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SPECTRAL ASYMPTOTICS FOR MAGNETIC SCHRÖDINGER OPERATOR WITH SLOWLY VARYING POTENTIAL

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Abstract

Consider the Schrödinger operator with constant magnetic field and smooth potential V : $H(\epsilon) = H + V(\epsilon x, \epsilon y)$, $H = D_x^2 + (D_y + \mu x)^2$, $(x, y) \in \Omega_d$, with Dirichlet boundary conditions. Here $\Omega_d = \Pi_{j=1}^d]-a_j, a_j[\times\mathbb{R}_y^d$. The spectral properties of two operators H and $H(\epsilon)$ are investigated. For ϵ small enough, we study the effect of the slowly varying potential $V(\epsilon x, \epsilon y)$. In particular, we derive asymptotic trace formula and we give an asymptotic expansion in powers of ϵ of the spectral shift function corresponding to $(H(\epsilon), H)$.

1. Introduction

The Hamiltonian for a system of d interacting electrons confined along the x-direction and free to move along the y-direction in the presence of magnetic and electric potentials is given by

$$H(\epsilon) := \sum_{j=1}^{d} D_{x_j}^2 + (D_{y_j} + \mu_j x_j)^2 + V(\epsilon x, \epsilon y), \; D_\nu = \frac{1}{i}\partial_\nu, \tag{1.1}$$

where $x = (x_1, \cdots, x_d) \in \Lambda_d := \Pi_{j=1}^d]-a_j, a_j[$, $y \in \mathbb{R}^d$, $\mu = (\mu_1, \cdots, \mu_d)$ with $\epsilon, a_j, \mu_j > 0$. The potential V is assumed to be smooth and real-valued. The non-perturbed operator

$$H = D_x^2 + (D_y + \mu x)^2 = \sum_{j=1}^{d} D_{x_j}^2 + (D_{y_j} + \mu_j x_j)^2$$

is defined on $\mathcal{H}_{\Omega_d}^D := \{u \in H^2(\Omega_d); u|_{\partial\Omega_d} = 0\}$, where $H^2(\Omega_d)$ stands for the second order Sobolev space on $\Omega_d := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; -a_j < x_j < a_j\} = \Lambda_d \times \mathbb{R}^d$. The Fourier transformation with respect to y reduces the spectral problem of H to an analysis of the eigenvalues $\{e_l(k)\}_{l=0}^\infty$ depending on $k = (k_1, \cdots, k_d)$ of the operator

$$H_0(k) = D_x^2 + (k + \mu x)^2 = \sum_{j=1}^{d} D_{x_j}^2 + (k_j + \mu_j x_j)^2,$$

on Λ_d with Dirichlet boundary condition.

When the electron moves freely in both directions (i.e. $a_j = \infty$, $H_\infty = H$ on $\mathbb{R}^{2d}$), the spectrum of H_∞ exhibits infinitely degenerate eigenvalues, the so called Landau levels. The two-dimensional version of (1.1) is generally considered to serve as a minimal model for the

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integer quantum Hall, and has therefore been intensively investigated by physicists, see for instance [19, 29].

When a_j is finite, the spectrum of H is absolutely continuous, and coincides with $[e_0(0), +\infty[$. The points $e_j(0)$ are thresholds in $\sigma(H)$, and tends to the Landau level when μ or a_j is large enough (see Proposition 3.1). The application of the $H(\epsilon)$ spectrum in the theory means that we take into consideration important factors like finite size of the Hall system and the presence of a crystal lattice or impurities, and so on, in it. If the scalar potential V tends to zero as $|y| \to \infty$, the essential spectra of $H(\epsilon)$ and H are the same, and discrete eigenvalues with finite multiplicities can arise in $]-\infty, e_0(0)[$. Moreover, it is reasonable to expect that the electric field creates embedded eigenvalues and resonances on the second sheet. The principal topic of this paper centers around the effect of the slowly varying decaying perturbation $V(\epsilon x, \epsilon y)$ on the non-perturbed operator H. Particular attention will be paid to the asymptotic behavior of the spectrum near the thresholds $e_j(0)$.

The spectrum of the non-perturbed Hamiltonian H on a bounded domain $\Omega \subset \mathbb{R}^2$ were considered by many others. The asymptotic behavior of the bottom of the spectrum of H as μ tends to infinity has been treated for different geometry of Ω (see [14] and the references cited therein). When Ω is the semi-infinite plane or the disk, the WKB approximations of the energies and the eigenfunctions are obtained in [28]. For the counting function of the number of eigenvalues of the two dimensional Schrödinger operator with magnetic field we refer to [23, 27] and the monographs [14, 16]. The nature of the spectrum of the operator $H(1)$ on the half plane with Dirichlet boundary condition was studied in [2]. Other exciting spectral properties of the 2D Schrödinger operator with crossed magnetic and electrical fields have been investigated in [4, 6, 18, 22, 26].

In [5] (see also [6]), Mourre's theory and the spectral shift function near the thresholds $e_j(0)$ were considered when $\epsilon = 1$ and $\Omega_1 =]-a, a[\times\mathbb{R}$. In [8], the W.K.B approximation method is used to study the dynamics and the bottom of the spectrum of the operator $H(\epsilon)$ on Ω_1. This method cannot used to describe all the spectrum of $H(\epsilon)$. On the other hand, the multi-dimensional case (i.e., Ω_d with $d > 1$) is more complicated, since the thresholds $e_j(0)$ are in general degenerates when $d > 1$. Here we present an unified approach and derive an explicit formula for the counting and spectral shift functions corresponding to H and $H(\epsilon)$. Our goal is to give a rigorous way to recover the spectrum of $H(\epsilon)$ on Ω_d, $(d \geq 1)$ near any energy level λ, by studying systems of pseudo-differential operators which have a principal symbol quite close to one of $e_j(\epsilon D_y) + V(0, y) - z$, where z is the spectral parameter and $e_j(k)$ is an eigenvalue of $H_0(k)$.

The main results of this paper are briefly summarized here. Sections 2 and 3 are devoted to the study of the non-perturbed operators $H_0(k)$ and H. We collect in Theorem 2.1 and Corollary 2.2 a few properties of the eigenvalues $e_j(k)$ and their corresponding eigenfunctions $\Psi_j(\cdot, k)$. We introduce some type of "density of states ρ", related to H (see (2.14)), and examine its regularity in Theorem 2.3. We show that $t \to \rho(t)$ is analytic except at the thresholds $e_j(0)$, and we give its asymptotic behavior near every point $e_j(0)$, $j = 0, 1, \cdots$. In section 3, we study the asymptotic behavior of $e_j(k)$ when μ tends to infinity. For $k = 0$, $j = 0$ and μ large enough, it is well known that $e_0(0) - 1 \sim 4\pi^{-\frac{1}{2}} a^2 \mu^{\frac{3}{2}} e^{-a^2\mu}$ (see [3]). In Proposition 3.1, we generalize this result for $j \in \mathbb{N}$ and $|k| << \mu$. The proof uses the parabolic cylinder functions.

In sections 4-7, we study the perturbed operator $H(\epsilon)$ when ϵ is small enough. First, we give a complete asymptotic expansion in powers of ϵ of $\mathrm{tr}(\Psi f(H(\epsilon)))$ where $f \in C_0^\infty(\mathbb{R})$ and Ψ is a multiplication operator by a real integrable function $\Psi(y) \in L^1(\mathbb{R}^d)$. In particular, we obtain a Weyl type asymptotics with optimal remainder estimates of the counting function of eigenvalues of $H(\epsilon)$ in any closed interval in $]-\infty, e_0(0)[$. To investigate the effect of the perturbation on the continuous spectrum of H, it is natural to study the spectral shift function (SSF for short). When V vanishes as $\|y\| \to \infty$ (see (4.1)), the SSF $\xi(\mu;\epsilon)$ related to $H(\epsilon)$ and H is well defined in the sense of distribution :

$$\mathrm{tr}[f(H(\epsilon)) - f(H)] = -\langle \xi'(\cdot;\epsilon), f(\cdot)\rangle = \int_{\mathbb{R}} \xi(\mu;\epsilon) f'(\mu) d\mu, \quad f \in C_0^\infty(\mathbb{R}). \tag{1.2}$$

The function $\xi(\mu;\epsilon)$ is fixed up to a constant by the formula (1.2), and we normalize $\xi(\mu;\epsilon)$ so that $\xi(\mu;\epsilon) = 0$ for $\mu < \inf(\sigma(H(\epsilon))$. The spectral shift function may be considered as a generalization of the eigenvalues counting function. It is one of important physical quantities in scattering theory, and it plays an important role in the study of the location of resonances in various scattering problems. We refer to [25] and references cited there for comprehensive information on related subjects.

Under assumption (4.1), we give in Theorem 4.3 a complete asymptotic expansion in powers of ϵ of the left hand side of (1.2), and in Theorem 4.4, we establish a complete asymptotic expansions in powers of ϵ for $\xi(\mu;\epsilon)$. The leading coefficients of these asymptotics are expressed in terms of the density ρ and the potential V (see (4.6) and (4.11)).

Let us provide a broad outline of the proof. Spectral properties of the free operator H follow from the direct integral decomposition (7.10). According to Theorem 2.1, we may write

$$H_0(k) = \sum_{j\geq 0} e_j(k)\pi_j(k),$$

where $\pi_j(k)u(x) = \langle u(\cdot), \Psi_j(\cdot,k)\rangle \Psi_j(x,k)$ is the projection on $\Psi_j(\cdot,k)$. By (2.6) and (2.7), the operators $e_j(D_y)$ and $\pi_j(D_y)$ are well defined as pseudo-differential operators. Thus, for instance, if $V(x,y) = V(y)$ is independent on x then

$$H(\epsilon) = H_0(D_y) + V(\epsilon y) = \sum_{j\geq 0} \left[e_j(D_y) + V(\epsilon y)\right]\pi_j(D_y).$$

Since V is bounded, and $\lim_{j\to\infty} e_j(k) = +\infty$ uniformly with respect to k, it follows by an elliptic argument that $(e_j(D_y) + V(\epsilon y) - z)$ is invertible for z in a bounded set and $j > N$, with N large enough. This allows one to reduce the spectral study of $H(\epsilon)$ on $L^2(\Omega_d)$ near z to the study of a system of ϵ-pseudo-differential operators on $L^2(\mathbb{R}^d_y)$, whose diagonal entries are $(e_j(\epsilon D_y) + V(y) - z)$, $j = 0, \cdots, N$ (see Propositions 6.1-6.2). Now, the main results follow from standard Theorems of functional calculus and micro-local analysis. When V depends on x, we use the fact that x is confined in a box, we then treat for ϵ small enough $V(\epsilon x, \epsilon y)$ as a perturbation of $V(0, \epsilon y)$.

Notations : We shall employ the following standard notations. Given a complex function f_h depending on a small positive parameter h, the relation $f_h = \mathcal{O}(h^N)$ means that there exist $C_N, h_N > 0$ such that $|f_h| \leq C_N h^N$ for all $h \in]0, h_N[$. The relation $f_h = \mathcal{O}(h^\infty)$ means that, for all $N \in \mathbb{N} := \{0, 1, 2, \ldots\}$, we have $f_h = \mathcal{O}(h^N)$. We write $f_h \sim \sum_{j=0}^\infty a_j h^j$ if, for each $N \in \mathbb{N}$, we have $f_h - \sum_{j=0}^{N} a_j h^j = \mathcal{O}(h^{N+1})$. We adopt the notation $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

Let $\mathcal{H}$ be a Hilbert space. The scalar product in $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle$. The set of linear bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{L}$ $(\mathcal{H}_1)$ in the case where $\mathcal{H}_1 = \mathcal{H}_2$.

2. The non-perturbed Hamiltonians $H_0(k)$ and H

In this section we study the non-perturbed operator $H_0(k)$ and H. In particular, we introduce an integrated density of states, ρ, corresponding to H.

The operator H is unitarily equivalent to

$$\mathcal{F}H\mathcal{F}^* = \int_{\mathbb{R}^d}^{\oplus} H_0(k)dk, \tag{2.1}$$

where $\mathcal{F}$ is the partial Fourier transform with respect to y given by

$$(\mathcal{F}u)(x,k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-iyk} u(x,y)dy,$$

and

$$H_0(k) = D_x^2 + (k + \mu x)^2, \tag{2.2}$$

is the operator defined on $\mathcal{H}_{\Lambda_d} := \{u \in H^2(\Lambda_d);\ u|_{\partial\Lambda_d} = 0\}$. In what follows, we will consider $\mathcal{H}_{\Lambda_d}$ as a Hilbert space equipped with the standard scalar product of $H^2(\Lambda_d)$.

We first consider the two dimensional case (i.e, $d = 1, \Omega_1 =]-a, a[\times\mathbb{R}$). From the Sturm-Liouville theory (see [21]), it is well-known that $H_0(k)$ has a simple discrete spectrum: $e_0(k) < e_1(k) < \cdots$. The change of variable $x \mapsto -x$ implies that $e_l(k) = e_l(-k)$. Since the eigenvalues are simple, an ordinary analytic perturbation theory shows that $e_l(k)$ (and the corresponding eigenfunction) are analytic functions in k (see [20, 24]).

Theorem 2.1. *The eigenvalue $e_j(k)$ satisfies :*

$$ke_j'(k) > 0 \quad (k \neq 0), \ \text{and} \ e_j'(0) = 0, \quad e_j''(0) > 0. \tag{2.3}$$

Moreover, for every fixed $j \in \mathbb{N}$ and any $a, \mu > 0$, the following properties hold :

$$e_j(k) = e_j(0) + \sum_{l=1}^{\infty} \alpha_{j,l} k^{2l} \ (k \to 0), \qquad \alpha_{j,1} > 0, \tag{2.4}$$

$$e_j(k) = k^2 - 2a\mu k + \nu_j (2\mu k)^{2/3}(1 + o(1)), \quad (k \to +\infty), \tag{2.5}$$

where $0 < \nu_0 < \nu_1 < \cdots < \nu_j < \cdots$ are the eigenvalues of the operator $D_x^2 + x$ on $\mathbb{R}^+$. The normalized eigenfunctions $\Psi_n(\cdot, k)$ corresponding to $e_n(k)$ can be chosen real-valued and analytic with respect to k satisfying :

$$\forall p \in \mathbb{N},\ \exists C_p,\ \text{such that} \ \int_{-a}^{a} \left(\partial_k^p \Psi_n(x,k)\right)^2 dx \leq C_p, \quad \|\Psi_n(\cdot,k)\|_{L^2(-a,a)} = 1. \tag{2.6}$$

For all $p \in \mathbb{N}$, there exists $C_p > 0$ such that

$$|\partial_k^p e_n(k)| \leq C_p(1 + |k|)^{2-p}. \tag{2.7}$$

Proof. The assertion (2.3) is proved in [15] (see Theorem 2 in [15]). Formula (2.4) follows from the fact that $e_j(k)$ is an even real analytic function with $e_j''(0) > 0$.

To prove (2.5), consider the operator $\tilde{H}(k) = D_x^2 + 2\mu xk + k^2$. Replacing x by $t = \mu(x+a)$ and rescaling $t \mapsto \lambda t/\mu$ (with $\lambda = (2\mu k)^{1/3}$) we transform $\tilde{H}(k)$ into $\lambda^2 G - 2a\mu k + k^2$, where

$$G = D_t^2 + t : L^2([0,b]) \to L^2([0,b]), \quad b = 2\lambda a,$$

is the Airy operator with Dirichlet boundary condition. The general solution of the equation $D_t^2 u(t) + tu(t) = 0$ can be written as a linear combination of the Airy functions :

$$u(t) = C_+ \mathrm{Ai}(t) + C_- \mathrm{Bi}(t).$$

We recall that $\mathrm{Bi}(t) = \mathrm{Ai}(e^{2\pi i/3}x)$. Using the fact that $v(t) = u(t - \nu_j)$ satisfies the equation $Gv = \nu_j v$, we deduce from the boundary conditions $v(0) = v(b) = 0$, the quantization condition on the eiguenvalues ν_j of the operator G

$$\mathrm{Ai}(-\nu_j) = \mathrm{Bi}(-\nu_j)\frac{\mathrm{Ai}(-\nu_j + b)}{\mathrm{Bi}(-\nu_j + b)}.$$

Since the right-hand side of the above equality tends to zero as b tends to $+\infty$, $-\nu_j$ are approximated (when $k \to +\infty$) by the zeros of the Airy function $\mathrm{Ai}(x)$. Consequently, the eigenvalues $\lambda_0(k) < \lambda_1(k) < \cdots$ of $\tilde{H}(k)$ satisfies

$$\lambda_j(k) = k^2 - 2a\mu k + \nu_j(2\mu k)^{2/3}(1 + o(1)) \quad (k \to +\infty). \tag{2.8}$$

Let A and B be self-adjoint operators that are bounded from below. We write $A \leq B$ if and only if $D(B) \subset D(A)$ and

$$(Au, u) \leq (Bu, u) \ \ \forall u \in D(B).$$

Using the above inequality and the fact that $x \in [-a, a]$, we obtain

$$H_0(k) - \mu^2 a^2 \leq \tilde{H}(k) = H_0(k) - \mu^2 x^2 \leq H_0(k),$$

which together with Theorem XIII.1 in [24] yields

$$e_j(k) - \mu^2 a^2 \leq \lambda_j(k) \leq e_j(k).$$

Thus (2.5) follows from (2.8) and the above inequality.

Next we prove (2.6). Let $\Psi_n(\cdot, k)$ be the normalized real-valued analytic function corresponding to $e_n(k)$. Since Ψ_n is real and $\|\Psi_n(\cdot, k)\| = 1$, it follows that

$$\frac{\partial}{\partial k}\int_{-a}^{a} \Psi_n(x,k)^2 dx = 0 = 2\int_{-a}^{a} \Psi_n(x,k)\frac{\partial}{\partial k}\Psi_n(x,k)dx. \tag{2.9}$$

Put $\widehat{H}(k) = H_0(k) - k^2$, and let Γ_n be a simple closed contour around $e_n(k) - k^2$ such that $\mathrm{dist}(\Gamma_n, \sigma(\widehat{H}(k))) \geq C > 0$ uniformly on k. Let $\Pi_n(k)$ be the orthogonal projection onto the eigenspace spanned by $\Psi_n(\cdot, k)$, that is for $u(x) \in \mathcal{H}_{\Lambda_2}$

$$\Pi_n(k)u(x) = \frac{1}{2\pi i}\int_{\Gamma_n} (\widehat{H}(k) - z)^{-1} dz = \langle\, u(\cdot)\,, \Psi_n(\cdot, k)\rangle\, \Psi_n(x, k). \tag{2.10}$$

From (2.9) we deduce that $\Pi_n(k)\partial_k\Psi_n(\cdot, k) = 0$. Combining this with the fact that $\Pi_n(k)\Psi_n(\cdot, k) = \Psi_n(\cdot, k)$ and using (2.10) as well as the fact that $\partial_k\widehat{H}(k) = 2\mu x$, we get

$$\partial_k \Psi_n(x,k) = \partial_k \Pi_n(k)\Psi_n(x,k) = \frac{-1}{2\pi i}\int_{\Gamma_n} (\widehat{H}(k)-z)^{-1} 2\mu x (\widehat{H}(k)-z)^{-1} dz \Psi_n(x,k), \tag{2.11}$$

which yields

$$\|\partial_k \Psi_n(\cdot,k)\| = \mathcal{O}(1)\|\Psi_n(\cdot,k)\| = \mathcal{O}(1).$$

We now proceed by induction using (2.11).

To prove (2.7), we differentiate the equality $(H_0(k) - e_n(k))\Psi_n(\cdot,k) = 0$ with respect to k we get

$$\Big(2(x+k) - e_n(k)\Big)\Psi_n(x,k) = \Big(H_0(k) - e_n(k)\Big)\partial_k \Psi_n(x,k).$$

Taking the product scalar of both sides of the above equality with $\Psi_n(\cdot,k)$ and using the self adjointeness of $H_0(k)$, as well as well as the fact that Ψ_n is real valued and normalized we obtain the formula

$$\partial_k e_n(k) = 2\int_{-a}^{a} x\Psi_n(x,k)^2 dx + 2k, \tag{2.12}$$

which yields (2.7) for $p = 1$. For $p \geq 2$, we differentiate (2.12) and we use (2.6). □

We return now to the general case $d \geq 1$. Let $\Big(e_l^j(k_j)\Big)_{l\in\mathbb{N}}$ and $\Big(\Psi_l^j(x_j,k_j)\Big)_{l\in\mathbb{N}}$ be the eigenvalues and eigenvectors of the operator $D_{x_j}^2 + (k_j + \mu_j x_j)^2$ given by Theorem 2.1. For $J = (j_1,\cdots,j_d) \in \mathbb{N}^d$ and $k = (k_1,\cdots,k_d) \in \mathbb{R}^d$, we denote

$$e_J(k) = e_{j_1}^1(k_1) + \cdots + e_{j_d}^d(k_d), \quad \Psi_J(x,k) = \Psi_{j_1}^1(x_1,k_1)\times\cdots\times\Psi_{j_d}^d(x_d,k_d). \tag{2.13}$$

By Theorem 2.1, we have

Corollary 2.2. *Fix $d \geq 1$. The spectrum of the operator $H_0(k)$ on $\{u \in H^2(\Lambda_d); u|_{\partial\Lambda_d} = 0\}$ is discrete and coincides with $\{e_J(k); J \in \mathbb{N}^d\}$. The family $(\Psi_J(\cdot,k))_{J\in\mathbb{N}^d}$ is an orthonormal basis in $L^2(\Lambda_d)$.*

According to Theorem 2.1, Corollary 2.2, and the theory of decomposable operators (see Theorem XIII. 85 in [24]) the spectrum of the operator $H = D_x^2 + (D_y + \mu x)^2$ with domain $\mathcal{H}_{\Omega_d}^D$ is absolutely continuous, and given by

$$\sigma(H) = \bigcup_{J\in\mathbb{N}^d}\bigcup_{k\in\mathbb{R}^d} e_J(k) = [e_0(0), +\infty[.$$

The points $e_J(0)$ are thresholds in $\sigma(H)$. From now on we denote this set by

$$\Sigma := \bigcup_{J\in\mathbb{N}^d} e_J(0) = \sigma(H_0(0)).$$

For $t_0 \in \Sigma$, we let $\mathcal{S}_{t_0} := \{J \in \mathbb{N}^d; e_J(0) = t_0\}$ and $m_{t_0} := \#\mathcal{S}_{t_0}$ be its multiplicity. In order to formulate our results on the trace formula and the asymptotics of the spectral shift function, we need to introduce the function $\rho : \mathbb{R} \to \mathbb{R}$ related to the non-perturbed H by

$$\rho(t) = \sum_{J\in\mathbb{N}^d}\int_{\{e_J(k)\leq t\}} \frac{dk}{(2\pi)^d}. \tag{2.14}$$

Obviously, $\rho(t) = 0$ for $t < e_0(0) = \inf\sigma(H)$. In an appendix, we shall prove that the function

$\rho(t)$ is analytic except near Σ. More precisely, we have

Theorem 2.3. *The function ρ is analytic except at Σ. Moreover, near any point $t_0 = e_J(0) \in \Sigma$, there exists analytic functions f and g such that :*

$$\rho(t) = f(t - t_0) + Y(t - t_0)g(\sqrt{t - t_0}),$$

for $|t - t_0|$ small enough with

$$g(t) \sim_{t\to 0} \sum_{J\in S_{t_0}} \frac{\mathrm{vol}(S^{d-1})}{d\sqrt{\det(\frac{\nabla^2 e_J(0)}{2})}} t^d.$$

Here $Y(t)$ is the Heaviside function and S^{d-1} stands for the unit sphere in $\mathbb{R}^d$.

Remark 2.4. Notice that the singularity and the behavior of ρ near $e_J(0)$ is similar to those of the integrated density of states, $\rho_0(t)$, of $-\Delta$ on $\mathbb{R}^d$ near $t = 0$. We recall that

$$\rho_0(t) = (2\pi)^{-d}\mathrm{vol}(B_{\mathbb{R}^d}(0, 1))Y(t)t^{d/2}.$$

3. Asymptotic behavior of eigenvalues of $H_0(k)$ for $\mu \gg 1$.

In this section, we investigate the asymptotic behavior of the eigenvalues of $H_0(k)$ when μ tend to infinity. Without any loss of generality we may assume that $d = 1$, (i.e, $\Omega_1 = [-a, a] \times \mathbb{R}$). For $d > 1$ we use (2.13). We set $e_j(k)$ and $\Psi_j(k)$ as the j−th eigenvalue and the j-th eigenfunction of $H_0(k)$, respectively. In the following proposition we give the asymptotic behavior of the eigenvalues $e_j(k)$ when μ tends to infinity.

Proposition 3.1. *Fix j and a, we have :*

$$(3.1)\quad e_j(k) - \mu(2j + 1) \underset{\mu\to\infty}{\sim} \frac{2\left(a\sqrt{2\mu}\right)^{2j+3}}{j!\sqrt{2\pi}} e^{-a^2\mu} e^{-k^2/\mu} \cosh(2ak) \times\left[1 + \frac{(2j + 1)k}{a\mu} \tanh(2ak) + j(2j + 1)\left(\frac{k}{a\mu}\right)^2 + o\left(\frac{k^2}{\mu^2}\right)\right],$$

uniformly for $|k| \ll \mu$.

Proof. Change of variable $x \to y - k/\mu$ transforms $H_0(k)$ to

$$\tilde{H}_0(k) = D_y^2 + \mu^2 y^2, \text{ on } \mathcal{H}_{[-a+k/\mu, a+k/\mu]},$$

and again employ the change of variable $y \to z/(\sqrt{2\mu})$, we have $H_0(k)$ is unitarily equivalent to

$$(3.2)\quad \check{H}_0(k) = 2\mu\left(D_x^2 + \frac{x^2}{4}\right), \text{ on } \check{\mathcal{H}}_{[-z_-, z_+]},$$

where

$$(3.3)\quad z_\pm := \sqrt{2\mu}\left(a \pm \frac{k}{\mu}\right).$$

Hence the eigenvalue problem for $H_0(k)$ can be reduced to the one for $\check{H}_0(k)$. Here let $u_\nu(x)$ be the solution of the Weber's equation

$$\left[D_x^2 + \frac{x^2}{4} - \left(\nu + \frac{1}{2}\right)\right] u_\nu(x) = 0, \tag{3.4}$$

with boundary condition $u_\nu(z_+) = u_\nu(-z_-) = 0$. Then $u_\nu(x)$ can be written as a linear combination of the parabolic cylinder functions $D_\nu(z)$ and $D_\nu(-z)$,

$$u_\nu(x) = A_1 D_\nu(x) + A_2 D_\nu(-x). \tag{3.5}$$

We recall that

$$D_\nu(z) = \frac{2^{\nu/2} e^{-z^2/4}}{\sqrt{\pi}} \left[\Gamma\left(\frac{\nu+1}{2}\right)\cos(\nu\pi/2) F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{2}\right) + \sqrt{2} z \Gamma\left(1 + \frac{\nu}{2}\right)\sin(\nu\pi/2) F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; \frac{z^2}{2}\right)\right],$$

where F_1 is the confluent hypergeometric function. For large $|z| \gg 1$, we have

$$D_\nu(z) = e^{-z^2/4} z^\nu \left[1 - \frac{\nu(1-\nu)}{2z^2} + \cdots\right], \quad z \gg 1, \tag{3.6}$$

and for $z \ll -1$,

$$D_\nu(z) = e^{-z^2/4} z^\nu \left[1 - \frac{\nu(1-\nu)}{2z^2} + \cdots\right] - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\nu\pi i} e^{z^2/4} z^{-\nu-1} \left[1 + \frac{(\nu+1)(\nu+2)}{2z^2} \pm \cdots\right]. \tag{3.7}$$

By the boundary condition $u_\nu(z_+) = u_\nu(-z_-) = 0$, we obtain from (3.5) the conditions on the energy spectrum :

$$D_\nu(z_+) D_\nu(z_-) - D_\nu(-z_+) D_\nu(-z_-) = 0. \tag{3.8}$$

Since $z_\pm$ tends to infinity as $\mu \to \infty$, it follows from (3.6) and (3.7) that

$$D_\nu(z_+) D_\nu(z_-) = e^{-(z_+^2 + z_-^2)/4} z_+^\nu z_-^\nu \left[1 + \mathcal{O}(z_\pm^{-2}) + \cdots\right]$$

and

$$\begin{aligned} D_\nu(-z_+) D_\nu(-z_-) &= e^{-(z_+^2 + z_-^2)/4} z_+^\nu z_-^\nu \left[1 + \mathcal{O}(z_\pm^{-2}) + \cdots\right] \\ &+ \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\nu\pi i} e^{z_+^2/4 - z_-^2/4} (z_+)^{-\nu-1} (z_-)^\nu \left[1 + \mathcal{O}(z_\pm^{-2}) + \cdots\right] \\ &+ \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\nu\pi i} e^{z_-^2/4 - z_+^2/4} (z_+)^\nu (z_-)^{-\nu-1} \left[1 + \mathcal{O}(z_\pm^{-2}) + \cdots\right] \\ &+ \left(\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\right)^2 e^{2\nu\pi i} e^{(z_+^2 + z_-^2)/4} (z_+ z_-)^{-\nu-1} \left[1 + \mathcal{O}(z_\pm^{-2}) + \cdots\right]. \end{aligned}$$

By (3.8), we have

$$\begin{aligned} & e^{z_+^2/4 - z_-^2/4} (z_+)^{-\nu-1} (z_-)^\nu \left[1 + \mathcal{O}(z_\pm^{-2}) + \cdots\right] \\ &+ e^{z_-^2/4 - z_+^2/4} (z_+)^\nu (z_-)^{-\nu-1} \left[1 + \mathcal{O}(z_\pm^{-2}) + \cdots\right] \\ &+ \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\nu\pi i} e^{(z_+^2 + z_-^2)/4} (z_+ z_-)^{-\nu-1} \left[1 + \mathcal{O}(z_\pm^{-2}) + \cdots\right] = 0. \end{aligned}$$

This implies

$$(3.9)\qquad \left(e^{-z_-^2/2}z_-^{2\nu+1}+e^{-z_+^2/2}z_+^{2\nu+1}\right)\left[1+\mathcal{O}(z_\pm^{-2})+\cdots\right]=-\frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{\nu\pi i}\left[1+\mathcal{O}(z_\pm^{-2})+\cdots\right].$$

Recall that,

$$\Gamma(1+z)\Gamma(-z)=-\frac{\pi}{\sin(\pi z)},\ \forall z\in\mathbb{C}\setminus\mathbb{Z}.$$

Combining this with (3.9) we get

$$(3.10)\qquad \left(z_+^{2\nu+1}e^{-z_+^2/2}+z_-^{2\nu+1}e^{-z_-^2/2}\right)\left(1+\mathcal{O}\left(\frac{1}{z_\pm^2}\right)\right)=\sqrt{\frac{2}{\pi}}\frac{e^{2\pi\nu i}-1}{2i}\Gamma(1+\nu)\left(1+\mathcal{O}\left(\frac{1}{z_\pm^2}\right)\right).$$

Now we look for $\nu=j+\alpha_j(\mu,k)$ for some fixed j, with $\alpha_j(\mu,k)$ tends to zero when μ tends to infinity. As a first approximation, it follows from (3.10) that

$$(3.11)\qquad z_+^{2j+1}e^{-z_+^2/2}+z_-^{2j+1}e^{-z_-^2/2}=\sqrt{2\pi}\Gamma(1+j)\alpha_j(\mu,k),$$

where we use

$$\frac{e^{2(j+\alpha_j)\pi i}-1}{\sqrt{2\pi}i}\sim\sqrt{2\pi}\alpha_j(\mu,k),\quad \text{as } \alpha_j(\mu,k)\to 0.$$

Thus by using (3.3) and (3.11),

$$\begin{aligned}
&\alpha_j(\mu,k)\sqrt{2\pi}\Gamma(1+j)\\
&=(\sqrt{2\mu})^{2j+1}\left(\left(a+\frac{k}{\mu}\right)^{2j+1}e^{-\mu(a+k/\mu)^2}+\left(a-\frac{k}{\mu}\right)^{2j+1}e^{-\mu(a-k/\mu)^2}\right)\\
&=\left(a\sqrt{2\mu}\right)^{2j+1}e^{-a^2\mu}e^{-k^2/\mu}\left[\left(1+\frac{k}{a\mu}\right)^{2j+1}e^{-2ak}+\left(1-\frac{k}{a\mu}\right)^{2j+1}e^{2ak}\right]\\
&=\left(a\sqrt{2\mu}\right)^{2j+1}e^{-a^2\mu}e^{-k^2/\mu}\left[\sum_{l=0}^{2j+1}\binom{2j+1}{l}\left(\frac{k}{a\mu}\right)^l\left(e^{-2ak}+(-1)^{2j-l}e^{2ak}\right)\right]\\
&=\left(a\sqrt{2\mu}\right)^{2j+1}e^{-a^2\mu}e^{-k^2/\mu}\\
&\quad\times\left[2\cosh(2ak)+\frac{2(2j+1)k}{a\mu}\sinh(2ak)+(2j)(2j+1)\left(\frac{k}{a\mu}\right)^2\cosh(2ak)+o\left(\frac{k^2}{\mu^2}\right)\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\alpha_j(\mu,k)\sqrt{2\pi}\Gamma(1+j)&=2\left(a\sqrt{2\mu}\right)^{2j+1}e^{-a^2\mu}e^{-k^2/\mu}\cosh(2ak)\\
&\quad\times\left[1+\frac{(2j+1)k}{a\mu}\tanh(2ak)+j(2j+1)\left(\frac{k}{a\mu}\right)^2+o\left(\frac{k^2}{\mu^2}\right)\right].
\end{aligned}$$

Notice that, according to (2.2), (3.4) and the unitary equivalence of $H_0(k)$ and $\check{H}_0(k)$, the eigenvalues of $H_0(k)$ satisfy

$$e_j(k)=2\mu\left(j+\frac{1}{2}+\alpha_j(\mu,k)\right).$$

Summing up and using that $\Gamma(1+j)=j!$ we obtain (3.1). □

REMARK 3.2. For $j = 0$ and $k = 0$, (3.1) were obtained in [3].

4. Perturbed Hamiltonian

Now, we investigate the effect of the slowly varying potential on the undisturbed operator spectrum. First, we give a complete asymptotic expansion in powers of ϵ of $\mathrm{tr}(\Psi f(H(\epsilon)))$ where $f \in C_0^\infty(\mathbb{R})$ and Ψ is an $L^1(\mathbb{R}_y^d)$-function. In particular, we obtain a Weyl type asymptotics with optimal remainder estimates of the counting function of eigenvalues of $H(\epsilon)$ below the essential spectra. Finally, we give a complete asymptotic expansion in powers of ϵ of the spectral shift function corresponding to $(H(\epsilon), H)$.

We suppose that V is smooth, and there exists $\delta \geq 0$ such that :

$$\forall \alpha, \beta \in \mathbb{N}^d, \ \exists C_{\alpha,\beta} \ \text{s.t} \ \ \sup_{x\in\Lambda_d} |\partial_x^\beta \partial_y^\alpha V(x,y)| \leq C_{\alpha,\beta} \langle y \rangle^{-\delta}. \tag{4.1}$$

By the Weyl criterion (see [17, 24]), if $\delta > 0$, the essential spectra of H and $H(\epsilon)$ are the same:

$$\sigma_{\mathrm{ess}}(H(\epsilon)) = \sigma_{\mathrm{ess}}(H) = \sigma(H) = [e_0(0), +\infty[.$$

First, we derive a local trace formula.

Theorem 4.1. *Assume* (4.1) *with* $\delta \geq 0$, *and let* Ψ *be a smooth function such that* $\partial_y^\alpha \Psi \in L^1(\mathbb{R}_y^d)$ *for* $|\alpha| \leq 2d + 1$. *Then for all* $f \in C_0^\infty(\mathbb{R})$, *the operator* $(\Psi f(H(\epsilon)))$ *is trace class and the following asymptotics hold :*

$$\mathrm{tr}\,(\Psi f(H(\epsilon))) \sim \sum_{j=0}^{\infty} a_j \epsilon^{-d+j}, \tag{4.2}$$

with

$$a_0 = -\iint_{\mathbb{R}^d \times \mathbb{R}_t} \Psi(y) f'(t) \rho(t - V(0,y)) dy dt. \tag{4.3}$$

Here $f(H(\epsilon))$ *is the operator given by the spectral theorem and* $\Psi : L^2(\Omega_d) \ni u \to \Psi(y)u(x,y) \in L^2(\Omega_d)$ *is the multiplication operator.*

Let $N([a,b];\epsilon)$ be the number of eigenvalues of $H(\epsilon)$ in $[a,b] \subset]-\infty, e_0(0)[$ counted with their multiplicity.

Corollary 4.2. *Assume that* V *tends to zero at infinity, and let* $f \in C_0^\infty(]-\infty, e_0(0)[;\mathbb{R})$. *We have*

$$\mathrm{tr}(f(H(\epsilon))) \sim \sum_{j=0}^{\infty} b_j \epsilon^{-d+j}, \tag{4.4}$$

with

$$b_0 = -\iint_{\mathbb{R}^d \times \mathbb{R}_t} f'(t) \rho(t - V(0,y)) dy dt. \tag{4.5}$$

In particular,

$$\lim_{\epsilon \searrow 0} \left[\epsilon^d N([a,b];\epsilon)\right] = \int_{\mathbb{R}^d} \left[\rho(b - V(0,y)) - \rho(a - V(0,y))\right] dy. \tag{4.6}$$

Theorem 4.3. *Assume* (4.1) *with* $\delta > d$. *For* $f \in C_0^\infty(\mathbb{R})$ *the operator* $f(H(\epsilon)) - f(H)$ *is trace class. Moreover, the following asymptotics holds*

$$\operatorname{tr}(f(H(\epsilon)) - f(H)) \sim \sum_{j=0}^{\infty} c_j \epsilon^{-d+j} \tag{4.7}$$

with

$$c_0 = \iint_{\mathbb{R}^d \times \mathbb{R}_t} f'(t)(\rho(t) - \rho(t - V(0,y)))\, dy dt. \tag{4.8}$$

The above theorem, enables us to define the spectral shift function $\xi(\cdot,\epsilon) \in \mathcal{D}'(\mathbb{R})$, related to the operators $H(\epsilon)$ and H (see (1.2)). Theorem 4.3 tel us that $\xi(\cdot,\epsilon)$ converges to $\int \rho(t) - \rho(t - V(0,y))dy$ in the sense of distribution. Under a non-trapping condition, the following result gives a pointwise asymptotic expansion in powers of ϵ of $\xi'(\cdot;\epsilon)$.

Theorem 4.4. *Fix* $\lambda > e_0(0)$ *with* $\lambda \notin \{e_1(0), e_2(0), \cdots\}$, *and assume that*

$$k \cdot \nabla e_j(k) - y \cdot \nabla_y V(0,y) \geq c > 0 \text{ in } \{(y,k) \in \mathbb{R}^{2d}; e_j(k) + V(0,y) = \lambda\}. \tag{4.9}$$

There exists $\eta > 0$ *such that the following complete asymptotic expansion holds uniformly on* $t \in]\lambda - \eta, \lambda + \eta[$*:*

$$\xi'(t,\epsilon) \sim \sum_{j=0}^{\infty} \kappa_j(t) \epsilon^{-d+j}, \tag{4.10}$$

with

$$\kappa_0(t) = \int (\rho'(t) - \rho'(t - V(0,y)))dy. \tag{4.11}$$

Comments. Let us briefly examine the above results and their generalizations.

- By (2.3), assumption (4.9) is satisfied under the following condition :

  $$-y\nabla_y V(0,y) \geq 0 \text{ and } -y\nabla_y V(0,y) > 0, \text{ on } \{y \in \mathbb{R}^d; V(0,y) = \lambda - e_j(0)\}.$$

- All results above will remain true if we substitute H by $H_W := H + W(x)$, where W is defined for $x \in \Lambda_d$. In this case, the ρ distribution of the abovementioned results is associated with operator H_W.
- If $W \neq 0$, the properties of the ρ distribution corresponding to the H_W operator will change. Indeed, it depends on the critical point of the eigenvalues $e_j(k)$ corresponding to the operator $H_W(k) = D_x^2 + (k + \mu x)^2 + W(x)$ on $L^2(\Lambda_d)$ (see Appendix A). In particular, the set of ρ singularities is not only the defined threshold Σ, but contains the critical values of the $e_j(k)$ eigenvalues. Note that statement (2.3) is generally not true for $W \neq 0$. Critical value can occur for $\lambda = e_j(k)$ with $\lambda > \inf_k e_j(k)$.
- Let $\mu \in \mathcal{D}'(\mathbb{R})$ be the distribution on $\mathbb{R}$ defined by

  $$\langle \mu, f \rangle = \int \Big[f(V(0,y)) - f(0)\Big] dy, \ f \in C_0^\infty(\mathbb{R}).$$

  As in [10, 12], using Theorem 4.4 and the definition of resonances by the analytic distortion method one prove that near any point $t \in \Sigma + \operatorname{singsupp}_a(\mu)$ there are at least $\mathcal{O}(\epsilon^{-d})$ resonances. Here $\operatorname{singsupp}_a(\mu)$ denotes the analytic singular support of the distribution μ.

5. Effective Hamiltonian

We need some basic result about pseudodifferential operators with operator-valued symbol (see [11] and the references cited therein). We shall consider a family of Hilbert space $\mathcal{A}_X$, $X = \mathbb{R}^{2d}$ satisfying :

$$\mathcal{A}_X = \mathcal{A}_Y, \forall X, Y \in \mathbb{R}^{2d}, \tag{5.1}$$

there exist $N \in \mathbb{N}$ and $C > 0$ such that for all $u \in \mathcal{A}_0$ and all $X, Y \in \mathbb{R}^{2d}$ we have

$$\|u\|_{\mathcal{A}_X} \leq C\langle X - Y\rangle^N \|u\|_{\mathcal{A}_Y}. \tag{5.2}$$

Notice that (5.1) means that only the norm of $\mathcal{A}_X$ depends on X, not on the space itself. Let $\mathcal{B}_X$ be a second family with the same properties. We say that $p \in C^\infty(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_0, \mathcal{B}_0))$ belongs to the symbol class $S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$ if for every $\alpha \in \mathbb{N}^{2d}$ there exists C_α such that

$$\|\partial_X^\alpha p\|_{\mathcal{L}(\mathcal{A}_X, \mathcal{B}_X)} \leq C_\alpha, \ \forall X \in \mathbb{R}^{2d}. \tag{5.3}$$

If p depends on a semi-classical parameter ϵ and possibly on other parameters as well, we require (5.3) to hold uniformly with respect to these parameters. For ϵ-dependent symbols, we say that $p(y, k; \epsilon)$ has an asymptotic expansion in powers of ϵ , and we write

$$p(y, k; \epsilon) \sim \sum_j p_j(y, k)\epsilon^j \text{ in } S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$$

if for every $N \in \mathbb{N}$, $\epsilon^{-N-1}\Big(p(y, k; \epsilon) - \sum_{j=0}^N p_j(y, k)\epsilon^j\Big) \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$.

We can then associate to p an ϵ-pseudo-differential operator

$$p^w(y, \epsilon D_y; \epsilon)u(y) = \iint e^{\frac{i}{\epsilon}(y-t)k} p(\frac{y+t}{2}, k; \epsilon)u(t)\frac{dtdk}{(2\pi\epsilon)^d}, \ u \in \mathcal{A}_0.$$

Here we use the Weyl quantization. Similarly to the scalar case, the following results hold.

Theorem 5.1. *Let $p \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$ where $\mathcal{A}_X, \mathcal{B}_X$ satisfy* (5.1) *and* (5.2) *then $p^w(y, \epsilon D_y, \epsilon)$ is uniformly continuous from $S(\mathbb{R}^d; \mathcal{A}_0)$ into $S(\mathbb{R}^d; \mathcal{B}_0)$.*

Theorem 5.2. *Assume $\mathcal{A}_X = \mathcal{A}_0$ and $\mathcal{B}_X = \mathcal{B}_0$ for all $X \in \mathbb{R}^{2d}$. If $p \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_0, \mathcal{B}_0))$ then $p^w(y, \epsilon D_y; \epsilon)$ is bounded from $L^2(\mathbb{R}^d, \mathcal{A}_0)$ into $L^2(\mathbb{R}^d, \mathcal{B}_0)$.*

Let $\mathcal{C}_X$ be a third Hilbert space which satisfies (5.1), (5.2).

Theorem 5.3. *Let $p \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{B}_X, \mathcal{C}_X))$, $q \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$. Then*

$$p^w(y, \epsilon D_y) \circ q^w(y, \epsilon D_y) = r^w(y, \epsilon D_y; \epsilon),$$

where r is given by

$$r(y, k; \epsilon) \sim \sum_{j=0} \frac{1}{j!}(\frac{i\epsilon}{2}\sigma(D_y, D_k; D_x, D_\xi))^j p(y, k)q(x, \xi)_{|x=y, k=\xi}. \tag{5.4}$$

5.1. Grushin problem: brief description. In this paragraph we recall the basic results about Grushin problem. Let H_1, H_2 and H_3 be three Hilbert spaces, and let $P \in \mathcal{L}(H_1, H_3)$ be self-adjoint. Assume that there exist $R_+ \in \mathcal{L}(H_1, H_2)$ and $R_- \in \mathcal{L}(H_2, H_3)$ such that the following operator

$$\mathcal{P}(z) = \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : H_1 \times H_2 \to H_3 \times H_2$$

is bijective for $z \in \Omega$. Here Ω is an open bounded set in $\mathbb{C}$. Let

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{\text{eff}}(z) \end{pmatrix}$$

be its inverse. We refer to the problem $\mathcal{P}(z)$ as a Grushin problem and the operator $E_{\text{eff}}(z)$ is called effective Hamiltonian. Notice that, an effective Hamiltonian is a Hamiltonian that acts in a reduced space and only describes a part of the eigenvalue spectrum of the true Hamiltonian P. Morally, effective Hamiltonians are much simpler than the true Hamiltonian and hence their eigensystems can often be determined analytically or with little effort numerically.

The following useful properties (relating the operator P and its effective Hamiltonian) are consequences of the identities $\mathcal{E} \circ \mathcal{P} = I$ and $\mathcal{P} \circ \mathcal{E} = I$:

$$(P - z) \text{ is invertible if and only if } E_{\text{eff}}(z) \text{ is invertible,} \tag{5.5}$$

$$\dim\ker(P - z) = \dim\ker(E_{\text{eff}}(z)), \tag{5.6}$$

$$(P - z)^{-1} = E(z) - E_+(z)E_{\text{eff}}^{-1}(z)E_-(z), \tag{5.7}$$

$$E_{\text{eff}}^{-1}(z) = -R_+(P - z)^{-1}R_-. \tag{5.8}$$

The last two equalities hold for all $\Im z \neq 0$. On the other hand, since $z \mapsto (P - z)$ is holomorphic, it follows that the operators $E(z), E_\pm(z)$ and $E_{\text{eff}}(z)$ are also holomorphic in $z \in \Omega$. Moreover, we have

$$\partial_z E_{\text{eff}}(z) = E_-(z)E_+(z). \tag{5.9}$$

This identity comes from the fact that $R_\pm$ are independent of z.

6. Spectral Reduction to an ϵ-pseudodifferential operator

Throughout this section we assume that V is independent on x. The proof of the general case is quite similar with minor modifications (see Remark 7.2). Fix an interval $I = [\alpha, \beta]$, and set

$$\mathbb{U} = \{J \in \mathbb{N}^d; e_J(k) \leq \beta + \|V\|_\infty\}.$$

According to Theorem 2.1 and Corollary 2.2, $e_J(0)$ (respectively $e_J(k)$) tends to infinity as $|J| \to \infty$ (respectively $|k| \to \infty$). Therefore $\mathbb{U}$ is finite. In what follows, $(\Psi_0(\cdot, k), \cdots, \Psi_{N-1}(\cdot, k))$ denotes the family $(\Psi_J(\cdot, k))_{J \in \mathbb{U}}$, where $N = \#\mathbb{U}$.

To shorten notation, we omit the index d in Ω_d and Λ_d. For $k \in \mathbb{R}^d$, let $\mathcal{H}_{\Lambda,k} = \mathcal{H}_\Lambda$ be the

Hilbert space with k-dependent norm: $\|u\|^2_{\Lambda,k} = \|u\|^2_{H^2(\Lambda)} + |k|^4\|u\|^2_{L^2(\Lambda)}$. We denote by $\mathbb{C}^N_k$ the space $\mathbb{C}^N$ equipped with norm $(1+|k|^2)|\cdot|_{\mathbb{C}^N}$.

By the change of variable $y \mapsto y/\epsilon$, the operator $H(\epsilon)$ is unitarily equivalent to

$$H_1 := H_{1,0} + V(y), \tag{6.1}$$

where

$$H_{1,0} := \sum_{j=1}^{d} D^2_{x_j} + (\epsilon D_{y_j} + \mu_j x_j)^2.$$

Let $G(y,k) = H_0(k) + V(y)$ be the linear bounded operator from $\mathcal{H}_\Lambda$ into $L^2(\Lambda)$, where $H_0(k)$ is given by (2.2). Obviously, $G \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{H}_{\Lambda,k}, L^2(\Lambda))$. Thus, by quantizing G we have

$$G(y, \epsilon D_y) = H_1.$$

More precisely, H_1 can be viewed as an ϵ-pseudo-differential operator on y with operator valued symbol $G(y,k)$.

For $k \in \mathbb{R}^d$, and $N \in \mathbb{N}^*$, define $R_+(k) : L^2(\Lambda) \to \mathbb{C}^N$, $R_-(k) = R^*_+(k) : \mathbb{C}^N \to L^2(\Lambda)$ by

$$R_+(k)u = (\langle u, \Psi_0(\cdot,k)\rangle, \cdots, \langle u, \Psi_{N-1}(\cdot,k)\rangle),$$
$$R_-(k)(c_1, \cdots, c_N) = \sum_{j=0}^{N-1} c_j \Psi_j(\cdot,k).$$

According to Corollary 2.2 the family $(\Psi_J(\cdot,k))_{J\in\mathbb{N}^d}$ is an orthonormal basis in $L^2(\Lambda)$. Hence, a simple computation yields

$$R_+(k)R_-(k) = I_{\mathbb{C}^N}, \tag{6.2}$$
$$R_-(k)R_+(k)u = \sum_{j=0}^{N-1} \langle u, \Psi_j(\cdot,k)\rangle \Psi_j(\cdot,k) =: \Pi_N u, \ \ \forall u \in L^2(\Lambda).$$

The following proposition reduces the spectral study of the operator $G(y,k) : \mathcal{H}_{\Lambda,k} \to L^2(\Lambda)$ near the energy z, to the study of an $N \times N$-square matrix $E_{\rm eff}(y,k,z)$.

Proposition 6.1. *Fix a bounded interval I. There exists $N \in \mathbb{N}^*$ such that for all $z \in I$ the operator*

$$\mathcal{P}(y,k) := \begin{pmatrix} G(y,k) - z & R_-(k) \\ R_+(k) & 0 \end{pmatrix} : \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N \to L^2(\Lambda) \times \mathbb{C}^N_k, \tag{6.3}$$

is bijective with bounded two-sided inverse

$$\mathcal{E}(y,k,z) := \begin{pmatrix} \widehat{G_N}(y,k,z) & R_-(k) \\ R_+(k) & E_{\rm eff}(y,k,z) \end{pmatrix}. \tag{6.4}$$

Here $\widehat{G_N}(y,k,z) = (G(y,k) - z)^{-1}(1 - \Pi_N)$ and $E_{\rm eff}(y,k,z)$ is the square diagonal matrix $(z - e_j(k) - V(0,y))\delta_{ij})_{0\le i,j\le N-1}$. Moreover

$$\mathcal{P} \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{H}_{\Lambda,k} \times \mathbb{C}^N; L^2(\Lambda) \times \mathbb{C}^N_k)). \tag{6.5}$$

(6.6) $$\mathcal{E} \in S^0(\mathbb{R}^{2d}; \mathcal{L}(L^2(\Lambda) \times \mathbb{C}^N_k; \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N)).$$

Proof. By construction, we have

$$e_J(k) + V(y) - z \geq c > 0,$$

uniformly for $(z, k, y) \in I \times \mathbb{R}^{2d}$ and $J \notin \mathbb{U}$. Thus, the operator

$$(G(y, k) - z)^{-1}(1 - \Pi_N) : L^2(\Lambda) \to \mathcal{H}_{\Lambda,Y},$$

is well-defined and uniformly bounded on $(z, y, k) \in I \times \mathbb{R}^{2d}$. Using (6.2), an easy computation shows that $\mathcal{P}(y, k) \circ \mathcal{E}(y, k, z) = I$ and $\mathcal{E}(y, k, z) \circ \mathcal{P}(y, k) = I$. On the other hand, it follows from (2.6) that $(y, k) \to R_-(k) \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathbb{C}^N; L^2(\Lambda))$ and $(y, k) \to R_+(k) \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{H}_{\Lambda,k}; \mathbb{C}^N))$. □

Proposition 6.2. *The operator*

(6.7) $$\mathcal{P} := \begin{pmatrix} G(y, \epsilon D_y) - z & R_-(\epsilon D_y) \\ R_+(\epsilon D_y) & 0 \end{pmatrix} : \mathcal{H}^D_\Omega \times H^2(\mathbb{R}^d; \mathbb{C}^N) \to L^2(\Omega) \times L^2(\mathbb{R}^d; \mathbb{C}^N),$$

is bijective with an inverse

$$\mathcal{E}(z; \epsilon) := \mathcal{E}^w(z; \epsilon) = \begin{pmatrix} E^w(y, \epsilon D_y, z; \epsilon) & E^w_+(y, \epsilon D_y, z; \epsilon) \\ E^w_-(y, \epsilon D_y, z; \epsilon) & E^w_{\text{eff}}(y, \epsilon D_y, z; \epsilon) \end{pmatrix},$$

uniformly bounded with respect to $z \in I$ and ϵ small enough. Moreover, $\mathcal{E}(z; \epsilon)$ depend holomorphically on z, and $\mathcal{E}(y, k, z; \epsilon)$ has an asymptotic expansion in $S^0(\mathbb{R}^{2d}; \mathcal{L}(L^2(\Lambda) \times \mathbb{C}^N_k; \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N))$, i.e.,

(6.8) $$\mathcal{E}(y, k, z; \epsilon) = \begin{pmatrix} E(y, k, z; \epsilon) & E_+(y, k, z; \epsilon) \\ E_-(y, k, z; \epsilon) & E_{\text{eff}}(y, k, z; \epsilon) \end{pmatrix} \sim \sum_{j=0}^{\infty} \mathcal{E}_j(y, k, z)\epsilon^j.$$

In particular $E_{\text{eff}}(y, k, z; \epsilon) \sim \sum_{j=0}^{\infty} E_{\text{eff},j}(y, k, z)\epsilon^j$ in $S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathbb{C}^N_k; \mathbb{C}^N))$. The leading terms $\mathcal{E}_0(y, k, z)$ and $E_{\text{eff},0}(y, k, z)$ are given by Proposition 6.1, i.e.,

$$\mathcal{E}_0(y, k, z) = \mathcal{E}(y, k, z; 0) \text{ and } E_{\text{eff},0}(y, k, z) = E_{\text{eff}}(y, k, z; 0).$$

Proof. The fact that $\mathcal{P}$ can be viewed as an ϵ-pseudodifferential operator valued symbol $\mathcal{P}(y, k)$ and Theorem 5.3 show that

(6.9) $$\mathcal{P}^w(y, \epsilon D_y) \circ \mathcal{E}^w(y, \epsilon D_y, z) = I + \epsilon \mathcal{R}^w(y, \epsilon D_y, z; \epsilon),$$

where $\mathcal{R}(y, k, z; \epsilon) \sim \sum_{j=0}^{\infty} \mathcal{R}_j(y, k, z)\epsilon^j$ in $S^0(\mathbb{R}^{2d}; \mathcal{L}(L^2(\Lambda) \times \mathbb{C}^N; L^2(\Lambda) \times \mathbb{C}^N))$. It follows from Theorem 5.2 that $\mathcal{R}^w(y, \epsilon D_y, z; \epsilon)$ is uniformly bounded for $z \in I$ and $|\epsilon| \leq 1$. Thus, for ϵ small enough the right hand side of (6.9) is invertible. On the other hand we know that if $P = p^w(y, k, \epsilon)$ is an invertible ϵ-pseudo-differential with $p(y, k; \epsilon) \sim \sum_{j=0}^{\infty} p_j(y, k)\epsilon^j$ then its inverse q^w is also an ϵ-pseudodifferential operator with $q(y, k; \epsilon) \sim \sum_{j=0}^{\infty} q_j(y, k)\epsilon^j$. Consequently, $\mathcal{E}^w(y, \epsilon D_y, z; \epsilon) := \mathcal{E}^w(y, \epsilon D_y, z) \circ (I + \epsilon \mathcal{R}^w(y, \epsilon D_y, z; \epsilon))^{-1}$ satisfies all the desired properties. □

REMARK 6.3. Let $\mathcal{E}_0(z)$ be the operator given by Proposition 6.2 corresponding to the non-perturbed operator H_0 (i.e., $V = 0$). Since $\mathcal{P}(y, k) = \mathcal{P}(k)$ is y-independent, we have

$$\mathcal{E}_0(z) = \begin{pmatrix} \widehat{G}_N(\epsilon D_y, z) & E_+^0(\epsilon D_y) \\ E_-^0(\epsilon D_y) & E_{\mathrm{eff}}^0(\epsilon D_y, z) \end{pmatrix},$$

where $E_+^0(k) = R_-(k), E_-^0(k) = R_+(k)$ and $E_{\mathrm{eff}}^0(k,z) = ((z - e_j(k))\delta_{ij})_{0\le i,j\le N-1}$

7. Proof of the main results

7.1. Proof of Theorem 4.1. In the following we fix a bounded interval I containing supp(f), and we apply Proposition 6.1 and Proposition 6.2 on I. For the simplicity of the notation we ignore the dependence of $E, E_+, E_-, E_{\mathrm{eff}}$ on (y, k, z, ϵ). We denote by E^0, E_+^0, E_-^0, E_{eff}^0 the operators given by Proposition 6.2 corresponding to the case $V = 0$ (see Remark 6.3). We shall sometimes use the same symbol for an ϵ-pseudodifferential operator and for its Weyl symbol.

Applying formulas (5.7) and (5.8) to Proposition 6.2 we obtain

$$(H_1 - z)^{-1} = E - E_+ E_{\mathrm{eff}}^{-1} E_-, \tag{7.1}$$

$$\partial_z E_{\mathrm{eff}} = E_- E_+. \tag{7.2}$$

Assume that $f \in C_0^\infty(\mathbb{R})$ is real-valued, we can construct an almost analytic extension $\tilde{f} \in C_0^\infty(\mathbb{C})$ of f satisfying the following properties (see [11]) :

$$\tilde{f}(z) = f(z), \forall z \in \mathbb{R}, \tag{7.3}$$

for all $N \in \mathbb{N}$ there exists C_N such that

$$|\frac{\partial \tilde{f}}{\partial \overline{z}}(z)| \le C_N |\Im z|^N. \tag{7.4}$$

Let H be any self-adjoint operator, the Dynkin-Helffer-Sjöstrand formula reads [11]:

$$f(H) = -\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \overline{z}}(z)(z - H)^{-1} L(dz), \text{ with } z = x + iy, \tag{7.5}$$

which yields

$$f(H_1) = -\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \overline{z}}(z)(z - H_1)^{-1} L(dz). \tag{7.6}$$

Here $L(dz)$ is the Lebesgue measure on the complex plane $\mathbb{C} \sim \mathbb{R}^2_{x,y}$.

Inserting (7.1) in the right hand side of (7.6) and using the fact that $z \to E^w(y, \epsilon D_y, z; \epsilon)$ is holomorphic, we get

$$f(H_1) = -\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \overline{z}}(z) E_+ E_{\mathrm{eff}}^{-1} E_- L(dz). \tag{7.7}$$

Here and in what follows we use the fact that $\int \overline{\partial}_z \tilde{f}(z) K(z) L(dz) = 0$ provided that $K(z)$ is holomorphic in a neighborhood of supp($\tilde{f}$). We recall that the principal symbol of E_{eff} is given by

$$E_{\mathrm{eff},0}(y, k, z) = ((z - V(y) - e_j(k))\delta_{i,j})_{0\le i,j\le N-1},$$

and that $e_j(k) \sim |k|^2$ at infinity from (2.5) in Theorem 2.1. For $j = 0, \cdots, N-1$, let $\tilde{e}_j(k)$ be a smooth function such that $\tilde{e}_j(k) = e_j(k)$ for $|k|$ large enough and

$$|z - V(y) - \tilde{e}_j(k)| \geq c_0(1 + |k|^2), \ \ \forall\ (z, y, k) \in \operatorname{supp}\tilde{f} \times \mathbb{R}^d \times \mathbb{R}^d. \tag{7.8}$$

Put

$$\tilde{E}_{\text{eff}}(y, k, z; \epsilon) = E_{\text{eff}}(y, k, z; \epsilon) + \tilde{E}_{\text{eff}}(y, k, z) - E_{\text{eff}}(y, k, z),$$

where $\tilde{E}_{\text{eff}}(y, k, z) = ((z - V(y) - \tilde{e}_j(k))\delta_{i,j})_{0\leq i,j\leq N-1}$. We conclude from (7.8) that $\tilde{E}_{\text{eff}}(y, k, z; \epsilon)$ is elliptic for ϵ small enough, hence that $\tilde{E}_{\text{eff}} := \tilde{E}^w_{\text{eff}}(y, \epsilon D_y, z; \epsilon)$ is invertible and holomorphic for $z \in \operatorname{supp}(\tilde{f})$, and finally that

$$\int \frac{\partial \tilde{f}}{\partial \bar{z}}(z) E_+ \tilde{E}^{-1}_{\text{eff}} E_- L(dz) = 0.$$

Combining the above equality with (7.7), we obtain

$$f(H_1) = -\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \bar{z}}(z) E_+ (E^{-1}_{\text{eff}} - \tilde{E}^{-1}_{\text{eff}}) E_- L(dz). \tag{7.9}$$

Let Ψ be as in Theorem 4.1. Writing $E^{-1}_{\text{eff}} - \tilde{E}^{-1}_{\text{eff}} = \tilde{E}^{-1}_{\text{eff}}(\tilde{E}_{\text{eff}} - E_{\text{eff}})E^{-1}_{\text{eff}}$ and using the fact that $\tilde{E}_{\text{eff}} - E_{\text{eff}} = ((e_j(k) - \tilde{e}_j(k))_{1\leq i,j\leq N}$ has a compact support, we deduce that the operator $\Psi\Big(E_+ \tilde{E}^{-1}_{\text{eff}}(\tilde{E}_{\text{eff}} - E_{\text{eff}})E^{-1}_{\text{eff}} E_-\Big)$ is trace class. Thus, by using the cyclicity of the trace we get

$$\begin{aligned} \operatorname{tr}\Big(\Psi f(H_1)\Big) &= -\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \bar{z}}(z) \operatorname{tr}\Big(E^{-1}_{\text{eff}} - \tilde{E}^{-1}_{\text{eff}}) E_- \Psi E_+\Big) L(dz) \\ &= \operatorname{tr}\Big(-\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \bar{z}}(z) E^{-1}_{\text{eff}} E_- \Psi E_+ L(dz)\Big). \end{aligned} \tag{7.10}$$

In the last equality we have used the fact the operator $\tilde{E}^{-1}_{\text{eff}} E_- \Psi E_+$ is holomorphic on $z \in \operatorname{supp}(\tilde{f})$.

According to Proposition 6.2 and Theorem 5.3 the operator $A = E_- \Psi E_+$ is an ϵ-pseudodifferential operator on $L^2(\mathbb{R}^d; \mathbb{C}^N)$ with $A = A^w(y, \epsilon D_y, z; \epsilon)$ where $A(y, k, z; \epsilon) \sim \sum_{j=0}^{\infty} A_j(y, k, z)\epsilon^j$ in $S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathbb{C}^N; \mathbb{C}^N))$. Moreover, from Proposition 6.1 we have $A_0(y, k, z) = \Psi(y)$.

The proof of the following lemma is similar to the one in [7].

Lemma 7.1. *Fix $\delta \in]0, 1/2[$. There exists $r \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N))$ such that $r(y, k; \epsilon) \sim \sum_{j=0}^{\infty} r_j(y, k)\epsilon^j$ and*

$$r^w(y, \epsilon D_y; \epsilon) = -\frac{1}{\pi} \int_{|\Im z|\geq \epsilon^\delta} \frac{\partial \tilde{f}}{\partial \bar{z}}(z) E^{-1}_{\text{eff}} E_- \Psi E_+ L(dz),$$

with

$$r_0(y, k) = -\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \bar{z}}(z) \Big(\Big(z - e_j(k) - V(y)\Big)^{-1} \delta_{i,j}\Big)_{0\leq i,j\leq N-1} L(dz)\Psi(y).$$

We now turn to the proof of Theorem 4.1. If we restrict the integral in the right hand side of (7.10) to the domain $|\Im z| \leq \epsilon^\delta$ then we get a term $\mathcal{O}(\epsilon^\infty)$ in trace norm. Here we have used the fact that $|\frac{\partial \tilde{f}}{\partial \bar{z}}(z)| = \mathcal{O}(|\Im z|^M)$ for all $M \in \mathbb{N}$ (see (7.4)). If we restrict our attention to the domain $|\Im z| \geq \epsilon^\delta$ then by Lemma 7.1 we get a complete asymptotic expansion in powers of ϵ, which yields (4.3). To finish the proof let us compute a_0. We have

$$a_0 = \iint \widehat{\mathrm{tr}}(r_0(y,k))\frac{dydk}{(2\pi)^d} = \sum_{j=0}^{N-1}\iint\left(-\frac{1}{\pi}\int\frac{\partial\tilde f}{\partial\overline{z}}(z)(z-e_j(k)-V(y))^{-1}L(dz)\right)\Psi(y)\frac{dydk}{(2\pi)^d}.$$

Here $\widehat{\mathrm{tr}}$ stands for the trace of square matrices. Since $\frac{1}{\pi}\overline{\partial}_z\frac{1}{z-z_0} = \delta(\cdot - z_0)$, it follows that $-\frac{1}{\pi}\int\frac{\partial\tilde f}{\partial\overline{z}}(z)(z-e_j(k)-V(y))^{-1}L(dz) = f(e_j(k)+V(y))$. Consequently,

$$a_0 = \sum_{j=0}^{N-1}\iint f(e_j(k)+V(y))\Psi(y)\frac{dydk}{(2\pi)^d} = \sum_j\iint f(e_j(k)+V(y))\Psi(y)\frac{dydk}{(2\pi)^d}.$$

In the above equality we have used the fact that $e_j(k)+V(y)\notin \mathrm{supp}(f)$ for $(y,k)\in\mathbb{R}^d\times\mathbb{R}^d$ and $j\notin\{0,\cdots,N-1\}$. Combining this with the obvious equality

$$\sum_j\int f(e_j(k)+V(y))\frac{dk}{(2\pi)^d} = -\sum_j\int f'(t)\int_{e_j(k)\le t-V(y)}dkdt = -\int f'(t)\rho(t-V(y))dt,$$

we get (4.3).

7.2. Proof of Corollary 4.2. Let f be as in Corollary 4.2, and fix $\eta>0$ small enough such that $\mathrm{supp}(f)\subset]-\infty, e_0(0)-\eta]$. Put $\omega_\eta := \{y\in\mathbb{R}^d; \exists(j,k)\in\mathbb{N}\times\mathbb{R}^d \text{ s.t. } e_j(k)+V(y)\in\mathrm{supp}(f)\}$. Since V tends to zero at infinity and $e_j(k)\ge e_j(0)$ for all j,k, it follows that ω_η is a compact set.

Let $\tilde V$ be a smooth function such that $\tilde V(y)\in[-\eta/2,\eta/2]$ for all $y\in\mathbb{R}^d$ and $\tilde V(y)=V(y)$ for $|y|$ large enough. Put

$$\tilde E_{\mathrm{eff}}(y,k,z;\epsilon) = E_{\mathrm{eff}}(y,k,z;\epsilon) + (\tilde V(y)-V(y))I_N.$$

By construction of $\tilde V$, we have

$$|z-e_j(k)-\tilde V(y)|\ge C(1+|k|^2),$$

uniformly on $(j,y,k)\in\mathbb{N}\times\mathbb{R}^{2d}$ and z in small complex neighborhood of $\mathrm{supp}(\tilde f)$.

Hence, the principal symbol $\tilde E_{\mathrm{eff}}(y,k,z) = ((z-\tilde V(y)-e_j(k))\delta_{i,j})_{0\le i,j\le N-1}$ of $\tilde E_{\mathrm{eff}}$ is elliptic. We can now proceed analogously to the proof of (7.9), and obtain

$$f(H_1) = -\frac{1}{\pi}\int\frac{\partial\tilde f}{\partial\overline{z}}(z)E_+(E_{\mathrm{eff}}^{-1}-\tilde E_{\mathrm{eff}}^{-1})E_-L(dz). \tag{7.11}$$

Let $\psi\in C_0^\infty(\mathbb{R}^d)$ be equal to one in a neighborhood of $\mathrm{supp}(\tilde V - V = \tilde E_{\mathrm{eff}}-E_{\mathrm{eff}})$. Writing $E_+(E_{\mathrm{eff}}^{-1}-\tilde E_{\mathrm{eff}}^{-1})E_- = E_+\tilde E_{\mathrm{eff}}^{-1}(\tilde E_{\mathrm{eff}}-E_{\mathrm{eff}})E_{\mathrm{eff}}^{-1}E_-$ and using the fact that $\mathrm{supp}(1-\psi)\cap\mathrm{supp}(\tilde V - V)=\emptyset$, we deduce from (7.11) and (5.4) that $\|(1-\psi)f(H_1)\|_{\mathrm{tr}} = \mathcal{O}(\epsilon^\infty)$. Consequently,

$$\mathrm{tr}(f(H_1)) = \mathrm{tr}(\psi f(H_1)) + \mathcal{O}(\epsilon^\infty), \tag{7.12}$$

which together with Theorem 4.1 yields (4.4) and (4.5). Notice that the right hand side of (7.12) is independent modulo $\mathcal{O}(\epsilon^\infty)$ of the choice of ψ, since $\psi=1$ near the characteristic set Σ_η of E_{eff}.

It remains to prove (4.6). For every small $\eta>0$, choose $\overline{f_\eta}, \underline{f_\eta}\in C_0^\infty(\mathbb{R};[0,1])$ with

$$1_{[a+\eta,b-\eta]}\le\underline{f_\eta}\le 1_{[a,b]}\le\overline{f_\eta}\le 1_{[a-\eta,b+\eta]}.$$

It then suffices to observe that

$$\operatorname{tr}\left[\underline{f_\eta}(H(\epsilon))\right] \le N([a,b];\epsilon) \le \operatorname{tr}\left[\overline{f_\eta}(H(\epsilon))\right],$$

which yields

$$\lim_{\eta\searrow 0}\lim_{\epsilon\searrow 0}\left((2\pi\epsilon)^d \operatorname{tr}\left[\underline{f_\eta}(H(\epsilon))\right]\right) \le \lim_{\epsilon\searrow 0}(2\pi\epsilon)^d N([a,b];\epsilon) \le \lim_{\eta\searrow 0}\lim_{\epsilon\searrow 0}\left((2\pi\epsilon)^d \operatorname{tr}\left[\overline{f_\eta}(H(\epsilon))\right]\right),$$

and to apply Theorem 4.1.

7.3. Proof of Theorem 4.3. We only mention the steps in the proof of Theorem 4.3 which are the same as in the proof of Theorem 4.1. Fix $z_0 < \inf(\sigma(H_j))$ $(j = 0,1)$, and let $m > d/2 + 1$. From the assumption (4.1) the operator $(H_1 - z_0)^{-m} - (H_0 - z_0)^{-m}$ is trace class. Therefore, $f(H_1) - f(H_0)$ is trace class for all $f \in C_0^\infty(\mathbb{R})$. In contrast to the proof of Theorem 4.1, we don't need to introduce the function Ψ, since $f(H_1) - f(H_0)$ is trace class.

As in the proof of (7.7), Proposition 6.2 and Remark 6.3 yield

$$f(H_0) = -\frac{1}{\pi}\int \frac{\partial \tilde f}{\partial \bar z}(z) E_+^0 (E_{\text{eff}}^0)^{-1} E_-^0 L(dz),$$

which together with (7.7) gives

$$\operatorname{tr}(f(H_1) - f(H_0)) = \operatorname{tr}\left(-\frac{1}{\pi}\int \frac{\partial \tilde f}{\partial \bar z}(z)\left[E_+^1 {E_{\text{eff}}}^{-1} E_-^1 - E_+^0 (E_{\text{eff}}^0)^{-1} E_-^0\right] L(dz)\right). \tag{7.13}$$

Next, analysis similar to that in the proof of (7.10) shows that

$$\operatorname{tr}(f(H_1) - f(H_0)) = \operatorname{tr}\left(-\frac{1}{\pi}\int \frac{\partial \tilde f}{\partial \bar z}(z)\left[{E_{\text{eff}}}^{-1} E_-^1 E_+^1 - (E_{\text{eff}}^0)^{-1} E_-^0 E_+^0\right] L(dz)\right). \tag{7.14}$$

According to (5.9), Proposition 6.2 and Remark 6.3, we have

$$\partial_z E_{\text{eff}} = E_-^1 E_+^1, \quad \partial_z E_{\text{eff}}^0 = E_-^0 E_+^0.$$

Combining this with (7.14), we obtain

$$\operatorname{tr}(f(H_1) - f(H_0)) = \operatorname{tr}\left(-\frac{1}{\pi}\int \frac{\partial \tilde f}{\partial \bar z}(z)\left[{E_{\text{eff}}}^{-1}\partial_z E_{\text{eff}} - (E_{\text{eff}}^0)^{-1}\partial_z E_{\text{eff}}^0\right] L(dz)\right). \tag{7.15}$$

We now apply the same arguments after Lemma 7.1, with (7.10) replaced by (7.15), to obtain Theorem 4.3.

7.4. Proof of Theorem 4.4. The starting point is formula (7.15). Let θ and g be C^∞-functions with compact support such that $\theta = 1$ near zero, $g = 1$ on $]\lambda - \eta, \lambda + \eta[$ and $\operatorname{supp}(g) \subset]\lambda - 2\eta, \lambda + 2\eta[$. We choose $\eta > 0$ small enough so that (4.9) holds on $]\lambda - 2\eta, \lambda + 2\eta[$. Applying (1.2) and (7.15) to the function $f(x) = g(x)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - x)$, we obtain

$$\begin{aligned}
&- \langle \xi'(\cdot;\epsilon), g(\cdot)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - \cdot)\rangle = \operatorname{tr}\left(g(H_1)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - H_1) - g(H_0)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - H_0)\right) \\
&\quad = \operatorname{tr}\left(-\frac{1}{\pi}\int \frac{\partial \tilde g}{\partial \bar z}(z)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - z)\left[{E_{\text{eff}}}^{-1}\partial_z E_{\text{eff}} - (E_{\text{eff}}^0)^{-1}\partial_z E_{\text{eff}}^0\right] L(dz)\right).
\end{aligned} \tag{7.16}$$

Here $\tilde g$ is an almost analytic extension of g, and $\mathcal{F}_\epsilon^{-1}$ is the semi-classical Fourier transform of θ :

$$(\mathcal{F}_\epsilon^{-1}\theta)(\tau) = \frac{1}{(2\pi\epsilon)} \int_{\mathbb{R}} e^{\frac{i}{\epsilon}t\tau}\theta(t)dt.$$

A symbol $(y, k) \to A(y, k, z) \in \mathcal{L}(\mathbb{C}^N; \mathbb{C}^N)$ is non-trapping at the energy $z = z_0$ if and only if there exists a scalar escape function $G \in C^\infty(\mathbb{R}^{2d}; \mathbb{R})$ such that

$$\exists C > 0, \quad \frac{\partial G}{\partial y} \cdot \frac{\partial A}{\partial k} - \frac{\partial G}{\partial k} \cdot \frac{\partial A}{\partial y} \geq C, \quad \forall (y, k) \text{ with } \det A(y, k, z_0) = 0.$$

According to Proposition 6.2, E_{eff} is an ϵ-pseudodifferential operator. On the other hand, the assumption (4.9) means that the classical symbol corresponding to E_{eff} is non-trapping. The asymptotic expansion with respect to ϵ of an integral similar to the right-hand side of the second equality in (7.16) have been studied by many authors (see [1, 9, 13, 11, 25] and the references given therein). In particular, under the assumption (4.9), it follows from the arguments in the proofs of Theorems 2.5 and 2.6 in [9] (see also [1]) that the left-hand side of (7.16) has a complete asymptotic expansion in powers of ϵ, and

$$\xi'(\tau, \epsilon)g(\tau) = \langle \xi'(\cdot; \epsilon), g(\cdot)(\mathcal{F}_\epsilon^{-1}\theta)(\tau - \cdot)\rangle + \mathcal{O}(\epsilon^\infty),$$

uniformly for $\tau \in]\lambda - 2\eta, \lambda + 2\eta[$. This implies (4.10). The explicit formula of $\kappa_0(t)$ follows from (4.8).

REMARK 7.2. We will now show how to treat the case when V depends on x. The only modification to be made is the proof of Proposition 6.1. Fix $m \in \mathbb{N}^*$. By Taylor's formula we have

$$V(\epsilon x, y) = V(0, y) + \sum_{|\alpha|=1}^{m} \frac{\epsilon^{|\alpha|}}{\alpha!} x^\alpha \frac{\partial^\alpha}{\partial x^\alpha} V(0, y) + \epsilon^{m+1}\mathcal{O}(1) =: V(0, y) + \epsilon W(x, y; \epsilon), \tag{7.17}$$

uniformly for $(x, y) \in \Omega_d$. Let $\mathcal{P}(y, k)$ and $\mathcal{E}(y, k, z)$ be the operators given in Proposition 6.1 corresponding to the operator $V(y) = V(0, y)$. Now, consider the Grushin problem related to $\tilde{G}(y, k, \epsilon) = G(y, k) + \epsilon W(x, y, \epsilon)$:

$$\tilde{\mathcal{P}}(y, k, , \epsilon) = \begin{pmatrix} \tilde{G}(y, k, \epsilon) - z & R_-(k) \\ R_+(k) & 0 \end{pmatrix} = \mathcal{P}(y, k) + \epsilon \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N \to L^2(\Lambda) \times \mathbb{C}_k^N.$$

Since $W(\cdot, y, \epsilon) : \mathcal{H}_{\Lambda,k} \to L^2(\Lambda)$ is uniformly bounded with respect to $y \in \mathbb{R}^d$ and $\epsilon \in [0, 1]$, it follows from Proposition 6.1 that, for ϵ small enough the operator $\tilde{\mathcal{P}}(y, k, , \epsilon)$ is bijective with bounded two-sided inverse

$$\tilde{\mathcal{E}}(y, k, z; \epsilon) := \begin{pmatrix} \widehat{G_N}(y, k, z; \epsilon) & E_+(k, z, \epsilon) \\ E_-(k, z, \epsilon) & E_{\text{eff}}(y, k, z; \epsilon) \end{pmatrix} = \left(I + \epsilon\mathcal{E}(y, k, z) \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}\right)^{-1} \mathcal{E}(y, k, z). \tag{7.18}$$

From (7.17) and the above equality it follows that, modulo $\mathcal{O}(\epsilon^{m+1})$, $\tilde{\mathcal{E}}(y, k, z; \epsilon)$ has an asymptotic expansion in powers of ϵ in $S^0(\mathbb{R}^{2d}; \mathcal{L}(L^2(\Lambda) \times \mathbb{C}_k^N; \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N))$. This gives Proposition 6.1 when V depends on (x, y).

We can now proceed analogously to the proof of the case $V = V(y)$.

Appendix A Proof of Theorem 2.3

Fix $J = (j_1, j_2, \cdots, j_d) \in \mathbb{N}^d$, and let $e_J(k) = e_{j_1}(k_1) + \cdots + e_{j_d}(k_d)$ be one eigenvalue of the operator $H_0(k)$. Set

$$\kappa(t) = \int_{\{k\in\mathbb{R}^d; e_J(k)\leq t\}} dk.$$

Lemma A.1. *The function κ is analytic in a neighborhood of $\mathbb{R} \setminus \{e_J(0)\}$.*

Proof. Fix $t_0 \neq e_J(0)$, and let ε be a small positive constant such that $\nabla e_J(k) \neq 0$ when $k \in \Sigma_\epsilon(t_0) := e_J^{-1}(]t_0 - \varepsilon, t_0 + \varepsilon[)$. Without any loss of generality we may assume that $\partial_{k_1} e_J(k) \neq 0$ for all $k \in \Sigma_\epsilon(t_0)$. By the change of variable $U : k \mapsto \tilde{k} = (e_J(k), k_2, ..., k_d)$, we have

$$\int_{\{k\in\Sigma_\epsilon(t_0)\,;\, e_J(k)\leq t\}} dk = \int_{\{\tilde{k}\in U(\Sigma_\epsilon(t_0))\,;\, \tilde{k}_1\leq t\}} \mathrm{Jac}(U^{-1}(\tilde{k}))d\tilde{k}.$$

Clearly the right-hand side of the above equality is analytic. Combining this with the fact that $\int_{\{k\in\mathbb{R}^d\setminus\Sigma_\epsilon(t_0)\,;\, e_J(k)\leq t\}} dk$ is constant for t near t_0, we get the lemma. □

Thus, the function ρ is analytic in a neighborhood of $\Sigma = \mathbb{R} \setminus \sigma(H_0(0))$. The remainder of the proof of Theorem 2.3 is a simple consequence of the following lemma.

Lemma A.2. *There exists an analytic function g with $g(s) \sim_{s\to 0} \frac{\mathrm{vol}(S^{d-1})}{d\sqrt{\det(\frac{\nabla^2 e_J(0)}{2})}} s^d$ such that*

$$\kappa(t) = Y(t - e_J(0))g(\sqrt{t - e_J(0)}),$$

for $|t - e_J(0)|$ small enough. Here $Y(t)$ is the Heaviside function, and S^{d-1} stands for the unit sphere in $\mathbb{R}^d$.

Proof. By Morse Lemma there exist a neighborhood $\mathcal{V}$ of $k = 0$, $\varepsilon > 0$ and a local analytic diffeomorphism $\mathcal{D} : \mathcal{V} \to B(0, \epsilon)$ satisfying $\mathcal{D}(k) = k + \mathcal{O}(k^2)$ such that

$$e_J \circ \mathcal{D}^{-1}(k) = e_J(0) + \frac{1}{2}\langle\nabla^2 e_J(0)k, k\rangle.$$

On the other hand, for $|t - e_J(0)|$ small enough we have

$$\{k \in \mathbb{R}^d; e_J(k) \leq t\} = \{k \in \mathcal{V}\,;\, e_J(k) \leq t\}.$$

Thus making the change of variable $k = \mathcal{D}^{-1}(\xi)$ and using polar coordinates, we obtain

$$\begin{aligned}\kappa(t) = \int_{\{k\in\mathcal{V}\,;\, e_J(k)\leq t\}} dk &= \left(\det\left(\frac{\nabla^2 e_J(0)}{2}\right)\right)^{-1/2} \int_{\{\xi\in B(0,\epsilon)\,;\, |\xi|^2\leq t-e_J(0)\}} \mathrm{Jac}(\mathcal{D}^{-1}(\xi))d\xi \\ &= \left(\det\left(\frac{\nabla^2 e_J(0)}{2}\right)\right)^{-1/2} \int_0^{\sqrt{\max(t-e_J(0),0)}} \int_{S^{d-1}} \mathrm{Jac}(\mathcal{D}^{-1}(r\omega))\, r^{d-1} dr d\omega,\end{aligned}$$

which yields the lemma since $\mathrm{Jac}(\mathcal{D}^{-1}(r\omega)) = 1 + \mathcal{O}(r)$. □

We now turn to the proof of Theorem 2.3. For $t_0 \in \Sigma$, we let $\mathcal{S}_{t_0} := \{J \in \mathbb{N}^d; e_J(0) = t_0\}$ and $m_{t_0} := \#\mathcal{S}_{t_0}$ be its multiplicity. Writing

$$\rho(t) = \underbrace{\sum_{(j_1,\cdots,j_d)\notin\Sigma_{t_0}} \int_{\{k\in\mathbb{R}^d; e_{j_1}(k_1)+\cdots+e_{j_d}(k_d)\le t\}} dk}_{(1)} + \underbrace{\sum_{(j_1,\cdots,j_d)\in\Sigma_{t_0}} \int_{\{k\in\mathbb{R}^d; e_{j_1}(k_1)+\cdots+e_{j_d}(k_d)\le e\}} dk}_{(2)}.$$

It follows from Theorem 2.1 that $\nabla_k e_J(k) = \nabla_k(e_{j_1}(k_1) + \cdots + e_{j_d}(k_d)) \neq 0$ on $\Sigma_\eta(t_0)$ for η small enough and $(j_1, \cdots, j_d) \notin \mathcal{S}_{t_0}$. Combining this with Lemma, we deduce that (1) is analytic for $|t - t_0|$ small enough. Thus applying Lemma A.2 to each term of (2) we get Theorem 2.3.

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