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GETZLER'S SYMBOL CALCULUS AND THE COMPOSITION OF DIFFERENTIAL OPERATORS ON CONTACT RIEMANNIAN MANIFOLDS

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Abstract

Following Getzler's idea from the geometric viewpoint as to symbol calculus on a spin manifold, we introduce a new symbol calculus of H-pseudodifferential operators on a contact Riemannian manifold with contact distribution H. An explicit formula for the top grading part of the symbol of the composite of H-differential operators is presented.

0. Introduction

On a spin manifold, Getzler [7] introduced a new symbol calculus of pseudodifferential operators by unifying two kinds of ideas: that of Widom ([14], [15]) about symbol calculus on Riemannian manifold and that of Alvarez-Gaumé ([1]) who used the Clifford variables to propose a suitable filtration of symbol space. Getzler's symbol calculus simplifies the calculation of the principal part, or the top grading part (cf. [4], (2.10)), of the composite of symbols ([7, Theorems 2.7 and 3.5]) and consequently provides a remarkably short proof of the Atiyah-Singer index theorem for the Dirac operator ([7, §3], [8], [1]).

In this paper, on a contact Riemannian manifold with contact distribution H, following Getzler's idea we will introduce a similar symbol calculus of H-pseudodifferential operators, which will turn out to be an effective tool for understanding the contact Riemannian structure from the viewpoint of calculus. The manifold possesses a canonical Spin^c structure, the Clifford variables associated with which provide similarly a filtration of symbol space, so that Getzler's idea can be applicable. The main result in this paper is Theorem 3.5, which expressly offers an explicit formula for the top grading part of the composite of polynomial symbols, that is, the symbols of H-differential operators. In the spin manifold case its counterpart is [7, Theorem 2.7], which was certified by using the Campbell-Hausdorff formula. To prove Theorem 3.5, we will employ not the CH formula but the formula (1.1), which gives an explicit expression of the connection coefficients of the hermitian Tanno connection. The CH formula is so daunting that Benameur-Heitsch [4], who applied Getzler's idea to the case of foliated spin manifold, used Atiyah-Bott-Patodi's formula [2, Proposition 3.7] for the proof of [4, Theorem 4.6] which is a foliation version of [7, Theorem 2.7]. Their idea certainly led us to the application of (1.1), but the proof itself of the formula (3.10), which is an essential part of Theorem 3.5, does not follow their strategy in [4, §4]. Our approach

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in this paper is quite straightforward and may be applied also to Getzler's case ([7]) and Benameur-Heitsch's case ([4]).

We want to comment briefly on an extension of Theorem 3.5 to general symbols. In the spin manifold case, a composition formula for general symbols was derived almost automatically from the one for polynomial symbols and Widom's formula ([14], [15]). It will be then natural to expect that, in the contact Riemannian manifold case, so can be a composition formula for general symbols from Theorem 3.5 and Beals-Greiner's formula ([3]). But the situation is not so simple and it seems to be difficult at present to extend Theorem 3.5 directly to the case of general symbol. In §4, using Beals-Greiner's formula we give another proof of Theorem 3.5, the method of which will be natural but may not work in the case of general symbol. The study of this paper started with a desire to investigate the Toeplitz operator (e.g. [9]), which is a pseudodifferential operator of degree 0, from the viewpoint of Getzler's symbol calculus, and even more effort toward the case of general symbols will be needed.

1. Preliminaries: contact Riemannian manifold and the canonical Spin^c structure

Let $M = (M, e^0, e_0, J, g)$ be a $(2n+1)$-dimensional contact Riemannian manifold. Here e^0 is a contact 1-form and e_0 is the unique vector field satisfying $e_0 \rfloor e^0 := e^0(e_0) = 1$, $e_0 \rfloor de^0 := de^0(e_0, \ ) = 0$, and (J, g) is a pair of $(1,1)$-tensor field and Riemannian metric satisfying $g(e_0, X) = e^0(X)$, $g(X, JY) = -de^0(X,Y) := -X(e^0(Y)) - Y(e^0(X)) + e^0([X,Y])$ and $J^2 X = -X + e^0(X)e_0$. Referring to [10], [11] and [12], we will briefly review some basic properties of the hermitian Tanno connection and the canonical Spin^c structure on M, which are tools crucial for our study.

We set $H = \ker e^0$, $H_\pm = \{X \in \mathbb{C}H \mid JX = \pm\sqrt{-1}X\}$ ($\mathbb{C}H := H \otimes \mathbb{C}$). Without the assumption that J is integrable (i.e., $[\Gamma(H_+), \Gamma(H_+)] \subset \Gamma(H_+)$), we will equip M with the hermitian Tanno connection ${}^\sharp\nabla$ ([10]), which is known to be appropriate for the study of such a manifold and is characterized by the following conditions:

$$
{}^\sharp\nabla e^0 = 0, \quad {}^\sharp\nabla g = 0, \quad {}^\sharp\nabla J = 0,
$$
$$
\pi_+ T({}^\sharp\nabla)(Z, W) = 0 \quad (Z \in H_+,\ W \in \mathbb{C}TM),
$$

where $T({}^\sharp\nabla)$ is the torsion tensor and $\pi_+ : \mathbb{C}TM = \mathbb{C}e_0 \oplus H_+ \oplus H_- \to H_+$ is the natural projection (cf. [10, Lemma 1.1], [12, §2]). We know that it coincides with the Tanaka-Webster connection ([6, §1.2]) provided that J is integrable. Near each point $\mathbb{P} \in M$, we always take a local unitary frame $e_\bullet^{\mathbb{C}} = (e_0^{\mathbb{C}}, e_1^{\mathbb{C}}, \ldots, e_n^{\mathbb{C}}, e_{\bar 1}^{\mathbb{C}}, \ldots, e_{\bar n}^{\mathbb{C}})$ of $\mathbb{C}TM$ ($e_0^{\mathbb{C}} := e_0$, $e_{\bar\alpha}^{\mathbb{C}} = \overline{e_\alpha^{\mathbb{C}}} \in H_-$, $g(e_\alpha^{\mathbb{C}}, e_{\bar\beta}^{\mathbb{C}}) = \delta_{\alpha\beta}$, $1 \le \alpha, \beta \le n$) which is ${}^\sharp\nabla$-parallel along all the ${}^\sharp\nabla$-geodesics from $\mathbb{P}$. Its dual frame is denoted by $e_{\mathbb{C}}^\bullet = (e_{\mathbb{C}}^0, e_{\mathbb{C}}^1, \ldots, e_{\mathbb{C}}^n, e_{\mathbb{C}}^{\bar 1}, \ldots, e_{\mathbb{C}}^{\bar n})$ (hence, $e_{\mathbb{C}}^0 = e^0$). We assume that the Greek indices $\alpha, \beta, \ldots$ vary from 1 to n, so that

$$
g = e_{\mathbb{C}}^0 \otimes e_{\mathbb{C}}^0 + \sum (e_{\mathbb{C}}^\alpha \otimes e_{\mathbb{C}}^{\bar\alpha} + e_{\mathbb{C}}^{\bar\alpha} \otimes e_{\mathbb{C}}^\alpha)
$$

and the connection ${}^\sharp\nabla$ can be expressed as

$$
{}^\sharp\nabla e_0^{\mathbb{C}} = 0, \quad {}^\sharp\nabla e_\beta^{\mathbb{C}} = \sum e_\alpha^{\mathbb{C}} \cdot \omega({}^\sharp\nabla)_\beta^\alpha,
$$
$$
{}^\sharp\nabla e_{\bar\beta}^{\mathbb{C}} = \sum e_{\bar\alpha}^{\mathbb{C}} \cdot \omega({}^\sharp\nabla)_{\bar\beta}^{\bar\alpha}, \quad \omega({}^\sharp\nabla)_{\bar\beta}^{\bar\alpha} = -\omega({}^\sharp\nabla)_\alpha^\beta.
$$

The associated orthonormal frames $e_\bullet = (e_0, e_1, \cdots, e_{2n})$, $e^\bullet = (e^0, e^1, \cdots, e^{2n})$ with respect to the underlying Riemannian metric are given by

$$e_\alpha = \frac{e_\alpha^{\mathbb{C}} + e_{\bar\alpha}^{\mathbb{C}}}{\sqrt{2}}, \quad e_{n+\alpha} = Je_\alpha, \quad e^\alpha = \frac{e^\alpha_{\mathbb{C}} + e^{\bar\alpha}_{\mathbb{C}}}{\sqrt{2}}, \quad e^{n+\alpha} = -Je^\alpha.$$

Certainly these frames are also ${}^\sharp\nabla$-parallel along the ${}^\sharp\nabla$-geodesics from $\mathbb{P}$. We denote the ${}^\sharp\nabla$-exponential map from $\mathbb{P}$ by $\exp = \exp_{\mathbb{P}} : T_{\mathbb{P}}M \to M$, and, as coordinates near $\mathbb{P}$, we always adopt the associated ${}^\sharp\nabla$-normal coordinates $x = {}^t(x_0, x_1 \ldots, x_{2n})$ with $\partial/\partial x_j = e_j$ at $0 = \mathbb{P}$. Then [10, (2.2)] says that there is a Taylor expansion

$$\omega({}^\sharp\nabla)^\alpha_\beta(\partial/\partial x_j) = -\sum_{\ell=1}^{\infty} \frac{\ell}{(\ell+1)!} \sum x_{j_1} \cdots x_{j_\ell} \frac{\partial^{\ell-1} F({}^\sharp\nabla)^\alpha_\beta(\partial/\partial x_j, \partial/\partial x_{j_1})}{\partial x_{j_2} \cdots \partial x_{j_\ell}}(0), \tag{1.1}$$

where $F({}^\sharp\nabla)$ is the curvature 2-form of ${}^\sharp\nabla$. Further, if we set

$$e_\bullet = (\partial/\partial x_\bullet) \cdot v_\bullet, \quad e^\bullet = (dx_\bullet) \cdot v^\bullet \qquad (\text{i.e., } e_j = \textstyle\sum v_{kj}\, \partial/\partial x_k, \text{ etc.}), \tag{1.2}$$

then the matrix-valued functions $v_\bullet$, $v^\bullet$ are also expanded explicitly: Let us set $z_0 = x_0$, $z_\alpha = (x_\alpha + i x_{n+\alpha})/\sqrt{2}$, $z_{\bar\alpha} = \overline{z_\alpha}$, $\partial/\partial z_0 = \partial/\partial x_0$, $\partial/\partial z_\alpha = (\partial/\partial x_\alpha - i\partial/\partial x_{n+\alpha})/\sqrt{2}$, $\partial/\partial z_{\bar\alpha} = \overline{\partial/\partial z_\alpha}$ and $(\partial/\partial z_\bullet) = (\partial/\partial z_0, \ldots, \partial/\partial z_n, \partial/\partial z_{\bar 1}, \ldots, \partial/\partial z_{\bar n})$. Then, in [10, (2.4)] we expressed accurately the Taylor expansions of the complex ones $V_\bullet$, $V^\bullet$ defined by $e^{\mathbb{C}}_\bullet = (\partial/\partial z_\bullet) \cdot V_\bullet$, $e^\bullet_{\mathbb{C}} = (dz_\bullet) \cdot V^\bullet$. For later use, we will record here the beginnings of that of $V_\bullet$, instead of that of $v_\bullet$:

$$V_\bullet = \begin{pmatrix} 1 & \underset{(0,\beta)\text{-th entry}}{z_{\bar\beta}\frac{i}{2}} & \underset{(0,\bar\beta)\text{-th entry}}{z_\beta \frac{-i}{2}} \\ \underset{(\alpha,0)\text{-th entry}}{\sum z_{\bar\gamma}\frac{\widetilde{\mathcal{T}}_{\bar\alpha 0\bar\gamma}}{2}} & E_n & \underset{(\alpha,\bar\beta)\text{-th entry}}{z_0\frac{-\widetilde{\mathcal{T}}_{\bar\alpha 0\bar\beta}}{2} + \sum z_{\bar\gamma}\frac{-\widetilde{\mathcal{T}}_{\bar\alpha\bar\gamma\bar\beta}}{2}} \\ \underset{(\bar\alpha,0)\text{-th entry}}{\sum z_\gamma \frac{\widetilde{\mathcal{T}}_{\alpha 0\gamma}}{2}} & \underset{(\bar\alpha,\beta)\text{-th entry}}{z_0\frac{-\widetilde{\mathcal{T}}_{\alpha 0\beta}}{2} + \sum z_\gamma \frac{-\widetilde{\mathcal{T}}_{\alpha\gamma\beta}}{2}} & E_n \end{pmatrix} + O(|z|^2), \tag{1.3}$$

where we set $\widetilde{\mathcal{T}}_{\alpha\gamma\beta} = g(T({}^\sharp\nabla)(e^{\mathbb{C}}_\gamma, e^{\mathbb{C}}_\beta), e^{\mathbb{C}}_\alpha)(0)$, etc.

Next, referring to [12, §2], let us recall that the hermitian vector bundle $(H, g|_H, J|_H)$ yields the canonical Spin^c structure over M with spinor bundle

$$\$^c = \wedge^{0,*}_H T^*M := \{\omega \in \wedge^* \mathbb{C}T^*M \mid X \lrcorner\, \omega = 0 \ (X \in \mathbb{R}e_0 \cup H_+)\}$$

accompanied with the Clifford action of $\mathbb{C}l(T^*M)$ given by

$$e^0_{\mathbb{C}} \diamond = (-1)^{*+1} i, \quad e^\alpha_{\mathbb{C}} \diamond = \sqrt{2}\, e^{\bar\alpha}_{\mathbb{C}} \wedge, \quad e^{\bar\alpha}_{\mathbb{C}} \diamond = -\sqrt{2}\, e^{\bar\alpha}_{\mathbb{C}} \vee, \tag{1.4}$$

where we set $e^{\bar\alpha}_{\mathbb{C}} \vee = e^{\mathbb{C}}_{\bar\alpha} \lrcorner$. Obviously the spinor metric coincides with the canonical one on the right hand side, i.e., $g^{\$^c} = g^{\wedge^{0,*}_H}$, and [12, Proposition 2.4] says that so does the spinor connection, that is,

$$\nabla^{\$^c} = {}^\sharp\nabla^{\wedge^{0,*}_H}, \quad \omega({}^\sharp\nabla^{\wedge^{0,*}_H}) := \sum \omega({}^\sharp\nabla)^\alpha_\beta \cdot e^{\bar\alpha}_{\mathbb{C}} \wedge e^{\bar\beta}_{\mathbb{C}} \vee.$$

Hence the curvature 2-form $F(\nabla^{\$^c}) = F({}^\sharp\nabla^{\wedge^{0,*}_H})$ is expressed as

$$F({}^\sharp\nabla^{\wedge^{0,*}_H})(X, Y) = \sum F({}^\sharp\nabla)^\alpha_\beta(X, Y)\, e^{\bar\alpha}_{\mathbb{C}} \wedge e^{\bar\beta}_{\mathbb{C}} \vee. \tag{1.5}$$

2. Intrinsic symbol spaces $\mathcal{S}_H^\infty$, $\mathcal{SC}_H^\infty$ and H-pseudodifferential operators

Let us take a hermitian vector bundle (E, g^E) over M with connection ∇^E and set

$$F = \$^c \otimes E = \wedge_H^{0,*} T^*M \otimes E$$

with canonically defined metric and connection (g^F, ∇^F). In this section, we will introduce two kinds of End(F)-valued symbol spaces and associated H-pseudodifferential operators.

We set

$$\mathcal{F}_m^H = \mathcal{F}_m^H(M; \mathrm{End}(F)) = \{f \in C^\infty(\pi^*\mathrm{End}(F)\backslash\{0\}) \mid f(\mathbb{P}, \lambda T) = \lambda^m f(\mathbb{P}, T)\},$$

where $\pi : T^*M \to M$ is the projection and λT denotes the Heisenberg dilation

$$T^*M = \mathbb{R}e^0 \oplus H^* \ni T = (T^0, T^H) \mapsto \lambda T := (\lambda^2 T^0, \lambda T^H).$$

By using the ∇^F-parallel transport along the ${}^\sharp\nabla$-geodesic from $\mathbb{P}'$ to $\mathbb{P}$

$$\mathcal{T}_{\mathbb{P}'}^{\mathbb{P}} = \mathcal{T}_{\nabla^F}(\mathbb{P}, \mathbb{P}') : F_{\mathbb{P}'} \to F_{\mathbb{P}}, \tag{2.1}$$

the bundle F is trivialized on a neighborhood $U_{\mathbb{P}}$ of $\mathbb{P}$ as

$$F\big|_{U_{\mathbb{P}}} \cong U_{\mathbb{P}} \times F_{\mathbb{P}}, \quad f(\mathbb{P}') \leftrightarrow (\mathbb{P}', \mathcal{T}_{\mathbb{P}'}^{\mathbb{P}} f(\mathbb{P}')). \tag{2.2}$$

Together with the trivialization

$$T^*U_{\mathbb{P}} \cong U_{\mathbb{P}} \times T_{\mathbb{P}}^*M = (U_{\mathbb{P}} \times \mathbb{R}^{2n+1}, (x, \sigma)), \;\; e^\bullet(x)\cdot\sigma \leftrightarrow (x, e^\bullet(0)\cdot\sigma) = (x, \sigma), \tag{2.3}$$

it induces naturally a local expression of $q \in C^\infty(\pi^*\mathrm{End}(F))$, which we denote by

$$q^{(\mathbb{P}, e^\bullet)}(x, \sigma) \in \mathrm{End}(F_{\mathbb{P}}).$$

The parallel transports for TM, T^*M, etc., are similarly defined and denoted also by $\mathcal{T}_{\mathbb{P}'}^{\mathbb{P}}$: therefore, $\mathcal{T}_{\exp(x)}^{\mathbb{P}} e^\bullet(x) = e^\bullet(0)$.

Let us define now one of the intrinsic symbol spaces, following the ideas due to Beals-Greiner ([3]) and Widom ([14], [15]), as

$$\begin{aligned} \mathcal{S}_H^m &= \mathcal{S}_H^m(M; \mathrm{End}(F)) \\ &= \Big\{ q \in C^\infty(\pi^*\mathrm{End}(F)) \,\Big|\, \text{there exist } q_k \in \mathcal{F}_k^H \ (k \le m) \text{ such that,} \\ &\qquad\qquad \text{for each } \mathbb{P}, \quad q \sim \sum_{k \le m} q_k \text{ at } \mathbb{P} \Big\}. \end{aligned}$$

Here "$q \sim \sum_{k\le m} q_k$ at $\mathbb{P}$" means that, for all multi-indices A, B and for all $N > 0$,

$$\left| \partial_x^A \partial_\sigma^B \Big(q^{(\mathbb{P},e^\bullet)} - \textstyle\sum_{k > m-N} q_k^{(\mathbb{P},e^\bullet)} \Big)(0, \sigma) \right| \le c_{ABN} |\sigma|_H^{m - |B|_H - N} \quad (|\sigma|_H \ge 1) \tag{2.4}$$

$$\Big(|\sigma|_H := \{ |\sigma_0|^2 + \textstyle\sum_{j\ge1} |\sigma_j|^4 \}^{1/4}, \quad |B|_H := 2B_0 + \sum_{j\ge1} B_j = B_0 + |B| \Big),$$

where $c_{ABN} = c_{ABN}(\mathbb{P}) > 0$ are bounded functions. Further we set $\mathcal{S}_H^\infty = \bigcup_m \mathcal{S}_H^m$, $\mathcal{S}_H^{-\infty} = \bigcap_m \mathcal{S}_H^m$ as usual.

Lemma 2.1. *The symbol space $\mathcal{S}_H^m$ coincides with the one given by Beals-Greiner* [3, §10].

Proof. It will suffice to show that (2.4) holds not only for $(0,\sigma)$ but also for any (y,σ). At the point $\mathbb{P}(y) := \exp_{\mathbb{P}}(e_\bullet(0)\cdot y)$, let us check the relation between the frame $(\partial/\partial x_\bullet, \partial/\partial\sigma_\bullet)$ induced from the coordinates (x,σ) centered at $\mathbb{P}$, and the frame $(\partial/\partial w_\bullet, \partial/\partial\eta_\bullet)$ induced from the ones (w,η) centered at $\mathbb{P}(y)$. We have

$$\begin{aligned}(2.5)\qquad &\mathbb{P}(x) = \mathbb{P}(y)(w) := \exp_{\mathbb{P}(y)}(e_\bullet(y)\cdot w):\ w = w(y,x) = O(|x-y|),\\ &e^\bullet(0)\cdot\eta = \mathcal{T}^{\mathbb{P}}_{\mathbb{P}(y)}\mathcal{T}^{\mathbb{P}(y)}_{\mathbb{P}(x)}\mathcal{T}^{\mathbb{P}(x)}_{\mathbb{P}}(e^\bullet(0)\cdot\sigma):\ \eta = \eta(y,x,\sigma) = a(y,x)\cdot\sigma,\ a(y,y) = E.\end{aligned}$$

Since ${}^\sharp\nabla e^0 = 0$, etc., obviously we have

$$a(y,x)^{-1} = {}^t a(y,x),\quad a(y,x)_{0k} = a(y,x)_{k0} = \begin{cases} 1 & (k=0),\\ 0 & (k\geq 1),\end{cases}$$

so that

$$\frac{\partial}{\partial x_i} = \sum \frac{\partial w_j}{\partial x_i}\frac{\partial}{\partial w_j} + \sum_{j,k,\ell\geq 1}\frac{\partial a_{jk}}{\partial x_i}a_{\ell k}\eta_\ell\frac{\partial}{\partial\eta_j},\qquad \frac{\partial}{\partial\sigma_i} = \begin{cases}\dfrac{\partial}{\partial\eta_0} & (i=0),\\ \displaystyle\sum_{j\geq 1} a_{ji}\frac{\partial}{\partial\eta_j} & (i\geq 1).\end{cases}$$

Consequently, for example we have

$$\begin{aligned}\partial_{x_i}\Big(q^{(\mathbb{P},e^\bullet)} - \sum_{k>m-N} q_k^{(\mathbb{P},e^\bullet)}\Big)(y,\sigma) &= \sum\frac{\partial w_j}{\partial x_i}\partial_{w_j}\Big(q^{(\mathbb{P}(y),e_y^\bullet)} - \sum_{k>m-N} q_k^{(\mathbb{P}(y),e_y^\bullet)}\Big)(0,\eta)\\ &\quad + \sum_{j,\ell\geq 1}\frac{\partial a_{j\ell}}{\partial x_i}\eta_\ell\,\partial_{\eta_j}\Big(q^{(\mathbb{P}(y),e_y^\bullet)} - \sum_{k>m-N} q_k^{(\mathbb{P}(y),e_y^\bullet)}\Big)(0,\eta),\end{aligned}$$

which satisfies such a condition as (2.4). Here the local expression $q^{(\mathbb{P}(y),e_y^\bullet)}$ is given by the trivialization centered at $\mathbb{P}(y)$ with ${}^\sharp\nabla$-parallel frame $e_y^\bullet := \mathcal{T}^{\mathbb{P}(y)(\cdot)}_{\mathbb{P}(y)}e^\bullet(y)$. □

By further consideration we notice that the estimate at (2.5) is refined into

$$(2.6)\qquad w = {}^t v^\bullet(y)(x-y) + O(|x-y|^2),\quad x = y + v_\bullet(y)w + O(|w|^2).$$

Next, let us introduce a class of pseudodifferential operators on M and associated intrinsic symbols. We adopt another trivialization

$$(2.7)\qquad T^*U_{\mathbb{P}} \cong U_{\mathbb{P}}\times T^*_{\mathbb{P}} = (U_{\mathbb{P}}\times\mathbb{R}^{2n+1},(x,\xi)),\ \ dx_\bullet(x)\cdot\xi \leftrightarrow (x, dx_\bullet(0)\cdot\xi) = (x,\xi),$$

which gives another local expression of $q\in C^\infty(\pi^*\mathrm{End}(F))$:

$$\begin{aligned}&q(x,\xi) := q^{(\mathbb{P},dx_\bullet)}(x,\xi) = q^{(\mathbb{P},e^\bullet)}(x,\sigma(x,\xi)) \in F_{\mathbb{P}},\\ &\text{hence, } q(\mathbb{P},\xi) := q(0,\xi) = q^{(\mathbb{P},e^\bullet)}(0,\sigma(0,\xi)) = q^{(\mathbb{P},e^\bullet)}(0,\xi).\end{aligned}$$

By (1.2) we know that the transition rule (cf. (2.7), (2.3)) between ξ and $\sigma = \sigma(x,\xi)$ is

$$(2.8)\qquad (dx_\bullet)(x)\cdot\xi = e^\bullet(x)\cdot\sigma(x,\xi):\quad \sigma(x,\xi) = v^\bullet(x)^{-1}\xi = {}^t v_\bullet(x)\xi.$$

For a symbol $p\in S^m_H$ and a smooth bump function ϕ on $M\times M$ which is supported in a small neighborhood of the diagonal set and is equal to 1 on a still smaller one, we define the H-pseudodifferential operator

$$\theta(p) = \theta^{\phi}(p) : \Gamma(F) \to \Gamma(F)$$

as follows: For $u \in \Gamma(F)$, at each $\mathbb{P} \in M$ we set

$$(\theta(p)u)(\mathbb{P}) = \frac{1}{(2\pi)^{2n+1}} \int_{T_{\mathbb{P}}M \times T^*_{\mathbb{P}}M \ni (x,\xi)} e^{-i\langle x,\xi\rangle} p(0,\xi)\, \overline{u}_{\mathbb{P}}(x)\, dx d\xi = \frac{1}{(2\pi)^{2n+1}} \int_{T^*_{\mathbb{P}}M \ni \xi} p(0,\xi)\, \widehat{\overline{u}_{\mathbb{P}}}(\xi)\, d\xi,$$

$$\overline{u}_{\mathbb{P}} := \Big(T_{\mathbb{P}}M \ni x \mapsto \phi(\mathbb{P}, \exp(x))\, \mathcal{T}^{\mathbb{P}}_{\exp(x)} u(\exp(x)) \in F_{\mathbb{P}}\Big),$$

where $\widehat{\overline{u}_{\mathbb{P}}}$ is the Fourier transform of $\overline{u}_{\mathbb{P}}$. The set of such operators is denoted by $Op\, \mathcal{S}^m_H$. By referring to [3, §10], it is certain that $Op\, \mathcal{S}^{-\infty}_H$ consists of operators with C^∞-kernels and, if we define the operator by using another bump function ψ, we have $\theta^{\phi}(p) = \theta^{\psi}(p)$ (mod $Op\, \mathcal{S}^{-\infty}_H$).

For an operator $P : \Gamma(F) \to \Gamma(F)$, according to Widom's idea ([14], [15]), **its intrinsic symbol** $\varsigma(P) \in C^\infty(T^*M,\, \pi^*\mathrm{End}(F))$ is now defined by

$$\varsigma(P)(\mathbb{P}, \xi) u_{\mathbb{P}} = P\Big(M \ni \exp(x) \mapsto e^{i\langle x,\xi\rangle}\, \phi(\mathbb{P}, \exp(x))\, \mathcal{T}^{\exp(x)}_{\mathbb{P}} u_{\mathbb{P}}\Big)\Big|_{x=0}$$

with $u_{\mathbb{P}} \in F_{\mathbb{P}}$. Then obviously we have

$$\varsigma(\theta(p)) = p \pmod{\mathcal{S}^{-\infty}_H}, \theta(\varsigma(\theta(p))) = \theta(p) \pmod{Op\, \mathcal{S}^{-\infty}_H}, \tag{2.9}$$

$$\mathcal{S}^m_H \,/\, \mathcal{S}^{-\infty}_H \ \underset{\varsigma}{\overset{\theta}{\rightleftharpoons}}\ Op\, \mathcal{S}^m_H \,/\, Op\, \mathcal{S}^{-\infty}_H.$$

In a way similar to the idea of Getzler ([7]) (and Alvarez-Gaumé ([1])), let us define here another symbol space $\mathcal{SC}^m_H$. By (1.4), we know that there are the identifications

$$\mathrm{End}(\not{S}^c) = \mathrm{End}(\wedge^{0,*}_H T^*M) \ \cong\ \mathbb{C}l(H^*) \ \cong\ \wedge^*\mathbb{C}H^* \ \subset \wedge^*\mathbb{C}T^*M,$$

$$e^{j_1} \diamond e^{j_2} \diamond \cdots \underset{(j_1<j_2<\cdots)}{\leftrightarrow} e^{j_1} \wedge e^{j_2} \wedge \cdots$$

$$\begin{matrix} e^{\alpha}_{\mathbb{C}} \diamond \\ e^{\bar\alpha}_{\mathbb{C}} \diamond \end{matrix} = \begin{matrix} \sqrt{2}\cdot e^{\bar\alpha}_{\mathbb{C}} \wedge \\ -\sqrt{2}\cdot e^{\bar\alpha}_{\mathbb{C}} \vee \end{matrix} \quad \leftrightarrow \quad \begin{matrix} e^{\alpha}_{\mathbb{C}} \\ e^{\bar\alpha}_{\mathbb{C}} \end{matrix}$$

$$\mathrm{End}(F) = \mathrm{End}(\wedge^{0,*}_H T^*M \otimes E) \ \cong\ \wedge^*\mathbb{C}H^* \otimes \mathrm{End}(E).$$

For example, referring to (1.5), we have

$$\begin{aligned} F({}^{\sharp}\nabla^{\wedge^{0,*}_H})(X,Y) &= -\frac{1}{2}\sum F({}^{\sharp}\nabla)^{\alpha}_{\beta}(X,Y)\, e^{\alpha}_{\mathbb{C}} \diamond e^{\bar\beta}_{\mathbb{C}} \\ &= \frac{1}{2}\sum F({}^{\sharp}\nabla)^{\bar\beta}_{\bar\alpha}(X,Y)\, e^{\alpha}_{\mathbb{C}} \wedge e^{\bar\beta}_{\mathbb{C}} =: \frac{1}{2} F({}^{\sharp}\nabla;\wedge)(X,Y), \end{aligned}$$

$$F(\nabla^F)(X,Y) = F({}^{\sharp}\nabla^{\wedge^{0,*}_H})(X,Y) + F(\nabla^E)(X,Y) = \frac{1}{2}F({}^{\sharp}\nabla;\wedge)(X,Y) + F(\nabla^E)(X,Y).$$

We set

$$\begin{aligned} \mathcal{SC}^m_H = \mathcal{SC}^m_H(M;\mathrm{End}(F)) &= \sum_{k=0}^{2n} \mathcal{S}^{m-k}_H(M; \wedge^k H^* \otimes \mathrm{End}(E))) \\ &:= \sum_{k=0}^{2n} \mathcal{S}^{m-k}_H \cap C^\infty(T^*M,\, \pi^*(\wedge^k H^* \otimes \mathrm{End}(E))), \end{aligned} \tag{2.10}$$

the element of which is said to **have grading** m (according to the naming in [4]). For

$p \in \mathcal{SC}_H^\infty$, the H-pseudodifferential operator $\theta(p) \in Op\,\mathcal{SC}_H^\infty$ is defined by regarding p as an element of $\mathcal{S}_H^\infty$ via the canonical identification

$$\mathcal{S}_H^\infty \leftrightarrow \mathcal{SC}_H^\infty$$
$$\sum \mathcal{S}_H^{m-k} \hookleftarrow \sum \mathcal{S}_H^{m-k}(M; \wedge^k H^* \otimes \mathrm{End}(E)) = \mathcal{SC}_H^m,$$

so that (2.9) holds also for $\mathcal{SC}_H^m$, etc.

3. A formula for the composition of polynomial intrinsic symbols $\in \mathcal{PC}_H^\infty$

In this section, we will focus on the space of polynomial intrinsic symbols, that is, the space of intrinsic symbols associated with H-differential operators,

$$\mathcal{PC}_H^m = \{p \in \mathcal{SC}_H^m \mid p(\mathbb{P}, \xi) \text{ is a polynomial in } \xi\},$$

and study the composition

$$\mathcal{PC}_H^\infty \times \mathcal{PC}_H^\infty \to \mathcal{PC}_H^\infty, \quad (p, q) \mapsto p \circ q := \varsigma(\theta(p) \circ \theta(q)). \tag{3.1}$$

As was mentioned in Introduction, Getzler [7, Theorem 2.7] (and Block-Fox [5, Theorem 2.1]) derived an explicit expression of such a composition on a spin manifold by means of the Campbell-Hausdorff formula, and so did Benameur-Heitsch [4, Theorem 4.6] on a foliated spin manifold but by means of Atiyah-Bott-Patodi's formula [2, Proposition 3.7]. Stimulated by the latter method, the author tries to examine the composition in the contact Riemannian case by using the following formula (cf. (1.1)): Let $(u_1, u_2, \ldots)$ be a local frame of F which is ∇^F-parallel along all the $^\sharp\nabla$-geodesics from $\mathbb{P}$ and let us set $\nabla^F u_{i_2} = \sum \omega(\nabla^F)_{i_2}^{i_1}(\partial/\partial x_j)\, u_{i_1} \otimes dx_j$. Then, at $x = 0$, the connection coefficients are expanded as

$$\omega(\nabla^F)_{i_2}^{i_1}(\partial/\partial x_j) = -\sum_{\ell=1}^{\infty} \frac{\ell}{(\ell+1)!} \sum x_{j_1} \cdots x_{j_\ell} \frac{\partial^{\ell-1} F(\nabla^F)_{i_2}^{i_1}(\partial/\partial x_j, \partial/\partial x_{j_1})}{\partial x_{j_2} \cdots \partial x_{j_\ell}}(0). \tag{3.2}$$

Let us start with calculating symbols of some H-differential operators.

Lemma 3.1. *For $X = X^0 + X^H = \sum X_j e_j \in \Gamma(TM = \mathbb{R}e_0 \oplus H)$, $\xi = \xi_0 + \xi_H = \sum \xi_j e^j(\mathbb{P}) \in T_{\mathbb{P}}^* M = \mathbb{R}e^0(\mathbb{P}) \oplus H_{\mathbb{P}}^*$, we have*

$$\varsigma(\nabla_X^F)(\mathbb{P}, \xi) = \langle iX_{\mathbb{P}}, \xi\rangle = \underbrace{\langle iX_{\mathbb{P}}^H, \xi_H\rangle}_{grading\ 1} + \underbrace{\langle iX_{\mathbb{P}}^0, \xi_0\rangle}_{grading\ 2} := \sum_{j\geq 1} iX_j(\mathbb{P})\,\xi_j + iX_0(\mathbb{P})\,\xi_0.$$

Proof. We have $\phi(\mathbb{P}, x) = 1$ near $x = 0$, so that we may ignore the bump function ϕ. Since

$$X\langle \exp^{-1}(x), \xi\rangle\Big|_{x=0} = \frac{d}{dt}\Big|_{t=0} \langle \exp^{-1}(\exp(tX_{\mathbb{P}})), \xi\rangle = \frac{d}{dt}\Big|_{t=0} t\langle X_{\mathbb{P}}, \xi\rangle = \langle X_{\mathbb{P}}, \xi\rangle \tag{3.3}$$

and obviously $\nabla_X^F(\mathcal{T}_{\mathbb{P}}^x u_{\mathbb{P}})\big|_{x=0} = 0$, we have

$$\begin{aligned}\varsigma(\nabla_X^F)(\mathbb{P}, \xi) u_{\mathbb{P}} &= \nabla_X^F\Big(e^{i\langle \exp^{-1}(x), \xi\rangle}\, \mathcal{T}_{\mathbb{P}}^x u_{\mathbb{P}}\Big)\Big|_{x=0} \\ &= X(i\langle \exp^{-1}(x), \xi\rangle)\Big|_{x=0} \cdot u_{\mathbb{P}} + \nabla_X^F(\mathcal{T}_{\mathbb{P}}^x u_{\mathbb{P}})\Big|_{x=0} = \langle iX_{\mathbb{P}}, \xi\rangle \cdot u_{\mathbb{P}},\end{aligned}$$

that is, the lemma is valid. □

Proposition 3.2. *We have*

$$\varsigma(\nabla^F_X\nabla^F_Y)(\mathbb{P},\xi) = \underbrace{\langle iX^H_{\mathbb{P}},\xi_H\rangle\langle iY^H_{\mathbb{P}},\xi_H\rangle + \frac{1}{4}F(^\sharp\nabla;\wedge)(X^H_{\mathbb{P}},Y^H_{\mathbb{P}})}_{grading\ 2} \tag{3.4}$$
$$+ \underbrace{\langle iX^0_{\mathbb{P}},\xi_0\rangle\langle iY^0_{\mathbb{P}},\xi_0\rangle}_{grading\ 4} + \underbrace{\langle iX^H_{\mathbb{P}},\xi_H\rangle\langle iY^0_{\mathbb{P}},\xi_0\rangle + \langle iX^0_{\mathbb{P}},\xi_0\rangle\langle iY^H_{\mathbb{P}},\xi_H\rangle}_{grading\ 3}$$
$$+ \underbrace{\frac{1}{4}F(^\sharp\nabla;\wedge)(X^0_{\mathbb{P}},Y^H_{\mathbb{P}}) + \frac{1}{4}F(^\sharp\nabla;\wedge)(X^H_{\mathbb{P}},Y^0_{\mathbb{P}})}_{grading\ 2}$$
$$+ \underbrace{iX_{\mathbb{P}}(Y_x\langle \exp^{-1}(x),\xi_H\rangle)}_{grading\ 1} + \underbrace{iX_{\mathbb{P}}(Y_x\langle \exp^{-1}(x),\xi_0\rangle)}_{grading\ 2} + \underbrace{\frac{1}{2}F(\nabla^E)(X_{\mathbb{P}},Y_{\mathbb{P}})}_{grading\ 0}$$

and

$$iX^H_{\mathbb{P}}(Y^H_x\langle \exp^{-1}(x),\xi_0\rangle) = \frac{\xi_0}{2i}\,de^0(X^H_{\mathbb{P}},Y^H_{\mathbb{P}}) = \frac{\xi_0}{2i}\,de^0(X_{\mathbb{P}},Y_{\mathbb{P}}), \tag{3.5}$$

where we know $de^0 = i\sum e^\alpha_{\mathbb{C}}\wedge e^{\bar\alpha}_{\mathbb{C}} = \sum e^\alpha\wedge e^{n+\alpha}$.

Proof. We have

$$\begin{aligned}
&\nabla^F_X\nabla^F_Y\Big(e^{i\langle \exp^{-1}(x),\xi\rangle}\,\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}}\Big)\Big|_{x=0}\\
&= \Big(i(\nabla^F_X\nabla^F_Y\langle \exp^{-1}(x),\xi\rangle)\,e^{i\langle \exp^{-1}(x),\xi\rangle}\,\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}}\\
&\quad + i(\nabla^F_X\langle \exp^{-1}(x),\xi\rangle)\,i(\nabla^F_Y\langle \exp^{-1}(x),\xi\rangle)\,e^{i\langle \exp^{-1}(x),\xi\rangle}\,\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}}\\
&\quad + i(\nabla^F_Y\langle \exp^{-1}(x),\xi\rangle)\,e^{i\langle \exp^{-1}(x),\xi\rangle}\nabla^F_X(\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}})\\
&\quad + i(\nabla^F_X\langle \exp^{-1}(x),\xi\rangle)\,e^{i\langle \exp^{-1}(x),\xi\rangle}\nabla^F_Y(\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}})\\
&\quad + e^{i\langle \exp^{-1}(x),\xi\rangle}\nabla^F_X\nabla^F_Y(\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}})\Big)\Big|_{x=0}\\
&= i\nabla^F_{X_{\mathbb{P}}}\nabla^F_Y\langle \exp^{-1}(x),\xi\rangle\,u_{\mathbb{P}} + \langle iX,\xi\rangle\langle iY,\xi\rangle u_{\mathbb{P}} + \nabla^F_X\nabla^F_Y(\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}})\Big|_{x=0}
\end{aligned}$$

and, by (3.2), (2.1) and (2.2), we have

$$\begin{aligned}
\nabla^F_X\nabla^F_Y(\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}})\Big|_{x=0} &= \nabla^F_X\nabla^F_Y\mathcal{T}^x_{\mathbb{P}}\Big(\sum a_{i_2}(\mathbb{P})u_{i_2}(\mathbb{P})\Big)\Big|_{x=0} \qquad (3.6)\\
&= \sum a_{i_2}(\mathbb{P})\Big(\nabla^F_X\nabla^F_Y u_{i_2}(x)\Big)\Big|_{x=0}\\
&= \sum a_{i_2}(\mathbb{P})\nabla^F_X\Big(\sum_{i_1,j}\omega(\nabla^F)^{i_1}_{i_2}(\partial/\partial x_j)dx_j(Y)\,u_{i_1}(x)\Big)\Big|_{x=0}\\
&= \sum_{i_1.i_2,j,j_1} a_{i_2}(\mathbb{P})\Big\{-\frac{1}{2}X(x_{j_1})F(\nabla^F)^{i_1}_{i_2}(\partial/\partial x_j,\partial/\partial x_{j_1})(0)\Big\}dx_j(Y)\,u_{i_1}(0)\\
&= -\frac{1}{2}\sum_{i_1,i_2,j} a_{i_2}(\mathbb{P})F(\nabla^F)^{i_1}_{i_2}(Y,X)(0)\,u_{i_1}(\mathbb{P})\\
&= -\frac{1}{2}\sum_{i_1,i_2,j} a_{i_2}(\mathbb{P})F(\nabla^F)^{i_1}_{i_2}(Y,X)(0)\cdot u_{i_1}\otimes u^*_{i_2}(u_{\mathbb{P}})
\end{aligned}$$

$$= -\frac{1}{2}F(\nabla^F)(Y, X)\,u_{\mathbb{P}} = \frac{1}{2}F(\nabla^F)(X, Y)\,u_{\mathbb{P}}$$

They imply

$$\begin{aligned}&\varsigma(\nabla^F_X\nabla^F_Y)(\mathbb{P}, \xi)\\ &= \langle iX, \xi\rangle\langle iY, \xi\rangle + \frac{1}{4}F(^\sharp\nabla; \wedge)(X, Y) + iX_{\mathbb{P}}(Y\langle \exp^{-1}(x), \xi\rangle) + \frac{1}{2}F(\nabla^E)(X, Y).\end{aligned}$$

Considering the gradings of the terms, thus we obtain (3.4). Next, (2.6) says

$$\begin{aligned}X_{\mathbb{P}}(Y_x\langle \exp^{-1}(x), \xi\rangle) &= \frac{d}{dt}\Big|_{t=0}\Big(Y_x\langle \exp^{-1}(x), \xi\rangle\Big|_{x=\exp(tX_{\mathbb{P}})}\Big)\\ &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\langle tX_{\mathbb{P}} + v_\bullet(tX_{\mathbb{P}})\cdot sY_x + \cdots,\ \xi\rangle = \frac{d}{dt}\Big|_{t=0}\langle v_\bullet(tX_{\mathbb{P}})\cdot Y_x,\ \xi\rangle\\ &= \langle \frac{d}{dt}\Big|_{t=0} v_\bullet(tX_{\mathbb{P}})\cdot Y_{\mathbb{P}} + \frac{d}{dt}\Big|_{t=0} Y_{\exp(tX_{\mathbb{P}})},\ \xi\rangle.\end{aligned}$$

Hence, by (1.3), we have

$$\begin{aligned}iX^H_{\mathbb{P}}(Y^H_x\langle \exp^{-1}(x), \xi_0\rangle &= \langle \frac{d}{dt}\Big|_{t=0} v_\bullet(tX^H_{\mathbb{P}})\cdot Y^H_{\mathbb{P}},\ \xi_0\rangle\\ &= \frac{i}{2}\sum\Big\{X_{n+\beta}(\mathbb{P})Y_\beta(\mathbb{P}) - X_\beta(\mathbb{P})Y_{n+\beta}(\mathbb{P})\Big\}\xi_0 = \frac{\xi_0}{2i}\,de^0(X_{\mathbb{P}}, Y_{\mathbb{P}}).\end{aligned}\tag{3.7}$$

Thus we obtain (3.5). □

Set

$$\mathscr{F}(^\sharp\nabla; \wedge) = F(^\sharp\nabla; \wedge) + \frac{2\xi_0}{i}\,de^0.\tag{3.8}$$

Then Proposition 3.2 yields

Corollary 3.3. *Denoting the grading of $\varsigma(\nabla^F_X)(\mathbb{P}, \xi)$ by m_X, we have*

$$\begin{aligned}(\varsigma(\nabla^F_X)\circ\varsigma(\nabla^F_Y))(\mathbb{P}, \xi) = \langle iX_{\mathbb{P}}, \xi\rangle\,\langle iY_{\mathbb{P}}, \xi\rangle + \frac{1}{4}\mathscr{F}(^\sharp\nabla; \wedge)(X_{\mathbb{P}}, Y_{\mathbb{P}})\\ + (\textit{terms of grading} < m_X + m_Y).\end{aligned}\tag{3.9}$$

This will suggest a formula for general polynomial symbols.

Definition 3.4. *For p, $q \in \mathcal{PC}^\infty_H$, we set*

$$\begin{aligned}&\mathscr{F}(^\sharp\nabla; \wedge)(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi'})p(\mathbb{P}, \xi)\wedge q(\mathbb{P}, \xi')\\ &= \sum_{i,j}\mathscr{F}(^\sharp\nabla; \wedge)(\partial/\partial x_i, \partial/\partial x_j)(\mathbb{P})\,\frac{\partial}{\partial\xi_i}p(\mathbb{P}, \xi)\wedge\frac{\partial}{\partial\xi'_j}q(\mathbb{P}, \xi')\\ &= \sum_{i,j}\Big\{\sum_{\alpha,\beta}F(^\sharp\nabla)^{\bar\beta}_{\bar\alpha}(\partial/\partial x_i, \partial/\partial x_j)(\mathbb{P})\,e^\alpha_{\mathbb{C}}(\mathbb{P})\wedge e^{\bar\beta}_{\mathbb{C}}(\mathbb{P})\wedge\\ &\qquad + \frac{2\xi_0}{\sqrt{-1}}\,de^0(\partial/\partial x_i, \partial/\partial x_j)(\mathbb{P})\Big\}\frac{\partial}{\partial\xi_i}p(\mathbb{P}, \xi)\wedge\frac{\partial}{\partial\xi'_j}q(\mathbb{P}, \xi'),\end{aligned}$$

$$e^{-\frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi'})}p(\mathbb{P}, \xi)\wedge q(\mathbb{P}, \xi')\Big|_{\xi'=\xi} = \sum_{j=0}^\infty\frac{1}{j!}\Big(-\frac{1}{4}\mathscr{F}(^\sharp\nabla; \wedge)(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi'})\Big)^j p(\mathbb{P}, \xi)\wedge q(\mathbb{P}, \xi')\Big|_{\xi'=\xi}.$$

Note that the summation in the last line is actually finite and the action of $\mathscr{F}({}^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})$ *may lower the grading* (*cf.* [13, Proposition 1.2(2)]).

The formula (3.9) then becomes

$$(\varsigma(\nabla^F_X)\circ\varsigma(\nabla^F_Y))(\mathbb{P},\xi) = e^{-\frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})}\varsigma(\nabla^F_X)(\mathbb{P},\xi)\wedge\varsigma(\nabla^F_Y)(\mathbb{P},\xi')\Big|_{\xi'=\xi} + (\text{terms of grading} < m_X + m_Y)$$

and the general one is given as follows.

Theorem 3.5. *There exists a series of bilinear differential operators*

$$a_k : \mathcal{PC}^\infty_H \times \mathcal{PC}^\infty_H \to \mathcal{PC}^\infty_H \quad (k = 0, 1, 2, \ldots)$$

such that

$$a_k(\mathcal{PC}^m_H, \mathcal{PC}^{m'}_H) \subset \mathcal{PC}^{m+m'-k}_H, \qquad p\circ q = \sum_{k=0}^{\infty} a_k(p,q), \tag{3.10}$$
$$a_0(p,q)(\mathbb{P},\xi) = e^{-\frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})} p(\mathbb{P},\xi)\wedge q(\mathbb{P},\xi')\Big|_{\xi'=\xi}.$$

In the proof certainly we will adopt Benameur-Heitsch's idea ([4]) to use the formulas (3.2) and (1.1), but the proof itself does not follow their strategy in [4, §4], which seems to have some difficulty in being applied to our case. Our approach in this section is quite straightforward and may be applied also to Getzler's case ([7]) and Benameur-Heitsch's case ([4]).

Now, for $p \in \mathcal{PC}^\infty_H$, the associated H-differential operator $\theta(p)$ can be written as a finite sum of operators such as $h\nabla^F_{X_1}\cdots\nabla^F_{X_N}$ ($h \in \Gamma(\mathrm{End}(F))$, $X_1,\ldots,X_N \in \Gamma(TM)$), so that the proof of Theorem 3.5 is reduced to the study of

$$\varsigma(h\nabla^F_{X_1}\nabla^F_{X_2}\cdots\nabla^F_{X_N}h'\nabla^F_{X'_1}\nabla^F_{X'_2}\cdots\nabla^F_{X'_{N'}})(\mathbb{P},\xi). \tag{3.11}$$

Accordingly, first let us prove the following:

Proposition 3.6. *For vector fields* X_1, X_2, …, X_N, *we have*

$$\begin{aligned}&\varsigma(\nabla^F_{X_1}\nabla^F_{X_2}\cdots\nabla^F_{X_N})(\mathbb{P},\xi)\\ &= \sum_{0\le 2k\le N}\sum\prod_{a=1}^{N-2k}\langle iX_{n_a},\xi\rangle(\mathbb{P})\prod_{b=1}^{k}\frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(X_{i_b},X_{j_b})(\mathbb{P})\\ &\qquad + (\textit{terms of grading} < \textstyle\sum m_{X_j}),\end{aligned} \tag{3.12}$$

$$\begin{pmatrix}\{n_1,\ldots,n_{N-2k},i_1,\ldots,i_k,j_1,\ldots,j_k\} = \{1,\ldots,N\},\\ n_1<\cdots<n_{N-2k},\\ j_1<\cdots<j_k,\ \ i_b<j_b\end{pmatrix}$$

where the subscript letters in the second line run in all the letters in the big parentheses.

Proof. On and after, $\equiv$ means that the top grading parts of both sides coincide. Since (3.3) and (3.5) say

$$X_1 e^{i\langle\exp^{-1}(x),\xi\rangle}\Big|_{x=0} = X_1\langle i\exp^{-1}(x),\xi\rangle\Big|_{x=0} = \langle iX_{1,\mathbb{P}},\xi\rangle,$$

$$
\begin{aligned}
&X_2X_1e^{i\langle \exp^{-1}(x),\xi\rangle}\Big|_{x=0}\\
&= X_2\langle i\exp^{-1}(x),\xi\rangle\Big|_{x=0}X_1\langle i\exp^{-1}(x),\xi\rangle\Big|_{x=0} + X_2X_1\langle i\exp^{-1}(x),\xi\rangle\Big|_{x=0}\\
&\equiv \langle iX_{2,\mathbb{P}},\xi\rangle\langle iX_{1,\mathbb{P}},\xi\rangle + \frac{\xi_0}{2i}de^0(X_{2,\mathbb{P}},X_{1,\mathbb{P}}),\\
&X_3X_2X_1e^{i\langle \exp^{-1}(x),\xi\rangle}\Big|_{x=0} = \langle iX_{3,\mathbb{P}},\xi\rangle\langle iX_{2,\mathbb{P}},\xi\rangle\langle iX_{1,\mathbb{P}},\xi\rangle\\
&\quad + X_3X_2\langle i\exp^{-1}(x),\xi\rangle\Big|_{x=0}\langle iX_{1,\mathbb{P}},\xi\rangle + \langle iX_2,\xi\rangle X_3X_1\langle i\exp^{-1}(x),\xi\rangle\Big|_{x=0}\\
&\quad + X_2X_1\langle i\exp^{-1}(x),\xi\rangle\Big|_{x=0}\langle iX_{3,\mathbb{P}},\xi\rangle + X_3X_2X_1\langle i\exp^{-1}(x),\xi\rangle\Big|_{x=0}\\
&\equiv \langle iX_{3,\mathbb{P}},\xi\rangle\langle iX_{2,\mathbb{P}},\xi\rangle\langle iX_{1,\mathbb{P}},\xi\rangle + \langle iX_{1,\mathbb{P}},\xi\rangle\frac{\xi_0}{2i}de^0(X_{3,\mathbb{P}},X_{2,\mathbb{P}})\\
&\quad + \langle iX_{2,\mathbb{P}},\xi\rangle\frac{\xi_0}{2i}de^0(X_{3,\mathbb{P}},X_{1,\mathbb{P}}) + \langle iX_{3,\mathbb{P}},\xi\rangle\frac{\xi_0}{2i}de^0(X_{2,\mathbb{P}},X_{1,\mathbb{P}}),
\end{aligned}
$$

inductively we know

$$
\begin{aligned}
&X_1X_2\cdots X_me^{i\langle \exp^{-1}(x),\xi\rangle}\Big|_{x=0}\\
&\equiv \sum_k \sum_{\substack{\{n_1,\dots,n_{m-2k},i_1,\dots,i_k,j_1,\dots,j_k\}=\{1,\dots,m\},\\ n_1<\cdots<n_{m-2k},\\ j_1<\cdots<j_k,\quad i_b<j_b}} \underbrace{\prod_{a=1}^{m-2k}\langle iX_{n_a},\xi\rangle(\mathbb{P})\cdot\prod_{b=1}^{k}\frac{\xi_0}{2i}de^0(X_{i_b},X_{j_b})(\mathbb{P}).}
\end{aligned}
$$

Hence, referring also to (3.6), we have

$$
\begin{aligned}
&\varsigma(\nabla^F_{X_1}\nabla^F_{X_2}\cdots\nabla^F_{X_N})(\mathbb{P},\xi)\,u_{\mathbb{P}} = \nabla^F_{X_1}\nabla^F_{X_2}\cdots\nabla^F_{X_N}\Big(e^{i\langle \exp^{-1}(x),\xi\rangle}\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}}\Big)\Big|_{x=0}\\
&= \sum_\ell \sum_{\substack{\{r_1,\dots,r_{N-2\ell},s_1,\dots,s_{2\ell}\}=\{1,\dots,N\},\\ r_1<\cdots<r_{N-2\ell},\quad s_1<\cdots<s_{2\ell}}} \underbrace{X_{r_1}X_{r_2}\cdots X_{r_{N-2\ell}}\big(e^{i\langle \exp^{-1}(x),\xi\rangle}\big)\Big|_{x=0}\nabla^F_{X_{s_1}}\nabla^F_{X_{s_2}}\cdots\nabla^F_{X_{s_{2\ell}}}(\mathcal{T}^x_{\mathbb{P}}u_{\mathbb{P}})\Big|_{x=0}} + \cdots\\
&\equiv \sum_{k,\ell}\sum_{\substack{\{n_1,\dots,n_{N-2k-2\ell},i_1,\dots,i_k,j_1,\dots,j_k,i'_1,\dots,i'_\ell,j'_1,\dots,j'_\ell\}\\ =\{1,\dots,N\},\\ n_1<\cdots<n_{N-2k-2\ell},\\ j_1<\cdots<j_k,\quad i_b<j_b,\\ j'_1<\cdots<j'_\ell,\quad i'_c<j'_c}} \underbrace{\prod_{a=1}^{N-2k-2\ell}\langle iX_{n_a},\xi\rangle(\mathbb{P}) \cdot\prod_{b=1}^{k}\frac{\xi_0}{2i}de^0(X_{i_b},X_{j_b})(\mathbb{P})\prod_{c=1}^{\ell}\frac{1}{4}F(^\sharp\nabla;\wedge)(X_{i'_c},X_{j'_c})(\mathbb{P})u_{\mathbb{P}}}
\end{aligned}
$$

$$\equiv \sum_{0\le 2k\le N} \sum \underbrace{\prod_{a=1}^{N-2k} \langle iX_{n_a}, \xi\rangle(\mathbb{P}) \prod_{b=1}^{k} \frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(X_{i_b}, X_{j_b})(\mathbb{P}) v_{\mathbb{P}}.}_{\text{grading} = \sum m_{X_j}}$$

$$\left(\begin{array}{c} \{n_1,\dots,n_{N-2k}, i_1,\dots,i_k, j_1,\dots,j_k\} = \{1,\dots,N\}, \\ n_1 < \cdots < n_{N-2k}, \\ j_1 < \cdots < j_k,\ \ i_b < j_b \end{array}\right)$$

That is, we get the formula (3.12). □

Let us prove Theorem 3.5.

Proof of Theorem 3.5. The top grading part of (3.11) is obviously equal to that of

$$h(\mathbb{P}) \wedge h'(\mathbb{P}) \wedge \varsigma(\nabla^F_{X_1}\nabla^F_{X_2}\cdots\nabla^F_{X_N}\,\nabla^F_{X'_1}\nabla^F_{X'_2}\cdots\nabla^F_{X'_{N'}})(\mathbb{P},\xi).$$

Hence, it suffices to examine (3.11) with $h = h' = 1$, and Proposition 3.6 implies that its top grading part is equal to that of

$$\sum_{k,k',\ell} \sum \prod_{a=1}^{N-2k-\ell} \langle iX_{n_a}, \xi\rangle(\mathbb{P}) \prod_{a'=1}^{N'-2k'-\ell} \langle iX'_{n'_{a'}}, \xi\rangle(\mathbb{P})$$
$$\cdot \prod_{c=1}^{\ell} \frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(X_{m_c}, X'_{m'_c})(\mathbb{P})$$
$$\cdot \prod_{b=1}^{k} \frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(X_{i_b}, X_{j_b})(\mathbb{P}) \prod_{b'=1}^{k'} \frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(X'_{i'_{b'}}, X'_{j'_{b'}})(\mathbb{P})$$

$$\left(\begin{array}{c} \text{grading} = \sum m_{X_j} + \sum m_{X'_{j'}} \\ \{n_1,\dots,n_{N-2k-\ell}, m_1,\dots,m_\ell, i_1,\dots,i_k, j_1,\dots,j_k\} = \{1,\dots,N\}, \\ n_1 < \cdots < n_{N-2k-\ell}, \\ j_1 < \cdots < j_k,\ \ i_b < j_b, \\ \{n'_1,\dots,n'_{N'-2k'-\ell}, m'_1,\dots,m'_\ell, i'_1,\dots,i'_{k'}, j'_1,\dots,j'_{k'}\} = \{1,\dots,N'\}, \\ n'_1 < \cdots < n'_{N'-2k'-\ell},\ \ m'_1 < \cdots < m'_\ell, \\ j'_1 < \cdots < j'_{k'},\ \ i'_b < j'_b \end{array}\right)$$

$$= \sum_{k,k',\ell} \frac{1}{\ell!}\Big(-\frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(e_m, e_{m'})\frac{\partial}{\partial\xi_m}\frac{\partial}{\partial\xi'_{m'}}\Big)^{\ell} \sum \prod_{a=1}^{N-2k} \langle iX_{n_a}, \xi\rangle(\mathbb{P}) \prod_{a'=1}^{N'-2k'} \langle iX'_{n'_{a'}}, \xi'\rangle(\mathbb{P})\Big|_{\xi'=\xi}$$
$$\cdot \prod_{b=1}^{k} \frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(X_{i_b}, X_{j_b})(\mathbb{P}) \prod_{b'=1}^{k'} \frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(X'_{i'_{b'}}, X'_{j'_{b'}})(\mathbb{P})$$
$$= e^{-\frac{1}{4}\mathscr{F}(^\sharp\nabla;\wedge)(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi'})}\varsigma(\nabla^F_{X_1}\cdots\nabla^F_{X_N})(\mathbb{P},\xi) \wedge \varsigma(\nabla^F_{X'_1}\cdots\nabla^F_{X'_{N'}})(\mathbb{P},\xi')\Big|_{\xi'=\xi}.$$

Thus we obtain the formula (3.10). By observing the above calculation, the other parts of the theorem will be obvious now. □

4. The Beals-Greiner formula and Theorem 3.5

Following Beals-Greiner's study ([3, (14.22)]), we know that the composition (3.1) can be written as

$$(p \circ q)(\mathbb{P}, \xi) = \frac{1}{(2\pi)^{2n+1}} \int_{T_{\mathbb{P}}M \times T^*_{\mathbb{P}}M \ni (x,\eta)} e^{-i\langle x,\eta\rangle} p^{(\mathbb{P},e^\bullet)}(0, \xi + \eta)\, q^{(\mathbb{P},e^\bullet)}(x, \sigma(x,\xi))\, dx d\eta. \tag{4.1}$$

This is, in fact, a kind of oscillatory integral (cf. [3, (12.17)–(12.19)]) as in the case of classical symbol calculus, and the equality means accurately that the both sides are equal modulo $\mathcal{SC}^{-\infty}_H$. Originally this is a formula for $p, q \in \mathcal{S}^\infty_H$, by which Beals-Greiner studied the asymptotic expansion of $p \circ q$ in $\mathcal{S}^\infty_H$ ([3, Theorems 14.1, 14.7 and 14.17]). In this section, returning to the formula (4.1) (for $\mathcal{SC}^\infty_H$), we want to propose another proof of Proposition 3.6, hence, of Theorem 3.5.

As was mentioned in Introduction, in the spin manifold case (Getzler [7], Block-Fox [5], Benameur-Heitsch [4]) a composition formula for general symbols was derived almost automatically from the one for polynomial symbols and Widom's formula ([14], [15]). It will be thus natural to expect that, in the contact Riemannian manifold case, so can be a composition formula for general intrinsic symbols $\in \mathcal{SC}^\infty_H$ from Theorem 3.5 and Beals-Greiner's formula ([3]). But the situation is not so simple. In [3], each term of the asymptotic expansion of the composite is expressed as an integral formula. For polynomial symbols each integral one can be changed into polynomial formula as we do in the bellow, but so cannot be for general symbols. Consequently it seems to be difficult at present to extend Theorem 3.5 to the case of general intrinsic symbol.

First, let us prove Corollary 3.3 by using the formula (4.1). Set $Y = \sum Y_j e_j = \sum \mathcal{Y}_j \partial/\partial x_j$, Then (2.8) implies

$$\varsigma(\nabla^F_Y)^{(\mathbb{P},dx_\bullet)}(x,\xi) = \sum i\mathcal{Y}_\ell(x)\xi_\ell + \omega(\nabla^F)(Y)(x),$$
$$\varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x,\sigma) = \sum i\mathcal{Y}_\ell(x) v^{\ell k}(x)\sigma_k + \omega(\nabla^F)(Y)(x) = \sum iY_k(x)\sigma_k + \omega(\nabla^F)(Y)(x).$$

Hence, we have

$$\begin{aligned}
&(\varsigma(\nabla^F_X) \circ \varsigma(\nabla^F_Y))(\mathbb{P},\xi)\\
&= \frac{1}{(2\pi)^{2n+1}} \int e^{-i\langle x,\eta\rangle} \varsigma(\nabla^F_X)^{(\mathbb{P},e^\bullet)}(0, \xi+\eta) \varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x, \sigma(x,\xi))\, dx d\eta\\
&= \frac{1}{(2\pi)^{2n+1}} \int e^{-i\langle x,\eta\rangle} \varsigma(\nabla^F_X)^{(\mathbb{P},e^\bullet)}(0, \xi) \varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x, \sigma(x,\xi))\, dx d\eta\\
&\qquad + \frac{1}{(2\pi)^{2n+1}} \int e^{-i\langle x,\eta\rangle} \sum i\mathcal{X}_\ell(0)\eta_\ell\, \varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x, \sigma(x,\xi))\, dx d\eta\\
&= \varsigma(\nabla^F_X)^{(\mathbb{P},e^\bullet)}(0,\xi)\, \varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(0, \sigma(0,\xi)) + \sum i\mathcal{X}_\ell(0) D_{x_\ell} \varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x, \sigma(x,\xi))\Big|_{x=0}\\
&= \langle iX, \xi\rangle(0)\, \langle iY, \xi\rangle(0) + \sum i\mathcal{X}_\ell(0) D_{x_\ell} \varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x, \sigma(x,\xi))\Big|_{x=0}
\end{aligned}$$

and, referring to (3.2), (1.3), (3.7) and (3.8), we have

$$\begin{aligned}
&D_{x_\ell} \varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x, \sigma(x,\xi))\Big|_{x=0}\\
&= D_{x_\ell} \varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x,\xi)\Big|_{x=0} + \sum \partial_{\sigma_k} \varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(0,\sigma)\Big|_{\sigma=\xi} D_{x_\ell}\sigma_k(x,\xi)\Big|_{x=0}
\end{aligned} \tag{4.2}$$

$$= \Big\{ \sum \partial_{x_\ell}(Y_k)(0)\,\xi_k - \frac{i}{2}F(\nabla^F)(\partial/\partial x_\ell, Y)(0)\Big\} + \sum Y_k(0)\,(\partial_{x_\ell}v_{jk})(0)\xi_j$$

$$= \sum \partial_{x_\ell}(Y_k)(0)\,\xi_k + \sum Y_k(0)\,(\partial_{x_\ell}v_{jk})(0)\xi_j - \frac{i}{2}F(\nabla^F)(\partial/\partial x_\ell, Y)(0),$$

$$\sum i\mathcal{X}_\ell(0)\Big\{ \sum Y_k(0)\,(\partial_{x_\ell}v_{0k})(0)\xi_0 - \frac{i}{2}F(\nabla^F)(\partial/\partial x_\ell, Y)(0)\Big\} \tag{4.3}$$

$$= i\xi_0 \sum X(v_{0k})(0)\,Y_k(0) + \frac{1}{2}F(\nabla^F)(X,Y)(0)$$

$$= \frac{i\xi_0}{2}\sum \Big\{\mathcal{X}_{n+\beta}(0)\,\mathcal{Y}_\beta(0) - \mathcal{X}_\beta(0)\,\mathcal{Y}_{n+\beta}(0)\Big\} + \frac{1}{2}F(\nabla^F)(X,Y)(0)$$

$$= \frac{\xi_0}{2i}\,de^0(X_{\mathbb{P}}, Y_{\mathbb{P}}) + \frac{1}{2}F(\nabla^F)(X,Y)(0)$$

$$= \frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(X,Y)(0) + \frac{1}{2}F(\nabla^E)(X,Y)(0).$$

Thus we know that the top grading part of $(\varsigma(\nabla^F_X)\circ\varsigma(\nabla^F_Y))(\mathbb{P},\xi)$ is equal to that of

$$\langle iX,\xi\rangle(0)\,\langle iY,\xi\rangle(0) + \frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(X,Y)(0).$$

That is, Corollary 3.3 was proved.

Next, let us prove Proposition 3.6. We assume that the formula (3.12) holds and investigate $\varsigma(\nabla^F_{X_1}\nabla^F_{X_2}\cdots\nabla^F_{X_N}\nabla^F_{X_{N+1}})(0,\xi)$. Set $Y = X_{N+1}$. Then the formula (4.1) yields

$$\varsigma(\nabla^F_{X_1}\nabla^F_{X_2}\cdots\nabla^F_{X_N}\nabla^F_{X_{N+1}})(0,\xi) = (\varsigma(\nabla^F_{X_1}\nabla^F_{X_2}\cdots\nabla^F_{X_N})\circ\varsigma(\nabla^F_Y))(0,\xi)$$

$$= \frac{1}{(2\pi)^{2n+1}}\int e^{-i\langle x,\eta\rangle}\varsigma(\nabla^F_{X_1}\nabla^F_{X_2}\cdots\nabla^F_{X_N})^{(\mathbb{P},e^\bullet)}(0,\xi+\eta)\varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x,\sigma(x,\xi))\,dxd\eta$$

$$\equiv \sum_{0\le 2k\le N}\sum\prod_{a=1}^{N-2k}\langle iX_{n_a,\mathbb{P}},\xi+D_x\rangle\prod_{b=1}^{k}\frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(X_{i_b},X_{j_b})(0,\xi_0+D_{x_0})$$
$$\varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x,\sigma(x,\xi))\Big|_{x=0}$$

$$\equiv \sum\prod_{a=1}^{N-2k}\langle iX_{n_a,\mathbb{P}},\xi\rangle\cdot\sum_{\ell=1}^{N-2k}\langle iX_{n_\ell,\mathbb{P}},\xi\rangle^{-1}\langle iX_{n_\ell,\mathbb{P}},D_x\rangle\varsigma(\nabla^F_Y)^{(\mathbb{P},e^\bullet)}(x,\sigma(x,\xi))\Big|_{x=0}$$
$$\cdot\prod_{b=1}^{k}\frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(X_{i_b},X_{j_b})(0,\xi_0)$$

$$\equiv \sum\prod_{a=1}^{N-2k}\langle iX_{n_a,\mathbb{P}},\xi\rangle$$
$$\cdot\sum_{\ell=1}^{N-2k}\langle iX_{n_\ell,\mathbb{P}},\xi\rangle^{-1}\Big\{\langle iX_{n_\ell,\mathbb{P}},\xi\rangle\langle iY_{\mathbb{P}},\xi\rangle + \frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(X_{n_\ell,\mathbb{P}},Y)(\mathbb{P})\Big\}$$
$$\cdot\prod_{b=1}^{k}\frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(X_{i_b},X_{j_b})(0,\xi_0)$$

$$= \sum_{0\le 2k\le N+1}\sum\prod_{a=1}^{N+1-2k}\langle iX_{n_a,\mathbb{P}},\xi\rangle\prod_{b=1}^{k}\frac{1}{4}\mathscr{F}({}^\sharp\nabla;\wedge)(X_{i_b},X_{j_b})(0,\xi_0).$$

Here the second $\equiv$ comes from the fact that

$$\frac{1}{(2\pi)^{2n+1}}\int e^{-i\langle x,\eta\rangle}\overbrace{\frac{\eta_0}{2i}\,de^0(X_{i_b,\mathbb{P}},X_{j_b,\mathbb{P}})}^{\text{grading}=2}\varsigma(\nabla_Y^F)^{(\mathbb{P},e^\bullet)}(x,\sigma(x,\xi))\,dx\eta$$
$$=\frac{1}{2i}\,de^0(X_{i_b,\mathbb{P}},X_{j_b,\mathbb{P}})\,D_{x_0}\varsigma(\nabla_Y^F)^{(\mathbb{P},e^\bullet)}(x,\sigma(x,\xi))\Big|_{x=0}$$

is of grading $< 2+m_Y$, and the third $\equiv$ is implied by (4.2) and (4.3). Inductively Proposition 3.6 is ascertained and, hence, we may remark that Theorem 3.5 is proved also by using the formula (4.1).

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