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OPTIMAL HARDY-TYPE INEQUALITIES FOR SCHRÖDINGER FORMS

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Abstract

We give a method to construct a critical Schrödinger form from the subcritical Schrödinger form by subtracting a suitable positive potential. The method enables us to obtain optimal Hardy-type inequalities.

Contents

1. Introduction

In [6], Devyver, Fraas and Pinchover give a method for obtaining *optimal* Hardy weights for second-order non-negative elliptic operators on non-compact Riemannian manifolds, in particular, they show that the criticality of Schrödinger forms is related to the *critical* Hardy weights. In [20] we give a method to construct a critical Schrödinger form from a transient Dirichlet form by subtracting a suitable positive potential. In other words, we give a method to construct critical Hardy weights for a transient Dirichlet form by applying the idea in [6]. In this paper, we will consider subcritical Schrödinger forms instead of transient Dirichlet forms, and extend the method for subcritical Schrödinger forms. As an application, we obtain a method to construct critical Hardy weights for Schrödinger forms. Moreover, we discuss the optimality of Hardy weights in the sense of [6], a stronger notion than the criticality, and give a condition for the critical Hardy weights being optimal ones.

Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $X = (P_x, X_t, \zeta)$ be an m-symmetric Hunt process. We assume that X is irreducible and resolvent doubly Feller, in addition, that X generates a regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E; m)$.

Denote by $\mathcal{K}_{loc}(X)$ the totality of local Kato measures (Definition 3.1 (1)). For a singed local Kato measure such that the positive (resp. negative) part μ^+ (resp. μ^-) of μ belongs to

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$\mathcal{K}_{loc}(X)$ ($\mu \in \mathcal{K}_{loc}(X) - \mathcal{K}_{loc}(X)$ in notation), we define a symmetric form by

$$\mathcal{E}^{\mu}(u,u) = \mathcal{E}(u,u) + \int_E u^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}) \cap C_0(E).$$

The regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ implies that a measure in $\mathcal{K}_{loc}(X)$ is Radon (Remark 3.2) and the form $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is well-defined. In the sequel, for a symmetric bilinear form $(a, \mathcal{D}(a))$ we simply write $a(u)$ for $a(u,u)$.

We suppose that $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is positive semi-definite:

$$\mathcal{E}^{\mu}(u) \geq 0 \left(\Longleftrightarrow \int_E u^2 d\mu^- \leq \mathcal{E}^{\mu^+}(u) \right), \; u \in \mathcal{D}(\mathcal{E}) \cap C_0(E). \tag{1}$$

Applying results in [1], we prove in [20] that $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is closable in $L^2(E;m)$. We denote the closure $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}^{\mu}))$ and call it *Schrödinger form* with potential μ. By the Radonness of μ^+, we see that $\mathcal{D}(\mathcal{E}) \cap L^2(E;\mu^+) \subset \mathcal{D}(\mathcal{E}^{\mu})$ and

$$\mathcal{E}^{\mu}(u) = \mathcal{E}(u) + \int_E \widetilde{u}^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}) \cap L^2(E;\mu^+).$$

Here $\widetilde{u}$ is a quasi-continuous version of u. In this paper, we always assume that every function u is represented by its quasi-continuous version if it admits.

The L^2-semigroup T_t^{μ} generated by $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}^{\mu}))$ is expressed by Feynman-Kac semigroup ([20, Theorem 4.2]): For a bounded Borel function f in $L^2(E;m)$

$$T_t^{\mu} f(x) = p_t^{\mu} f(x) \left(:= E_x \left(e^{-A_t^{\mu}} f(X_t) \right) \right), \; m\text{-a.e. } x. \tag{2}$$

Here $A_t^{\mu} = A_t^{\mu^+} - A_t^{\mu^-}$ and $A_t^{\mu^+}$ (resp. $A_t^{\mu^-}$) is the positive continuous additive functional with Revuz measure μ^+ (resp. μ^-). We suppose that $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}^{\mu}))$ is *subcritical*, that is, there exists the Green function $R^{\mu}(x,y)$ such that for a positive Borel function f

$$\int_0^{\infty} p_t^{\mu} f(x) dt = \int_E R^{\mu}(x,y) f(y) dm(y), \; \forall x \in E.$$

Let $\mathcal{K}_{loc}^{\mu}(X)$ be the set of local Kato measures such that for any compact set $K \subset E$

$$R^{\mu}(1_K \nu) u(x) = \int_E R^{\mu}(x,y) 1_K(y) d\nu(y) \in L^{\infty}(E;m). \tag{3}$$

For a non-trivial measure ν in $\mathcal{K}_{loc}^{\mu}(X)$ define measures ν^{μ} and μ^{ν} by

$$\nu^{\mu} = \frac{\nu}{R^{\mu}\nu} \tag{4}$$

and

$$\mu^{\nu} = \mu - \nu^{\mu}. \tag{5}$$

We will show in Corollary 4.2 and Lemma 4.3 below that μ^{ν} belongs to $\mathcal{K}_{loc}(X) - \mathcal{K}_{loc}(X)$ and $(\mathcal{E}^{\mu^{\nu}}, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is still positive semi-definite

$$\mathcal{E}^{\mu^{\nu}}(u) = \mathcal{E}^{\mu}(u) - \int_E u^2 d\nu^{\mu} \geq 0, \; u \in \mathcal{D}(\mathcal{E}) \cap C_0(E). \tag{6}$$

In other words, the measure ν^{μ} is a *Hardy weight* for $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}^{\mu}))$. As remarked above, $(\mathcal{E}^{\mu^{\nu}}, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is closable and its closure defines a new Schrödinger form $(\mathcal{E}^{\mu^{\nu}}, \mathcal{D}(\mathcal{E}^{\mu^{\nu}}))$.

Let $\mathcal{C}$ be the totality of compact sets of E. We then obtain the following main result in this paper: If a non-trivial positive measure ν in $\mathcal{K}^{\mu}_{loc}(X)$ satisfies that

$$\sup_{K\in\mathcal{C}} \iint_{K\times K^c} R^{\mu}(x,y)\,\nu(dx)\nu(dy) < \infty, \tag{7}$$

then $(\mathcal{E}^{\mu^{\nu}}, \mathcal{D}(\mathcal{E}^{\mu^{\nu}}))$ turns out to be a *critical* Schrödinger form. Here K^c is the complement of K. More precisely, the function $R^{\mu}\nu$ is a ground state of $(\mathcal{E}^{\mu^{\nu}}, \mathcal{D}(\mathcal{E}^{\mu^{\nu}}))$, that is, $R^{\mu}\nu$ belongs to the *extended Schrödinger space* $\mathcal{D}_e(\mathcal{E}^{\mu^{\nu}})$ of $(\mathcal{E}^{\mu^{\nu}}, \mathcal{D}(\mathcal{E}^{\mu^{\nu}}))$ (see Section 2 for the definition of the extended Schrödinger space) and $\mathcal{E}^{\mu^{\nu}}(R^{\mu}\nu) = 0$. As a corollary, we see that ν^{μ} is a *critical* Hardy weight for $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}^{\mu}))$ in the sense that there exists no non-trivial positive function ψ such that

$$\int_E u^2 d\,(\nu^{\mu} + \psi m) \le \mathcal{E}^{\mu}(u),\ u \in \mathcal{D}(\mathcal{E}) \cap C_0(E). \tag{8}$$

In particular, if $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient and $\mu \equiv 0$, then every $\nu \in \mathcal{K}_{loc}(X)$ satisfies (3) by replacing $R^{\mu}(x,y)$ with the 0-resolvent $R(x,y)$ of X. Indeed, since $1_K\nu$ is Green-tight, $1_K\nu \in \mathcal{K}_{\infty}(X)$ (Definition 3.1 (2)), the condition (3) is derived from [3, Proposition 2.2]. As a result, for any $\nu \in \mathcal{K}_{loc}$ the next Hardy-type inequality follows:

$$\int_E u^2 \frac{d\nu}{R\nu} \le \mathcal{E}(u),\ u \in \mathcal{D}(\mathcal{E}) \cap C_0(E). \tag{9}$$

The inequality (9) is proved in Fitzsimmons [7] (see also [2]). Moreover, we see that if the measure $\nu/R\mu$ is a critical Hardy weight for the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ if ν satisfies (7) obtained by replacing $R^{\mu}(x,y)$ with $R(x,y)$.

As stated above, the function $R^{\mu}\nu$ belongs to $\mathcal{D}_e(\mathcal{E}^{\mu^{\nu}})$ under the condition (7). Lemma 4.3 below tells us that $\mathcal{D}_e(\mathcal{E}^{\mu})$ is included in $\mathcal{D}_e(\mathcal{E}^{\mu^{\nu}})$ and $R^{\mu}\nu$ does not belong to $\mathcal{D}_e(\mathcal{E}^{\mu})$ in general. If ν satisfies the stronger condition than (7),

$$\iint_{E\times E} R^{\mu}(x,y)\nu(dx)\nu(dy) < \infty,$$

i.e., ν is of finite energy with respect to R^{μ}, then $R^{\mu}\nu$ belongs to $L^2(E;\nu^{\mu})$ because

$$\int_E (R^{\mu}\nu)^2 d\nu^{\mu} = \int_E R^{\mu}\nu d\nu = \iint_{E\times E} R^{\mu}(x,y)\nu(dx)\nu(dy) < \infty.$$

Moreover, $R^{\mu}\nu$ belongs to $\mathcal{D}_e(\mathcal{E}^{\mu})$ by Lemma 4.8 below. Hence, $\mathcal{E}^{\mu}(R^{\mu}\nu)$ is finite and thus

$$\mathcal{E}^{\mu^{\nu}}(R^{\mu}\nu) = 0 \Longleftrightarrow \frac{\mathcal{E}^{\mu}(R^{\mu}\nu)}{\int_E (R^{\mu}\nu)^2 d\nu^{\mu}} = 1.$$

Noting that by (6)

$$\inf_{u\in\mathcal{D}_e(\mathcal{E}^{\mu})} \frac{\mathcal{E}^{\mu}(u)}{\int_E u^2 d\nu^{\mu}} \ge 1, \tag{10}$$

we see $R^{\mu}\nu$ is a minimizer for the left hand side of (10). In this case, the Schrödinger form $(\mathcal{E}^{\mu^{\nu}}, \mathcal{D}(\mathcal{E}^{\mu^{\nu}}))$ is said to be *positive-critical* ([6, Definition 4.8]).

On the other hand, if ν is not of finite energy,

(11) $$\iint_{E\times E} R^{\mu}(x,y)\nu(dx)\nu(dy) = \infty,$$

then $R^{\mu}\nu$ does not belong to $L^2(E;\nu^{\mu})$ and $(\mathcal{E}^{\mu^{\nu}},\mathcal{D}(\mathcal{E}^{\mu^{\nu}}))$ is *null-critical* in the sense of [6].

The measure ν^{μ} is called *optimal at infinity* if for any $K \in \mathcal{C}$

$$\lambda \int_E u^2 d\nu^{\mu} \le \mathcal{E}^{\mu}(u), \quad u \in \mathcal{D}(\mathcal{E}) \cap C_0(K^c),$$

then $\lambda \le 1$. We see from [12, Corollary 3.4] (or [14, Theorem 3]) that if for any $K \in \mathcal{C}$

$$\iint_{K\times E} R^{\mu}(x,y)\nu(dx)\nu(dy) < \infty,$$

i.e., $R^{\mu}\nu$ is locally integrable, then the null-criticality implies the optimality at infinity. In generally, if for any $K \in \mathcal{C}$

(12) $$\iint_{K^c\times E} R^{\mu}(x,y)\nu(dx)\nu(dy) = \infty,$$

then the optimality at infinity holds. Devyver, Fraas and Pinchover [6], where they call a Hardy-type inequality *optimal* if a Hardy weight is critical, null-critical and optimal at infinity. Noting that (12) implies (11), we can conclude that if a measure ν satisfies (3), (7) and (12), then the measure ν^{μ} is an optimal Hardy-weight for $(\mathcal{E}^{\mu},\mathcal{D}(\mathcal{E}^{\mu}))$ in the sense of [6].

2. Extended Schrödinger spaces

Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(E;m)$ (c.f. [9, p.6]). We denote by $u \in \mathcal{D}_{loc}(\mathcal{E})$ if for any relatively compact open set D there exists a function $v \in \mathcal{D}(\mathcal{E})$ such that $u = v$ m-a.e. on D. We assume that $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is irreducible (c.f. [9, p.40, p.55]).

We call a positive Borel measure μ on E *smooth* if it satisfies

(i) μ charges no set of zero capacity,

(ii) there exists an increasing sequence $\{F_n\}$ of closed sets such that

a) $\mu(F_n) < \infty$, $n = 1,2,\ldots,$

b) $\lim_{n\to\infty} \mathrm{Cap}(K \setminus F_n) = 0$ for any compact set K.

We denote by $\mathcal{S}$ the totality of smooth measures.

For a signed smooth Radon measure $\mu = \mu^+ - \mu^- \in \mathcal{S} - \mathcal{S}$ define a symmetric form on $L^2(E;m)$ by

(13) $$\mathcal{E}^{\mu}(u,v) = \mathcal{E}(u,v) + \int_E uv\,d\mu, \quad u,v \in \mathcal{D}(\mathcal{E}) \cap C_0(E).$$

We assume that $(\mathcal{E}^{\mu},\mathcal{D}(\mathcal{E}) \cap C_0(E))$ is positive semi-definite:

(14) $$\mathcal{E}^{\mu}(u) \ge 0 \left(\Longleftrightarrow \int_E u^2 d\mu^- \le \mathcal{E}^{\mu^+}(u)\right), \ u \in \mathcal{D}(\mathcal{E}) \cap C_0(E).$$

When $(\mathcal{E}^{\mu},\mathcal{D}(\mathcal{E}) \cap C_0(E))$ is closable, we denote by $(\mathcal{E}^{\mu},\mathcal{D}(\mathcal{E}^{\mu}))$ its closure and call it *Schrödinger form* with potential μ.

A densely defined, closed, positive semi-definite symmetric bilinear form $(a, \mathcal{D}(a))$ is said to be *positive preserving* if for $u \in \mathcal{D}(a)$, $|u|$ belongs to $\mathcal{D}(a)$ and $a(|u|) \leq a(u)$. It follow from [5, Lemma 1.3.4] that the form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive preserving because $\mathcal{E}^\mu(|u|) \leq \mathcal{E}^\mu(u)$ for $u \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$. As a result, we see from [17, Proposition 2] that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ has the *Fatou property*, i.e., if $\{u_n\} \subset \mathcal{D}(\mathcal{E}^\mu)$ satisfies $\sup_n \mathcal{E}^\mu(u_n) < \infty$ and $u_n \to u \in \mathcal{D}(\mathcal{E}^\mu)$ m-a.e., then $\liminf_{n\to\infty} \mathcal{E}^\mu(u_n) \geq \mathcal{E}^\mu(u)$. Hence, following [16], we can define a space $\mathcal{D}_e(\mathcal{E}^\mu)$ in the way similar to the extended Dirichlet space: An m-measurable function u with $|u| < \infty$ m-a.e. is said to be in $\mathcal{D}_e(\mathcal{E}^\mu)$ if there exists an $\mathcal{E}^\mu$-Cauchy sequence $\{u_n\} \subset \mathcal{D}(\mathcal{E}^\mu)$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\mathcal{D}_e(\mathcal{E}^\mu)$ the *extended Schrödinger space* of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ and the sequence $\{u_n\}$ an *approximating sequence* of u. For $u \in \mathcal{D}_e(\mathcal{E}^\mu)$ and an approximating sequence $\{u_n\}$ of u, we define

$$\mathcal{E}^\mu(u) = \lim_{n\to\infty} \mathcal{E}^\mu(u_n). \tag{15}$$

We define the criticality and subcriticality of Schrödinger forms in the way similar to the recurrence and transience of Dirichlet forms.

DEFINITION 2.1. Let $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ be a positive semi-definite Schrödinger form.

(1) $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is said to be *subcritical* if there exists a bounded function g in $L^1(E; m)$ strictly positive m-a.e. such that

$$\int_E |u| g dm \leq \sqrt{\mathcal{E}^\mu(u)}, \quad u \in \mathcal{D}_e(\mathcal{E}^\mu). \tag{16}$$

(2) $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is said to be *critical* if there exists a function ϕ in $\mathcal{D}_e(\mathcal{E}^\mu)$ strictly positive m-a.e. such that $\mathcal{E}^\mu(\phi) = 0$. The function ϕ is said to be the *ground state*.

Define the operator G^μ by

$$G^\mu f(x) = \int_0^\infty T_t^\mu f(x) dt \ (\leq +\infty)$$

for a positive function f. Here T_t^μ is the L^2-semigroup on $L^2(E; m)$ generated by $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$.

Lemma 2.2 ([20, Lemma 2.3])**.** *Let g be the function in* Definition 2.1 (1). *Then $G^\mu g$ belongs to $\mathcal{D}_e(\mathcal{E}^\mu)$.*

REMARK 2.3. It is recently proved in [15, Theorem A.3] that if the semigroup T_t^μ is expressed using a density $p_t^\mu(x, y)$, $T_t^\mu f(x) = \int_E p_t^\mu(x, y) f(y) dm(y)$, then $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical or critical.

REMARK 2.4. We see from the inequality (16) that if $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is subcritical, then $(\mathcal{D}(\mathcal{E}^\mu), \mathcal{E}^\mu(\cdot,\cdot))$ is a Hilbert space.

3. Probabilistic representation of Schrödinger semigroups

Let $X = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \{P_x\}_{x\in E}, \{X_t\}_{t\geq 0}, \zeta)$ be the symmetric Hunt process generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmented filtration and ζ is the lifetime of X. Denote by $\{p_t\}_{t\geq 0}$ and $\{R_\alpha\}_{\alpha\geq 0}$ the semigroup and resolvent of X:

$$p_t f(x) = E_x(f(X_t)), \qquad R_\alpha f(x) = \int_0^\infty e^{-\alpha t} p_t f(x) dt.$$

Then $p_t f(x) = T_t f(x)$ m-a.e., $R_\alpha f(x) = \int_0^\infty T_t f(x) dt$ m-a.e., where T_t is the L^2-semigroup on $L^2(E;m)$ generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. In the sequel, we assume that X satisfies, in addition, the next condition:

Feller Property (F). For each $t > 0$, $p_t(C_\infty(E)) \subset C_\infty(E)$ and for each $f \in C_\infty(E)$ and $x \in E$, $\lim_{t\to 0} p_t f(x) = f(x)$, where $C_\infty(E)$ is the space of continuous functions on E vanishing at infinity.

Resolvent Strong Feller Property (RSF). For each $\alpha > 0$, $R_\alpha(\mathcal{B}_b(E)) \subset C_b(E)$, where $\mathcal{B}_b(E)$(resp. $C_b(E)$) is the space of bounded Borel (resp. continuous) functions on E.

Following [11], a Hunt process is said to be *resolvent doubly Feller* if it enjoys both the Feller property and resolvent strong Feller property. We see from **(RSF)** that the resolvent kernel $R_\alpha(x, dy)$ of X has a non-negative jointly measurable density $R_\alpha(x, y)$ with respect to m: For $x \in E$ and $f \in \mathcal{B}_b(E)$

$$R_\alpha f(x) = \int_E R_\alpha(x, y) f(y) m(dy).$$

Moreover, $R_\alpha(x, y)$ is α-excessive in x and in y ([9, Lemma 4.2.4]). We simply write $R(x, y)$ for $R_0(x, y)(:= \lim_{\alpha\to 0} R_\alpha(x, y))$. For a measure μ, we define the α-potential of μ by

$$R_\alpha \mu(x) = \int_E R_\alpha(x, y)\mu(dy), \quad \alpha \geq 0.$$

Let S_{00} be the set of positive Borel measures μ such that $\mu(E) < \infty$ and $R_1\mu$ is bounded. We call a Borel measure μ on E *smooth measure in the strict sense* if there exists a sequence $\{E_n\}$ of Borel sets increasing to E such that for each n, $1_{E_n}\mu \in \mathcal{S}_{00}$ and for any $x \in E$

$$P_x(\lim_{n\to\infty} \sigma_{E\setminus E_n} \geq \zeta) = 1,$$

where $\sigma_{E\setminus E_n}$ is the first hitting time of $E \setminus E_n$. We denote by $\mathcal{S}^1$ the set of smooth measures in the strict sense.

DEFINITION 3.1. Let $\mu \in \mathcal{S}^1$.

(1) μ is said to be in the *Kato class* of X ($\mathcal{K}(X)$ in abbreviation) if

$$\lim_{\alpha\to\infty} \|R_\alpha \mu\|_\infty = 0.$$

μ is said to be in the *local Kato class* ($\mathcal{K}_{loc}(X)$ in abbreviation) if for any compact set K, $1_K \cdot \mu$ belongs to $\mathcal{K}(X)$. (2) Suppose that X is transient. A measure μ is said to be in the class $\mathcal{K}_\infty(X)$ if for any $\epsilon > 0$, there exists a compact set $K = K(\epsilon)$

$$\sup_{x\in E} \int_{K^c} R(x, y)\mu(dy) < \epsilon.$$

μ in $\mathcal{K}_\infty(X)$ is called *Green-tight*.

Remark 3.2. It is known in [19, Theorem 3.1] that for a measure μ in $\mathcal{K}(X)$ and $\alpha > 0$

$$\int_E u^2 d\mu \leq \|R_\alpha \mu\|_\infty \, \mathcal{E}_\alpha(u), \quad u \in \mathcal{D}(\mathcal{E}). \tag{17}$$

By the regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and the inequality (17), a measure μ in $\mathcal{K}(X)$ is Radon, and so is a measure μ in $\mathcal{K}_{loc}(X)$. As a result, $\mathcal{D}(\mathcal{E}) \cap L^2(E; \mu^+) \subset \mathcal{D}(\mathcal{E}^\mu)$ and

$$\mathcal{E}^\mu(u) = \mathcal{E}(u) + \int_E u^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}) \cap L^2(E; \mu^+).$$

If $\mu \in \mathcal{K}_\infty(X)$, then $\|R\mu\|_\infty < \infty$ by [3, Proposition 2.2] and [11, Lemma 4.1], and the equation (17) is meaningful for $\alpha = 0$:

$$\int_E u^2 d\mu \leq \|R\mu\|_\infty \, \mathcal{E}(u), \quad u \in \mathcal{D}_e(\mathcal{E}). \tag{18}$$

We denote by A_t^μ the PCAF corresponding to $\mu \in \mathcal{S}^1$.

Theorem 3.3 ([20, Theorem 4.2]). *Let $\mu = \mu^+ - \mu^- \in \mathcal{K}_{loc}(X) - \mathcal{K}_{loc}(X)$. If $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is positive semi-definite, then it is closable. Moreover, the semigroup T_t^μ generated by the closure $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is expressed as*

$$T_t^\mu f(x) = p_t^\mu f(x) = E_x\left(e^{-A_t^\mu} f(X_t)\right) \quad m\text{-a.e.}$$

Remark 3.4. By [9, Theorem 4.2.4], the transition semigroup p_t of X is expressed using transition probability density $p_t(x, y)$, as a result, T_t^μ is also expressed by a kernel $p_t^\mu(x, y)$ by Theorem 3.3. Hence, as discussed in Remark 2.3, $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is either critical or subcritical.

4. Criticality and Hardy-type inequalities

We maintain the setting in Section 3 and fix a measure $\mu \in \mathcal{K}_{loc}(X) - \mathcal{K}_{loc}(X)$. Though this section, we assume that $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ is positive semi-definite and subcritical. By the subcriticality of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$, $(\mathcal{D}_e(\mathcal{E}^\mu), \mathcal{E}^\mu(\cdot,\cdot))$ becomes a Hilbert space. The α-order resolvent kernel $\{R_\alpha^\mu(x, y)\}_{\alpha>0}$ of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ can be constructed in the same manner as [9, Lemma 4.2.4] and the Green kernel, i.e., 0-order resolvent kernel $R^\mu(x, y)$ is defined by $R^\mu(x, y) = \lim_{\alpha \to 0} R_\alpha^\mu(x, y)$. The potential of a positive measure ν is defined by

$$R^\mu \nu(x) = \int_E R^\mu(x, y) \nu(dy).$$

Lemma 4.1. *Let ν be a non-trivial positive measure in $\mathcal{K}_{loc}(X)$. Then for any compact set K*

$$\inf_{x \in K} R^\mu \nu(x) > 0.$$

Proof. For any compact set K, take a relatively compact domain G such that $K \subset G$ and $\nu(G) > 0$. Consider the subprocess $X^{\mu^+} = (\{P_x^{\mu^+}\}_{x \in E}, \{X_t\}_{t \geq 0}, \zeta)$ defined by

$$P_x^{\mu^+}(B; t < \zeta) = \int_{B \cap \{t < \zeta\}} e^{-A_t^{\mu^+}} dP_x, \quad B \in \mathfrak{F}_t.$$

Then X^{μ^+} has Properties (**F**) and (**RSF**) by [13, Corollary 6.1], and so the part process $X^{\mu^+, G}$ of X^{μ^+} on G has Property (**RSF**) by [13, Theorem 3.1]. Furthermore, $X^{\mu^+, G}$ is irreducible because G is a domain.

Since the measure ν^G, the restriction of ν to G, is in the Green-tight Kato class of $X^{\mu^+,G}$, $\nu^G \in \mathcal{K}_\infty(X^{\mu^+,G})$, $R^{\mu^+,G}\nu(= R^{\mu^+,G}\nu^G)$ is bounded by [3, Proposition 2.4] on G. Moreover it is continuous on G. Indeed, by Property (**RSF**) of $X^{\mu^+,G}$, $R_\alpha^{\mu^+,G}(R^{\mu^+,G}\nu) \in C_b(G)$ and $\|R_\alpha^{\mu^+,G}\nu\|_\infty \to 0$ as $\alpha \to \infty$ because of $\nu^G \in \mathcal{K}(X^{\mu,G})$. Hence, $R^{\mu^+,G}\nu \in C_b(G)$ because the resolvent equation implies

$$\|R^{\mu^+,G}\nu - \alpha R_\alpha^{\mu^+,G}(R^{\mu^+,G}\nu)\|_\infty = \|R_\alpha^{\mu^+,G}\nu\|_\infty \to 0,\ \alpha \to \infty.$$

By the irreducibility and $\nu(G) > 0$, $R^{\mu^+,G}\nu(x) > 0$ for each $x \in E$, and thus $\inf_{x\in K} R^{\mu^+,G}\nu(x) > 0$. On account of $R^\mu\nu(x) \geq R^{\mu^+,G}\nu(x)$, we have this lemma. □

By Lemma 4.1, we have the next corollary.

Corollary 4.2. *For a non-trivial positive measure $\nu \in \mathcal{K}_{loc}(X)$, the measure $\nu/R^\mu\nu$ belongs to $\mathcal{K}_{loc}(X)$.*

We define the subclass $\mathcal{K}_{loc}^\mu(X)$ of $\mathcal{K}_{loc}(X)$ by

$$\mathcal{K}_{loc}^\mu(X) = \{\nu \in \mathcal{K}_{loc}(X) \mid \text{For any } K \subset C,\ \|R^\mu(1_K\nu)\|_\infty < \infty.\},$$

where C is the totality of compact sets of E. If $\mu = 0$, then $\mathcal{K}_{loc}^\mu(X)$ equals $\mathcal{K}_{loc}(X)$ because $1_K\nu \in \mathcal{K}_\infty(X)$ and $\|R(1_K\nu)\|_\infty < \infty$.

Lemma 4.3. *Let ν be a non-trivial measure in $\mathcal{K}_{loc}^\mu(X)$. Then*

$$\int_E \phi^2 \frac{d\nu}{R^\mu\nu} \leq \mathcal{E}^\mu(\phi),\quad \phi \in \mathcal{D}(\mathcal{E}) \cap C_0(E).$$

Proof. Let $\{K_n\}$ be a increasing sequence of compact sets such that $K_n \subset \mathring{K}_{n+1}$ and $K_n \uparrow E$. We fix the sequence $\{K_n\}$. For $0 < \epsilon < 1$, define $\mu_n^\epsilon = \mu^+ - \epsilon\mu_n^-$, where $\mu_n^-(\cdot) := \mu^-(K_n \cap \cdot)$. The positive semi-definiteness of $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ implies that

$$\epsilon \int_E \phi^2 d\mu_n^- \leq \epsilon\, \mathcal{E}^{\mu^+}(\phi),$$

and

$$(1-\epsilon)\,\mathcal{E}^{\mu^+}(\phi) \leq \mathcal{E}^{\mu^+}(\phi) - \epsilon \int_E \phi^2 d\mu_n^- = \mathcal{E}^{\mu_n^\epsilon}(\phi) \leq \mathcal{E}^{\mu^+}(\phi), \tag{19}$$

which implies

$$\mathcal{D}_e(\mathcal{E}^{\mu_n^\epsilon}) = \mathcal{D}_e(\mathcal{E}^{\mu^+})(\subset \mathcal{D}_e(\mathcal{E})). \tag{20}$$

Let $\nu_m = \nu(\cdot \cap K_m)$. We may suppose that ν_1 is non-trivial and $R^{\mu_n^\epsilon}\nu_1(x)$ is bounded below by a positive constant on each compact set $K \subset E$. Noting $\nu_m \in \mathcal{K}_\infty(X)$, we see from (18) and (19) that

$$\begin{aligned}\int_E |\phi| d\nu_m &\leq \nu(K_m)^{1/2}\left(\int_E \phi^2 d\nu_m\right)^{1/2} \leq \nu(K_m)^{1/2}\|R\nu_m\|_\infty^{1/2} \cdot \mathcal{E}(\phi)^{1/2}\\ &\leq C\mathcal{E}^{\mu^+}(\phi)^{1/2} \leq C'\,\mathcal{E}^{\mu_n^\epsilon}(\phi)^{1/2}.\end{aligned}$$

Hence $R^{\mu_n^\epsilon}\nu_m$ belongs to $\mathcal{D}_e(\mathcal{E}^{\mu_n^\epsilon})$ and

$$\mathcal{E}^{\mu_n^\epsilon}(R^{\mu_n^\epsilon}\nu_m,\phi)=\int_E\phi\,d\nu_m=\int_E R^{\mu_n^\epsilon}\nu_m\cdot\phi\frac{d\nu_m}{R^{\mu_n^\epsilon}\nu_m},$$

which implies

$$\mathcal{E}^{\mu_n^\epsilon-\nu_m/R^{\mu_n^\epsilon}\nu_m}(R^{\mu_n^\epsilon}\nu_m,\phi)=0,\ \ \phi\in\mathcal{D}(E)\cap C_0(E).$$

Note that $R^{\mu_n^\epsilon}\nu_m$ is in $\mathcal{D}_e(\mathcal{E})$ by (20) and in $L^\infty(E,m)$ by $R^{\mu_n^\epsilon}\nu_m\le R^\mu\nu_m$. Moreover, it is bounded below by a positive constant on each compact set by Lemma 4.1. We then see from Lemma 4.5 and Lemma 4.6 below that

$$\mathcal{E}^{\mu_n^\epsilon-\nu_m/R^{\mu_n^\epsilon}\nu_m}(\phi)\ge0,\ \ \phi\in\mathcal{D}(\mathcal{E})\cap C_0(E),$$

and

$$\mathcal{E}^{\mu_n^\epsilon}(\phi)-\int_E\phi^2\frac{d\nu_m}{R^\mu\nu}\ge\mathcal{E}^{\mu_n^\epsilon}(\phi)-\int_E\phi^2\frac{d\nu_m}{R^{\mu_n^\epsilon}\nu_m}=\mathcal{E}^{\mu_n^\epsilon-\nu_m/R^{\mu_n^\epsilon}\nu_m}(\phi)\ge0.$$

Since

$$\begin{aligned}\mathcal{E}^{\mu_n^\epsilon}(\phi)-\int_E\phi^2\frac{d\nu_m}{R^\mu\nu}&\xrightarrow{m\to\infty}\ \mathcal{E}^{\mu_n^\epsilon}(\phi)-\int_E\phi^2\frac{d\nu}{R^\mu\nu}\\&\xrightarrow{\epsilon\to 1}\ \mathcal{E}^{\mu_n^1}(\phi)-\int_E\phi^2\frac{d\nu}{R^\mu\nu}\\&\xrightarrow{n\to\infty}\ \mathcal{E}^{\mu}(\phi)-\int_E\phi^2\frac{d\nu}{R^\mu\nu},\end{aligned}$$

we have this lemma. □

Lemma 4.3 leads us to an extension of the inequality (17).

Corollary 4.4. *It holds that*

$$\int_E\phi^2d\nu\le\|R^\mu\nu\|_\infty\,\mathcal{E}^\mu(\phi),\ \ \phi\in\mathcal{D}(\mathcal{E})\cap C_0(E).$$

Lemma 4.5. *Let $u\in\mathcal{D}_e(\mathcal{E})\cap L^\infty(E;m)$ is bounded below by a positive constant on each compact set. Then φ/u belongs to $\mathcal{D}(\mathcal{E})$ for any $\varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E)$.*

Proof. Let $\varphi\in\mathcal{D}(\mathcal{E})\cap C_0(E)$ and s suppose that $u\ge c>0$ on supp$[\varphi]$. Let $\{u_n\}\subset\mathcal{D}(\mathcal{E})\cap C_0(E)$ be an approximating sequence of u. We may suppose $\sup_n\|u_n\|_\infty\le\|u\|_\infty$ Then since by [9, Theorem 1.4.2 (ii)]

$$\mathcal{E}(u_n\varphi)^{1/2}\le\|u_n\|_\infty\mathcal{E}(\varphi)^{1/2}+\|\varphi\|_\infty\mathcal{E}(u_n)^{1/2},$$

we have $\sup_n\mathcal{E}(u_n\varphi)<\infty$. On account of [18, 1.6.1'], $u\varphi$ is in $\mathcal{D}_e(\mathcal{E})$ and so in $\mathcal{D}(\mathcal{E})$ because $\mathcal{D}_e(\mathcal{E})\cap L^2(E;m)=\mathcal{D}(\mathcal{E})$.

Since for $(x,y)\in$ supp$[\varphi]\times$ supp$[\varphi]$

$$\left|\frac{\varphi(x)}{u(x)}\right|\le c^{-1}|\varphi(x)|$$

$$\left|\frac{\varphi(x)}{u(x)}-\frac{\varphi(y)}{u(y)}\right|\le2c^{-1}|\varphi(x)-\varphi(y)|+c^{-2}|u(x)\varphi(x)-u(y)\varphi(y)|,$$

we have this lemma by the same argument as in the proof of [9, Theorem 6.3.2]. □

[8, Theorem 10.2] yields the next lemma.

Lemma 4.6. *Let* $\mu = \mu^+ - \mu^- \in \mathcal{K}_{loc}(X) - \mathcal{K}(X)$ *and* $u \in \mathcal{D}_e(\mathcal{E}) \cap L^\infty(E;m)$ *be a function bounded below by a positive constant on each compact. If u satisfies* $\mathcal{E}^\mu(u,\varphi) = 0$ *for any* $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$, *then* $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ *is positive semi-definite.*

Proof. The function u is a *generalized eigenfunction* corresponding to the *generalized eigenvalue 0* in [8, Definition 9.1]. Note that by Lemma 4.5, φ/u is a bounded function in $\mathcal{D}(\mathcal{E}^\mu)$ with compact support. Then, applying [8, Theorem 10.2], we have

$$\mathcal{E}^\mu(\varphi) = \mathcal{E}^\mu(u(\varphi/u)) = \int_{E\times E} u(x)u(y)d\Gamma(\varphi/u) \geq 0,$$

where $\Gamma(\varphi/u)$ is the positive measure on $E \times E$ defined in [8, Subsection 3.2]. □

Lemma 4.7. *Let* $\nu \in \mathcal{K}^\mu_{loc}(X)$ *and* $\nu_m = \nu(\cdot \cap K_m)$. *Then* $R^\mu\nu_m$ *belongs to* $\mathcal{D}_e(\mathcal{E}^\mu)$ *for any* m.

Proof. Since for $\phi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$

$$\int_E |\phi| d\nu_m \leq \nu(K_m)^{1/2}\left(\int_E \phi^2 d\nu_m\right)^{1/2} \leq \mu(K_m)^{1/2}\|R^\mu\nu_m\|_\infty^{1/2}\mathcal{E}^\mu(\phi)^{1/2}$$

by Corollary 4.4 and $\|R^\mu\nu_m\|_\infty < \infty$ by $\nu \in \mathcal{K}^\mu_{loc}(X)$, we have this lemma. □

Lemma 4.8. *If* $\nu \in \mathcal{K}^\mu_{loc}(X)$ *is of finite energy with respect to* $R^\mu(x,y)$,

$$\iint_{E\times E} R^\mu(x,y)\nu(dx)\nu(dy) < \infty, \tag{21}$$

then $R^\mu\nu$ *belongs to* $\mathcal{D}_e(\mathcal{E}^\mu)$.

Proof. Since $R^\mu\nu_m \in \mathcal{D}_e(\mathcal{E}^\mu) \uparrow R^\mu\nu(x)$ for any $x \in E$ as $m \to \infty$ and

$$\begin{aligned}\sup_m \mathcal{E}^\mu(R^\mu\nu_m) &= \sup_m \int_E R^\mu\nu_m d\nu_m = \sup_m \iint_{K_m\times K_m} R^\mu(x,y)\nu(dx)\nu(dy)\\ &\leq \iint_{E\times E} R^\mu(x,y)\nu(dx)\nu(dy) < \infty.\end{aligned}$$

By Banach-Saks Theorem (cf.[4, Theorem A.4.1]) there exists a subsequence $\{K_{m_l}\} \subset \{K_m\}$ such that

$$\frac{R^\mu\nu_{m_1} + R^\mu\nu_{m_2} + \cdots + R^\mu\nu_{m_l}}{l} = R^\mu\left(\frac{(1_{K_{m_1}} + 1_{K_{m_2}} \cdots + 1_{K_{m_l}})}{l}\nu\right) \longrightarrow R^\mu\nu$$

with $\mathcal{E}^\mu$-strongly, and thus Lemma 4.7 implies this lemma. □

For $\mu \in \mathcal{K}_{loc}(X) - \mathcal{K}_{loc}(X)$ and $\nu \in \mathcal{K}^\mu_{loc}(X)$, define

$$\nu^\mu = \frac{\nu}{R^\mu\nu}, \qquad \mu^\nu = \mu - \nu^\mu. \tag{22}$$

Then μ^ν is in $\mathcal{K}_{loc}(X) - \mathcal{K}_{loc}(X)$ by Corollary 4.2 and $(\mathcal{E}^{\mu^\nu}, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ is positive semi-definite by Lemma 4.3. Hence by [20, Theorem 4.2] we can define the Schrödinger form with potential μ^ν, the closure $(\mathcal{E}^{\mu^\nu}, \mathcal{D}(\mathcal{E}^{\mu^\nu}))$ of $(\mathcal{E}^{\mu^\nu}, \mathcal{D}(\mathcal{E}) \cap C_0(E))$ and its extended Schrödinger space $\mathcal{D}_e(\mathcal{E}^{\mu^\nu})$.

Lemma 4.9. *If $u \in \mathcal{D}_e(\mathcal{E}^{\mu^+})$, then*

$$\mathcal{E}^{\mu^\nu}(u) = \mathcal{E}^{\mu}(u) - \int_E u^2 d\nu^\mu.$$

Proof. Noting $u \in \mathcal{D}_e(\mathcal{E})$, there exists an $\mathcal{E}^{\mu^+}$-Cauchy sequence $\{u_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that $u_n \to u$ q.e. Since $\mathcal{E}^{\mu^\nu}(u) \leq \mathcal{E}^{\mu}(u) \leq \mathcal{E}^{\mu^+}(u)$, $u \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$, $\{u_n\}$ is also an approximating sequence of u in $\mathcal{D}_e(\mathcal{E}^\mu)$ and $\mathcal{D}_e(\mathcal{E}^{\mu^\nu})$. In particular, u is in $\mathcal{D}_e(\mathcal{E}^\mu) \subset \mathcal{D}_e(\mathcal{E}^{\mu^\nu})$, and thus $u \in L^2(E; \nu^\mu)$ by Lemma 4.3. Hence we have

$$\mathcal{E}^{\mu^\nu}(u) = \lim_{n\to\infty} \mathcal{E}^{\mu^\nu}(u_n) = \lim_{n\to\infty} \left(\mathcal{E}^{\mu}(u_n) - \int_E u_n^2 d\nu^\mu\right) = \mathcal{E}^{\mu}(u) - \int_E u^2 d\nu^\mu. \qquad \square$$

Lemma 4.10. *It holds that*

$$\mathcal{E}^{\mu^\nu}(R^\mu \nu_m) = \mathcal{E}^{\mu}(R^\mu \nu_m) - \int_E (R^\mu \nu_m)^2 d\nu^\mu.$$

Proof. Let $\{\epsilon_n\}$ be a positive sequence such that $\epsilon_n \uparrow 1$ as $n \to \infty$ and denote by μ'_n the measure $\mu_n^{\epsilon_n}$ defined in Lemma 4.3. Put $u_n = R^{\mu'_n}\nu_m$. Then u_n is in $\mathcal{D}_e(\mathcal{E}^{\mu^+})$ as shown in the proof of Lemma 4.3. Since

$$\mathcal{E}^{\mu}(u_n) \leq \mathcal{E}^{\mu'_n}(u_n) = \int_E u_n d\nu_m \leq \int_E R^\mu \nu_m d\nu_m < \infty,$$

There exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$v_k := \frac{u_{n_1} + u_{n_2} + \cdots + u_{n_k}}{k} \in \mathcal{D}_e(\mathcal{E}^{\mu^+})$$

is an approximating sequence of $R^\mu\nu_m$ in $\mathcal{D}_e(\mathcal{E}^\mu)$ and $v_k(x) \uparrow R^\mu\nu_m(x)$ for any $x \in E$.

Noting that $\{v_k\}$ is also an approximating sequence of $R^\mu\nu_m$ in $\mathcal{D}_e(\mathcal{E}^{\mu^\nu})$, we have by Lemma 4.9

$$\mathcal{E}^{\mu^\nu}(R^\mu \nu_m) = \lim_{k\to\infty} \mathcal{E}^{\mu^\nu}(v_k) = \lim_{k\to\infty} \left(\mathcal{E}^{\mu}(v_k) - \int_E v_k^2 d\nu^\mu\right) = \mathcal{E}^{\mu}(R^\mu \nu_m) - \int_E (R^\mu \nu_m)^2 d\nu^\mu. \qquad \square$$

Let $\mathcal{K}_C^\mu$ be the set of measures in $\mathcal{K}_{loc}^\mu(X)$ satisfying (7). For $\nu \in \mathcal{K}_C^\mu$ there exists a sequence $\{K_m\}_{m=1}^\infty \subset C$ such that $K_m \uparrow E$ and

$$\sup_m \iint_{E\times E} R^\mu(x,y)\nu_m(dx)\nu_m^c(dy) < \infty, \tag{23}$$

where $\nu_m^c(A) = \nu(K_m^c \cap A)$. If a measures $\nu \in \mathcal{K}_{loc}^\mu(X)$ of finite energy with respect to R^μ, then it satisfies (23).

Lemma 4.11. *If $\nu \in \mathcal{K}_C^\mu$, then $R^\mu\nu$ is in $\mathcal{D}_e(\mathcal{E}^{\mu^\nu})$.*

Proof. For $\nu \in \mathcal{K}_C^\mu$

$$\int_E R^\mu \nu_m d\nu = \int_E R^\mu \nu_m d\nu_m + \int_E R^\mu \nu_m d\nu_m^c < \infty$$

because

$$\int_E R^\mu \nu_m d\nu_m = \mathcal{E}^\mu(R^\mu \nu_m) < \infty$$

by Lemma 4.7.

By Lemma 4.10 we have

$$\begin{aligned}\mathcal{E}^{\mu^\nu}(R^\mu \nu_m) &= \mathcal{E}^\mu(R^\mu \nu_m) - \int_E (R^\mu \nu_m)^2 d\nu^\mu \\ &= \int_E R^\mu \nu_m d\nu_m - \int_E (R^\mu \nu_m)^2 d\nu^\mu \\ &= \int_E R^\mu \nu_m d\nu - \int_E R^\mu \nu_m d\nu_m^c - \int_E \frac{(R^\mu \nu_m)^2}{R^\mu \nu_m + R^\mu \nu_m^c} d\nu.\end{aligned}$$

The right hand side equals

$$\begin{aligned}&\int_E \left(\frac{R^\mu \nu_m (R^\mu \nu_m + R^\mu \nu_m^c) - (R^\mu \nu_m)^2}{R^\mu \nu_m + R^\mu \nu_m^c} \right) d\nu - \int_E R^\mu \nu_m d\nu_m^c \\ &= \int_E \frac{R^\mu \nu_m R^\mu \nu_m^c}{R^\mu \nu_m + R^\mu \nu_m^c} d\nu - \int_E R^\mu \nu_m d\nu_m^c \\ &= \int_E \frac{R^\mu \nu_m R^\mu \nu_m^c}{R^\mu \nu_m + R^\mu \nu_m^c} d\nu_m + \int_E \left(\frac{R^\mu \nu_m R^\mu \nu_m^c}{R^\mu \nu_m + R^\mu \nu_m^c} - R^\mu \nu_m \right) d\nu_m^c.\end{aligned} \tag{24}$$

Since

$$\frac{R^\mu \nu_m R^\mu \nu_m^c}{R^\mu \nu_m + R^\mu \nu_m^c} \leq R^\mu \nu_m^c, \quad \frac{R^\mu \nu_m R^\mu \nu_m^c}{R^\mu \nu_m + R^\mu \nu_m^c} \leq R^\mu \nu_m,$$

the right hand side of (24) is less than or equal to $\int_E R^\mu \nu_m^c d\nu_m$. Therefore, we see from (23) that

$$\sup_m \mathcal{E}^{\mu^\nu}(R^\mu \nu_m) \leq \sup_m \int_E R^\mu \nu_m^c d\nu_m < \infty.$$

Since $R^\mu \nu_m \to R^\mu \nu$, this lemma follows from Lemma 4.7. □

The next lemma is obtained in the same argument as in [20, Lemma 5.3].

Lemma 4.12. *For* $\nu \in \mathcal{K}_C^\mu$

$$\mathcal{E}^{\mu^\nu}(R^\mu \nu, \varphi) = 0, \quad \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E).$$

Proof. Since $\sup_m \mathcal{E}^{\mu^\nu}(R^\mu \nu_m) < \infty$, there exists a subsequence $\{K_{m_l}\} \subset \{K_m\}$ such that

$$R^\mu \left(\frac{(1_{K_{m_1}} + 1_{K_{m_2}} \cdots + 1_{K_{m_l}})}{l} \nu \right) \longrightarrow R^\mu \nu$$

$\mathcal{E}^{\mu^\nu}$-strongly.

Let $\phi_l := (1_{K_{m_1}} + 1_{K_{m_2}} \cdots + 1_{K_{m_l}})/l$. For a fixed $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$ we can assume $\mathrm{supp}[\varphi] \subset K_{m_1}$. By the same argument as in Lemma 4.10, we have

$$\mathcal{E}^{\mu^\nu}(R^\mu(\phi_l \nu) + \varphi) = \mathcal{E}^\mu(R^\mu(\phi_l \nu) + \varphi) - \int_E (R^\mu(\phi_l \nu) + \varphi)^2 d\nu^\mu,$$

and thus

$$\mathcal{E}^{\mu^\nu}(R^\mu(\phi_l\nu),\varphi)=\mathcal{E}^\mu(R^\mu(\phi_l\nu),\varphi)-\int_E R^\mu(\phi_l\nu)\varphi d\nu^\mu.$$

Hence

$$\begin{aligned}\mathcal{E}^{\mu^\nu}(R^\mu\nu,\varphi)&=\lim_{l\to\infty}\mathcal{E}^{\mu^\nu}(R^\mu(\phi_l\nu),\varphi)\\&=\lim_{l\to\infty}\left(\mathcal{E}^\mu(R^\mu(\phi_l\nu),\varphi)-\int_E R^\mu(\phi_l\nu)\varphi d\nu^\mu\right).\end{aligned}$$

Note that $R^\mu(\phi_l\nu)\in\mathcal{D}_e(\mathcal{E}^\mu)$ by Lemma 4.7. Then since

$$\lim_{l\to\infty}\mathcal{E}^\mu(R^\mu(\phi_l\nu),\varphi)=\lim_{l\to\infty}\int_E\varphi\phi_l d\nu=\int_E\varphi d\nu$$

and by the monotone convergence theorem

$$\lim_{l\to\infty}\int_E R^\mu(\phi_l\nu)\varphi d\nu^\mu=\int_E R^\mu\nu\cdot\varphi\frac{d\nu}{R^\mu\nu}=\int_E\varphi d\nu,$$

we have this lemma. □

The next theorem is an extension of [20, Theorem 5.4].

Theorem 4.13. *If $\nu\in\mathcal{K}_C^\mu$, then $R^\mu\nu$ is a ground state of $(\mathcal{E}^{\mu^\nu},\mathcal{D}(\mathcal{E}^{\mu^\nu}))$, consequently, $(\mathcal{E}^{\mu^\nu},\mathcal{D}(\mathcal{E}^{\mu^\nu}))$ is critical.*

Proof. Since $R^\mu\nu$ belongs to $\mathcal{D}_e(\mathcal{E}^{\mu^\nu})$, there exists a sequence $\{\varphi_n\}\subset\mathcal{D}(\mathcal{E})\cap C_0(E)$ such that φ_n converges $\mathcal{E}^{\mu^\nu}$-strongly to $R^\mu\nu$. Hence

$$\mathcal{E}^{\mu^\nu}(R^\mu\nu)=\lim_{n\to\infty}\mathcal{E}^{\mu^\nu}(R^\mu\nu,\varphi_n)=0$$

by Lamma 4.12. □

Corollary 4.14. *There exists no non-trivial positive function ψ such that*

$$\int_E u^2 d\left(\nu^\mu+\psi m\right)\leq\mathcal{E}^\mu(u,u),\ u\in\mathcal{D}(\mathcal{E})\cap C_0(E). \tag{25}$$

Proof. If (25) holds, then

$$\int_E u^2\psi dm\leq\mathcal{E}^{\mu^\nu}(u)=0,\ u\in\mathcal{D}(\mathcal{E})\cap C_0(E).$$

Since $R^\mu\nu$ is in $\mathcal{D}_e(\mathcal{E}^{\mu^\nu})$, there exists an approximating sequence $\{u_n\}\subset\mathcal{D}(\mathcal{E})\cap C_0(E)$. We then have

$$\int_E(R^\mu\nu)^2\psi dm\leq\varliminf_{n\to\infty}\int_E u_n^2\psi dm\leq\varliminf_{n\to\infty}\mathcal{E}^{\mu^\nu}(u_n)=\mathcal{E}^{\mu^\nu}(R^\mu\nu)=0,$$

and so $\psi=0$ m-a.e. because $R^\mu\nu>0$ by the irreducibility of X. □

Corollary 4.14 tells us that ν^μ is a *critical* Hardy weight for $(\mathcal{E}^\mu,\mathcal{D}(\mathcal{E}^\mu))$ ([6], [10]).

A Hardy weight ν^μ is called *optimal at infinity* if for any $K\in C$

$$\lambda\int_E u^2 d\nu^\mu\leq\mathcal{E}^\mu(u),\quad u\in\mathcal{D}(\mathcal{E})\cap C_0(K^c),$$

then $\lambda \le 1$.

Lemma 4.15. *If $\nu \in \mathcal{K}_C^\mu$ satisfies that*

$$\iint_{K^c \times E} R^\mu(x,y)\nu(dx)\nu(dy) = \infty \text{ for any } K \in \mathcal{C}, \tag{26}$$

then ν^μ is optimal at infinity.

Proof. Denote $h = R^\mu \nu$. Since h is a ground state of $(\mathcal{E}^{\mu^\nu}, \mathcal{D}(\mathcal{E}^{\mu^\nu}))$ by Theorem 4.13, h is $p_t^{\mu^\nu}$-invariant, $p_t^{\mu^\nu} h = h$, where $p_t^{\mu^\nu}$ is the semigroup associated with $(\mathcal{E}^{\mu^\nu}, \mathcal{D}(\mathcal{E}^{\mu^\nu}))$. Denote by $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ the Dirichlet form generated by h-transform of $(\mathcal{E}^{\mu^\nu}, \mathcal{D}(\mathcal{E}^{\mu^\nu}))$:

$$\mathcal{E}^h(u) = \mathcal{E}^{\mu^\nu}(uh), \quad u \in \mathcal{D}(\mathcal{E}^h) = \{u \mid uh \in \mathcal{D}(\mathcal{E}^{\mu^\nu})\}.$$

Since h is in $\mathcal{D}_e(\mathcal{E}^{\mu^\nu})$, there exists a sequence $\{h_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that $0 \le h_n \uparrow h$ and $\mathcal{E}^{\mu^\nu}(h - h_n) \to 0$ as $n \to \infty$. Then $\{g_n := h_n/h\}$ is an approximating sequence of $1 \in \mathcal{D}_e(\mathcal{E}^h)$.

Suppose that there exist $F \in \mathcal{C}$ and $\epsilon > 0$ such that for any $u \in \mathcal{D}(\mathcal{E}) \cap C_0(F^c)$

$$\mathcal{E}^\mu(u) \ge (1+\epsilon) \int_{F^c} u^2 d\nu^\mu, \quad u \in \mathcal{D}(\mathcal{E}) \cap C_0(F^c). \tag{27}$$

Let $G_1,\ G_2$ be relatively compact open set such that $F \subset G_1 \subset \overline{G}_1 \subset G_2 \subset \overline{G}_2 \subset E$. Let φ be a function in $\mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ on $x \in \overline{G}_1$ and $\mathrm{supp}[\varphi] \subset G_2$. Put $\psi = (1 - \varphi)$. Then $h_n\psi \in \mathcal{D}(\mathcal{E}) \cap C_0(F^c)$, and so by (27)

$$\epsilon \int_E (h_n\psi)^2 \frac{d\nu}{h} \le \mathcal{E}^{\mu^\nu}(h_n\psi). \tag{28}$$

Then we have by [9, Theorem 1.4.2 (ii)]

$$\mathcal{E}^{\mu^\nu}(h_n\psi) = \mathcal{E}^h(\frac{h_n}{h}\psi) \le 2\left(\mathcal{E}^h(h_n/h) + \mathcal{E}^h(\psi)\right),$$

and so

$$\sup_n \int_E (h_n\psi)^2 \frac{d\nu}{h} \le \frac{2}{\epsilon}\left(\sup_n \mathcal{E}^h(h_n/h) + \mathcal{E}^h(\psi)\right) < \infty$$

on account of (28). Hence

$$\int_{\overline{G}_2^c} h d\nu = \int_{\overline{G}_2^c} \lim_{n\to\infty} (h_n\psi)^2 \frac{d\nu}{h} \le \varliminf_{n\to\infty} \int_E (h_n\psi)^2 \frac{d\nu}{h} < \infty,$$

and thus

$$\iint_{\overline{G}_2^c \times E} R^\mu(x,y) d\nu(x) d\nu(y) = \int_{\overline{G}_2^c} h d\nu < \infty,$$

which is contradictory to (26). □

If $\nu \in \mathcal{K}_C^\mu$ satisfies the inequality (26), then the ground state $R^\mu\nu$ of $(\mathcal{E}^{\mu^\nu}, \mathcal{D}(\mathcal{E}^{\mu^\nu}))$ does not belong to $L^2(E; \mu^\nu)$ and so ν^μ is a null-critical Hardy weight for $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$. Therefore, we have

Theorem 4.16. *If* $\nu \in \mathcal{K}_C^{\mu}$ *satisfies*

$$\iint_{K^c \times E} R^{\mu}(x,y)\nu(dx)\nu(dy) = \infty \ \textit{ for any } K \in \mathcal{C},$$

then the measure ν^{μ} *defined in (22) is a optimal Hardy weight for* $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}^{\mu}))$.

REMARK 4.17. The measure $\nu(dx) := |x|^{-(d+\alpha)/2}dx$ satisfies (26) with respect to the Green kernel $|x-y|^{\alpha-d}$, $\alpha < d$, the 0-resolvent of the symmetric α-stable process because $(|y|^{\alpha-d} * |y|^{-(d+\alpha)/2})(x) = C|x|^{(\alpha-d)/2}$ and $|x|^{(\alpha-d)/2} \cdot |x|^{-(d+\alpha)/2} = |x|^{-d}$; however ν satisfies (23) ([20, Example 5.6]). Hence ν is an optimal Hardy weight for the Dirichlet form of symmetric α-stable process.

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