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EQUIVARIANT HOLOMORPHIC EMBEDDINGS FROM THE COMPLEX PROJECTIVE LINE INTO COMPLEX GRASSMANNIAN OF 2-PLANES

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Abstract

Using gauge theory, we classify SU(2)-equivariant holomorphic embeddings from $\mathbf{C}P^1$ with the Fubini–Study metric into Grassmann manifold $Gr_{N-2}(\mathbf{C}^N)$. It is shown that the moduli spaces of those embeddings are identified with the gauge equivalence classes of non-flat invariant connections satisfying semi-positivity on the vector bundles given by *extensions* of line bundles. A topology on the moduli is obtained by means of L^2-inner product on Dolbeault cohomology group to which the extension class belongs. The compactification of the moduli is provided with geometric meaning from viewpoint of maps.

1. Introduction

Among many advances in the theory of holomorphic isometric embeddings of the complex projective line $\mathbf{C}P^1$ into Grassmann manifolds, one of prominent results is shown by E. Calabi, namely, a rigidity of holomorphic isometric embeddings of $\mathbf{C}P^1$ into complex projective spaces [2]. All those embeddings turn out to be equivariant under SU(2)-actions. Hence a natural problem arises when replacing the target by general Grassmannians.

However, there exist non-equivariant holomorphic isometric embeddings into general Grassmannians. Even in the case of equivariant maps, it does not seem to be far from the complete classification. Despite of the situation, Peng and Xu classify all SU(2)-equivariant minimal immersions of $\mathbf{C}P^1$ into complex Grassmannians of two-planes from Lie theoretic viewpoint in [6].

In the present paper, we adopt another viewpoint—gauge theory—in particular, differential geometry of vector bundles with connections. Over a Grassmann manifold $Gr_p(\mathbf{C}^N)$ of p-planes in N-dimensional complex vector space $\mathbf{C}^N$, there exists a holomorphic vector bundle called the *universal quotient bundle* $Q \to Gr_p(\mathbf{C}^N)$ (see §2). If we fix a Hermitian inner product on $\mathbf{C}^N$, $Gr_p(\mathbf{C}^N)$ is provided with a Kähler structure of Fubini–Study type. Moreover, the universal quotient bundle is also equipped with a Hermitian metric. We have a compatible connection on $Q \to Gr_p(\mathbf{C}^N)$ with the fibre metric and the holomorphic structure of the bundle, which is called the Hermitian connection. Then $Gr_p(\mathbf{C}^N)$ is a compact Hermitian symmetric space and $Q \to Gr_p(\mathbf{C}^N)$ is a homogeneous vector bundle. The Hermitian connection on $Q \to Gr_p(\mathbf{C}^N)$ is an invariant connection. Since $\mathbf{C}^N$ can be regarded as the space of holomorphic sections of $Q \to Gr_p(\mathbf{C}^N)$ by Borel–Weil theorem, we obtain the evaluation map $Gr_p(\mathbf{C}^N) \times \mathbf{C}^N \to Q$, which is an SU($N$)-equivariant bundle map.

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When f is a holomorphic map from a Kähler manifold M into $Gr_p(\mathbf{C}^N)$, the pull-back vector bundle of the universal quotient bundle is also a holomorphic vector bundle $f^*Q \to M$. By the pull-back of the evaluation map, $\mathbf{C}^N$ gives rise to holomorphic sections of $f^*Q \to M$. Thus a holomorphic map of M into a Grassmannian is recovered by the vector bundle and a subspace of the holomorphic sections of the bundle. If $p = 1$, then the well-known Kodaira embedding is induced by a positive line bundle and the space of holomorphic sections. These are the typical examples of *classifying maps* [1]. Since a classifying map is constructed by a vector bundle and a finite dimensional subspace of sections of the bundle, it is quite natural to consider a holomorphic vector bundle and a space of holomorphic sections when considering holomorphic maps into Grassmannians.

Let f be an SU(2)-equivariant holomorphic map from $\mathbf{C}P^1$ into $Gr_p(\mathbf{C}^N)$. Then $f^*Q \to \mathbf{C}P^1$ has an SU(2)-action and the pull-back connection is also an invariant connection. A subspace $\mathbf{C}^N$ is an SU(2)-module and the pull-back of the evaluation map is SU(2)-equivariant. Thus an SU(2)-equivariant holomorphic map is constructed by the vector bundle with an SU(2)-action and a subspace of the holomorphic sections of the bundle with an invariant Hermitian inner product, which is an SU(2)-module. We shall tackle this program in the case that $p = N - 2$.

In §2, we review an invariant connection and geometry of the complex Grassmannian mainly, to fix notation. In §3, invariant connections are realized through the *extension* of vector bundles. Using sheaf theory including sheaf cohomology groups, we show a key result in which we classify holomorphic vector bundles of rank 2 with SU(2)-actions and invariant connections over $\mathbf{C}P^1$ (Theorem 3.2). After introducing (semi-)positivity of vector bundles with a Hermitian metric, we obtain the classification of homogeneous semi-positive vector bundles of rank 2 on $\mathbf{C}P^1$ (Corollary 3.3).

In the final section, we introduce an induced map by a holomorphic vector bundle and a space of holomorphic sections. Then the *compatibility* condition with a Hermitian inner product and the induced connection is the main subject to be considered. In the holomorphic category, the compatibility condition is easily handled, due to the uniqueness of the Hermitian connection. We obtain three types of equivariant holomorphic maps, one of which constitutes a one parameter family. This one parameter family has a natural topology induced from L^2-inner product on the space of extension classes regarded as the corresponding Dolbeault cohomology group and the action of covariant constant gauge transformations. The compactification of the family enables us to combine the family with the other type of equivariant holomorphic maps. Finally, we classify equivariant holomorphic maps from $\mathbf{C}P^1$ into Grassmann manifold $Gr_{N-2}(\mathbf{C}^N)$ as induced maps (Theorem 4.6). As a result, it turns out that the moduli spaces of those maps are identified with the set of non-flat invariant connections modulo gauge equivalence on the vector bundles of rank 2 on $\mathbf{C}P^1$ with semi-positivity (Theorem 4.7).

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2. Preliminaries

2.1. Holomorphic vector bundles. Let (V, h) be a pair of a holomorphic vector bundle $V \to M$ over a Kähler manifold M and a Hermitian metric h on $V \to M$. We call (V, h) a Hermitian (vector) bundle. We have a unique connection compatible with h and the holomorphic vector bundle structure, which is called the *Hermitian connection* [4]. Then (V_1, h_1) is said to be holomorphically isomorphic to (V_2, h_2) if there exists a bundle isomorphism preserving the metrics and the Hermitian connection.

We introduce a group action on a vector bundle over a manifold M with a G-action. Let $\pi_V : V \to M$ be a vector bundle over M with structure group K, where K is a compact Lie group. A compact Lie group G acts on M and $V \to M$ on the left in such a way that

- π_V is G-equivariant, and
- the G action on $V \to M$ commutes with the K action. More precisely, the G action induces linear isomorphisms preserving the K structure on the fibres of $V \to M$.

Then the vector bundle $V \to M$ is said to have a G-action (compatible with the G-action on M). When $V \to M$ has a G-action, the space of sections $\Gamma(V)$ inherits an action of G.

Let $V \to M$ be a vector bundle with a G-action. If a connection ∇ on $V \to M$ is invariant under the G-action, then ∇ is called an *invariant connection*. One of the typical examples is given on a symmetric space G/K with the standard decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively. On the principal fibre bundle $G \to G/K$ with K as the fibre, the canonical connection is defined by taking the horizontal subspace as $L_g\mathfrak{m}$, where $g \in G$ and L_g means the left translation by g. Then a homogeneous vector bundle $G \times_K V_0 \to G/K$, where V_0 is a K-representation, admits an obvious G-action and the canonical connection is an invariant connection.

We usually suppose that M is a Kähler manifold and (V, h) is a Hermitian vector bundle. In this case, the group action is supposed to preserve the fibre metric and the Hermitian connection throughout this paper.

2.2. Geometry of Grassmannians. We follow [5] in which the details of the theory may be found. We denote by $Gr_p(\mathbf{C}^N)$ a complex Grassmannian of p-planes in $\mathbf{C}^N$. The tautological vector bundle is denoted by $S \to Gr_p(W)$. By definition, we have a bundle injection $i_S : S \to \underline{\mathbf{C}^N}$, where $\underline{\mathbf{C}^N} \to Gr_p(\mathbf{C}^N)$ is a trivial bundle of fibre $\mathbf{C}^N$. Then, the quotient vector bundle $Q \to Gr_p(\mathbf{C}^N)$ with natural projection $\pi_Q : \underline{\mathbf{C}^N} \to Q$ is called the *universal quotient bundle*. By the natural projection π_Q, $\mathbf{C}^N$ can also be regarded as a subspace of $\Gamma(Q)$ which is the space of sections of $Q \to Gr_p(\mathbf{C}^N)$. Then π_Q is also called an *evaluation map*. The (holomorphic) tangent bundle $T_{1,0} \to Gr_p(\mathbf{C}^N)$ is identified with $S^* \otimes Q \to Gr_p(\mathbf{C}^N)$.

Next, we fix a Hermitian inner product on $\mathbf{C}^N$. It gives orthogonal projections and so, we obtain two bundle homomorphisms: $\pi_S : \underline{\mathbf{C}^N} \to S$, and $i_Q : Q \to \underline{\mathbf{C}^N}$. Then the vector bundles $S, Q \to Gr_p(\mathbf{C}^N)$ are equipped with fibre metrics h_S and h_Q, respectively. We can also induce a Kähler structure on $Gr_p(\mathbf{C}^N)$ induced by h_S and h_Q using the identification of $T_{1,0} \to Gr_p(\mathbf{C}^N)$ with $S^* \otimes Q \to Gr_p(\mathbf{C}^N)$.

A section t of $Q \to Gr_p(\mathbf{C}^N)$ is regarded as a $\mathbf{C}^N$-valued function $i_Q(t)$. Then the differential $di_Q(t)$ can be decomposed into two components:

$$di_Q(t) = \pi_S di_Q(t) + \pi_Q di_Q(t).$$

Indeed, $\pi_Q di_Q(t)$ is a connection denoted by $\nabla^Q t$. The other term $\pi_S di_Q(t)$ denoted by Kt is called *the second fundamental form* in the sense of Kobayashi [4], which turns out to be a 1-form with values in $\mathrm{Hom}(Q, S) \cong Q^* \otimes S$.

In a similar way, the second fundamental form $H := \pi_Q di_S$ is defined, which is a 1-form with values in $\mathrm{Hom}(S, Q) \cong S^* \otimes Q$.

If $Gr_p(\mathbf{C}^N)$ is regarded as the homogeneous space $\mathrm{U}(N)/\mathrm{U}(p) \times \mathrm{U}(N-p)$, then $Q \to Gr_p(\mathbf{C}^N)$ is expressed as $\mathrm{U}(N) \times_{\mathrm{U}(p)\times\mathrm{U}(N-p)} \mathbf{C}^{N-p}$, where $\mathbf{C}^{N-p}$ is the standard representation of $\mathrm{U}(N-p)$. Then the induced connection ∇^Q is the canonical connection, which induces a holomorphic vector bundle structure of $Q \to Gr_p(\mathbf{C}^N)$. Moreover, $\pi_Q : \underline{\mathbf{C}^N} \to Q$ is an equivariant homomorphism.

We will need the following property of the second fundamental forms in the next section.

Lemma 2.1 ([4], see also [5]). *The second fundamental forms H and K satisfy $h_Q(Hu, v) = -h_S(u, Kv)$, where $u \in S$ and $v \in Q$.*

If $f : M \to Gr_p(\mathbf{C}^N)$ is a smooth map from a Riemannian manifold M, then we pull-back the fibre metric and the connection on $Q \to Gr_p(\mathbf{C}^N)$ to obtain a fibre metric and a connection on the pull-back bundle $f^*Q \to M$. The second fundamental forms also pull back, which are denoted by the same symbols H and K. A bundle homomorphism $A \in \Gamma(\mathrm{End}\, f^*Q)$ is defined as the trace of the composite of the second fundamental forms $HK \in (T^* \otimes T^* \otimes \mathrm{End}\, f^*Q)$. The bundle endomorphism $A \in \Gamma(\mathrm{End}\, f^*Q)$ is called the *mean curvature operator* of $f : M \to Gr_p(\mathbf{C}^N)$.

3. Invariant connections on the complex projective line

3.1. Examples. Let $[z_1 : z_2]$ be the homogeneous co-ordinates on the complex projective line $\mathbf{C}P^1$. Two open subsets of $\mathbf{C}P^1$ denoted by U_1 and U_2 are defined as

$$U_i = \left\{[z_1 : z_2] \in \mathbf{C}P^1 \;\middle|\; z_i \neq 0\right\}, \quad i = 1, 2.$$

We denote the inhomogeneous co-ordinates by $z := z_2/z_1$ on U_1 and $w := z_1/z_2$ on U_2. A $(1, 0)$ form θ is defined on U_1 as

$$\theta = \frac{1}{1 + |z|^2} dz.$$

The dual vector field of θ is denoted by Z, which is of type $(1, 0)$:

$$Z = (1 + |z|^2)\frac{\partial}{\partial z}.$$

For simplicity, a Riemannian metric of the Fubini–Study type is defined in such a way that $|Z|^2 = 1$. Thus, the Kähler form is, by definition, expressed as $\sqrt{-1}\theta \wedge \overline{\theta}$.

We regard $\mathbf{C}P^1$ as a homogeneous space $\mathrm{SU}(2)/\mathrm{U}(1)$. Let $\mathcal{O}(-1) \to \mathbf{C}P^1$ be the tautological vector bundle which, by definition, is a subbundle of a trivial bundle $\underline{\mathbf{C}^2} \to \mathbf{C}P^1$ of rank 2. Consequently, we have an exact sequence of vector bundles:

$$0 \to \mathcal{O}(-1) \xrightarrow{i} \underline{\mathbf{C}^2} \xrightarrow{\pi} \mathcal{O}(1) \to 0, \tag{1}$$

where $\mathcal{O}(1) \to \mathbf{C}P^1$ is the hyperplane bundle. Then $\mathbf{C}^2$ is considered as the standard representation of $\mathrm{SU}(2)$. Hence $\mathbf{C}^2$ has an invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ and an invariant complex symplectic form ω. If a unitary basis $\{e_1, e_2\}$ of $\mathbf{C}^2$ satisfies $\omega(e_1, e_2) = 1$, then $\{e_1, e_2\}$ is called the standard basis of $\mathbf{C}^2$. We fix a standard basis $\{e_1, e_2\}$ of $\mathbf{C}^2$ once and for all in this paper. And the homogeneous co-ordinates are supposed to be compatible with $\{e_1, e_2\}$. In other words, if the equivalence class $[g] \in \mathbf{C}P^1$ represented by $g \in \mathrm{SU}(2)$ corresponds to the line represented by $[z_1 : z_2]$, then we have $ge_1 = z_1e_1 + z_2e_2$, up to a constant multiple. Thus g which corresponds to $z \in U_1$ may be represented using the basis $\{e_1, e_2\}$ as

$$g = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \quad \text{on } U_1. \tag{2}$$

Consequently, $\mathcal{O}(-1) \to \mathbf{C}P^1$ has an SU(2)-action under which i is an equivariant bundle homomorphism.

From (1), $\mathbf{C}^2$ can be regarded as the space of holomorphic sections of $\mathcal{O}(1) \to \mathbf{C}P^1$. Thus e_1 and e_2 induce holomorphic sections of $\mathcal{O}(1) \to \mathbf{C}P^1$ denoted by $\tilde{t}_1$ and $\tilde{t}_2$, respectively. If $\mathcal{O}(1) \to \mathbf{C}P^1$ is regarded as homogeneous vector bundle $SU(2) \times_{U(1)} \mathbf{C}e_2$ and the orthogonal projection is denoted by $\pi_2 : \mathbf{C}^2 \to \mathbf{C}e_2$, then, by definition, we obtain

$$\tilde{t}_i([z_1 : z_2]) = \left[g, \pi_2(g^{-1}e_i)\right], \quad i = 1, 2.$$

Notice that $\tilde{t}_1$ and $\tilde{t}_2$ satisfy $\tilde{t}_1 = -z\tilde{t}_2$. The identification between $\mathcal{O}(1) \to \mathbf{C}P^1$ and the orthogonal complement of $\mathcal{O}(-1) \to \mathbf{C}P^1$ in $\underline{\mathbf{C}^2} \to \mathbf{C}P^1$ provides us with the induced Hermitian metric denoted by h_1 on $\mathcal{O}(1) \to \mathbf{C}P^1$:

$$h_1(\tilde{t}_1, \tilde{t}_1) = \frac{|z|^2}{1+|z|^2}, \quad h_1(\tilde{t}_1, \tilde{t}_2) = \frac{-z}{1+|z|^2}, \quad h_1(\tilde{t}_2, \tilde{t}_2) = \frac{1}{1+|z|^2}.$$

Hence, $\mathcal{O}(1) \to \mathbf{C}P^1$ also has an SU(2)-action under which π is an equivariant bundle homomorphism and h_1 is an invariant metric.

Next, we take holomorphic frame fields $\tilde{s}_1 \in \Gamma(\mathcal{O}(-1)|_{U_1})$ and $\tilde{s}_2 \in \Gamma(\mathcal{O}(-1)|_{U_2})$ on U_1 and U_2, respectively:

$$i(\tilde{s}_1) := \begin{pmatrix} 1 \\ z \end{pmatrix} \quad \text{and} \quad i(\tilde{s}_2) := \begin{pmatrix} w \\ 1 \end{pmatrix}.$$

Then, we have $\tilde{s}_2 = w\tilde{s}_1$. In the same manner, we induce a Hermitian metric h_{-1} on $\mathcal{O}(-1) \to \mathbf{C}P^1$ which is an invariant metric. A connection ∇ on $\mathcal{O}(-1) \to \mathbf{C}P^1$ is induced from the product connection d on $\underline{\mathbf{C}^2} \to \mathbf{C}P^1$:

$$\nabla \tilde{s}_1 = i^* di(\tilde{s}_1) = \bar{z}\theta \tilde{s}_1,$$

where i^* is the adjoint homomorphism of i. Since d and the induced Hermitian metric are invariant under the action of SU(2), ∇ is also an invariant connection.

The tautological vector bundle $\mathcal{O}(-1) \to \mathbf{C}P^1$ is dual to $\mathcal{O}(1) \to \mathbf{C}P^1$. This relation is described by the complex symplectic form ω on $\mathbf{C}^2$ in the following way. We use the adjoint homomorphism π^* of π to regard $v \in \mathcal{O}(1)$ as a vector $\pi^*(v)$ in $\mathbf{C}^2$. If $u \in \mathcal{O}(-1)_x$ and $v \in \mathcal{O}(1)_x$, where $x \in \mathbf{C}P^1$, then $\omega(i(u), \pi^*(v))$ gives a perfect pairing. Since

$$\pi^*(\tilde{t}_1) = \frac{-z}{1+|z|^2} \begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix}, \quad \text{and} \quad \pi^*(\tilde{t}_2) = \frac{1}{1+|z|^2} \begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix},$$

we have

$$\omega(i(\tilde{s}_1), \pi^*(\tilde{t}_1)) = -z, \quad \text{and} \quad \omega(i(\tilde{s}_1), \pi^*(\tilde{t}_2)) = 1.$$

Thus $\tilde{t}_2$ is the dual frame field of $\tilde{s}_1$ on U_1. Consequently, we obtain the induced connection on $\mathcal{O}(1) \to \mathbf{C}P^1$:

$$\nabla \tilde{t}_2 = -\bar{z}\theta \tilde{t}_2, \quad \nabla \tilde{t}_1 = -\theta \tilde{t}_2,$$

which is also an invariant connection.

Using the normalization of $\tilde{s}_1$ and $\tilde{t}_2$, we define unitary frame fields of $\mathcal{O}(-1) \to \mathbf{C}P^1$ and $\mathcal{O}(1) \to \mathbf{C}P^1$ on U_1, which are denoted by s_1 and t_1, respectively. Then, we have

$$\nabla s_1 = \omega_{-1} s_1, \quad \nabla t_1 = \omega_1 t_1,$$

where, $\omega_1 = -\omega_{-1} = \frac{1}{2}\left(z\bar{\theta} - \bar{z}\theta\right)$. Thus the curvature form R_{-1} and R_1 is expressed as

$$R_{-1} s_1 = -\theta \wedge \bar{\theta} s_1, \quad R_1 t_1 = \theta \wedge \bar{\theta} t_1.$$

Notice that s_1 and t_1 are invariant under $g \in \mathrm{SU}(2)$ represented in (2) on U_1 and so, the relevant forms are invariant forms.

The second fundamental forms associated with (1) are defined as πdi and $i^* d\pi^*$, which are indeed 1-forms with values in bundles:

$$\begin{aligned}\pi di(s_1) =& \frac{1}{|\tilde{s}_1|}\pi di(\tilde{s}_1) = \frac{1}{|\tilde{s}_1|}\left\langle \begin{pmatrix} 0 \\ dz \end{pmatrix}, \pi^*(t_1) \right\rangle t_1 \\ =& \frac{1}{|\tilde{s}_1||\tilde{t}_2|(1+|z|^2)}\left\langle \begin{pmatrix} 0 \\ dz \end{pmatrix}, \begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix} \right\rangle t_1 = \theta t_1.\end{aligned}$$

Since $i^* d\pi^* = -(\pi di)^*$ from Lemma 2.1, we have $\pi^* di^*(t_1) = -\bar{\theta} s_1$.

Under the identification of $\mathcal{O}(-2) \to \mathbf{C}P^1$ with the canonical bundle $T^{1,0}\mathbf{C}P^1 \to \mathbf{C}P^1$ and $\mathcal{O}(2) \to \mathbf{C}P^1$ with $T_{1,0}\mathbf{C}P^1 \to \mathbf{C}P^1$, respectively, we get

$$\theta = s_1 \otimes s_1, \quad \text{and} \quad Z = t_1 \otimes t_1.$$

Since $\nabla(s_1 \otimes s_1) = -2\omega_1 s_1 \otimes s_1$, $\nabla(t_1 \otimes t_1) = 2\omega_1 t_1 \otimes t_1$, and the induced metric is the same as the Riemannian metric, it follows that

$$\nabla \theta = -2\omega_1 \theta, \quad \text{and} \quad \nabla Z = 2\omega_1 Z.$$

In particular, we can describe the Riemannian curvature R_2:

$$R_2 Z = 2\theta \wedge \bar{\theta} Z,$$

and so, the scalar curvature is 2.

Using the tensor product of line bundles, we can find an SU(2)-action on $\mathcal{O}(k) \to \mathbf{C}P^1$ ($k \in \mathbf{Z}$), an invariant Hermitian metric h_k and an invariant connection ∇^k on $\mathcal{O}(k) \to \mathbf{C}P^1$ as the Hermitian connection. Since $\mathcal{O}(k) \to \mathbf{C}P^1$ is also a homogeneous vector bundle, $\mathcal{O}(k) \to \mathbf{C}P^1$ has the canonical connection. We will show that ∇^k is nothing but the canonical connection as follows. It follows from Kodaira vanishing theorem that the holomorphic vector bundle structure on $\mathcal{O}(k) \to \mathbf{C}P^1$ is unique. In addition, the Hermitian Yang–Mills connection is also unique up to gauge transformation on $\mathcal{O}(k) \to \mathbf{C}P^1$. Since the curvature form of an invariant connection is a constant multiple of the Kähler form, an invariant connection on a line bundle over $\mathbf{C}P^1$ is the Hermitian Yang–Mills connection. (See [3] for the definition of the Hermitian Yang–Mills connection.) We deduce that the canonical connection on $\mathcal{O}(k) \to \mathbf{C}P^1$ is a unique invariant connection.

The above observation yields that a direct sum $(\mathcal{O}(k) \oplus \mathcal{O}(l), h_k \oplus h_l)$ $(k, l \in \mathbf{Z})$ of line bundles on $\mathbf{C}P^1$ has an SU(2)-action preserving the metric and the Hermitian connection.

Since the Hermitian connection is the direct sum of the canonical connections, it is also an invariant connection.

However, we can find other invariant connections on a vector bundle of rank 2 over $\mathbf{C}P^1$, which is explained in the next subsection.

3.2. Extension. (See [3, §10.2.1] for general argument of the extension of holomorphic vector bundles from the differential-geometric viewpoint.) First of all, we begin with a remark. The sheaf cohomology groups of a sheaf $\mathcal{S}$ on $\mathbf{C}P^1$ are denoted by $H^i(\mathcal{S})$. We do not distinguish vector bundles from the corresponding locally free sheaves.

We consider an extension of $\mathcal{O}(1) \to \mathbf{C}P^1$ by $\mathcal{O}(-1) \to \mathbf{C}P^1$. The extension class is, by definition, an element of $H^1(\mathcal{O}(1)^* \otimes \mathcal{O}(-1)) = H^1(\mathcal{O}(-2))$. It follows from Bott–Borel–Weil theorem that $H^1(\mathcal{O}(-2))$ is identified with a trivial representation space $\mathbf{C}$ as SU(2)-module. Thus such an extension is determined by $a \in \mathbf{C}$ and we have an exact sequence of vector bundles:

$$0 \to \mathcal{O}(-1) \to V_a \xrightarrow{\pi_a} \mathcal{O}(1) \to 0, \tag{3}$$

where $V_a \to \mathbf{C}P^1$ has the metric $h = h_{-1} \oplus h_1$ and the second fundamental form corresponds to the Dolbeault representative $-\bar{a}\bar{\theta} \otimes s_1 \otimes s_1 \in H^1(\mathcal{O}(-2))$ on U_1. Since $V_a \to \mathbf{C}P^1$ is isomorphic to the direct sum of line bundles $\mathcal{O}(-1) \oplus \mathcal{O}(1) \to \mathbf{C}P^1$ as C^∞-bundle, we can use frames s_1 and t_1 on U_1 to express the Hermitian connection ∇^a on (V_a, h):

$$\nabla^a s_1 = \omega_{-1} s_1 + a\theta t_1, \quad \nabla^a t_1 = -\bar{a}\bar{\theta} s_1 + \omega_1 t_1.$$

Notice that $V_a \to \mathbf{C}P^1$ has a non-trivial action of SU(2) which is induced from the action on $\mathcal{O}(-1) \oplus \mathcal{O}(1) \to \mathbf{C}P^1$. Since all forms involved in connection forms are invariant, the connection ∇^a is an invariant connection.

If a is not a real number, say $a = re^{\sqrt{-1}\psi}$, then we make use of a constant gauge transformation

$$g = \begin{pmatrix} \exp\left(-\frac{\sqrt{-1}\psi}{2}\right) & 0 \\ 0 & \exp\left(\frac{\sqrt{-1}\psi}{2}\right) \end{pmatrix}$$

to obtain a connection $\nabla^r = g^{-1}\nabla^a g$. Hence, to describe an invariant connection up to gauge transformation, we can assume that a is a non-negative real number.

With this understood, the curvature form R^a is expressed as:

$$R^a s_1 = \left(a^2 - 1\right) s_1 \theta \wedge \bar{\theta}, \quad R^a t_1 = \left(1 - a^2\right) t_1 \theta \wedge \bar{\theta}.$$

We take an associated long exact sequence of cohomology groups with (3) to conclude that $\pi_a : H^0(V_a) \cong H^0(\mathcal{O}(1)) = \mathbf{C}^2$. Since the SU(2)-action on $V_a \to \mathbf{C}P^1$ preserves the connection, and so the holomorphic structure, $H^0(V_a)$ inherits an SU(2)-action. Moreover, we can deduce that $\pi_a : H^0(V_a) \cong H^0(\mathcal{O}(1))$ is an equivariant map, because $\pi_a : V_a \to \mathcal{O}(1)$ is an equivariant map. Indeed, a direct computation yields

Lemma 3.1. *Let $V_a \to \mathbf{C}P^1$ be a holomorphic vector bundle of which the holomorphic structure is induced by ∇^a. Then*

$$\tilde{u}_1^a = \left(1 + |z|^2\right)^{-\frac{1}{2}} (a s_1 - z t_1) = \frac{a}{1 + |z|^2} \tilde{s}_1 + \tilde{t}_1,$$

$$\tilde{u}_2^a = \left(1+|z|^2\right)^{-\frac{1}{2}}(a\bar{z}s_1 + t_1) = \frac{a\bar{z}}{1+|z|^2}\tilde{s}_1 + \tilde{t}_2$$

are holomorphic sections of $V_a \to \mathbf{C}P^1$ *satisfying* $\pi_a(u_i^a) = \tilde{t}_i$, $i = 1, 2$.

Proof. By definition of ∇^a, we have

$$\nabla^a \tilde{u}_1^a = (a^2-1)\theta\tilde{t}_2, \quad \nabla^a \tilde{u}_2^a = (a^2-1)\bar{z}\theta\tilde{t}_2. \qquad \square$$

We can easily see that

$$h\left(\tilde{u}_1^a, \tilde{u}_1^a\right) = \frac{a^2+|z|^2}{1+|z|^2},\ h\left(\tilde{u}_1^a, \tilde{u}_2^a\right) = \frac{(a^2-1)z}{1+|z|^2},\ h\left(\tilde{u}_2^a, \tilde{u}_2^a\right) = \frac{a^2|z|^2+1}{1+|z|^2}.$$

If $a = 0$, then (V_0, h) is $(\mathcal{O}(-1)\oplus\mathcal{O}(1), h)$, in other words, $V_0 \to \mathbf{C}P^1$ is the *orthogonal* direct sum of the indicated line bundles. If $a \neq 0$, then $\left\{\tilde{u}_1^a, \tilde{u}_2^a\right\}$ is a *global holomorphic frame* of $V_a \to \mathbf{C}P^1$ and $V_a \cong \mathcal{O}\oplus\mathcal{O}$ as holomorphic vector bundle. Since $\tilde{u}_1$ and $\tilde{u}_2$ give a global parallel *unitary* frame in the case that $a = 1$, (V_1, h_1) is holomorphically isomorphic to a flat bundle $(\mathcal{O}\oplus\mathcal{O}, h_0\oplus h_0)$ and the Hermitian connection of (V_1, h_1) is the product connection.

Since $V_a \to \mathbf{C}P^1$ $(a \neq 0)$ is holomorphically isomorphic to $V_1 \to \mathbf{C}P^1$, we can find a complex gauge transformation $\phi : V_1 \to V_a$ such that ∇^1 is complex gauge equivalent to ∇^a. (See [3, p.210] for the action of complex gauge transformation to connections. Here, the action respects the Hermitian metric.) To find ϕ, we denote by Φ_a the connection form of ∇^a on U_1:

$$\Phi_a = \begin{pmatrix} \omega_{-1} & -a\bar{\theta} \\ a\theta & \omega_1 \end{pmatrix}.$$

Next we define a constant complex gauge transformation ϕ as

$$\phi := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, we have

$$\phi^{-1}\Phi_a(\overline{Z})\phi = \Phi_1(\overline{Z}), \quad \text{and} \quad \phi^*\Phi_a(Z)\phi^{*^{-1}} = \Phi_1(Z).$$

Thus, ∇^a $(a \neq 0)$ is complex gauge equivalent to ∇^1 under ϕ.

For our purpose, we would like to fix the space of holomorphic sections. To do so, we pull-back the Hermitian metric on $V_a \to \mathbf{C}P^1$:

$$\phi^* h = a^2 h_{-1} + h_1.$$

If we use $h_a := \phi^* h$ as a Hermitian metric on $V_1 \to \mathbf{C}P^1$ and holomorphic sections of $V_1 \to \mathbf{C}P^1$

$$\tilde{u}_1 = \left(1+|z|^2\right)^{-\frac{1}{2}}(s_1 - zt_1), \quad \tilde{u}_2 = \left(1+|z|^2\right)^{-\frac{1}{2}}(\bar{z}s_1 + t_1),$$

then we obtain

$$h_a(\tilde{u}_1, \tilde{u}_1) = \frac{a^2+|z|^2}{1+|z|^2},\ h_a(\tilde{u}_1, \tilde{u}_2) = \frac{(a^2-1)z}{1+|z|^2},\ h_a(\tilde{u}_2, \tilde{u}_2) = \frac{a^2|z|^2+1}{1+|z|^2},$$

and

$$\phi(\tilde{u}_1) = \tilde{u}_1^a, \quad \phi(\tilde{u}_2) = \tilde{u}_2^a.$$

From this point of view, the holomorphic structure does not change on $V_1 \to \mathbf{C}P^1$, but the Hermitian metric varies and so, the Hermitian connection also varies. Since R^a and ϕ are of the diagonal form and so, they commute, the curvature form of the pull-back connection denoted by the same symbol ∇^a does not change.

If we take a tensor product of (V_1, h_a), $(a \neq 0)$ with $(\mathcal{O}(k), h_k)$, then we obtain a Hermitian bundle $(V_1(k) := V_1 \otimes \mathcal{O}(k), h_a \otimes h_k)$ on $\mathbf{C}P^1$. An SU(2)-action is induced on $(V_1(k), h_a \otimes h_k)$, which preserves the metric and the Hermitian connection.

If k is positive, then the curvature form R_a of the Hermitian connection denoted by the same symbol ∇^a on $(V_1(k), h_a \otimes h_k)$ is provided with

$$\begin{cases} R^a s_1 \otimes t_1^k = (a^2 - 1 + k) s_1 \otimes t_1^k \theta \wedge \overline{\theta}, \\ R^a t_1 \otimes t_1^k = (1 - a^2 + k) t_1 \otimes t_1^k \theta \wedge \overline{\theta}. \end{cases} \tag{4}$$

3.3. Classification of invariant connections.

Theorem 3.2. *Let $\mathbf{C}P^1$ be the projective line and (V, h) a Hermitian vector bundle on $\mathbf{C}P^1$ of rank 2 with an* SU(2)*-action. If the* SU(2)*-action preserves h and the holomorphic vector bundle structure, then (V, h) is holomorphically isomorphic to $(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2), h_{k_1} \oplus h_{k_2})$, where k_1 and k_2 are integers or $(V_1(k), h_a \otimes h_k)$ for an integer k and a non-negative real number a. These bundle isomorphisms can be taken to preserve the group actions.*

Proof. Taking tensor product with a line bundle of an appropriate degree with the canonical connection, if necessary, we can assume that $H^0(V(-1)) = 0$ and $H^0(V) \neq 0$ without loss of generality, from a theorem of Grothendieck. Then for any point $p \in \mathbf{C}P^1$, we use a section $t_p \in H^0(\mathcal{O}(1))$ which vanishes at p to obtain a sequence of sheaves:

$$0 \to V(-1) \xrightarrow{\otimes t_p} V \to Q \to 0,$$

where Q is the quotient sheaf. Taking a long exact sequence of cohomology groups, we get

$$0 \to H^0(V(-1)) \to H^0(V) \to V_p \to \cdots,$$

where $H^0(V) \to V_p$ is obtained as the evaluation of sections at p. It follows from our assumption that $H^0(V) \to V_p$ is injective.

Next, since the SU(2)-action on $V \to \mathbf{C}P^1$ preserves the holomorphic structure of the bundle, $H^0(V)$ also has an SU(2)-action.

Suppose that the SU(2)-action on $H^0(V)$ is trivial. If $t \in H^0(V)$ is not a zero section, then t is a nowhere vanishing section, because SU(2)-action covers the transitive action on the base manifold. Hence, there exists a trivial line bundle with an SU(2)-action compatible with the SU(2)-action on $\mathbf{C}P^1$, which is a subbundle of $V \to \mathbf{C}P^1$. Thus we have an exact sequence of vector bundles:

$$0 \to \mathcal{O} \xrightarrow{i} V \to \mathcal{O}(l) \to 0,$$

where $\mathcal{O}(l) \to \mathbf{C}P^1$ denotes the quotient line bundle. Since the bundle map i is an equivariant injection, $\mathcal{O}(l) \to \mathbf{C}P^1$ has the canonical connection as the induced connection. From our assumption, the degree l of the quotient bundle must be non-positive. Since Bott–Borel–Weil theorem yields that $H^1(\mathrm{Hom}(\mathcal{O}(l),\mathcal{O})) = H^1(\mathcal{O}(-l)) = 0$, the extension class of $\mathcal{O}(l) \to \mathbf{C}P^1$ by $\mathcal{O} \to \mathbf{C}P^1$ is zero. Consequently, $V \to \mathbf{C}P^1$ is a direct sum of line bundles as holomorphic vector bundle with an invariant Hermitian metric, and so the connection is also the direct sum of the canonical connections. This bundle isomorphism can be taken to preserve group actions, due to the equivariance of the relevant homomorphisms.

Suppose that the SU(2)-action on $H^0(V)$ is non-trivial. The injectivity of $H^0(V) \to V_p$ implies that $H^0(V)$ is the standard representation of SU(2) and $H^0(V)$ is isomorphic to V_p for an arbitrary $p \in \mathbf{C}P^1$. Considering the evaluation homomorphism, $V \to \mathbf{C}P^1$ is isomorphic to $\underline{H^0(V)} \to \mathbf{C}P^1$ as holomorphic bundle. Hence if $V \to \mathbf{C}P^1$ is regarded as a trivial bundle $\mathbf{C}P^1 \times \mathbf{C}^2$, the action of $g \in \mathrm{SU}(2)$ on $V \to \mathbf{C}P^1$ is described as $g(x, v) = (gx, gv)$. Since $H^0(V)$ is isomorphic to $H^0(\mathcal{O}(1))$ as SU(2)-module, we obtain an equivariant bundle homomorphism ev_1:

$$ev_1 : V \to \mathcal{O}(1), \quad ev_1(x, v) = v(x),$$

where $v \in \mathbf{C}^2$ is now considered as a holomorphic section of $\mathcal{O}(1) \to \mathbf{C}P^1$. Thus, computing the Chern class, we see that $V \to \mathbf{C}P^1$ is obtained as the extension of $\mathcal{O}(1) \to \mathbf{C}P^1$ by $\mathcal{O}(-1) \to \mathbf{C}P^1$:

$$0 \to \mathcal{O}(-1) \to V \to \mathcal{O}(1) \to 0.$$

Since the evaluation ev_1 is an equivariant homomorphism, $\mathcal{O}(-1) \to \mathbf{C}P^1$ is also equipped with the canonical connection as the induced connection. Hence, $V \to \mathbf{C}P^1$ is gauge equivalent to $V_a \to \mathbf{C}P^1$. The equivariance implies that the isomorphism can be taken to preserve the group actions. □

Suppose that $V \to M$ is a holomorphic vector bundle on a Kähler manifold M. Then $V \to M$ is semi-positive if we have a Hermitian metric h on $V \to M$ such that for each $x \in M$,

$$h(R(Z,\overline{Z})v, v) \geqq 0 \quad \forall Z \in T_{1,0}M_x\backslash\{0\}, \forall v \in V_x\backslash\{0\}, \tag{5}$$

where R is the curvature of the Hermitian connection (see, for example, [4]). In this paper, the pair (V, h) is called *semi-positive* if the Hermitian connection of (V, h) satisfies (5).

Corollary 3.3. *Let $(V, h) \to \mathbf{C}P^1$ be a non-flat semi-positive holomorphic vector bundle of rank* 2. *If (V, h) has an* SU(2)*-action preserving h and the holomorphic vector bundle structure, then the invariant connection on (V, h) is indexed by a pair of non-negative integers (k_1, k_2), where $0 \leqq k_1 < k_2$, or a pair of a positive integer and a real number $\{k, a\}$, where $0 < a \leqq \sqrt{k+1}$.*

Proof. The pair (k_1, k_2) represents a direct sum $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$ with the canonical metrics. The semi-positivity yields that $k_1 \geqq 0$.

The pair $\{k, a\}$ stands for $(V_1(k), h_a \otimes h_k)$. Then (4) gives $0 < a \leqq \sqrt{k+1}$ by the semi-positivity. □

4. Classification

Let $Gr_n(\mathbf{C}^{n+2})$ be a complex Grassmannian of n-subspaces in $\mathbf{C}^{n+2}$. We fix a Hermitian inner product on $\mathbf{C}^{n+2}$ to obtain a Kähler structure on $Gr_n(\mathbf{C}^{n+2})$ as in §2. Then the Kähler form ω_Q on $Gr_n(\mathbf{C}^{n+2})$ satisfies

$$\operatorname{trace} R = -\sqrt{-1}\omega_Q,$$

where, R is the curvature two-form of the canonical connection on the universal quotient bundle.

Denote by ω_0 a Kähler form on $\mathbf{C}P^1$ and by R_1 the curvature two-form of the canonical connection on $\mathcal{O}(1) \to \mathbf{C}P^1$. We have $R_1 = -\sqrt{-1}\omega_0$.

Let f be a map from $\mathbf{C}P^1$ into $Gr_n(\mathbf{C}^{n+2})$. Then the pull-back bundle of the universal quotient bundle is regarded as a complex vector bundle with the induced connection whose curvature form denoted by the same symbol R satisfies

$$\operatorname{trace} R = -\sqrt{-1} f^*\omega_Q.$$

DEFINITION 4.1. Let f be a map from $\mathbf{C}P^1$ into $Gr_n(\mathbf{C}^{n+2})$. A map $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+2})$ is called an *equivariant* map, if we have a group homomorphism $\rho : \mathrm{SU}(2) \to \mathrm{U}(n+2)$ such that $f(gx) = \rho(g)f(x)$, where $x \in \mathbf{C}P^1$, $g \in \mathrm{SU}(2)$ and $\rho(g)$ is now regarded as a holomorphic isometry of $Gr_n(\mathbf{C}^{n+2})$.

If f is an equivariant map, then we have an integer l such that

$$f^*\omega_Q = l\omega_0, \tag{6}$$

because both forms are invariant forms which represent the Chern classes of line bundles $\wedge^2 f^*Q \to \mathbf{C}P^1$ and $\mathcal{O}(l) \to \mathbf{C}P^1$ and the Picard group of $\mathbf{C}P^1$ is $\mathbf{Z}$. If an equivariant map $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+2})$ satisfies (6), then f is called a map of *degree* l. When f is an equivariant map, $f^*Q \to \mathbf{C}P^1$ has an SU(2)-action and the induced connection on $f^*Q \to \mathbf{C}P^1$ is an invariant connection. Moreover the pull-back of the evaluation map $f^*ev : \underline{\mathbf{C}^{n+2}} \to f^*Q$ is an equivariant epimorphism, where $\underline{\mathbf{C}^{n+2}} \to \mathbf{C}P^1$ is a trivial bundle with fibre $\mathbf{C}^{n+2}$.

Proposition 4.2. *Let $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+2})$ be an equivariant holomorphic map of degree l. Then the pull-back of the universal quotient bundle $f^*Q \to \mathbf{C}P^1$ with the induced metric is holomorphically isomorphic to $(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2), h_{k_1} \otimes h_{k_2})$, where $0 \leqq k_1 \leqq k_2$ and $k_1 + k_2 = l$, or $(V_1(k), h_a \otimes h_k)$ for a positive real number a, where $2k = l$ and $k \geqq 0$.*

Proof. Since f is a holomorphic map, $f^*Q \to \mathbf{C}P^1$ is a holomorphic vector bundle of rank 2. The equivariance of f implies that $f^*Q \to \mathbf{C}P^1$ has an SU(2)-action with an invariant Hermitian metric. Then Theorem 3.2 determines $f^*Q \to \mathbf{C}P^1$ up to a gauge equivalence.

In addition, $\mathbf{C}^{n+2}$ induces holomorphic sections of $f^*Q \to \mathbf{C}P^1$. Since the evaluation map is an epimorphism, it follows that relevant integers are non-negative. □

To classify equivariant holomorphic maps, we introduce the induced map by a holomorphic vector bundle and the space of holomorphic sections of the bundle. (For a general argument, see [5].) Let $V \to \mathbf{C}P^1$ be a holomorphic vector bundle of rank 2. It is said that $\mathbf{C}^N \subset H^0(V)$ *globally generates* $V \to \mathbf{C}P^1$, if the evaluation map $ev : \underline{\mathbf{C}^N} \to V$ is an epimorphism. In the case that $\mathbf{C}^N \subset H^0(V)$ globally generates $V \to \mathbf{C}P^1$, we have a map $f : \mathbf{C}P^1 \to Gr_{N-2}(\mathbf{C}^N)$ called the *induced* map from $(V \to \mathbf{C}P^1, \mathbf{C}^N)$, which is defined as

$$f(x) = \operatorname{Ker} ev_x \subset \mathbf{C}^N, \quad x \in \mathbf{C}P^1.$$

Conversely, every holomorphic map $f : \mathbf{C}P^1 \to Gr_{N-2}(\mathbf{C}^N)$ can be recognized as the induced map from $(f^*Q \to \mathbf{C}P^1, \mathbf{C}^N)$, where $Q \to Gr_{N-2}(\mathbf{C}^N)$ is the universal quotient bundle.

Next, our main concern is the space of holomorphic sections of $V_1(k) \to \mathbf{C}P^1$, where k is positive.

Let $S^l\mathbf{C}^2$ be the l-th symmetric product of $\mathbf{C}^2$ with the induced Hermitian inner product. It is well-known that $S^l\mathbf{C}^2$ is an irreducible SU(2)-module.

We put a symmetric product of degree l as

$$e_1^{l-p}e_2^p := e_1 \otimes e_1 \otimes \cdots \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_2 + \cdots + e_2 \otimes e_2 \otimes \cdots \otimes e_2 \otimes e_1 \otimes \cdots \otimes e_1.$$

Hence, the induced Hermitian inner product denoted by $\langle \cdot, \cdot \rangle_l$ satisfies

$$\left\langle e_1^{l-p}e_2^p, e_1^{l-q}e_2^q \right\rangle_l = \delta_{pq}\binom{l}{p}, \quad \text{where} \quad \delta_{pq} \text{ is the Kronecker delta.} \tag{7}$$

Borel–Weil theorem yields that $H^0(\mathcal{O}(k)) \cong S^k\mathbf{C}^2$ and $H^1(\mathcal{O}(k))$ vanishes. From the exact sequence of vector bundles:

$$0 \to \mathcal{O}(k-1) \to V_1(k) \to \mathcal{O}(k+1) \to 0,$$

we obtain

$$0 \to S^{k-1}\mathbf{C}^2 \to H^0(V_1(k)) \to S^{k+1}\mathbf{C}^2 \to 0.$$

By dimension count, $H^0(V_1(k))$ is identified with $H^0(V_1) \otimes H^0(\mathcal{O}(k))$. Thus $H^0(V_1(k))$ is spanned by

$$\tilde{u}_i \otimes \tilde{t}_1^{k-p}\tilde{t}_2^p, \quad i = 1, 2 \quad \text{and} \quad p = 0, \cdots, k.$$

Notice that $H^0(V_1)$ and $H^0(\mathcal{O}(1))$ can be regarded as the standard representation of SU(2). Lemma 3.1 and the definition of $\tilde{t}_i$ yield that e_i corresponds to $\tilde{u}_i$ and $\tilde{t}_i$, respectively, where $i = 1, 2$.

Next, we fix an invariant Hermitian inner product denoted by $\langle \cdot, \cdot \rangle_{V_1}$ on $H^0(V_1)$ under which $\{\tilde{u}_1, \tilde{u}_2\}$ is a unitary basis. We also fix an SU(2)-invariant Hermitian inner product on $H^0(\mathcal{O}(k)) \cong S^k\mathbf{C}^2$ in the same manner as (7), which is denoted by the same symbol. An invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on $H^0(V_1) \otimes H^0(\mathcal{O}(k))$ is defined as $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{V_1} \otimes \langle \cdot, \cdot \rangle_k$.

Clebsch–Gordan formula yields that

$$H^0(V_1(k)) \cong \mathbf{C}^2 \otimes S^k\mathbf{C}^2 \cong S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2. \tag{8}$$

This decomposition is given by the symmetric product and the contraction by the complex symplectic form ω. Under the identification e_i with $\tilde{u}_i$ and $\tilde{t}_i$, $\tilde{u}_1 \otimes \tilde{t}_1^{q-1}\tilde{t}_2^{k-(q-1)} + \tilde{u}_2 \otimes \tilde{t}_1^q\tilde{t}_2^{k-q}$, $q = 0, \ldots, k+1$ of $\mathbf{C}^2 \otimes S^k\mathbf{C}^2$ can be regarded as a symmetric tensor. Hence, for $q = 0, \ldots, k+1$, $S^{k+1}\mathbf{C}^2 \subset H^0(V_1(k))$ is spanned by

$$v_+^q := \tilde{u}_1 \otimes \tilde{t}_1^{q-1}\tilde{t}_2^{k-(q-1)} + \tilde{u}_2 \otimes \tilde{t}_1^q \tilde{t}_2^{k-q}. \tag{9}$$

Considering weights and the orthogonality between $S^{k\pm1}\mathbf{C}^2$, we see that for $p = 0, \dots, k-1$, $S^{k-1}\mathbf{C}^2 \subset H^0(V_1(k))$ is spanned by

$$v_-^p := \binom{k}{p}^{-1} \tilde{u}_1 \otimes \tilde{t}_1^p \tilde{t}_2^{k-p} - \binom{k}{p+1}^{-1} \tilde{u}_2 \otimes \tilde{t}_1^{p+1}\tilde{t}_2^{k-(p+1)}. \tag{10}$$

Since the evaluation map plays a crucial role to identify the induced map, we describe the evaluation map using explicit expression of sections in each case. Due to the equivariance, the evaluation is provided only at one point. Let $o \in \mathbf{C}P^1$ be a reference point corresponding to $[e]$, where e is the unit of SU(2).

Borel–Weil theorem imply that $H^0(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2)) = S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2$. An evaluation map $ev : \underline{S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2} \to \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$ is given by $ev = ev_{k_1} \oplus ev_{k_2}$, where $ev_{k_i} : \underline{S^{k_i}\mathbf{C}^2} \to \mathcal{O}(k_i)$ is an evaluation for $i = 1, 2$. Since $\tilde{t}_1(o) = 0$ and $h_1(\tilde{t}_2(o), \tilde{t}_2(o)) = 1$, it follows that

$$\operatorname{Ker} ev_{k_{i o}} = \operatorname{Span}\left\langle \tilde{t}_1^p \tilde{t}_2^{k_i - p} \,\middle|\, 1 \leqq p \leqq k_i \right\rangle.$$

As already seen, $H^0(V_1(k))$ is equivalent to $S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$ as SU(2)-module. Suppose that $a > 0$. We denote the evaluation map by $ev^1 : \underline{S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2} \to V_1(k)$. In addition to the behaviour of $\tilde{t}_i(o)$, since $\{\tilde{u}_1, \tilde{u}_2\}$ is a global frame of $V_1 \to \mathbf{C}P^1$, it follows from (9) and (10) that

$$v_-^p, \quad p = 1, \dots, k-1, \quad \text{and} \quad v_+^q, \quad q = 2, \dots, k+1$$

are in $\operatorname{Ker} ev_o^1$. Since $-v_-^0 + v_+^1 = (k^{-1} + 1)\tilde{u}_2 \otimes \tilde{t}_1 \tilde{t}_2^{k-1}$, we deduce that $-v_-^0 + v_+^1$ is also in $\operatorname{Ker} ev_o^1$. By dimension count, we conclude that

$$\operatorname{Ker} ev_o^1 = \operatorname{Span}\left\langle v_-^p, v_+^q, -v_-^0 + v_+^1 \,\middle|\, 1 \leqq p \leqq k-1,\ 2 \leqq q \leqq k+1 \right\rangle.$$

Next, we consider the induced connection on the pull-back of the universal quotient bundle by the induced map.

First of all, we begin with the general theory. Let $f : \mathbf{C}P^1 \to Gr_{N-2}(\mathbf{C}^N)$ be the induced holomorphic map from $(V \to \mathbf{C}P^1, \mathbf{C}^N)$, where $V \to \mathbf{C}P^1$ is a holomorphic vector bundle of rank 2. Suppose that $V \to \mathbf{C}P^1$ has a Hermitian metric h and an SU(2)-action which preserves the metric h and the holomorphic vector bundle structure. We can deduce that $V \to \mathbf{C}P^1$ has a unique invariant Hermitian connection ∇. Suppose that $\mathbf{C}^N$ is an SU(2)-module such that the evaluation map $ev : \underline{\mathbf{C}^N} \to V$ is equivariant under the SU(2)-actions. Then f is an equivariant map. We fix an SU(2)-invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{C}^N$. Then the pull-back of the universal quotient bundle has the induced metric. If the induced metric is the same as h, then $\langle \cdot, \cdot \rangle$ is called to be *compatible* with h. Then the uniqueness of the Hermitian connection implies that the induced connection is the same as the invariant connection ∇. In our case, the compatibility condition is easily checked, because of the equivariance. Let $o \in \mathbf{C}P^1$ be the reference point and U(1) the isotropy subgroup of SU(2). Then V_o the fibre of $V \to \mathbf{C}P^1$ at o is regarded as a U(1)-module. We take the orthogonal complement $\operatorname{Ker}^\perp ev_o$ of $\operatorname{Ker} ev_o \subset \mathbf{C}^N$. Then the compatibility condition is expressed as

$$\langle \tilde{v}_1, \tilde{v}_2 \rangle = h(\tilde{v}_1(o), \tilde{v}_2(o)),$$

where $\tilde{v}_1, \tilde{v}_2 \in \operatorname{Ker}^\perp ev_o$. The image of the adjoint homomorphism ev^* of ev at o is identified with $\mathrm{Ker}^{\perp} ev_o$. Since ev is equivariant, $\mathrm{Ker}^{\perp} ev_o$ is also regarded as a U(1)-module which is equivalent to V_o as representation. Hence $\mathbf{C}^N$ has the same weights as ones of V_o.

From Theorem 3.2, the candidates for (V, h) are $(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2), h_{k_1} \oplus h_{k_2})$ and $(V_1(k), h_a \otimes h_k)$. We will examine the compatibility condition in each case.

First of all, we begin with $(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2), h_{k_1} \oplus h_{k_2})$.

Proposition 4.3. *Let* $f : \mathbf{C}P^1 \to Gr_{N-2}(\mathbf{C}^N)$ *be the induced holomorphic map from* $(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \to \mathbf{C}P^1, \mathbf{C}^N)$*, where* $\mathbf{C}^N$ *is a submodule of the* SU(2)*-representation space* $H^0(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2))$*. We put an invariant Hermitian inner product on* $\mathbf{C}^N$*.*

Then, the induced connection is the direct product of the canonical connections if and only if $\mathbf{C}^N$ *is equivalent to* $S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2$ *with a Hermitian inner product* $\langle \cdot, \cdot \rangle_{k_1} \oplus \langle \cdot, \cdot \rangle_{k_2}$ *as* SU(2)*-module.*

Proof. Let $f : \mathbf{C}P^1 \to Gr_{N-2}(\mathbf{C}^N)$ be the induced holomorphic map from $(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \to \mathbf{C}P^1, \mathbf{C}^N)$. Since $\mathbf{C}^N$ inherits an SU(2)-action induced by the group action on $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \to \mathbf{C}P^1$, the evaluation map is equivariant. Borel–Weil theorem yields that $\mathbf{C}^N$ is a submodule of $S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2$. Suppose that $k_1 \neq k_2$. If $\mathbf{C}^N$ is a proper submodule, then the evaluation is not an epimorphism by Schur's lemma. This is a contradiction. Suppose that $k_1 = k_2$. Then $\mathbf{C}^N$ has $-k_1$ as weight with multiplicity two. However, $S^{k_1}\mathbf{C}^2$ has $-k_1$ with multiplicity 1 as weight. Hence $\mathbf{C}^N$ is $S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2$ in both cases. As already seen, the evaluation $ev : \underline{S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2} \to \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$ is given by $ev = ev_{k_1} \oplus ev_{k_2}$. Consequently, we may check the compatibility condition for each $i = 1, 2$.

Since $S^{k_i}\mathbf{C}^2$ is an irreducible module, an invariant Hermitian inner product is unique up to a constant multiple. We therefore deduce that $\{\tilde{t}_1^{\,p}\tilde{t}_2^{\,k_i-p} \mid p = 0, 1, \ldots, k_i\}$ is a unitary basis of $S^{k_i}\mathbf{C}^2$. Since $\mathrm{Ker}\, ev_{k_{i\,o}}$ is spanned by $\tilde{t}_1^{\,p}\tilde{t}_2^{\,k_i-p}$, where $p = 1, \ldots, k_i$, we see that $\tilde{t}_2^{\,k_i}$ is orthogonal to $\mathrm{Ker}\, ev_o$. Thus the compatibility condition is satisfied only in the case that the Hermitian inner product is $\langle \cdot, \cdot \rangle_{k_i}$ on $S^{k_i}\mathbf{C}^2$. $\square$

We denote by f_{k_i} the induced map from $(\mathcal{O}(k_i) \to \mathbf{C}P^1, S^{k_i}\mathbf{C}^2)$, where $S^{k_i}\mathbf{C}^2$ has an invariant Hermitian inner product $\langle \cdot, \cdot \rangle_{k_i}$. Then the induced map f_d from $(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2) \to \mathbf{C}P^1, S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2)$, where $S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2$ has an invariant Hermitian inner product $\langle \cdot, \cdot \rangle_{k_1} \oplus \langle \cdot, \cdot \rangle_{k_2}$, is described by

$$f_d(x) = f_{k_1} \oplus f_{k_2} = \mathrm{Ker}\, ev_{k_{1\,x}} \oplus \mathrm{Ker}\, ev_{k_{2\,x}},$$

which is called of *direct sum* type. Using equivariance, f_d is also expressed as

$$f_d([g]) = g\, \mathrm{Ker}\, ev_{k_{1\,o}} \oplus g\, \mathrm{Ker}\, ev_{k_{2\,o}}, \qquad g \in \mathrm{SU}(2).$$

Since f_d is determined up to an isomorphism $\mathbf{C}^N$ with $S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2$ preserving the Hermitian inner products and the group actions, f_d is unique up to the composite of a holomorphic isometry of $Gr_{N-2}(\mathbf{C}^N)$.

We have already defined an invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on $H^0(V_1(k))$ as $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{V_1} \otimes \langle \cdot, \cdot \rangle_k$. If the Hermitian inner product $\langle \cdot, \cdot \rangle$ is restricted to the subspace $S^{k\pm 1}\mathbf{C}^2$ of $H^0(V_1(k))$, then we obtain an invariant Hermitian inner product denoted by $\langle \cdot, \cdot \rangle|_{k\pm 1}$, respectively.

Proposition 4.4. *Let $f : \mathbf{C}P^1 \to Gr_{N-2}(\mathbf{C}^N)$ be the induced holomorphic map from $(V_1(k) \to \mathbf{C}P^1, \mathbf{C}^N)$. Suppose that $\mathbf{C}^N$ is a submodule of the* SU(2)*-representation space $H^0(V_1(k))$. We put an invariant Hermitian inner product on $\mathbf{C}^N$.*

Then, the induced connection is ∇_a $(a \neq 0)$ if and only if $\mathbf{C}^N$ is equivalent as SU(2)*-module to*

(1) *$S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$ with a Hermitian inner product $\frac{a^2 k}{k+1-a^2}\langle\cdot,\cdot\rangle|_{k-1} \oplus \langle\cdot,\cdot\rangle|_{k+1}$ for $0 < a < \sqrt{k+1}$, or*

(2) *$S^{k+1}\mathbf{C}^2$ with a Hermitian inner product $\langle\cdot,\cdot\rangle|_{k+1}$ for $a = \sqrt{k+1}$.*

Proof. Let $f : \mathbf{C}P^1 \to Gr_{N-2}(\mathbf{C}^N)$ be the induced holomorphic map from $(V_1(k) \to \mathbf{C}P^1, \mathbf{C}^N)$. Since $\mathbf{C}^N$ inherits an SU(2)-action induced by the group action on $V_1(k) \to \mathbf{C}P^1$, the evaluation map is equivariant. It follows from (8) that $\mathbf{C}^N$ is a subspace of $S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$. Then $\mathbf{C}^N$ has $-k+1$ and $-k-1$ as weights. On the one hand, $S^{k-1}\mathbf{C}^2$ has $-k+1$ with multiplicity 1 as weight and $-k-1$ is not a weight of $S^{k-1}\mathbf{C}^2$. On the other hand, $S^{k+1}\mathbf{C}^2$ has $-k+1$ and $-k-1$ as weight, each of which has multiplicity 1. Hence we conclude that $\mathbf{C}^N$ is $S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$ or $S^{k+1}\mathbf{C}^2$.

Suppose that $\mathbf{C}^N = S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$. Schur's lemma yields that an invariant Hermitian inner product on $\mathbf{C}^N$ is of the form $b\langle\cdot,\cdot\rangle|_{k-1} \oplus c\langle\cdot,\cdot\rangle|_{k+1}$ for some positive real numbers b and c, which is denoted by $\langle\cdot,\cdot\rangle_b$. (Though we omit c in this notation, the reason will be clear in the next paragraph.)

Since $\langle\cdot,\cdot\rangle|_{k+1}$ is an invariant Hermitian inner product on $S^{k+1}\mathbf{C}^2$ and $S^{k+1}\mathbf{C}^2$ is an irreducible SU(2)-module, $\langle\cdot,\cdot\rangle|_{k+1}$ is the unique invariant Hermitian inner product up to a constant multiple. Hence $\tilde{u}_2 \otimes \tilde{t}_2^k \in S^{k+1}\mathbf{C}^2$ is orthogonal to $\mathrm{Ker}\, ev_o^1$. By definition, we have that

$$\langle \tilde{u}_2 \otimes \tilde{t}_2^k, \tilde{u}_2 \otimes \tilde{t}_2^k \rangle_b = c.$$

Since $\tilde{u}_2(o) = t_1(o)$ and $\tilde{t}_2(o) = t_1(o)$, it follows for $a \neq 0$ that

$$h_a \otimes h_k \left(\tilde{u}_2(o) \otimes \tilde{t}_2^k(o), \tilde{u}_2(o) \otimes \tilde{t}_2^k(o) \right) = 1.$$

Thus we deduce that $c = 1$.

Next, we abbreviate v_-^0 and v_+^1 to v_- and v_+, respectively. We examine the compatibility condition for $\alpha v_- + \beta v_+$, where α and β are complex numbers. The compatibility condition requires

$$\alpha v_- + \beta v_+ \in \mathrm{Ker}\, ev_{a_o}^{\perp},$$

and

$$\langle \alpha v_- + \beta v_+, \alpha v_- + \beta v_+ \rangle_b = h_a \otimes h_k \left(\alpha v_-(o) + \beta v_+(o), \alpha v_-(o) + \beta v_+(o) \right).$$

Since $\langle\cdot,\cdot\rangle|_{k\pm1}$ is an invariant Hermitian inner product on $S^{k\pm1}\mathbf{C}^2$ and $S^{k\pm1}\mathbf{C}^2$ is an irreducible SU(2)-module, the condition $\alpha v_- + \beta v_+ \in \mathrm{Ker}\, ev_o^{1\perp}$ is equivalent to a condition that

$$\langle \alpha v_- + \beta v_+, -v_- + v_+ \rangle_b = -b\alpha(1 + k^{-1}) + \beta(k+1) = 0.$$

Thus $\beta = k^{-1} b\alpha$. From the definition of $\langle\cdot,\cdot\rangle_b$, we have

$$\langle \alpha v_- + \beta v_+, \alpha v_- + \beta v_+ \rangle_b = b|\alpha|^2 k^{-1}(k+1) + |\beta|^2(k+1).$$

On the other hand, since $v_-(o) = v_+(o) = s_1 \otimes t_1^k(o)$, we get

$$h_a \otimes h_k\,(\alpha v_-(o) + \beta v_+(o), \alpha v_-(o) + \beta v_+(o)) = |\alpha + \beta|^2 a^2.$$

Hence the compatibility condition implies that

$$b = \frac{a^2 k}{k + 1 - a^2}.$$

In the case that $\mathbf{C}^N = S^{k+1}\mathbf{C}^2$, an invariant Hermitian inner product on $\mathbf{C}^N$ is unique up to a constant multiple by Schur's lemma. Therefore we take $c\langle\cdot,\cdot\rangle|_{k+1}$ as an invariant Hermitian inner product on $\mathbf{C}^N$. Since $\operatorname{Ker} ev_o^1$ is spanned by v_+^q ($q = 2, \ldots, k+1$), it follows that $\operatorname{Ker} ev_o^{1\perp}$ is spanned by

$$v_+^0 = \tilde{u}_2 \otimes \tilde{t}_2^k, \quad \text{and} \quad v_+^1 = \tilde{u}_1 \otimes \tilde{t}_2^k + \tilde{u}_2 \otimes \tilde{t}_1^1 \tilde{t}_2^{k-1},$$

which are orthogonal. Consequently, the compatibility condition requires

$$c\langle v_+^0, v_+^0\rangle|_{k+1} = h_a \otimes h_k\,(\tilde{u}_2(o) \otimes \tilde{t}_2(o), \tilde{u}_2(o) \otimes \tilde{t}_2(o)) = 1$$

and

$$c\langle v_+^1, v_+^1\rangle|_{k+1} = h_a \otimes h_k \left(s_1 \otimes \tilde{t}_1^k(o), s_1 \otimes \tilde{t}_1^k(o)\right) = a^2.$$

It follows that $c = 1$ and $a^2 = 1 + k$. □

Remark 1. We have found an induced map f for $0 < a \leqq \sqrt{k+1}$, under which the induced connection on $f^*Q \to \mathbf{C}P^1$ is gauge equivalent to ∇^a on $V_1(k) \to \mathbf{C}P^1$. If $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+2})$ is a holomorphic map satisfying $f^*Q \cong V_1(k)$, then the mean curvature operator coincides with the induced curvature contracted with the Kähler form up to the sign ([5, Proposition 4.4]). From (4), we see that the eigenvalues of the mean curvature operator are $-(k - 1 + a^2)$ and $-(k + 1 - a^2)$. Since the mean curvature operator is a non-positive Hermitian endomorphism [5, Lemma 3.2], we have $a \leqq \sqrt{k+1}$. In summary, $a \leqq \sqrt{k+1}$ is a necessary condition for the existence of holomorphic map with the pull-back connection being gauge equivalent to ∇^a.

We describe the induced map f_a from $(V_1(k) \to \mathbf{C}P^1, S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2)$, where $0 < a < \sqrt{k+1}$. The Hermitian inner product on $S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$ given in Proposition 4.4 is denoted by $\langle\cdot,\cdot\rangle_b$:

$$\langle\cdot,\cdot\rangle_b := b\langle\cdot,\cdot\rangle|_{k-1} \oplus \langle\cdot,\cdot\rangle|_{k+1} = \frac{a^2 k}{k + 1 - a^2}\langle\cdot,\cdot\rangle|_{k-1} \oplus \langle\cdot,\cdot\rangle|_{k+1}.$$

The equivariance of f_a yields that

$$f_a([g]) = \operatorname{Ker} ev_{[g]}^1 = g \operatorname{Ker} ev_o^1, \quad g \in \mathrm{SU}(2),$$

where

$$\operatorname{Ker} ev_o^1 = \operatorname{Span}\left\langle v_-^p, v_+^q, -v_-^0 + v_+^1 \;\middle|\; 1 \leqq p \leqq k-1,\ 2 \leqq q \leqq k+1 \right\rangle.$$

Since f_a is determined up to an isomorphism of $\mathbf{C}^N$ with $S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$ preserving the SU(2) structure, f_a is unique up to the composite of a holomorphic isometry of $Gr_{N-2}(\mathbf{C}^N)$.

To describe $f_{\sqrt{k+1}}$, the evaluation ev^1 is now restricted to $S^{k+1}\mathbf{C}^2$. Then, the induced map $f_{\sqrt{k+1}}$ from the pair $(V_{\sqrt{k+1}} \to \mathbf{C}P^1, S^{k+1}\mathbf{C}^2)$, where $S^{k+1}\mathbf{C}^2$ has $\langle\cdot,\cdot\rangle|_{k+1}$ as Hermitian inner product, is expressed as

$$f_{\sqrt{k+1}}([g]) = \operatorname{Ker} ev^1|_{S^{k+1}\mathbf{C}^2[g]} = g \operatorname{Ker} ev^1|_{S^{k+1}\mathbf{C}^2 o}, \quad g \in \mathrm{SU}(2).$$

Since $f_{\sqrt{k+1}}$ is determined up to an isomorphism $\mathbf{C}^N$ with $S^{k+1}\mathbf{C}^2$ preserving the SU(2) structure, $f_{\sqrt{k+1}}$ is unique up to the composite of a holomorphic isometry of $Gr_{N-2}(\mathbf{C}^N)$.

REMARK 2. When $a = 1$, $V_1(k) \to \mathbf{C}P^1$ is holomorphically isomorphic to the orthogonal direct sum $\mathcal{O}(k) \oplus \mathcal{O}(k) \to \mathbf{C}P^1$ with the canonical connection. The group action on $\mathcal{O}(k) \oplus \mathcal{O}(k) \to \mathbf{C}P^1$ is obtained by forgetting the group action on $V_1 \to \mathbf{C}P^1$ from the action on $V_1(k) \to \mathbf{C}P^1$. This means that $\tilde{u}_i$ has weight 0 in the case of the group action on $\mathcal{O}(k) \oplus \mathcal{O}(k) \to \mathbf{C}P^1$. Using $\tilde{u}_i$, the compatible Hermitian inner product on $H^0(\mathcal{O}(k)) \oplus H^0(\mathcal{O}(k))$ is expressed as $\langle\cdot,\cdot\rangle_k \oplus \langle\cdot,\cdot\rangle_k = \langle\cdot,\cdot\rangle_{V_1} \otimes \langle\cdot,\cdot\rangle_k$. Thus, $H^0(V_1(k))$ and $H^0(\mathcal{O}(k) \oplus \mathcal{O}(k))$ have the same Hermitian inner product. Since the evaluation map is independent of the group actions, we have the same induced map, which is equivariant under both group actions.

We would like to understand f_a as a deformation of f_1. For this end, we fix a Hermitian inner product $\langle\cdot,\cdot\rangle_{V_1} \otimes \langle\cdot,\cdot\rangle_k$ on $\mathbf{C}^{2k+2} = S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$ and vary the evaluation. Recall that

$$\operatorname{Ker} ev^1_o = \operatorname{Span}\left\langle v_-^p, v_+^q, -v_-^0 + v_+^1 \;\middle|\; 1 \leqq p \leqq k-1,\, 2 \leqq q \leqq k+1 \right\rangle.$$

Next, we introduce a Hermitian transformation T_a of $\mathbf{C}^{2k+2}$ as

$$T_a := \begin{pmatrix} \frac{\sqrt{k+1-a^2}}{a\sqrt{k}} I_{S^{k-1}\mathbf{C}^2} & O \\ O & I_{S^{k+1}\mathbf{C}^2} \end{pmatrix}$$

to obtain $\langle T_a\cdot, T_a\cdot\rangle_b = \langle\cdot,\cdot\rangle_{b=1} = \langle\cdot,\cdot\rangle_{V_1} \otimes \langle\cdot,\cdot\rangle_k$.

Since a Hermitian inner product $\langle\cdot,\cdot\rangle_b$ on $\mathbf{C}^{2k+2}$ provides a complex Grassmannian with a Kähler metric, a Kähler manifold $Gr_{2k}(\mathbf{C}^{2k+2})$ is denoted by $(Gr_{2k}(\mathbf{C}^{2k+2}), \langle\cdot,\cdot\rangle_b)$. With this understood, T_a gives a holomorphic isometry of $(Gr_{2k}(\mathbf{C}^{2k+2}), \langle\cdot,\cdot\rangle_b)$ into $(Gr_{2k}(\mathbf{C}^{2k+2}), \langle\cdot,\cdot\rangle_{b=1})$ as $U \mapsto T_a^{-1}U$, where U is a $2k$-dimensional subspace in $\mathbf{C}^{2k+2}$. By the composition, we can describe the induced map denoted by the same symbol $f_a : \mathbf{C}P^1 \to \left(Gr_{2k}(\mathbf{C}^{2k+2}), \langle\cdot,\cdot\rangle_{b=1}\right)$ as

$$f_a([g]) = T_a^{-1} g \operatorname{Ker} ev^1_o = g T_a^{-1} \operatorname{Ker} ev^1_o, \quad g \in \mathrm{SU}(2), \tag{11}$$

where,

$$T_a^{-1}\operatorname{Ker} ev^1_o = \operatorname{Span}\left\langle v_-^p, v_+^q, -a\sqrt{k}v_-^0 + \sqrt{k+1-a^2}\,v_+^1 \;\middle|\; 1 \leqq p \leqq k-1,\, 2 \leqq q \leqq k+1 \right\rangle. \tag{12}$$

The map $f_{\sqrt{k+1}}$ is also explained as a deformation of f_1. If $S^{k+1}\mathbf{C}^2$ is considered as a subspace of $\mathbf{C}^{2k+2}$, then $Gr_k(S^{k+1}\mathbf{C}^2)$ is realized as a totally geodesic submanifold of $\left(Gr_{2k}(\mathbf{C}^{2k+2}), \langle\cdot,\cdot\rangle_{b=1}\right)$. Because the Hermitian inner product on $S^{k+1}\mathbf{C}^2$ is unchanged. From the viewpoint of vector bundle, $Gr_k(S^{k+1}\mathbf{C}^2)$ is the zero set of sections, which belong to $S^{k-1}\mathbf{C}^2 \subset H^0\left(Gr_{2k}(\mathbf{C}^{2k+2}); Q\right)$, of the universal quotient bundle over $Gr_{2k}(\mathbf{C}^{2k+2})$. Notice that $\operatorname{Ker} T_{\sqrt{k+1}}$ is nothing but $S^{k-1}\mathbf{C}^2$, and so, $T_{\sqrt{k+1}}$ determines the totally geodesic embedding. The evaluation ev^1 restricted to $S^{k+1}\mathbf{C}^2$ satisfies

$$\operatorname{Ker} ev^1|_{S^{k+1}\mathbf{C}^2 o} = \operatorname{Span}\left\langle v_+^q \,\middle|\, 2 \leqq q \leqq k+1 \right\rangle.$$

Though $T_{\sqrt{k+1}}$ is not invertible, a subspace $T^{-1}_{\sqrt{k+1}} \operatorname{Ker} ev^1_o$ of $\mathbf{C}^{2k+2}$ can be defined by putting $a^2 = k+1$ in (12). It follows that

$$\operatorname{Ker} ev^1|_{S^{k+1}\mathbf{C}^2 o} = T^{-1}_{\sqrt{k+1}} \operatorname{Ker} ev^1_o \cap S^{k+1}\mathbf{C}^2.$$

By the composition with the totally geodesic embedding, the induced map from $(V_{\sqrt{k+1}} \to \mathbf{C}P^1, S^{k+1}\mathbf{C}^2)$ is regarded as a map into Grassmannian $\left(Gr_{2k}(\mathbf{C}^{2k+2}), \langle\cdot,\cdot\rangle_{b=1}\right)$, which is described as

$$f_{\sqrt{k+1}}([g]) = gT^{-1}_{\sqrt{k+1}} \operatorname{Ker} ev^1_o, \quad g \in SU(2).$$

Hence $f_{\sqrt{k+1}}$ is called of *degenerate* type.

Next, we compactify the moduli space. Since $T_a \in \operatorname{Aut}(\mathbf{C}^{2k+2})$, where $\mathbf{C}^{2k+2} = S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$, is invertible in the case that $0 < a < \sqrt{k+1}$, f_a is well-defined as (11), which is considered as the deformation of f_1 for $a \in (0, \sqrt{k+1})$. We can equip the moduli space $(0, \sqrt{k+1})$ with a natural topology as follows. The L^2-inner product provides the Dolbeault cohomology group $H^1(\mathcal{O}(-2))$ with a topology. Since $[0,\infty)$ is the quotient of $H^1(\mathcal{O}(-2))$ by S^1- (or constant gauge group) action from our description of the extension, $[0,\infty)$ has the induced topology. Then, $(0, \sqrt{k+1})$ is indeed an open interval of $[0,\infty)$. Hence the closed interval $[0, \sqrt{k+1}]$ is considered as the natural compactification of $(0, \sqrt{k+1})$ from the induced topology. We give a geometric interpretation to the compactification of the moduli.

Since $V_0(k) \cong \mathcal{O}(k-1) \oplus \mathcal{O}(k+1)$, the induced map f_0 is expected to be of the direct sum type. On the one hand, the Hermitian transform T_a is blown-up, and so T_a seems to be of no use for our purpose. On the other hand, putting $a = 0$ in (12), we obtain

$$T_0^{-1} \operatorname{Ker} ev^1_o = \operatorname{Span}\left\langle v_-^p, v_+^q \,\middle|\, 1 \leqq p \leqq k-1,\ 1 \leqq q \leqq k+1 \right\rangle,$$

which is the same as $\operatorname{Ker} ev_{k-1_o} \oplus \operatorname{Ker} ev_{k+1_o}$. Notice that we fix Hermitian inner products $\langle,\rangle_{V_1} \otimes \langle,\rangle_k$ on $\mathbf{C}^{2k+2}$ and $\langle,\rangle_{k-1} \oplus \langle,\rangle_{k+1}$ on $S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2 \cong H^0(\mathcal{O}(k-1)) \oplus H^0(\mathcal{O}(k+1))$. Under the identification e_i with $\tilde{u}_i$ and $\tilde{t}_i$, we define a contraction operator $C : \mathbf{C}^{2k+2} \to S^{k-1}\mathbf{C}^2$ as

$$C(v_+^q) = 0, \quad \text{and} \quad C(v_-^p) = \sqrt{\frac{k+1}{k}} \binom{k-1}{p}^{-1} \tilde{t}_1^p \tilde{t}_2^{k-1-p},$$

and a symmetrization operator $S : \mathbf{C}^{2k+2} \to S^{k+1}\mathbf{C}^2$ as

$$S(v_+^q) = \tilde{t}_1^q \tilde{t}_2^{k+1-q} \quad \text{and} \quad S(v_-^p) = 0.$$

Then $C \oplus S : \mathbf{C}^{2k+2} \to S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$ preserves the Hermitian inner products. Under the identification $\mathbf{C}^{2k+2} \cong S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$ by $C \oplus S$, f_0 is regarded as a map of the direct sum type $f_{k-1} \oplus f_{k+1}$.

In the case that $a^2 = k+1$, we have a map of degenerate type and a totally geodesic embedding of $Gr_k(S^{k+1}\mathbf{C}^2)$ into $Gr_{2k}(\mathbf{C}^{2k+2})$ specified by $\operatorname{Ker} T_{\sqrt{k+1}}$ as already seen.

Thus, our compactification of the moduli space is naturally interpreted from subspaces $T_a^{-1} \operatorname{Ker} ev_o$ and $\operatorname{Ker} T_a$.

To state the main theorem, we define the fullness of a map.

DEFINITION 4.5 ([5]). Let $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+2})$ be a map. Then every element of $\mathbf{C}^{n+2}$ gives a section of $f^*Q \to CP^1$ by the pull-back of section, and so we have a linear map $F : \mathbf{C}^{n+2} \to \Gamma(f^*Q)$. If the linear map F has a trivial kernel, then f is called a *full* map.

Finally, to present the main theorem in terms of representation theory, we define a unitary basis of $S^k\mathbf{C}^2$ as

$$w_p^k = \sqrt{\binom{k}{p}^{-1}} e_1^p e_2^{k-p}, \quad p = 0, 1, \dots, k.$$

Theorem 4.6. *Let $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+2})$ be an equivariant full holomorphic embedding of degree l. Then, l is positive and one of the following three cases holds.*

(1) *We have $n = l$. There exist non-negative integers k_1 and k_2 satisfying $k_1 + k_2 = l$. The vector space $\mathbf{C}^{n+2}$ is identified with $S^{k_1}\mathbf{C}^2 \oplus S^{k_2}\mathbf{C}^2$ as* SU(2)*-module. The map f is congruent to f_d defined as*

$$f_d([g]) = gU_{k_1} \oplus gU_{k_2}, \quad U_{k_i} = \mathrm{Span}\left\langle w_p^{k_i} \,\middle|\, 1 \leqq p \leqq k_i \right\rangle, i = 1, 2.$$

(2) *We have $n = l$. There exist a positive integer k satisfying $2k = l$ and $a \in (0, \sqrt{k+1})$. The vector space $\mathbf{C}^{n+2}$ is identified with $S^{k-1}\mathbf{C}^2 \oplus S^{k+1}\mathbf{C}^2$ as* SU(2)*-module. The map f is congruent to f_a defined as*

$$f_a([g]) = gU_a,$$
$$U_a = \mathrm{Span}\left\langle w_p^{k-1}, w_q^{k+1}, -aw_0^{k-1} + \sqrt{k+1-a^2}w_1^{k+1} \,\middle|\, 1 \leqq p \leqq k-1,\ 2 \leqq q \leqq k+1 \right\rangle.$$

(3) *We have $2n = l$. There exists a positive integer k satisfying $2k = l$. The vector space $\mathbf{C}^{n+2}$ is identified with $S^{k+1}\mathbf{C}^2$ as* SU(2)*-module. The map f is congruent to $f_{\sqrt{k+1}}$ defined as*

$$f_{\sqrt{k+1}}([g]) = gU, \quad U = \mathrm{Span}\left\langle w_q^{k+1} \,\middle|\, 2 \leqq q \leqq k+1 \right\rangle.$$

Proof. Let $f : \mathbf{C}P^1 \to Gr_n(\mathbf{C}^{n+2})$ be an equivariant full holomorphic map of degree l. Since f is a holomorphic embedding, l is positive. It follows from the equivariance that $f^*Q \to \mathbf{C}P^1$ is a holomorphic vector bundle of rank 2 with an invariant connection. By fullness of the map, $\mathbf{C}^{n+2}$ is regarded as a subspace of the space of holomorphic sections of $f^*Q \to \mathbf{C}P^1$. Thus f is considered as the induced map from $\left(f^*Q \to \mathbf{C}P^1, \mathbf{C}^{n+2}\right)$. Propositions 4.2, 4.3, 4.4 and the successive classification of induced maps imply the result. □

From Theorem 4.6 and Corollary 3.3, we can conclude

Theorem 4.7. *The set of equivariant full holomorphic embeddings of $\mathbf{C}P^1$ into Grassmannians of two-planes are identified with the set of non-flat invariant connections modulo gauge equivalence on the vector bundles of rank two on $\mathbf{C}P^1$ with semi-positivity.*

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