that $\boldsymbol{u}_\varepsilon$ two-scale converges to $\boldsymbol{u}$ in $\Omega\times S$, so we can identify the limits and conclude that the whole family satisfies $\{\|T_\varepsilon^s[\boldsymbol{u}_\varepsilon] - \boldsymbol{u}\|_{L^2(\Omega\setminus\mathcal{N}(\xi\varepsilon))}\} \to 0$. This implies (5.9). □

Next, we claim that there exists $\zeta \geq \xi$ such that

$$\int_{\Omega_\varepsilon^f\setminus\mathcal{N}(\zeta\varepsilon)} |\boldsymbol{u}_\varepsilon(x) - \boldsymbol{u}(x) - \boldsymbol{v}\,(x, x/\varepsilon)\,|^2\,dx \xrightarrow{\varepsilon\downarrow 0} 0. \tag{5.10}$$

Proof of (5.10)**.** Here we face a technical difficulty: we have $\boldsymbol{v} \in L^2(\Omega, H_0^1(F))$, but we do not know whether $\boldsymbol{v}(x,y)$ admits weak derivatives in $L^2(\Omega)$ with respect to the x variable. For this reason we need to mollify $\boldsymbol{v}$: let $\rho \in \mathcal{D}(\mathbf{R}^3)$ with $\mathrm{supp}(\rho) \subset B(0,\xi)$ and $\int \rho = 1$; we set $\rho_\varepsilon(x) := 1/\varepsilon^3\, \rho(x/\varepsilon)$ and we define

$$\boldsymbol{v}_\varepsilon(x,y) := \int_\Omega \boldsymbol{v}(x-x',y)\rho_\varepsilon(x')\,dx'.$$

This function is well defined in $\Omega \setminus \mathcal{N}(\xi\varepsilon)$ and it is smooth in this domain. Now, using the extension Lemma 3.6, we set

$$\boldsymbol{w}_\varepsilon(x) := \begin{cases} \boldsymbol{u}_\varepsilon(x) - T_\varepsilon^s[\boldsymbol{u}_\varepsilon](x) - \boldsymbol{v}_\varepsilon\,(x,x/\varepsilon) & \text{if } x \in (\Omega \setminus \mathcal{N}(\xi\varepsilon)) \cap \Omega_\varepsilon^f, \\ 0 & \text{if } x \in (\Omega \setminus \mathcal{N}(\xi\varepsilon)) \cap \Omega_\varepsilon^s, \end{cases}$$

and we decompose:

$$\boldsymbol{u}_\varepsilon(x) - \boldsymbol{u}(x) - \boldsymbol{v}\,(x,x/\varepsilon) = \boldsymbol{w}_\varepsilon(x) + \{T_\varepsilon^s[\boldsymbol{u}_\varepsilon](x) - \boldsymbol{u}(x)\} + \{\boldsymbol{v}_\varepsilon - \boldsymbol{v}\}\,(x,x/\varepsilon)\,.$$

For the second term, we have seen in the proof of (5.9) that

$$\int_{\Omega\setminus\mathcal{N}(\xi\varepsilon)} |T_\varepsilon^s[\boldsymbol{u}_\varepsilon] - \boldsymbol{u}|^2 \to 0.$$

For the last term, we deduce

$$\int_{\Omega\setminus\mathcal{N}(\xi\varepsilon)} |\boldsymbol{v}_\varepsilon - \boldsymbol{v}|^2\,(x,x/\varepsilon)\;dx \to 0$$

from the decomposition (5.6) and the standard fact that $\|\rho_\varepsilon \star \theta - \theta\|_{L^2(\Omega\setminus\mathcal{N}(\xi\varepsilon))} \to 0$ for any $\theta \in L^2(\Omega)$. So we only have to bound $\boldsymbol{w}_\varepsilon$.

By construction, $\boldsymbol{w}_\varepsilon \in H^1(\Omega \setminus \mathcal{N}(\xi\varepsilon))$ and $\boldsymbol{w}_\varepsilon \equiv 0$ on $\Omega_\varepsilon^s \cap (\Omega \setminus \mathcal{N}(\xi\varepsilon))$, so we can apply the Poincaré inequality of Lemma 3.2 in the set

$$\tilde{Q}_\varepsilon := \bigcup \Big\{\varepsilon(\tilde{Y}+k) : k \in \mathbf{Z}^3,\, B(\varepsilon(k+\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)),\varepsilon) \subset \Omega \setminus \mathcal{N}(\xi\varepsilon)\Big\}.$$

We have

$$\int_{\tilde{Q}_\varepsilon} |\boldsymbol{w}_\varepsilon|^2 \;\le\; C \int_{\Omega\setminus\mathcal{N}(\xi\varepsilon)} |\varepsilon \boldsymbol{G}(\boldsymbol{w}_\varepsilon)|^2.$$

Next, we write

$$\begin{aligned} \varepsilon\boldsymbol{G}(\boldsymbol{w}_\varepsilon)(x) = {}& \{\varepsilon\boldsymbol{G}(\boldsymbol{u}_\varepsilon) - \boldsymbol{G}_y(\boldsymbol{v})\,(x,x/\varepsilon)\} \\ & + \{\boldsymbol{G}_y(\boldsymbol{v} - \boldsymbol{v}_\varepsilon)\}\,(x,x/\varepsilon) - \varepsilon\boldsymbol{G}_x(\boldsymbol{v}_\varepsilon)\,(x,x/\varepsilon) - \varepsilon\boldsymbol{G}(T_\varepsilon^s[\boldsymbol{u}_\varepsilon])(x). \end{aligned} \tag{5.11}$$

First, taking the imaginary part of (5.4), we deduce from (5.5) that

$$\int_{\Omega_\varepsilon^f} |\varepsilon\boldsymbol{G}(\boldsymbol{u}_\varepsilon)(x) - \boldsymbol{G}_y(\boldsymbol{v}\,(x,x/\varepsilon)\,|^2\,dx \;\to\; 0, \tag{5.12}$$

so the first term of (5.11) goes to 0 in $L^2(\Omega)$. Now, from the decomposition (5.6) and the standard properties of mollifiers, we have

$$\int_{\Omega\setminus\mathcal{N}(\xi\varepsilon)} \left|\boldsymbol{G}_y(\boldsymbol{v}-\boldsymbol{v}_\varepsilon)\right|^2(x,x/\varepsilon)\;dx \;\to\; 0.$$

Similarly, since for any $\theta \in L^2(\Omega)$ we have $\|(\varepsilon\nabla\rho_\varepsilon)\star\theta\|_{L^2(\Omega\setminus\mathcal{N}(\xi\varepsilon))} \to 0$, we deduce from the definition of $\boldsymbol{v}_\varepsilon$ that

$$\int_{\Omega\setminus\mathcal{N}(\xi\varepsilon)} \left|\varepsilon\boldsymbol{G}_x(\boldsymbol{v}_\varepsilon)\right|^2(x,x/\varepsilon)\;dx \;\to\; 0.$$

Now, we have seen that $\{\|T^s_\varepsilon[\boldsymbol{u}_\varepsilon]\|_{H^1(\Omega\setminus\mathcal{N}(\xi\varepsilon))}\}$ is bounded, so in particular, we have

$$\int_{\Omega\setminus\mathcal{N}(\xi\varepsilon)} |\varepsilon\boldsymbol{G}(T^s_\varepsilon[\boldsymbol{u}_\varepsilon])|^2(x)\,dx \;\to\; 0.$$

Collecting these estimates, we obtain $\|\boldsymbol{w}_\varepsilon\|_{L^2(\tilde{Q}_\varepsilon)} \to 0$, which yields (5.10) for $\zeta = \xi+1$, since $\Omega\setminus\mathcal{N}((\xi+1)\varepsilon)\subset\tilde{Q}_\varepsilon$. □

Finally, we prove

$$\int_{\mathcal{N}(\zeta\varepsilon)} |\boldsymbol{u}_\varepsilon - \tilde{\boldsymbol{u}}_\varepsilon|^2(x)\,dx \;\xrightarrow{\varepsilon\downarrow 0}\; 0. \tag{5.13}$$

Proof of (5.13)**.** First, we notice that $\boldsymbol{u}\in L^2(\Omega)$ and $\boldsymbol{v}\in L^2(\Omega, C(F))$, so, since the volume of the domain of integration goes to 0, we have

$$\int_{\mathcal{N}(\zeta\varepsilon)} |\tilde{\boldsymbol{u}}_\varepsilon|^2(x)\,dx \;\to\; 0.$$

Then, using Lemma 3.3 we have the estimate

$$\begin{aligned}
\int_{\mathcal{N}(\zeta\varepsilon)} |\boldsymbol{u}_\varepsilon|^2(x)\,dx &\le C\Big(\int_{\mathcal{N}(2\zeta\varepsilon)\setminus\mathcal{N}(\zeta\varepsilon)} |\boldsymbol{u}_\varepsilon|^2 + \int_{\mathcal{N}(2\zeta\varepsilon)} |\varepsilon\boldsymbol{G}(\boldsymbol{u}_\varepsilon)|^2\Big)\\
&\le 2C\Big(\int_{\Omega\setminus\mathcal{N}(\zeta\varepsilon)} |\boldsymbol{u}_\varepsilon-\tilde{\boldsymbol{u}}_\varepsilon|^2 + \int_{\mathcal{N}(2\zeta\varepsilon)} |\tilde{\boldsymbol{u}}_\varepsilon|^2 + \varepsilon^2\int_{\Omega^s_\varepsilon} |\boldsymbol{G}(\boldsymbol{u}_\varepsilon)|^2\\
&+ \int_{\Omega^f_\varepsilon} |\varepsilon\boldsymbol{G}(\boldsymbol{u}_\varepsilon)(x) - \boldsymbol{G}_y(\boldsymbol{v})(x,x/\varepsilon)|^2\,dx + \int_{\mathcal{N}(2\zeta\varepsilon)\cap\Omega^f_\varepsilon} |\boldsymbol{G}_y(\boldsymbol{v}(x,x/\varepsilon)|^2\,dx\Big).
\end{aligned}$$

The first term goes to 0 by (5.9) and (5.10), the third term goes to 0 by (5.1), and the fourth term goes to 0 by (5.12). Finally, since $\boldsymbol{u}, \boldsymbol{G}(\boldsymbol{u}) \in L^2(\Omega)$ and $\boldsymbol{v}, \boldsymbol{G}_y(\boldsymbol{v}) \in L^2(\Omega, C(F))$, the second and last terms go to 0 because the volume of the domain of integration goes to 0. □

Collecting (5.9), (5.10), and (5.13), we get (5.8), which together with (5.5) yields the convergences of Theorem 2.12.

Step 3: a priori estimates (*proof of part* b) *of Theorem* 2.1)*.* We prove by contradiction the hypothesis (5.1). Let $(\boldsymbol{h}^n)$ be a sequence such that $\|\boldsymbol{h}^n\|_{H^1(\Omega)} \to 0$, and let $(\boldsymbol{u}_{\varepsilon_n})_n$ be a sequence of solutions of (2.1)–(2.2) where $\boldsymbol{h} = \boldsymbol{h}^n$ such that

$$\|\boldsymbol{u}_{\varepsilon_n}\|^2_{L^2(\Omega)} + \|\boldsymbol{G}(\boldsymbol{u}_{\varepsilon_n})\|^2_{L^2(\Omega^s_{\varepsilon_n})} + \varepsilon_n^2\|\boldsymbol{G}(\boldsymbol{u}_{\varepsilon_n})\|^2_{L^2(\Omega^f_{\varepsilon_n})} = 1. \tag{5.14}$$

We first remark the fact that $\lim_{n\to\infty}\|\boldsymbol{u}_{\varepsilon_n}\|^2_{L^2(\Omega)} = 0$ contradicts (5.14). Indeed, writing the variational formulation (2.1)–(2.2) with $\boldsymbol{\psi} = \boldsymbol{u}_{\varepsilon_n}$, we get

$$\begin{aligned} - 2\mu\|\boldsymbol{G}(\boldsymbol{u}_{\varepsilon_n})\|^2_{L^2(\Omega^s_{\varepsilon_n})} - \lambda\|\nabla\cdot\boldsymbol{u}_{\varepsilon_n}\|^2_{L^2(\Omega^s_{\varepsilon_n})} - i\frac{\omega\eta{\varepsilon_n}^2}{2}\|\boldsymbol{G}(\boldsymbol{u}_{\varepsilon_n})\|^2_{L^2(\Omega^f_{\varepsilon_n})} \\ + \omega^2\rho_s\|\boldsymbol{u}_{\varepsilon_n}\|^2_{L^2(\Omega^s_{\varepsilon_n})} + \omega^2\rho_f\|\boldsymbol{u}_{\varepsilon_n}\|^2_{L^2(\Omega^f_{\varepsilon_n})} = F_{\varepsilon_n}(\boldsymbol{h}^n, \boldsymbol{u}_{\varepsilon_n}). \end{aligned} \tag{5.15}$$

The Cauchy–Schwarz inequality, along with equation (5.14), imply the property $|F_{\varepsilon_n}(\boldsymbol{h}^n, \boldsymbol{u}_{\varepsilon_n})| \le \|\boldsymbol{h}\|_{H^1(\Omega)}$. Then, separating real and imaginary parts of (5.15) we obtain the following inequalities:

$$\varepsilon^2\|\boldsymbol{G}(\boldsymbol{u}_{\varepsilon_n})\|^2_{L^2(\Omega^f_{\varepsilon_n})} \le \alpha\|\boldsymbol{h}^n\|_{H^1(\Omega)} \;\to\; 0, \tag{5.16}$$

$$\|\boldsymbol{G}(\boldsymbol{u}_{\varepsilon_n})\|^2_{L^2(\Omega^s_{\varepsilon_n})} \le \beta\left(\|\boldsymbol{h}^n\|_{H^1(\Omega)} + \|\boldsymbol{u}_{\varepsilon_n}\|^2_{L^2(\Omega^f_{\varepsilon_n})} + \|\boldsymbol{u}_{\varepsilon_n}\|^2_{L^2(\Omega^s_{\varepsilon_n})}\right), \tag{5.17}$$

where α and β depend only on the physical data, i.e., Lamé coefficients λ and μ, frequency ω, viscosity η, and both densities ρ_f and ρ_s. Then, $\varepsilon_n^2\|\boldsymbol{G}(\boldsymbol{u}_{\varepsilon_n})\|^2_{L^2(\Omega^f_{\varepsilon_n})}$ clearly tends to 0. Thus if $\|\boldsymbol{u}_{\varepsilon_n}\|_{L^2(\Omega)}$ tends to 0, by (5.17) $\|\boldsymbol{G}(\boldsymbol{u}_{\varepsilon_n})\|_{L^2(\Omega^s_{\varepsilon_n})}$ also does and the contradiction follows.

Now, from Step 1, the two-scale convergences (2.6)–(2.7)–(2.8) hold and the limits satisfy the homogenized problem (2.9)–(2.13). Since $\|\boldsymbol{h}^n\|_{H^1(\Omega)}$ goes to 0, by uniqueness of the homogenized problem (established in Section 4.3), we have $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}_1, p) = 0$ if $\omega \notin N$. Moreover, (5.14) and the strong convergence results of Step 2 yield

$$\|\boldsymbol{u}_{\varepsilon_n}\|_{L^2(\Omega^s_{\varepsilon_n})} \xrightarrow{\varepsilon_n\downarrow 0} 0 \quad \text{and} \quad \|\boldsymbol{u}_{\varepsilon_n}\|_{L^2(\Omega^f_{\varepsilon_n})} \xrightarrow{\varepsilon_n\downarrow 0} 0, \tag{5.18}$$

which provides a contradiction with (5.14). Thus the *a priori* estimates (5.1) are true. Eventually, Step 1 provides a bound on the pressure field. This establishes part b) of Theorem 2.1.

Conclusion. Finally, Theorem 2.1 is proved (recall that Theorem 2.1.a) was proved in Section 4.1). In particular, the assumption (5.1) is true under the hypotheses of Theorem 2.1. Then, by Step 1, the two-scale limits exist and solve the homogenized problem (that is, Theorem 2.6 holds) and by Step 2 the strong convergence results set in Theorem 2.12 hold.

We now only have to check that the well-posedness result, Theorem 2.10.c), also holds. The uniqueness of the solution of the homogenized problem has been proved in Section 4.3 for any $\omega \notin N$, and the existence of the solution is a consequence of the two-scale convergence. In order to obtain the bounds (2.17) we first use the lower semicontinuity of the L^2-norm under two-scale convergence to get

$$\|p\|_{L^2(\Omega)} + \|\boldsymbol{G}(\boldsymbol{v})\|_{L^2(\Omega\times F)} + \|\boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{G}_y(\boldsymbol{u}_1)\|_{L^2(\Omega\times S)} \leq C(\Omega)\|\boldsymbol{h}\|_{H^1(\Omega)}. \tag{5.19}$$

Now, using the Korn inequality and inequality (4.12), we have the estimate

$$\|\boldsymbol{u}\|_{H^1(\Omega)} + \|\boldsymbol{G}_y(\boldsymbol{u}_1)\|_{L^2(\Omega\times S)} \leq C\left(\|\boldsymbol{G}(\boldsymbol{u}) + \boldsymbol{G}_y(\boldsymbol{u}_1)\|_{L^2(\Omega\times S)}\right),$$

and the estimate (5.19) provides the desired bound on $\|\boldsymbol{G}_y(\boldsymbol{u}_1)\|_{L^2(\Omega\times S)}$ and $\|\boldsymbol{u}\|_{H^1(\Omega)}$.

The estimate on ∇p is given by (5.3). Finally, the estimate on p_1 follows from (4.6). The bound (2.17) is proved. □

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