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THE HAUSDORFF DIMENSION OF SOME PLANAR SETS WITH UNBOUNDED DIGITS

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Abstract

We consider some parameterized planar sets with unbounded digits. We investigate these sets by using the method of "transversality", which is the main tool in investigating self-similar sets with overlaps. We calculate the Hausdorff dimension of these sets for typical parameters in some region with respect to the 2-dimensional Lebesgue measure. In addition, we estimate the local dimension of the exceptional set of parameters.

Contents

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1. Introduction

1.1. Planar sets generated by pairs of linear maps. We consider the following planar sets $A(\lambda)$ for $\lambda \in \mathbb{D}^*$, where $\mathbb{D}^* := \{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$:

$$A(\lambda) := \left\{ \sum_{j=0}^{\infty} a_j \lambda^j : a_j \in \{0, 1\} \right\}.$$

These sets have fractal structure. Indeed, the sets $A(\lambda)$ are generated by the iterated function systems $\{\lambda z, \lambda z + 1\}$ on the complex plane. For the general theory of the iterated function system (for short, IFS), see [4]. In order to discuss these sets, we introduce a set of functions $\mathcal{F}$ and a set of zeros in $\mathbb{D}^*$ for functions in $\mathcal{F}$:

$$\mathcal{F} := \left\{ f(\lambda) = 1 + \sum_{j=1}^{\infty} a_j \lambda^j : a_j \in \{-1, 0, 1\} \right\},$$
$$\mathcal{M} := \{\lambda \in \mathbb{D}^* : \text{there exists } f \in \mathcal{F} \text{ such that } f(\lambda) = 0\}.$$

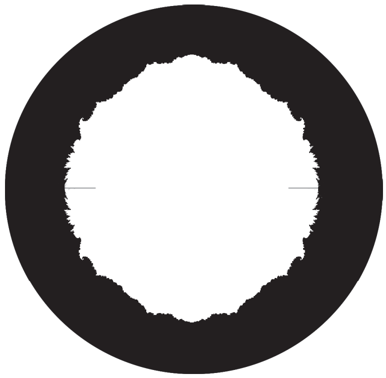

Fig. 1. $\mathcal{M}$

The set $\mathcal{M}$ is known as *the Mandelbrot set for pairs of linear maps* (see [1], [2] and Fig. 1). Note that

$$\left\{ \lambda \in \mathbb{D}^* : \frac{1}{\sqrt{2}} < |\lambda| < 1 \right\} \subset \mathcal{M} \subset \left\{ \lambda \in \mathbb{D}^* : \frac{1}{2} < |\lambda| < 1 \right\} \tag{1}$$

(see [16, p. 538 (6)]).

We set $f_1(z) = \lambda z$ and $f_2(z) = \lambda z + 1$. We say that the IFS $\{f_1, f_2\}$ satisfies *the open set condition* if there exists a non-empty bounded open set V such that $f_1(V) \cap f_2(V) = \emptyset$ and $f_i(V) \subset V$ for all $i \in \{1, 2\}$. If λ is not an element of $\mathcal{M}$, the corresponding IFS satisfies the open set condition, and hence we have that the Hausdorff dimension of $A(\lambda)$ is equal to $-\log 2/\log|\lambda|$ (see [4, Theorem 9.3]). However, in general, it is difficult to estimate the Hausdorff dimension of $A(\lambda)$ if λ is an element of $\mathcal{M}$. We set

$$\tilde{\mathcal{M}} := \{\lambda \in \mathbb{D}^* : \text{there exists } f \in \mathcal{F} \text{ such that } f(\lambda) = f'(\lambda) = 0\} \ (\subset \mathcal{M}).$$

For any set $A \subset \mathbb{C}$, we denote by $\dim_H(A)$ the Hausdorff dimension of A with respect to the

Euclidean norm $|\cdot|$. We denote by $\mathcal{L}$ the 2-dimensional Lebesgue measure. The following holds by [16, Theorem 2.2] and [17, Proposition 2.7].

Theorem 1.1.

$$\dim_H(A(\lambda)) = \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 0 < |\lambda| < 1/\sqrt{2}\}; \tag{2}$$

$$\mathcal{L}(A(\lambda)) > 0 \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 1/\sqrt{2} < |\lambda| < 1\} \backslash \tilde{\mathcal{M}}. \tag{3}$$

REMARK 1.2. 1. It is well known that $\dim_H(A(\lambda)) \le \log 2/-\log|\lambda|$ for all λ (see [4, Proposition 9.6]).

2. In [16, Theorem 2.2], Solomyak deals with more general self-similar sets in the plane. However, the statement of the result are essentially same as in Theorem 1.1.

3. The proof of [17, Proposition 2.7] essentially depends on [3, Theorem 2].

The local dimension of the exceptional set of parameters is estimated as the following.

Theorem 1.3 ([11, Theorem 8.2])**.** *For any* $0 < r < R < 1/\sqrt{2}$,

$$\dim_H\left(\left\{\lambda \in \mathbb{D}^* : r < |\lambda| < R,\ \dim_H(A(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \le \frac{\log 2}{-\log R} < 2.$$

REMARK 1.4. Solomyak proved that $\dim_H(A(\lambda)) < \log 2/-\log|\lambda|$ for λ in a dense subset of $\{\lambda \in \mathbb{D}^* : 0 < |\lambda| < 1/\sqrt{2}\}$ in [16, Proposition 2.3].

For further results about dimensions and measures on $A(\lambda)$, see [17].

1.2. Planar sets with unbounded digits. In this paper, we consider the following sets $A_0(\lambda)$ for $\lambda \in \mathbb{D}^*$:

$$A_0(\lambda) := \left\{\sum_{j=0}^{\infty} a_j\lambda^j : a_j \in \{0, p_j\}\right\},$$

where $1 \le p_j \in \mathbb{R}$ for all $j \in \mathbb{N}_0$, $p_j \to \infty$ as $j \to \infty$ and $\{p_j\}_{j=0}^{\infty}$ satisfies the condition

$$\frac{p_{j+1}}{p_j} \to 1 \text{ as } j \to \infty.$$

Note that the sets $A_0(\lambda)$ depend on the sequence $\{p_j\}_{j=0}^{\infty}$ and these sets are well-defined by the above condition (see Remark 3.1).

We are motivated by the theory of the non-autonomous iterated function system (for short, NIFS). Here, an NIFS is some family of contracting maps $\{f_{1,j}, f_{2,j}, \dots, f_{n_j,j}\}_{j=0}^{\infty}$. As examples of studies of NIFSs on **a compact metric space**, see [5], [13]. Inui [6] gave the methods to construct "the limit set" of an NIFS on **a complete metric space**. The set $A_0(\lambda)$ is the limit set of the NIFS $\{f_{1,j}, f_{2,j}\} := \{\lambda z, \lambda z + p_j\}_{j=0}^{\infty}$ as the following.

Theorem 1.5 ([6, Theorem 1.11])**.** *Let* $\mathcal{K}(\mathbb{C})$ *be the set of all non-empty compact subsets of* $\mathbb{C}$ *and let* d_H *be the Hausdorff distance on* $\mathcal{K}(\mathbb{C})$*. We define* $A_0(\lambda) = \left\{\sum_{j=0}^{\infty} a_j\lambda^j : a_j \in \{0, p_j\}\right\}$*. For each* $j \in \mathbb{N}_0$*, we define the map* $F_j : \mathcal{K}(\mathbb{C}) \to \mathcal{K}(\mathbb{C})$ *by*

$$F_j(A) := f_{1,j}(A) \cup f_{2,j}(A)$$

for $A \in \mathcal{K}(\mathbb{C})$. Then for any $A \in \mathcal{K}(\mathbb{C})$,

$$\lim_{j\to\infty} d_H(F_0 \circ F_1 \circ \cdots \circ F_j(A), A_0(\lambda)) \to 0.$$

Note that there does not exist a compact subset $X \subset \mathbb{C}$ such that for each j, $f_{2,j}(X) \subset X$ since the set of digits $\{p_j : j \in \mathbb{N}_0\}$ is **not bounded**. One of the aims in this paper is to establish some methods to estimate the Hausdorff dimension of limit sets of NIFSs on **a non-compact metric space** via studying examples. We give the main results, which are analogues of Theorem 1.1 and Theorem 1.3.

Main result A (Theorem 5.11)**.**

$$\dim_H(A_0(\lambda)) = \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 0 < |\lambda| < 1/\sqrt{2}\};$$

$$\mathcal{L}(A_0(\lambda)) > 0 \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 1/\sqrt{2} < |\lambda| < 1\}\backslash\tilde{\mathcal{M}}.$$

Main result B (Theorem 5.14)**.** *For any* $0 < R < 1/\sqrt{2}$,

$$\dim_H\left(\left\{\lambda \in \mathbb{D}^* : 0 < |\lambda| < R,\ \dim_H(A_0(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \le \frac{\log 2}{-\log R} < 2.$$

In order to prove our results, we use the method of "transversality". Here, for a parameterized family of functions, the "transversality" means a condition which controls the way the functions depend on parameters. Usually, we call the set of parameters "the transversality region". The method of transversality is used for self-similar sets with overlaps (e.g., [12], [16], [8], [9]), for self-similar measures (e.g., [15]) and for some general family of functions (e.g., [14], [10], [18]). Note that their setting depend on the compactness of the whole space. Hence we cannot apply their framework or methods to our setting since the set of digits $\{p_j : j \in \mathbb{N}_0\}$ is not bounded.

1.3. A strategy for the proof of the main results. In Section 3, we define a metric $\rho_{n,m}$ (see Definition 3.3) on a symbolic space I^∞ so that the Hausdorff dimension of I^∞ is equal to 1 with respect to $\rho_{n,m}$ for each $m, n \in \mathbb{N}_0$ (see Proposition 3.5). For each $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{D}^*$, we define $A_n(\lambda) = \left\{\sum_{j=0}^{\infty} a_j\lambda^j : a_j \in \{0, p_{n+j}\}\right\}$. For each $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{D}^*$, we define the address map $\pi_{n,\lambda} : I^\infty \to \mathbb{C}$ (see Definition 3.6) so that $\pi_{n,\lambda}(I^\infty) = A_n(\lambda)$. For each $n \in \mathbb{N}_0$, we define a set of double zeros of some power series $\tilde{\mathcal{M}}_n$ related to the address map $\pi_{n,\lambda}$ so that $\bigcap_{n\ge 0}\tilde{\mathcal{M}}_n = \tilde{\mathcal{M}}$ (see Definition 3.10 and Lemma 3.12). Then for each $\lambda \in \mathbb{D}^*$, there exists $m_0 \in \mathbb{N}$ such that $\pi_{n,\lambda}$ is $(-\log|\lambda|/\log 2)$-Hölder continuous with respect to ρ_{n,m_0} (see Lemma 3.14), which implies the upper estimation of the Hausdorff dimension of $A_0(\lambda)$.

In Section 4, we give some lemmas in order to estimate the Hausdorff dimension. In addition, we give a technical lemma for the transversality (Lemma 4.10).

In Section 5, we give the key lemmas (Lemmas 5.6 and 5.7), which imply the lower estimation of the Hausdorff dimension of $A_n(\lambda)$ for typical parameters λ with respect to $\mathcal{L}$ on $\mathbb{D}^*\backslash\tilde{\mathcal{M}}_n$ (Theorem 5.8) and the estimation of local dimension of the exceptional set of parameters (Theorem 5.14). Here, we use $\dim_H(A_0(\lambda)) = \dim_H(A_n(\lambda))$, $\mathcal{L}(A_0(\lambda)) \ge |\lambda|^{2n}\mathcal{L}(A_n(\lambda))$ (Corollary 3.8) and $\bigcap_{n\ge 0}\tilde{\mathcal{M}}_n = \tilde{\mathcal{M}}$ (Lemma 3.12).

2. Notation and conventions

- $\mathbb{N} := \{1, 2, 3, ...\}$.
- $\mathbb{N}_0 := \{0, 1, 2, ...\}$.
- $\mathbb{R}$: the set of all real numbers.
- $\mathbb{C}$: the set of all complex numbers.
- Usually, we identify $\mathbb{C}$ with $\mathbb{R}^2$. For $\lambda \in \mathbb{C}$, we denote by $|\lambda|$ the Euclidean norm of $\lambda \in \mathbb{R}^2$.
- $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.
- $\mathbb{D}^* := \{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$.
- For any set $A \subset \mathbb{C}$, we denote by $\dim_H(A)$ the Hausdorff dimension of A with respect to the Euclidean norm $|\cdot|$.
- $\mathcal{L}$: the 2-dimensional Lebesgue measure on $\mathbb{C}$.
- For each $j \in \mathbb{N}_0$, let $G_j \subset \mathbb{R}$. Let $\lambda \in \mathbb{D}^*$. We use $\left\{\sum_{j=0}^{\infty} a_j\lambda^j : a_j \in G_j\right\}$ to denote $\left\{\sum_{j=0}^{\infty} a_j\lambda^j : \text{for each } j \in \mathbb{N}_0, a_j \in G_j\right\}$.
- If X and Y are topological spaces, and $f : X \to Y$ is any Borel measurable map, then for any Borel measure μ on X, we define $f\mu$ as the push-forward measure $\mu \circ f^{-1}$.
- Let X be a topological space, let X_0 be a Borel measurable subspace of X and let m be a Borel measure on X_0. If we set $\tilde{m}(B) := m(B \cap X_0)$ for any Borel subset $B \subset X$, then $\tilde{m}$ is a Borel measure on X. We also denote by m the measure $\tilde{m}$.
- Let (X, d) be a metric space and let x be a point in X. For any $r > 0$, we denote by $B(x, r)$ the set $\{y \in X : d(x, y) < r\}$. For any set $A \subset X$, we denote by $\mathrm{cl}(A)$ the topological closure of A.

3. Preliminaries

3.1. On the symbolic space. We deal with the digits $\{p_j\}_{j=0}^{\infty}$ satisfying the following conditions:

- For each $j \in \mathbb{N}_0$, $p_j \geq 1$;
- $p_j \to \infty$ as $j \to \infty$;
- $p_{j+1}/p_j \to 1$ as $j \to \infty$.

The above conditions imply the following.

REMARK 3.1. 1. For each $n \in \mathbb{N}$, $p_{j+n}/p_j \to 1$ as $j \to \infty$.
2. Let $a > 1$ and $b > 0$. For each $n \in \mathbb{N}$, $(p_{j+n})^b/a^j \to 0$ as $j \to \infty$.

We set $I := \{0, 1\}$. For each $\omega = \omega_0\omega_1 \cdots \in I^\infty$ and $k \in \mathbb{N}$, we set $\omega|_k := \omega_0\omega_1 \cdots \omega_{k-1} \in I^k$. For each $\omega = \omega_0\omega_1 \cdots \omega_{k-1} \in I^k$, we denote by $[\omega]$ the set $\{\tau \in I^\infty : \tau_0 = \omega_0, \tau_1 = \omega_1, ..., \tau_{k-1} = \omega_{k-1}\}$. For each $\omega = \omega_0\omega_1 \cdots, \tau = \tau_0\tau_1 \cdots \in I^\infty$, we define $|\omega \wedge \tau| := \inf\{j \in \mathbb{N}_0 : \omega_j \neq \tau_j\}$.

Proposition 3.2. *Let $m, n \in \mathbb{N}_0$. Then there exists minimum $j_{n,m} \in \mathbb{N}_0$ such that for all $j_1 \geq j_2 \geq j_{n,m}$, $(p_{j_1+n})^m/2^{j_1} \leq (p_{j_2+n})^m/2^{j_2}$.*

Proof. Since for each $n \in \mathbb{N}_0$, $(p_{j+1+n})^m/(p_{j+n})^m \to 1$ as $j \to \infty$, there exists $k_{n,m} \in \mathbb{N}_0$ such that for each $j \geq k_{n,m}$,

$$2 \geq \frac{(p_{j+1+n})^m}{(p_{j+n})^m}.$$

Hence for any $j_1 = j_2 + l \geq j_2 \geq k_{n,m}$,

$$2 \geq \frac{(p_{j_2+1+n})^m}{(p_{j_2+n})^m},\ 2 \geq \frac{(p_{j_2+2+n})^m}{(p_{j_2+1+n})^m},\ \ldots,\ 2 \geq \frac{(p_{j_2+l+n})^m}{(p_{j_2+(l-1)+n})^m}.$$

Thus we have that

$$\frac{2^{j_1}}{2^{j_2}} = 2^l \geq \frac{(p_{j_1+n})^m}{(p_{j_2+n})^m}.$$ □

By Proposition 3.2, we define the metric $\rho_{n,m}$ on I^∞ as the following.

DEFINITION 3.3. Let $m, n \in \mathbb{N}_0$. We define the metric $\rho_{n,m}$ on I^∞ by

$$\rho_{n,m}(\omega, \tau) := \begin{cases} K_{n,m} & (|\omega \wedge \tau| \leq j_{n,m}) \\ \dfrac{(p_{|\omega\wedge\tau|+n})^m}{2^{|\omega\wedge\tau|}} & (|\omega \wedge \tau| > j_{n,m}) \end{cases}$$

for each $\omega, \tau \in I^\infty$. Here, $K_{n,m} = (p_{j_{n,m}+n})^m/2^{j_{n,m}}$.

REMARK 3.4. 1. The metric space $(I^\infty, \rho_{n,m})$ is a compact metric space for each $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$.

2. $\rho_{n,0}(\omega, \tau) = 1/2^{|\omega\wedge\tau|}$ for each $\omega, \tau \in I^\infty$.

Let X be a metric space endowed with a metric ρ. Let $A \subset X$. We define $|A|_\rho := \sup\{\rho(x, y) : x, y \in A\}$. For each $t \geq 0$ and $\delta > 0$, we set

$$\mathcal{H}^t_{\rho,\delta}(A) := \inf\left\{\sum_{i=1}^{\infty} |U_i|^t_\rho : A \subset \bigcup_{i=1}^{\infty} U_i, |U_i| \leq \delta \text{ for } U_i \subset X\right\}.$$

We define *the t-dimensional Hausdorff outer measure of A with respect to ρ* as

$$\mathcal{H}^t_\rho(A) := \lim_{\delta\to 0} \mathcal{H}^t_{\rho,\delta}(A) \in [0, \infty].$$

For any set $A \subset X$, we define *the Hausdorff dimension of A with respect to ρ* as

$$\dim_\rho(A) := \sup\{t \geq 0 : \mathcal{H}^t_\rho(A) = \infty\} = \inf\{t \geq 0 : \mathcal{H}^t_\rho(A) = 0\}.$$

We compute the Hausdorff dimension of I^∞ with respect to $\rho_{n,m}$ as the following.

Proposition 3.5. *For each $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$,* $\dim_{\rho_{n,m}}(I^\infty) = 1$.

Proof. Let μ be a probability measure on I^∞ such that

$$\mu([\omega_0\omega_1 \cdots \omega_{j-1}]) = \frac{1}{2^j}$$

for each $\omega_0\omega_1 \cdots \omega_{j-1} \in I^j$ (μ is the $(1/2, 1/2)$-Bernoulli measure on I^∞). Fix $m \in \mathbb{N}_0$. Then we have that for any $\omega \in I^j$ with $j > j_{n,m}$,

$$\mu\left(\left\{\tau \in I^\infty : \rho_{n,m}(\omega,\tau) \le \frac{(p_{j+n})^m}{2^j}\right\}\right) = \mu([\omega_0\omega_1\cdots\omega_{j-1}]) = \frac{1}{2^j}$$
$$\le \left|\left\{\tau \in I^\infty : \rho_{n,m}(\omega,\tau) \le \frac{(p_{j+n})^m}{2^j}\right\}\right|^1_{\rho_{n,m}} \left(= \frac{(p_{j+n})^m}{2^j}\right)$$

By the mass distribution principle (see [4, p. 67]), we have that $1 \le \dim_{\rho_{n,m}}(I^\infty)$.

We prove that for each $m \in \mathbb{N}_0$, $\dim_{\rho_{n,m}}(I^\infty) \le 1$. For any $\epsilon > 0$ and $j > j_{n,m}$, since the family of sets $\{[\omega]\}_{\omega \in I^j}$ is a covering for I^∞, we have that

$$\mathcal{H}^{1+\epsilon}_{\rho_{n,m},(p_{j+n})^m/2^j}(I^\infty) \le \sum_{\omega \in I^j} |[\omega]|^{1+\epsilon}_{\rho_{n,m}} = 2^j \frac{(p_{j+n})^{m(1+\epsilon)}}{2^{j(1+\epsilon)}} \to 0 \text{ as } j \to \infty.$$

Hence we have that $\mathcal{H}^{1+\epsilon}_{\rho_{n,m}}(I^\infty) = 0$ and hence $\dim_{\rho_{n,m}}(I^\infty) \le 1 + \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that $\dim_{\rho_{n,m}}(I^\infty) \le 1$.

Hence we have proved our proposition. □

3.2. Address maps. We now define address maps as follows.

DEFINITION 3.6. For each $\lambda \in \mathbb{D}^*$ and $n \in \mathbb{N}_0$, we define the address map $\pi_{n,\lambda} : I^\infty \to \mathbb{C}$ by

$$\pi_{n,\lambda}(\omega) := \sum_{j=0}^{\infty} p_{n+j}\omega_j\lambda^j$$

($\omega = \omega_0\omega_1 \cdots \in I^\infty$). Note that this map is well-defined.

Then we have that

$$\pi_{n,\lambda}(I^\infty) = \left\{\sum_{j=0}^{\infty} a_j\lambda^j : a_j \in \{0, p_{n+j}\}\right\}.$$

In particular, $A_0(\lambda) = \pi_{0,\lambda}(I^\infty)$. Below we set $A_n(\lambda) := \pi_{n,\lambda}(I^\infty)$. We give the following proposition.

Proposition 3.7. *For each $n \in \mathbb{N}_0$, if we set $\phi_{n,\lambda}(z) := \lambda z, \varphi_{n,\lambda}(z) := \lambda z + p_n$, then*

$$A_n(\lambda) = \phi_{n,\lambda}(A_{n+1}(\lambda)) \cup \varphi_{n,\lambda}(A_{n+1}(\lambda)).$$

Proof.

$$\phi_{n,\lambda}(A_{n+1}(\lambda)) \cup \varphi_{n,\lambda}(A_{n+1}(\lambda)) = \left\{\lambda\left(\sum_{j=0}^{\infty} p_{n+j+1}\omega_j\lambda^j\right) + 0 : \omega_j \in \{0,1\}\right\}$$
$$\cup \left\{\lambda\left(\sum_{j=0}^{\infty} p_{n+j+1}\omega_j\lambda^j\right) + p_n : \omega_j \in \{0,1\}\right\}$$
$$= \left\{\sum_{j=0}^{\infty} p_{n+j}\omega_j\lambda^j : \omega_j \in \{0,1\}\right\} = A_n(\lambda).$$ □

Corollary 3.8.

$$\dim_H(A_0(\lambda)) = \dim_H(A_n(\lambda));$$
$$\mathcal{L}(A_0(\lambda)) \geq |\lambda|^{2n}\mathcal{L}(A_n(\lambda)).$$

Proof. By Proposition 3.7, we have that for each $n \in \mathbb{N}_0$,

$$\begin{aligned}\dim_H(A_n(\lambda)) &= \max\{\dim_H(\phi_{n,\lambda}(A_{n+1}(\lambda))), \dim_H(\varphi_{n,\lambda}(A_{n+1}(\lambda)))\} \\ &= \max\{\dim_H(A_{n+1}(\lambda)), \dim_H(A_{n+1}(\lambda))\} = \dim_H(A_{n+1}(\lambda))\end{aligned}$$

and

$$\mathcal{L}(A_n(\lambda)) \geq \mathcal{L}(\phi_{n,\lambda}(A_{n+1}(\lambda))) = |\lambda|^2\mathcal{L}(A_{n+1}(\lambda)). \qquad \square$$

3.3. Sets of some power series. In this subsection, we introduce sets of some power series and the sets of double zeros. For each $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we set

$$G_{n,j} := \bigcup_{m \geq n} \left\{ \frac{-p_{m+j}}{p_m}, 0, \frac{p_{m+j}}{p_m} \right\} \cup \{-1, 1\}.$$

For each $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$, the set $G_{n,j}$ is a compact subset in $\mathbb{R}$ since p_{m+j}/p_m tends to 1 as $m \to \infty$. If we set $b_{n,j} := \max G_{n,j} < \infty$, there exists $m_{n,j} \geq n$ such that $b_{n,j} = p_{m_{n,j}+j}/p_{m_{n,j}}$.

Lemma 3.9.

$$\lim_{j\to\infty} \frac{1}{j} \log b_{n,j} = 0.$$

Proof.

$$\begin{aligned}\log b_{n,j} &= \log \frac{p_{m_{n,j}+j}}{p_{m_{n,j}}} \\ &= \log\left(\frac{p_{m_{n,j}+1}}{p_{m_{n,j}}} \frac{p_{m_{n,j}+2}}{p_{m_{n,j}+1}} \frac{p_{m_{n,j}+3}}{p_{m_{n,j}+2}} \cdots \frac{p_{m_{n,j}+j}}{p_{m_{n,j}+(j-1)}} \right) \\ &= \sum_{k=0}^{j-1} \log \frac{p_{(m_{n,j}+k)+1}}{p_{m_{n,j}+k}}.\end{aligned}$$

For any $\epsilon > 0$, there exists $j_1 \in \mathbb{N}$ such that for any $j \geq j_1$,

$$\log \frac{p_{j+1}}{p_j} < \epsilon$$

since $p_{j+1}/p_j \to 1$ as $j \to \infty$. In addition, there exists $j_2 \in \mathbb{N}$ with $j_2 \geq j_1$ such that for any $j \geq j_2$,

$$\frac{(j_1+1)}{j} \log \frac{p_{m_{n,1}+1}}{p_{m_{n,1}}} < \epsilon.$$

Since $p_{m+1}/p_m \leq p_{m_{n,1}+1}/p_{m_{n,1}}$ for any $m \geq n$, we have that for any $j \geq j_2$,

$$0 \leq \frac{1}{j}\log b_{n,j} = \frac{1}{j}\left(\sum_{k=0}^{j_1} \log \frac{p_{(m_{n,j}+k)+1}}{p_{m_{n,j}+k}} + \sum_{k=j_1+1}^{j} \log \frac{p_{(m_{n,j}+k)+1}}{p_{m_{n,j}+k}} \right)$$

$$\leq \frac{(j_1+1)}{j}\log\frac{p_{m_{n,1}+1}}{p_{m_{n,1}}} + \frac{(j-j_1)\epsilon}{j} < 2\epsilon. \qquad \square$$

By Lemma 3.9, the function

$$\lambda \mapsto C_n(\lambda) := \sum_{j=0}^{\infty} b_{n,j}|\lambda|^j$$

is well-defined on $\mathbb{D}$. We define the following sets.

Definition 3.10. For each $n \in \mathbb{N}_0$, we set

$$\mathcal{F}_n := \left\{ f(\lambda) = \pm 1 + \sum_{j=1}^{\infty} a_{n,j}\lambda^j : a_{n,j} \in G_{n,j} \right\},$$
$$\tilde{\mathcal{M}}_n := \{\lambda \in \mathbb{D}^* \ : \text{there exists } f \in \mathcal{F}_n \text{ such that } f(\lambda) = f'(\lambda) = 0\},$$
$$\mathcal{F} := \left\{ f(\lambda) = \pm 1 + \sum_{j=1}^{\infty} a_j\lambda^j : a_j \in \{-1,0,1\} \right\},$$
$$\tilde{\mathcal{M}} := \{\lambda \in \mathbb{D}^* \ : \text{there exists } f \in \mathcal{F} \text{ such that } f(\lambda) = f'(\lambda) = 0\}.$$

Remark 3.11. For any $n \in \mathbb{N}_0$, the sets $\mathcal{F}_n$ and $\mathcal{F}$ are compact subsets of the space of holomorphic functions on $\mathbb{D}$ endowed with the compact open topology.

Lemma 3.12.

$$\bigcap_{n\geq 0} \tilde{\mathcal{M}}_n = \tilde{\mathcal{M}}.$$

Proof. Since for all $n \in \mathbb{N}_0$,

$$\mathcal{F}_n \supset \mathcal{F}$$

we have that

$$\bigcap_{n\geq 0} \tilde{\mathcal{M}}_n \supset \tilde{\mathcal{M}}.$$

Fix $z_0 \in \bigcap_{n\geq 0} \tilde{\mathcal{M}}_n$. Then for each $n \in \mathbb{N}_0$, there exists $f_n \in \mathcal{F}_n$ such that $f_n(z_0) = f_n'(z_0) = 0$. Here,

$$f_n(\lambda) = 1 + \sum_{j=1}^{\infty} \alpha_{n,j}\lambda^j,$$

where

$$\alpha_{n,j} = \frac{p_{m_{n,j}+j}a_{n,j}}{p_{m_{n,j}}} \text{ or } a_{n,j}$$

$(a_{n,j} \in \{-1,0,1\}, m_{n,j} \geq n$ for each $j \in \mathbb{N})$. For each $n \in \mathbb{N}_0$, we set

$$g_n(\lambda) := 1 + \sum_{j=1}^{\infty} a_{n,j}\lambda^j \in \mathcal{F}.$$

Then there exists a sub-sequence $\{g_{n_k}\}$ and $g \in \mathcal{F}$ s.t.

$$g_{n_k} \to g \text{ on every compact subset of } \mathbb{D} \text{ as } k \to \infty$$

since $\mathcal{F}$ is compact.

Then we have that

$$|f_{n_k}(z_0) - g_{n_k}(z_0)| = \left| \left(1 + \sum_{j=1}^{\infty} \alpha_{n_k,j} {z_0}^j \right) - \left(1 + \sum_{j=1}^{\infty} a_{n_k,j} {z_0}^j \right) \right| \leq \sum_{j=1}^{\infty} |\alpha_{n_k,j} - a_{n_k,j}||z_0|^j.$$

Since $f_{n_k}(z_0) = 0$ and the last term tends to 0 as $k \to \infty$, we have that

$$g(z_0) = 0.$$

In addition,

$$|f'_{n_k}(z_0) - g'_{n_k}(z_0)| = \left| \left(\sum_{j=1}^{\infty} j\alpha_{n_k,j} {z_0}^{j-1} \right) - \left(\sum_{j=1}^{\infty} j a_{n_k,j} {z_0}^{j-1} \right) \right| \leq \sum_{j=1}^{\infty} j|\alpha_{n_k,j} - a_{n_k,j}||z_0|^{j-1}.$$

Since $f'_{n_k}(z_0) = 0$ and the last term tends to 0 as $k \to \infty$, we have that

$$g'(z_0) = 0.$$

Hence we have that $z_0 \in \tilde{\mathcal{M}}$. □

3.4. The upper estimation of the Hausdorff dimension.

Proposition 3.13. *Let $n \in \mathbb{N}_0$. For any $\omega \neq \tau \in I^\infty$ and for any $\lambda \in \mathbb{D}^*$, there exists $f_{n,\omega,\tau} \in \mathcal{F}_n$ such that*

$$\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau) = \lambda^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n} f_{n,\omega,\tau}(\lambda).$$

Proof. For each $\omega \neq \tau \in I^\infty$,

$$\begin{aligned}
\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau) &= \sum_{j=0}^{\infty} p_{n+j}\omega_j\lambda^j - \sum_{j=0}^{\infty} p_{n+j}\tau_j\lambda^j \\
&= \sum_{j=|\omega\wedge\tau|}^{\infty} p_{n+j}(\omega_j - \tau_j)\lambda^j \\
&= \lambda^{|\omega\wedge\tau|} \sum_{j=0}^{\infty} p_{|\omega\wedge\tau|+n+j}(\omega_{|\omega\wedge\tau|+j} - \tau_{|\omega\wedge\tau|+j})\lambda^j \\
&= \lambda^{|\omega\wedge\tau|} \sum_{j=0}^{\infty} p_{|\omega\wedge\tau|+n+j} a_j \lambda^j \quad (a_0 \in \{-1, 1\}, a_j \in \{-1, 0, 1\} \text{ for } j \in \mathbb{N}) \\
&= \lambda^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n} \sum_{j=0}^{\infty} \frac{p_{|\omega\wedge\tau|+n+j}}{p_{|\omega\wedge\tau|+n}} a_j \lambda^j.
\end{aligned}$$

Since $p_{|\omega\wedge\tau|+n}/p_{|\omega\wedge\tau|+n}a_0 \in \{-1, 1\}$ and for each $j \in \mathbb{N}$, $p_{|\omega\wedge\tau|+n+j}/p_{|\omega\wedge\tau|+n}a_j \in G_{n,j}$, we have that $f_{n,\omega,\tau}(\lambda) := \sum_{j=0}^{\infty} p_{|\omega\wedge\tau|+n+j}/p_{|\omega\wedge\tau|+n}a_j\lambda^j \in \mathcal{F}_n$. Then we have proved our proposition. □

Lemma 3.14. *Let $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$. For any $\omega, \tau \in I^\infty$ with $|\omega \wedge \tau| > j_{n,m}$ and for any $\lambda \in \mathbb{D}^*$ with $|\lambda| \le 1/\sqrt[m]{2}$,*

$$|\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| \le C_n(\lambda)\rho_{n,m}(\omega,\tau)^{\frac{-\log|\lambda|}{\log 2}},$$

where $C_n(\lambda) := \sum_{j=0}^{\infty} b_{n,j}|\lambda|^j < \infty$, $b_{n,j} := \max G_{n,j}$.

Proof. By Proposition 3.13, there exists $f_{n,\omega,\tau} \in \mathcal{F}_n$ such that

$$|\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| = |\lambda|^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n}|f_{n,\omega,\tau}(\lambda)| = \left(\frac{1}{2^{|\omega\wedge\tau|}}\right)^{\frac{-\log|\lambda|}{\log 2}} p_{|\omega\wedge\tau|+n}|f_{n,\omega,\tau}(\lambda)|.$$

Since $|\lambda| \le 1/\sqrt[m]{2}$,

$$p_{|\omega\wedge\tau|+n} \le (p_{|\omega\wedge\tau|+n})^{m\frac{-\log|\lambda|}{\log 2}}.$$

Hence we have that

$$\begin{aligned}\left(\frac{1}{2^{|\omega\wedge\tau|}}\right)^{\frac{-\log|\lambda|}{\log 2}} p_{|\omega\wedge\tau|+n}|f_{n,\omega,\tau}(\lambda)| &\le \left(\frac{1}{2^{|\omega\wedge\tau|}}\right)^{\frac{-\log|\lambda|}{\log 2}} (p_{|\omega\wedge\tau|+n})^{m\frac{-\log|\lambda|}{\log 2}}|f_{n,\omega,\tau}(\lambda)| \\ &\le C_n(\lambda)\rho_{n,m}(\omega,\tau)^{\frac{-\log|\lambda|}{\log 2}}.\end{aligned}$$

□

Theorem 3.15. *Let $n \in \mathbb{N}_0$. Then for any $\lambda \in \mathbb{D}^*$,*

$$\dim_H(A_n(\lambda)) \le \frac{\log 2}{-\log|\lambda|}.$$

Proof. Fix $\lambda \in \mathbb{D}^*$. Since $1/\sqrt[m]{2} \to 1$ as $m \to \infty$, there exists m_0 such that $|\lambda| \le 1/\sqrt[m_0]{2}$. By Lemma 3.14, for any $\omega, \tau \in I^\infty$ with $|\omega \wedge \tau| > j_{n,m_0}$,

$$|\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| \le C_n(\lambda)\rho_{n,m_0}(\omega,\tau)^{\frac{-\log|\lambda|}{\log 2}}.$$

Hence we have that

$$\dim_H(A_n(\lambda)) \le \frac{\log 2}{-\log|\lambda|}\dim_{\rho_{n,m_0}}(I^\infty) = \frac{\log 2}{-\log|\lambda|}$$

by Proposition 3.5 (see [4, Proposition 3.3]). □

4. Some lemmas

4.1. Frostman's Lemma and an inverse Frostman's Lemma.

DEFINITION 4.1 (FROSTMAN MEASURE). Let m be a Borel measure on $\mathbb{R}^d$. Let $t \ge 0$. Let E be a Borel subset of $\mathbb{R}^d$. We say that m is a Frostman measure on E with exponent t if $0 < m(E) < \infty$ and there exists a constant $C = C_t > 0$ such that for each $x \in \mathbb{R}^d$ and for each $r > 0$, $m(B(x,r)) \le Cr^t$.

Let $\mathcal{H}^t$ be the t-dimensional Hausdorff outer measure on $\mathbb{R}^d$ with respect to $|\cdot|$. We give the following lemma, which is known as Frostman's Lemma.

Lemma 4.2 ([4, Corollary 4.12]). *Let E be a Borel subset of $\mathbb{R}^d$ with $\mathcal{H}^t(E) > 0$. Then there exists a Frostman measure on E with exponent t.*

Corollary 4.3. *Let $0 < t \leq 2$. For each $x \in \mathbb{R}^2$ and for each $r > 0$, there exists a Frostman measure m on $B(x, r)$ with exponent t.*

Proof. If $0 < t < 2$, by Lemma 4.2, there exists a Frostman measure m on $B(x, r)$ with exponent t since $\mathcal{H}^t(B(x, r)) = \infty$. If $t = 2$, we set $m = \mathcal{L}$. □

DEFINITION 4.4 (s-ENERGY OF MEASURES). Let m be a Borel measure on $\mathbb{R}^d$. For any $s \geq 0$, we define the *s-energy* of m as

$$I_s(m) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x - y|^s} dm(x)dm(y).$$

We give the following lemma, which is known as an inverse Frostman's Lemma.

Lemma 4.5 ([4, Theorem 4.13])**.** *Let A be a Borel subset of $\mathbb{R}^d$ with $m(A) > 0$. If $I_s(m) < \infty$, then $\dim_H(A) \geq s$.*

4.2. Differentiation of measures. Let $d \in \mathbb{N}$. Let μ and m be Borel measures on $\mathbb{R}^d$ such that $\mu(G) < \infty$ and $\lambda(G) < \infty$ for any compact subset G. We say that the measure μ is absolutely continuous with respect to the measure m if $m(A) = 0$ implies $\mu(A) = 0$ for all Borel subsets A.

DEFINITION 4.6. The *lower derivative* of μ with respect to m at a point $x \in \mathbb{R}^d$ is defined by

$$\underline{D}(\mu, m, x) := \liminf_{r \to 0} \frac{\mu(B(x, r))}{m(B(x, r))}.$$

Note that the function $x \mapsto \underline{D}(\mu, m, x)$ is Borel measurable. For the details of differentiation of measures, see [7, p. 36]. The lower derivatives of measures are related to the absolute continuity of measures by the following.

Lemma 4.7 ([7, 2.12 Theorem])**.** *Let μ and m be Borel measures on $\mathbb{R}^n$ such that $\mu(G) < \infty$ and $m(G) < \infty$ for any compact subset G. Then μ is absolutely continuous with respect to m if and only if $\underline{D}(\mu, m, x) < \infty$ for μ a.e. $x \in \mathbb{R}^n$.*

4.3. A technical lemma for the transversality. We give a technical lemma for the transversality condition. In order to prove it, we give some definition and remark.

DEFINITION 4.8. Let G be a compact subset of $\mathbb{R}^d$. We say that a family of balls $\{B(x_i, r_i)\}_{i=1}^k$ in $\mathbb{R}^d$ is a *packing for G* if for each $i \in \{1, ..., k\}$, $x_i \in G$ and for each $i, j \in \{1, ..., k\}$ with $i \neq j$, $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$.

REMARK 4.9. Let G be a compact subset of $\mathbb{R}^d$, let $r > 0$ and let $\{B(x_i, r)\}_{i=1}^k$ be a family of balls in $\mathbb{R}^d$. If $\{B(x_i, r)\}_{i=1}^k$ is a packing for G, then there exists $N \in \mathbb{N}$ which only depends on G and r such that $k \leq N$.

Proof. There exists a finite covering $\{B(y_j, r/2)\}_{j=1}^N$ for G since G is compact. Here, N only depends on G and r. Since $x_i \in G$ for each i, there exists j_i such that $x_i \in B(y_{j_i}, r/2)$.

Since $\{B(x_i, r)\}_{i=1}^k$ is a disjoint family, if $i \neq l \in \{1, ..., k\}$, then $j_i \neq j_l$. Thus $k \leq N$. □

We now give a slight variation of [16, Lemma 5.2].

Lemma 4.10. *Let $\mathcal{H}$ be a compact subset of the space of holomorphic functions on $\mathbb{D}$. We set*

$$\tilde{\mathcal{M}}_{\mathcal{H}} := \{\lambda \in \mathbb{D}^* \ : \textit{there exists } f \in \mathcal{H} \text{ such that } f(\lambda) = f'(\lambda) = 0\}.$$

Let G be a compact subset of $\mathbb{D}^* \backslash \tilde{\mathcal{M}}_{\mathcal{H}}$. Let $t \geq 0$ and let $\mathcal{L}^t$ be a Frostman measure on G with exponent t. Then there exists $K > 0$ such that for any $f \in \mathcal{H}$ and for any $r > 0$,

$$\mathcal{L}^t(\{\lambda \in G : |f(\lambda)| \leq r\}) \leq Kr^t. \tag{4}$$

Proof. Since $\mathcal{H}$ is compact and the set $\tilde{\mathcal{M}}_H$ is the set of possible double zeros, we have that there exists $\delta = \delta_G > 0$ such that for any $f \in \mathcal{H}$,

$$|f(\lambda)| < \delta \Rightarrow |f'(\lambda)| > \delta \ \text{ for } \lambda \in G. \tag{5}$$

We assume that $r < \delta$, otherwise (4) holds with $K = \mathcal{L}^t(G)/\delta^t$. Let

$$\Delta_r := \{\lambda \in G : |f(\lambda)| \leq r\}.$$

Let $\mathrm{Co}(G)$ be the convex hull of G. We set $M = M_G := \sup\{|g''(\lambda)| \in [0, \infty) : \lambda \in \mathrm{Co}(G), g \in \mathcal{H}\}$. Since $\mathrm{Co}(G)$ is compact and $\mathcal{H}$ is compact, $M < \infty$. Fix $z_0 \in \Delta_r$. By Taylor's formula, for $z \in G$,

$$|f(z) - f(z_0)| = \left| f'(z_0)(z - z_0) + \int_{z_0}^{z} (z - \xi) f''(\xi) d\xi \right|,$$

where the integration is performed along the straight line path from z_0 to z. Then $|f'(z_0)| > \delta$ by (5). Hence

$$|f(z) - f(z_0)| \geq |f'(z_0)||z - z_0| - M|z - z_0|^2 > \delta|z - z_0| - M|z - z_0|^2.$$

Now if we set

$$A_{z_0,r} := \left\{ z \in \mathbb{D}^* : \frac{4r}{\delta} < |z - z_0| < \frac{\delta}{2M} \right\},$$

then for any $z \in A_{z_0,r}$,

$$\delta|z - z_0| - M|z - z_0|^2 = |z - z_0|(\delta - M|z - z_0|) > \frac{4r}{\delta}\frac{\delta}{2} = 2r,$$

and $|f(z)| \geq |f(z) - f(z_0)| - |f(z_0)| > r$. It follows that the annulus $A_{z_0,r}$ does not intersect Δ_r.

Assume that $4r/\delta \leq \delta/4M$, otherwise (4) holds with $K = \mathcal{L}^t(G)(16M/\delta^2)^t$. Then the disc $B(z_0, \delta/4M)$ centered at z_0 with the radius $\delta/4M$ covers $\Delta_r \cap \{z : |z - z_0| < \delta/2M\}$. Then fix $z_1 \in \Delta_r \backslash \{z : |z - z_0| < \delta/2M\}$. Since the annulus $A_{z_1,r}$ does not intersect Δ_r, $B(z_1, \delta/4M)$ covers $(\Delta_r \backslash \{z : |z - z_0| < \delta/2M\}) \cap \{z : |z - z_1| < \delta/2M\}$ and $B(z_0, \delta/4M) \cap B(z_1, \delta/4M) = \emptyset$. If we repeat the procedure, we get a finite covering $\{B(z_i, \delta/4M)\}_{i=0}^k$ for Δ_r since Δ_r is compact. Then $\{B(z_i, \delta/4M)\}_{i=0}^k$ is packing for G. By Remark 4.9, there exists $N \in \mathbb{N}$ which only depends on $\mathcal{H}$ and G such that $k \leq N$. Since the annulus $A_{z_i,r}$ does not intersect Δ_r for each $i \in \{0, ..., k\}$, $\{B(z_i, 4r/\delta)\}_{i=0}^k$ is also a covering for Δ_r. Hence we have

$$\mathcal{L}^t(\Delta_r) \leq \mathcal{L}^t\left(\bigcup_{i=0}^{k}\{B(z_i, 4r/\delta)\}\right) = \sum_{i=0}^{k} \mathcal{L}^t(\{B(z_i, 4r/\delta)\}) \leq NC\left(\frac{4r}{\delta}\right)^t = NC\left(\frac{4}{\delta}\right)^t r^t,$$

where C denotes a constant which appears in the definition of $\mathcal{L}^t$. If we set $K := NC(4/\delta)^t$, we get the desired inequality. □

5. Proofs of main results

5.1. The lower estimation of the Hausdorff dimension for typical parameters. For each $n \in \mathbb{N}_0$, we endow I^∞ with the metric $\rho_{n,0}$ (for the definition of $\rho_{n,0}$, see Definition 3.3). Since the metric $\rho_{n,0}$ does not depend on n, we set $\rho_0 := \rho_{n,0}$. We consider the address maps $\pi_{n,\lambda} : (I^\infty, \rho_0) \to \mathbb{C}$ for $\lambda \in \mathbb{D}^*$. We set $A_n(\lambda) := \pi_{n,\lambda}(I^\infty)$. Fix $\delta > 0$. Then for any $\lambda, \eta \in B(0, \delta) \cap \mathbb{D}^*$ and any $\omega = \omega_0\omega_1 \cdots \in I^\infty$,

$$\begin{aligned}
|\pi_{n,\lambda}(\omega) - \pi_{n,\eta}(\omega)| &\leq \sum_{j=0}^{\infty} p_{n+j}\omega_j|\lambda^j - \eta^j| \\
&\leq \sum_{j=0}^{\infty} p_{n+j}|\lambda - \eta|(|\lambda|^{j-1} + |\lambda|^{j-2}|\eta| + \cdots + |\lambda||\eta|^{j-2} + |\eta|^{j-1}) \\
&\leq \sum_{j=0}^{\infty} j p_{n+j}|\lambda - \eta|\delta^{j-1}.
\end{aligned}$$

Hence we have the following.

REMARK 5.1. Let $\lambda \in \mathbb{D}^*$. If $\lambda_j \to \lambda$ as $j \to \infty$, then $\pi_{n,\lambda_j}(\cdot)$ uniformly converges to $\pi_{n,\lambda}(\cdot)$ on I^∞. In particular, the sequence of sets $\{A_n(\lambda_j)\}_{j=1}^\infty$ converges to $A_n(\lambda)$ in the Hausdorff metric.

By Proposition 3.13, if we set $C_n(\lambda) := \sum_{j=0}^\infty b_{n,j}|\lambda|^j < \infty$, where $b_{n,j} := \max G_{n,j}$,

$$|\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| \leq |\lambda|^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n} C_n(\lambda)$$

for any $\omega, \tau \in I^\infty$. If $\rho_0(\omega_j, \omega) = 1/2^{|\omega_j\wedge\omega|} \to 0$ as $j \to \infty$, then $|\lambda|^{|\omega_j\wedge\omega|} p_{|\omega_j\wedge\omega|+n} \to 0$. Hence for each $\lambda \in \mathbb{D}^*$, the map $\omega \mapsto \pi_{n,\lambda}(\omega)$ is continuous on I^∞. We set $\alpha : \mathbb{D}^* \to [0, \infty)$ by

$$\alpha(\lambda) := \frac{-\log|\lambda|}{\log 2}.$$

For any compact subset $G \subset \mathbb{D}^*$, we set $\alpha_G := \sup\{\alpha(\lambda) : \lambda \in G\}$. We set $U_n := \mathbb{D}^* \setminus \tilde{\mathcal{M}}_n$ (for the definition of $\tilde{\mathcal{M}}_n$, see Definition 3.10).

Lemma 5.2. *Let G be a compact subset of U_n and let $\mathcal{L}^t$ be a Frostman measure on G with exponent t for some $t > 0$. Then there exists $K_{n,G} > 0$ such that for any $r > 0$ and any $\omega \neq \tau \in I^\infty$,*

$$\mathcal{L}^t(\{\lambda \in G : |\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| \leq r\}) \leq K_{n,G}\rho_0(\omega, \tau)^{-t\alpha_G} r^t.$$

Proof. By Proposition 3.13, for any $\omega \neq \tau \in I^\infty$, there exists $f_{n,\omega,\tau} \in \mathcal{F}_n$ such that $\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau) = \lambda^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n} f_{n,\omega,\tau}(\lambda)$. Hence for any $r > 0$,

$$\{\lambda \in G : |\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| \le r\} = \left\{\lambda \in G : |f_{n,\omega,\tau}(\lambda)| \le \rho_0(\omega,\tau)^{-\alpha(\lambda)} \frac{1}{p_{|\omega\wedge\tau|+n}} r\right\}.$$

Since $\mathcal{F}_n$ is a compact subset of the space of holomorphic functions on $\mathbb{D}$, by Lemma 4.10 we have that for any compact subset $G \subset \mathbb{D}^* \backslash \tilde{\mathcal{M}}_n$, there exists $K_{n,G} > 0$ such that for any $r > 0$,

$$\begin{aligned}\mathcal{L}^t(\{\lambda \in G : |\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| \le r\}) &= \mathcal{L}^t\left(\left\{\lambda \in G : |f_{n,\omega,\tau}(\lambda)| \le \rho_0(\omega,\tau)^{-\alpha(\lambda)} \frac{1}{p_{|\omega\wedge\tau|+n}} r\right\}\right) \\ &\le K_{n,G}\rho_0(\omega,\tau)^{-t\alpha(\lambda)} \frac{1}{(p_{|\omega\wedge\tau|+n})^t} r^t \\ &\le K_{n,G}\rho_0(\omega,\tau)^{-t\alpha_G} r^t. \qquad \square\end{aligned}$$

Let μ be the $(1/2, 1/2)$-Bernoulli measure on I^∞. We set $\nu_{n,\lambda} = \pi_{n,\lambda}\mu$. This is a Borel probability measure on $\pi_{n,\lambda}(I^\infty) = A_n(\lambda)$, since the map $\omega \mapsto \pi_{n,\lambda}(\omega)$ is continuous on I^∞.

Lemma 5.3. *Let $0 \le s < 1$. Then*

$$\int_{I^\infty}\int_{I^\infty} \rho_0(\omega,\tau)^{-s}\, d\mu(\omega)d\mu(\tau) < \infty.$$

Proof. For any $i \in I$, we set

$$\tilde{i} := \begin{cases} 1 & (i = 0) \\ 0 & (i = 1). \end{cases}$$

Then

$$\begin{aligned}\int_{I^\infty}\int_{I^\infty} \rho_0(\omega,\tau)^{-s}\, d\mu(\omega)d\mu(\tau) &= \int_{I^\infty}\int_{I^\infty} 2^{s|\omega\wedge\tau|}\, d\mu(\omega)d\mu(\tau) \\ &= \int_{I^\infty}\sum_{j=0}^{\infty}\int_{\{\omega:|\omega\wedge\tau|=j\}} 2^{s|\omega\wedge\tau|}\, d\mu(\omega)d\mu(\tau) \\ &= \int_{I^\infty}\sum_{j=0}^{\infty} 2^{sj}\mu([\tau_0\tau_1\cdots\tau_{j-1}\tilde{\tau}_j])d\mu(\tau) \\ &= \frac{1}{2}\int_{I^\infty}\sum_{j=0}^{\infty} 2^{(s-1)j} d\mu(\tau) \\ &= \frac{1}{2}\int_{I^\infty} \frac{1}{1-2^{(s-1)}} d\mu(\tau) \\ &= \frac{1}{2}\frac{1}{1-2^{(s-1)}}. \qquad \square\end{aligned}$$

Lemma 5.4. *Let $\lambda \in \mathbb{D}^*$. Let $s_1 \ge s_2 \ge 0$. If*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u - v|^{-s_2}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \infty,$$

then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u - v|^{-s_1}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \infty.$$

Proof. Since for any Borel subset $B \subset \mathbb{R}^2$ with $B \cap A_n(\lambda) = \emptyset$, $\nu_{n,\lambda}(B) = 0$, we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-s_1}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \int_{A_n(\lambda)}\int_{A_n(\lambda)} |u-v|^{-s_1}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v).$$

If we set $D := \sup_{u,v\in A_n(\lambda)} |u-v| < \infty$, then we have

$$\begin{aligned}\int_{A_n(\lambda)}\int_{A_n(\lambda)} |u-v|^{-s_1}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) &= \int_{A_n(\lambda)}\int_{A_n(\lambda)} D^{-s_1}\left(\frac{|u-v|}{D}\right)^{-s_1} d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)\\ &\geq \int_{A_n(\lambda)}\int_{A_n(\lambda)} D^{-s_1}\left(\frac{|u-v|}{D}\right)^{-s_2} d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)\\ &= D^{-s_1+s_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-s_2}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)\\ &= \infty.\end{aligned}$$

□

Lemma 5.5. *The function*

$$\lambda \mapsto \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-1/\alpha(\lambda)}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)$$

is Borel measurable on $\mathbb{D}^*$.

Proof. For any $\lambda \in \mathbb{D}^*$,

$$\begin{aligned}\Phi(\lambda) &:= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-1/\alpha(\lambda)}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)\\ &= \int_{I^\infty}\int_{I^\infty} |\pi_{n,\lambda}(\omega)-\pi_{n,\lambda}(\tau)|^{-1/\alpha(\lambda)}\, d\mu(\omega)d\mu(\tau).\end{aligned}$$

Fix a sequence $\{\lambda_j\}_{j=1}^\infty \to \lambda$ as $j \to \infty$. Then $|\pi_{n,\lambda_j}(\omega)-\pi_{n,\lambda_j}(\tau)|^{-1/\alpha(\lambda_j)} \to |\pi_{n,\lambda}(\omega)-\pi_{n,\lambda}(\tau)|^{-1/\alpha(\lambda)} \in (0,\infty]$ as $j\to\infty$ for each $\omega,\tau \in I^\infty$ by Remark 5.1 and the continuity of α. By Fatou's Lemma,

$$\begin{aligned}&\int_{I^\infty}\int_{I^\infty} |\pi_{n,\lambda}(\omega)-\pi_{n,\lambda}(\tau)|^{-1/\alpha(\lambda)}\, d\mu(\omega)d\mu(\tau)\\ &= \int_{I^\infty}\int_{I^\infty} \liminf_{j\to\infty} |\pi_{n,\lambda_j}(\omega)-\pi_{n,\lambda_j}(\tau)|^{-1/\alpha(\lambda_j)}\, d\mu(\omega)d\mu(\tau)\\ &\leq \liminf_{j\to\infty}\int_{I^\infty}\int_{I^\infty} |\pi_{n,\lambda_j}(\omega)-\pi_{n,\lambda_j}(\tau)|^{-1/\alpha(\lambda_j)}\, d\mu(\omega)d\mu(\tau).\end{aligned}$$

Hence the function $\lambda \mapsto \Phi(\lambda)$ is lower semi-continuous, and hence Borel measurable. □

We give key lemmas as the following.

Lemma 5.6. *Let $0 < t \leq 2$. For any $\lambda_0 \in U_n \cap \{\lambda \in \mathbb{D}^* : 1/\alpha(\lambda) < t\}$ and any $\epsilon > 0$, there exists $\delta > 0$ such that for any Frostman measure $\mathcal{L}^t$ on $B(\lambda_0,\delta)$ with exponent t,*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-(1/\alpha(\lambda_0)-\epsilon)}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) < \infty$$

for $\mathcal{L}^t$-a.e. λ in $B(\lambda_0,\delta)$.

Proof. Fix $\lambda_0 \in U_n \cap \{\lambda \in \mathbb{D}^* : 1/\alpha(\lambda) < t\}$ and any $\epsilon > 0$. There exists $\delta > 0$ such that $1/\alpha(\lambda_0) - \epsilon < 1/\alpha_{\mathrm{cl}(B(\lambda_0,\delta))}$ since α is continuous. Below, we set $s = 1/\alpha(\lambda_0) - \epsilon$ and $G := \mathrm{cl}(B(\lambda_0, \delta))$. Then

$$\int_{I^\infty}\int_{I^\infty} \rho_0(\omega,\tau)^{-s\alpha_G}\, d\mu(\omega)d\mu(\tau) < \infty$$

by Lemma 5.3 since $s\alpha_G < 1$. If we prove

$$\mathcal{S} := \int_G \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-s}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)d\mathcal{L}^t(\lambda) < \infty,$$

we get the desired result. By changing variables and Fubini's Theorem,

$$\mathcal{S} = \int_{I^\infty}\int_{I^\infty}\int_G |\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)|^{-s}\, d\mathcal{L}^t(\lambda)d\mu(\omega)d\mu(\tau).$$

By using Lemma 5.2 and $\mathcal{L}^t(G) < \infty$, we have that for any $r > 0$ and any $\omega, \tau \in I^\infty$,

$$\mathcal{L}^t(\{\lambda \in G : |\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| \le r\}) \le \mathrm{Const.}\min\{1, \rho_0(\omega,\tau)^{-t\alpha_G} r^t\}.$$

Here, we set Const. := $\max\{1, \mathcal{L}^t(G)\}K_{n,G}$, where $K_{n,G}$ comes from Lemma 5.2. Then by using that $s < t$, we obtain

$$\begin{aligned}
\int_G |\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)|^{-s}\, d\mathcal{L}^t(\lambda) &= \int_0^\infty \mathcal{L}^t(\{\lambda \in G : |\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)|^{-s} \ge x\})\, dx \\
&\le \mathrm{Const.}\int_0^\infty \min\{1, \rho_0(\omega,\tau)^{-t\alpha_G} x^{-t/s}\}\, dx \\
&= \mathrm{Const.}\left(\int_0^{\rho_0(\omega,\tau)^{-s\alpha_G}} 1\, dx + \rho_0(\omega,\tau)^{-t\alpha_G}\int_{\rho_0(\omega,\tau)^{-s\alpha_G}}^\infty x^{-t/s}\, dx\right) \\
&= \mathrm{Const.}'\rho_0(\omega,\tau)^{-s\alpha_G}.
\end{aligned}$$

Here, we set $\mathrm{Const.}' := \left(\mathrm{Const.} + \frac{1}{t/s-1}\right)$. Hence we have $\mathcal{S} < \infty$. □

Lemma 5.7. *For any $\lambda_0 \in U_n \cap \{\lambda \in \mathbb{D}^* : 1/\alpha(\lambda) > 2\}$, there exists $\delta > 0$ such that*

$$\mathcal{L}(A_n(\lambda)) > 0$$

for $\mathcal{L}$-a.e. λ in $B(\lambda_0, \delta)$.

Proof. Fix any $\lambda_0 \in U_n \cap \{\lambda \in \mathbb{D}^* : 1/\alpha(\lambda) > 2\}$ and any $\epsilon > 0$ with $(1-\epsilon)/\alpha(\lambda_0) > 2$. Then by Lemma 5.3,

$$\int_{I^\infty}\int_{I^\infty} \rho_0(\omega,\tau)^{-(1-\epsilon)}\, d\mu(\omega)d\mu(\tau) < \infty.$$

There exists $\delta > 0$ such that $(1-\epsilon)/\alpha_{\mathrm{cl}(B(\lambda_0,\delta))} > 2$ since α is continuous. It suffices to prove that $\nu_{n,\lambda}$ is absolutely continuous with respect to $\mathcal{L}$ for $\mathcal{L}$-a.e. λ in $B(\lambda_0, \delta)$. We set $G = \mathrm{cl}(B(\lambda_0, \delta))$. Let

$$\underline{D}(\nu_{n,\lambda}, u) := \liminf_{r\to 0} \frac{\nu_{n,\lambda}(B(u,r))}{\mathcal{L}(B(u,r))}$$

be the lower derivative of $\nu_{n,\lambda}$ with respect to $\mathcal{L}$ at the point u. If we show that

$$\mathcal{S} := \int_G \int_{\mathbb{R}^2} \underline{D}(\nu_{n,\lambda}, u)\, d\nu_{n,\lambda} d\mathcal{L}(\lambda) < \infty,$$

then for $\mathcal{L}$-a.e. $\lambda \in G$ we have $\underline{D}(\nu_{n,\lambda}, u) < \infty$ for $\nu_{n,\lambda}$-a.e. u and hence $\nu_{n,\lambda}$ is absolutely continuous by Lemma 4.7. By Fatou's Lemma,

$$\mathcal{S} \leq \text{Const.} \liminf_{r \to 0} r^{-2} \int_G \int_{\mathbb{R}^2} \nu_{n,\lambda}(B(u,r))\, d\nu_{n,\lambda}(u) d\mathcal{L}(\lambda).$$

Then

$$\begin{aligned}\int_{\mathbb{R}^2} \nu_{n,\lambda}(B(u,r))\, d\nu_{n,\lambda}(u) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \chi_{B(u,r)}(v)\, d\nu_{n,\lambda}(v) d\nu_{n,\lambda}(u) \\ &= \int_{I^\infty} \int_{I^\infty} \chi_{\{\tau \in I^\infty : |\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| \leq r\}}\, d\mu(\tau) d\mu(\omega),\end{aligned}$$

where χ_A is the characteristic function with respect to the set A. By Fubini's Theorem, integrating with respect to λ,

$$\mathcal{S} \leq \text{Const.} \liminf_{r \to 0} r^{-2} \int_{I^\infty} \int_{I^\infty} \mathcal{L}(\{\lambda \in G : |\pi_{n,\lambda}(\omega) - \pi_{n,\lambda}(\tau)| \leq r\})\, d\mu(\omega)\mu(\tau).$$

By using Lemma 5.2, we have that

$$\mathcal{S} \leq \text{Const.}' \int_{I^\infty} \int_{I^\infty} \rho_0(\omega, \tau)^{-2\alpha_G}\, d\mu(\omega) d\mu(\tau),$$

which is finite since $2\alpha_G < 1 - \epsilon$ by Lemma 5.3. □

Theorem 5.8. *Let* $n \in \mathbb{N}_0$.

(i) $\dim_H(A_n(\lambda)) \geq \dfrac{\log 2}{-\log|\lambda|}$ for $\mathcal{L}$-a.e. $\lambda \in \{\lambda \in \mathbb{D}^* : 0 < |\lambda| < 1/\sqrt{2}\} \setminus \tilde{\mathcal{M}}_n$.

(ii) $\mathcal{L}(A_n(\lambda)) > 0$ for $\mathcal{L}$-a.e. $\lambda \in \{\lambda \in \mathbb{D}^* : 1/\sqrt{2} < |\lambda| < 1\} \setminus \tilde{\mathcal{M}}_n$.

Proof. We first prove (i). We set $V_n := \{\lambda \in \mathbb{D}^* : 0 < |\lambda| < 1/\sqrt{2}\} \setminus \tilde{\mathcal{M}}_n$. Fix $k \in \mathbb{N}$. We prove

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u - v|^{-(1/\alpha(\lambda) - 1/k)}\, d\nu_{n,\lambda}(u) d\nu_{n,\lambda}(v) < \infty \tag{6}$$

for $\mathcal{L}$-a.e. λ in V_n.

Suppose that (6) does not hold. Then there exists a Lebesgue density point $\lambda_0 \in V_n$ of the set

$$\left\{\lambda \in V_n : \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u - v|^{-(1/\alpha(\lambda) - 1/k)}\, d\nu_{n,\lambda}(u) d\nu_{n,\lambda}(v) = \infty\right\}.$$

Then there exists $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$,

$$\mathcal{L}\left(\left\{\lambda \in B(\lambda_0, \delta) : \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u - v|^{-(1/\alpha(\lambda) - 1/k)}\, d\nu_{n,\lambda}(u) d\nu_{n,\lambda}(v) = \infty\right\}\right) > 0.$$

By the continuity of the function $\lambda \mapsto 1/\alpha(\lambda)$, if δ is small enough, then $1/\alpha(\lambda) - 1/k < 1/\alpha(\lambda_0) - 1/2k$ for each $\lambda \in B(\lambda_0, \delta)$. Hence for all sufficiently small δ, by Lemma 5.4, we have that

$$\mathcal{L}\left(\left\{\lambda \in B(\lambda_0, \delta) : \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-(1/\alpha(\lambda_0)-1/2k)}\, d\nu_{n,\lambda}(u) d\nu_{n,\lambda}(v) = \infty\right\}\right) > 0.$$

This however contradicts Lemma 5.6 since $\mathcal{L}$ is a Frostman measure on $B(\lambda_0, \delta)$ with exponent 2. Thus we have proved (6). By Lemma 4.5, we have that

$$\dim_H(A_n(\lambda)) \geq \frac{\log 2}{-\log|\lambda|} - \frac{1}{k} \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 0 < |\lambda| < 1/\sqrt{2}\}\setminus\tilde{\mathcal{M}}_n.$$

By letting $k \to \infty$, we prove (i).

Statement (ii) follows from Lemma 5.7 in a similar way. □

Corollary 5.9.

$$\dim_H(A_0(\lambda)) \geq \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 0 < |\lambda| < 1/\sqrt{2}\}\setminus\tilde{\mathcal{M}};$$

$$\mathcal{L}(A_0(\lambda)) > 0 \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 1/\sqrt{2} < |\lambda| < 1\}\setminus\tilde{\mathcal{M}}.$$

Proof. By Theorem 5.8 and Corollary 3.8, we have that

$$\dim_H(A_0(\lambda)) \geq \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 0 < |\lambda| < 1/\sqrt{2}\}\setminus\tilde{\mathcal{M}}_n;$$

$$\mathcal{L}(A_0(\lambda)) > 0 \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 1/\sqrt{2} < |\lambda| < 1\}\setminus\tilde{\mathcal{M}}_n.$$

By Lemma 3.12, letting $n \to \infty$, we get our corollary. □

We use the following theorem in order to prove our main result.

Theorem 5.10 ([17, Proposition 2.7])**.** *A power series of the form* $1 + \sum_{j=1}^{\infty} a_j z^j$, *with* $a_j \in [-1, 1]$, *cannot have a non-real double zero of modulus less than* $2\times 5^{-5/8} \approx 0.73143$ ($> 1/\sqrt{2}$).

Finally, we get the following theorem by using Theorem 3.15, Corollary 5.9 and Theorem 5.10.

Theorem 5.11.

$$\dim_H(A_0(\lambda)) = \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 0 < |\lambda| < 1/\sqrt{2}\};$$

$$\mathcal{L}(A_0(\lambda)) > 0 \text{ for } \mathcal{L}\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* : 1/\sqrt{2} < |\lambda| < 1\}\setminus\tilde{\mathcal{M}}.$$

5.2. The estimation of local dimension of the exceptional set of parameters. Recall that $U_n = \mathbb{D}^*\setminus\tilde{\mathcal{M}}_n$ and $\alpha(\lambda) = -\log|\lambda|/\log 2$ for $\lambda \in \mathbb{D}^*$. Note that $\bigcup_{n\in\mathbb{N}_0} U_n = \mathbb{D}^*\setminus\tilde{\mathcal{M}}$ by Lemma 3.12.

Lemma 5.12. *Let G be a compact subset of* U_n. *Then we have*

$$\dim_H\left(\left\{\lambda \in G : \dim_H(A_n(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \leq \sup_{\lambda\in G} \frac{\log 2}{-\log|\lambda|}.$$

Proof. We set $s_G := \sup_{\lambda\in G} \log 2/-\log|\lambda|$. By the countable stability of the Hausdorff dimension, it suffices to prove that for each $k \in \mathbb{N}$,

$$\dim_H\left(\left\{\lambda \in G : \dim_H(A_n(\lambda)) < \frac{\log 2}{-\log|\lambda|} - \frac{1}{k}\right\}\right) \leq s_G.$$

Since G is compact, it is enough to prove that for each $\lambda \in G$, there exists $\delta > 0$ such that

$$\dim_H\left(\left\{\lambda \in B(\lambda,\delta) : \dim_H(A_n(\lambda)) < \frac{\log 2}{-\log|\lambda|} - \frac{1}{k}\right\}\right) \leq s_G.$$

Suppose that this is false, that is, there exists $\lambda_0 \in G$ such that for any $\delta > 0$,

$$\dim_H\left(\left\{\lambda \in B(\lambda_0,\delta) : \dim_H(A_n(\lambda)) < \frac{\log 2}{-\log|\lambda|} - \frac{1}{k}\right\}\right) > s_G.$$

By the continuity of the function $\lambda \mapsto \log 2/-\log|\lambda|$, there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$,

$$\dim_H\left(\left\{\lambda \in B(\lambda_0,\delta) : \dim_H(A_n(\lambda)) < \frac{\log 2}{-\log|\lambda_0|} - \frac{1}{2k}\right\}\right) > s_G.$$

Take $\delta_1 > 0$ with $\delta_1 < \delta_0$ so that Lemma 5.6 holds with $t = s_G$ and $\epsilon = 1/2k$. By Lemma 4.5, we have

$$\begin{aligned}&\left\{\lambda \in B(\lambda_0,\delta_1) : \dim_H(A_n(\lambda)) < \frac{\log 2}{-\log|\lambda_0|} - \frac{1}{2k}\right\}\\ &\subset \left\{\lambda \in B(\lambda_0,\delta_1) : \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-(1/\alpha(\lambda_0)-1/2k)}\, d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \infty\right\} =: E.\end{aligned}$$

By Lemma 5.5, the set E is a Borel subset of $\mathbb{D}^*$. Since $\mathcal{H}^{s_G}(E) > 0$, by Lemma 4.2, there exists a Frostman measure $\mathcal{L}^{s_G}$ on E with exponent s_G. However this contradicts Lemma 5.6 since $\mathcal{L}^{s_G}$ is also a Frostman measure on $B(\lambda_0,\delta_1)$ with exponent s_G. □

Theorem 5.13. *Let G be a compact subset of $\mathbb{D}^*\backslash\tilde{\mathcal{M}}$. Then we have*

$$\dim_H\left(\left\{\lambda \in G : \dim_H(A_0(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \leq \sup_{\lambda\in G}\frac{\log 2}{-\log|\lambda|}.$$

Proof. Since $\bigcup_{n\in\mathbb{N}_0} U_n = \mathbb{D}^*\backslash\tilde{\mathcal{M}}$, there exists $n_0 \in \mathbb{N}_0$ such that $G \subset U_n$. By Lemma 5.12, we have

$$\dim_H\left(\left\{\lambda \in G : \dim_H(A_n(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \leq \sup_{\lambda\in G}\frac{\log 2}{-\log|\lambda|}.$$

By Corollary 3.8, we have that

$$\dim_H\left(\left\{\lambda \in G : \dim_H(A_0(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \leq \sup_{\lambda\in G}\frac{\log 2}{-\log|\lambda|}.$$ □

Theorem 5.14. *For any* $0 < R < 1/\sqrt{2}$,

$$\dim_H\left(\left\{\lambda \in \mathbb{D}^* : 0<|\lambda|<R,\ \dim_H(A_0(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \leq \frac{\log 2}{-\log R} < 2.$$

Proof. Let $0 < r < R < 1/\sqrt{2}$. If $R \leq 1/2$, by (1) and since $\tilde{\mathcal{M}} \subset \mathcal{M}$,

$$\{\lambda \in \mathbb{D}^* : r < |\lambda| < R\}\backslash\tilde{\mathcal{M}} = \{\lambda \in \mathbb{D}^* : r < |\lambda| < R\}.$$

For each $k \in \mathbb{N}$, we set $G_k := \{\lambda \in \mathbb{D}^* : r + 1/k \leq |\lambda| \leq R - 1/k\}$. Then G_k is a compact subset of $\mathbb{D}^* \backslash \tilde{\mathcal{M}}$ and $\bigcup_{k \in \mathbb{N}} G_k = \{\lambda \in \mathbb{D}^* : r < |\lambda| < R\}$. By Theorem 5.13 and the countable stability of the Hausdorff dimension, we have that

$$\dim_H \left(\left\{ \lambda \in \mathbb{D}^* : r < |\lambda| < R,\ \dim_H(A_0(\lambda)) < \frac{\log 2}{-\log |\lambda|} \right\} \right) \leq \frac{\log 2}{-\log R}.$$

If $1/2 < R \leq 1/\sqrt{2}$, by Theorem 5.10,

$$\{\lambda \in \mathbb{D}^* : r < |\lambda| < R\} \backslash \tilde{\mathcal{M}} = \{\lambda \in \mathbb{D}^* \backslash \mathbb{R} : r < |\lambda| < R\} \cup (\{\lambda \in \mathbb{R} : r < |\lambda| < R\} \backslash \tilde{\mathcal{M}}).$$

For each $k \in \mathbb{N}$, we set

$$\begin{aligned} G_k := &\{\lambda \in \mathbb{D}^* : r + 1/k \leq |\lambda| \leq R - 1/k, \mathrm{Im}(\lambda) \geq 1/k\} \\ &\cup \{\lambda \in \mathbb{D}^* : r + 1/k \leq |\lambda| \leq R - 1/k, \mathrm{Im}(\lambda) \leq -1/k\}, \end{aligned}$$

where $\mathrm{Im}(\lambda)$ denotes the imaginary part of λ. Then G_k is a compact subset of $\mathbb{D}^* \backslash \tilde{\mathcal{M}}$ and $\bigcup_{k \in \mathbb{N}} G_k = \{\lambda \in \mathbb{D}^* \backslash \mathbb{R} : r < |\lambda| < R\}$. By Theorem 5.13 and the countable stability of the Hausdorff dimension, we have that

$$\dim_H \left(\left\{ \lambda \in \mathbb{D}^* \backslash \mathbb{R} : r < |\lambda| < R,\ \dim_H(A_0(\lambda)) < \frac{\log 2}{-\log |\lambda|} \right\} \right) \leq \frac{\log 2}{-\log R}.$$

Since $\dim_H(\mathbb{R}) = 1 < \log 2 / -\log R$, we have that

$$\dim_H \left(\left\{ \lambda \in \mathbb{D}^* : r < |\lambda| < R,\ \dim_H(A_0(\lambda)) < \frac{\log 2}{-\log |\lambda|} \right\} \right) \leq \frac{\log 2}{-\log R}.$$

By the countable stability of the Hausdorff dimension, we have that

$$\dim_H \left(\left\{ \lambda \in \mathbb{D}^* : 0 < |\lambda| < R,\ \dim_H(A_0(\lambda)) < \frac{\log 2}{-\log |\lambda|} \right\} \right) \leq \frac{\log 2}{-\log R}. \qquad \square$$

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