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ROOTED ORDER ON MINIMAL GENERATORS OF POWERS OF SOME COVER IDEALS

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Abstract

We define a total order, which we call rooted order, on minimal generating set of $J(P_n)^s$ where $J(P_n)$ is the cover ideal of a path graph on n vertices. We show that each power of a cover ideal of a path has linear quotients with respect to the rooted order. Along the way, we characterize minimal generating set of $J(P_n)^s$ for $s \geq 3$ in terms of minimal generating set of $J(P_n)^2$. We also discuss the extension of the concept of rooted order to chordal graphs. Computational examples suggest that such order gives linear quotients for powers of cover ideals of chordal graphs as well.

1. Introduction

Let $S = \mathbb{k}[x_1, \dots, x_n]$ be the polynomial ring over a field $\mathbb{k}$ and let G be a finite simple graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$. The *cover ideal* of G is a squarefree monomial ideal of S defined by

$$J(G) = \bigcap_{\{x_i, x_j\} \in E(G)} (x_i, x_j).$$

The cover ideal $J(G)$ is the Alexander dual of the well-known edge ideal of G. Cover ideals and their powers were studied in many articles, see for example [2, 5, 6, 7, 8, 9, 12, 15, 19, 20, 21] Herzog, Hibi and Ohsugi [15] showed that if G is a Cohen-Macaulay chordal graph, then all powers of the cover ideal of G have linear resolutions. Moreover, they proposed the following conjecture:

Conjecture 1.1 ([15, Conjecture 2.5]). *All powers of the vertex cover ideal of a chordal graph are componentwise linear.*

Francisco and Van Tuyl [9] showed that cover ideals of chordal graphs are componentwise linear. For a graded ideal $I \subset S$, being componentwise linear is an algebraic property which requires that for all j, the ideal $I_{\langle j \rangle}$, generated by all homogeneous polynomials of degree j belonging to I, has a linear resolution. Later, it was proved that chordal graphs in fact have stronger combinatorial properties such as being shellable [22] and vertex decomposable [23]. In [20] it was proved that powers of cover ideals of Cohen-Macaulay chordal graphs have linear quotients. A graph G is called Cohen-Macaulay if the quotient ring $S/I(G)$ is Cohen-Macaulay, where $I(G)$ denotes the edge ideal of G. It is well-known [13, Lemma 9.1.10] that the cover ideal of a Cohen-Macaulay graph is generated in single degree. Since

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cover ideal of a path graph can have minimal generators of different degrees, paths are not necessarily Cohen-Macaulay. In fact, using the recursive description of the minimal generating set of $J(P_n)$ in Lemma 2.2, one can show that P_n is not Cohen-Macaulay for $n \geq 5$.

In a recent paper [19] Kumar and Kumar proved Conjecture 1.1 for all trees. Their main tool is a result from [8] which says that for any graph G the polarization of k^{th} symbolic power of $J(G)$ is the cover ideal of some graph denoted by G_k. Since symbolic powers and ordinary powers of cover ideal of bipartite graphs coincide [11], their approach is to show that G_k is vertex decomposable when G is a tree. Although trees contain the class of path graphs, the methods in [19] cannot be applied to non-bipartite chordal graphs.

The main goal of this paper is to make a contribution to the problem in Conjecture 1.1 and bring up an idea that is applicable to all chordal graphs. We introduce the notion of rooted order (Definition 2.7 and Definition 5.5) and we show that all powers of the cover ideal of a path graph have linear quotients with respect to such order (Theorem 4.3). Our results build on and extend the analogous results presented in [6] from second powers to all powers. We analyze the minimal generating set of $J(P_n)^s$ in relation to rooted order. An interesting byproduct we obtain in the process is Corollary 3.12 which characterizes the minimal generators of $J(P_n)^s$ for $s \geq 3$ in terms of those of the second power. Although we focus on the class of path graphs, the notion of rooted order naturally generalizes to chordal graphs. In fact, examples we tested on chordal graphs led us to question if one can always find a rooted order which gives linear quotients for powers of their cover ideals. We discuss this in Section 5 and we think that the techniques developed in this article may be helpful to further explore the problem at hand in a more general framework.

2. Preliminaries

Let $S = \mathbb{k}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{k}$ and let I be a monomial ideal. We denote the set of minimal generators of I by $G(I)$. We say I has *linear quotients* if there exists an order $u_1, \ldots, u_k$ on the elements of $G(I)$ such that for every $i = 2, \ldots, k$ the colon ideal $(u_1, \ldots, u_{i-1}) : (u_i)$ is generated by some variables. To simplify our notation, for any pair of monomials u and v we will write

$$u : v = \frac{u}{\gcd(u, v)}.$$

If M is a subset of S and u is a monomial, then we define a new subset uM by

$$uM = \{um : m \in M\}.$$

Similarly, if $L = v_1, \ldots, v_t$ is a list (or sequence) of monomials, then uL denotes a new list obtained from L by multiplying each term by u. In other words,

$$uL = uv_1, \ldots, uv_t.$$

To keep our notation simple and also to distinguish lists from ideals we will not put parentheses around lists.

Let G be a finite simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. A set C of vertices of G is called a *vertex cover* if $e \cap C \neq \emptyset$ for every edge $e \in E(G)$. A vertex cover C is called *minimal* if no proper subset of C forms a vertex cover for G. The *cover* *ideal* of G is denoted by $J(G)$ and it is defined by

$$J(G) = \bigcap_{\{x_i,x_j\}\in E(G)} (x_i, x_j).$$

The set of minimal generators of $J(G)$ is given by

$$G(J(G)) = \{x_{i_1} \dots x_{i_k} : \{x_{i_1}, \dots, x_{i_k}\} \text{ is a minimal vertex cover of } G\}.$$

If the graph G has no edges, then $J(G) = (1)$.

If A is a subset of vertices of G, then $G \setminus A$ denotes the graph which is obtained from G by removing the vertices in A. We call a graph *chordal* if it has no induced cycle of length greater than 3. We say x_i is a *neighbor* of x_j if $\{x_i, x_j\} \in E(G)$. The set of all neighbors of x_i is denoted by $N(x_i)$. The *closed neighborhood* of x_i is denoted by $N[x_i]$ and it is equal to the union $N(x_i) \cup \{x_i\}$. Every chordal graph has a vertex whose closed neighbourhood induces a complete graph and such vertex is called a *simplicial vertex*.

A *path* on vertices $x_1, \dots, x_n$ is denoted by P_n. Throughout the paper we will assume that edges of P_n are labelled as

$$E(G) = \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}\}.$$

Definition 2.1 (**Rooted list** [6, Definition 2.2]). The *rooted list* of P_n, denoted by $\mathcal{R}(P_n)$, is recursively defined by the following formulas:

- $\mathcal{R}(P_1) = 1$
- $\mathcal{R}(P_2) = x_1, x_2$
- $\mathcal{R}(P_3) = x_2, x_1x_3$
- for $n \geq 4$, if $\mathcal{R}(P_{n-2}) = u_1, \dots, u_r$ and $\mathcal{R}(P_{n-3}) = v_1, \dots, v_s$ then

$$\mathcal{R}(P_n) = x_{n-1}u_1, \dots, x_{n-1}u_r, x_nx_{n-2}v_1, \dots, x_nx_{n-2}v_s.$$

The motivation for this definition is the next lemma.

Lemma 2.2. *Let $\mathcal{R}(P_n) = u_1, \dots, u_q$. Then*

(1) $G(J(P_n)) = \{u_1, \dots, u_q\}$.
(2) *$J(P_n)$ has linear quotients with respect to $u_1, \dots, u_q$.*

Proof. Follows from [6, Lemma 2.1] and the recursive definition of rooted list. □

Based on the lemma above, a total order on minimal generators of $J(P_n)$ was defined.

Definition 2.3 (**Rooted order** [6, Definition 2.2]). Let $\mathcal{R}(P_n) = u_1, \dots, u_q$. The *rooted order*, denoted by $>_{\mathcal{R}}$, is a total order on $G(J(P_n))$ such that $u_i >_{\mathcal{R}} u_j$ when $i < j$.

Definition 2.4. Let $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$ be two elements in $\mathbb{Z}^n$. Then we write $\mathbf{u} >_{\text{lex}} \mathbf{v}$ if the first non-zero entry in $\mathbf{u} - \mathbf{v}$ is positive.

The following is a general version of Definition 2.4 in [6].

Definition 2.5 (**s-fold product, maximal expression**). Let $I = (u_1, \dots, u_q)$. We say that $M = u_1^{a_1} \dots u_q^{a_q}$ is an *s-fold product* of minimal generators of I if each a_i is a non-negative integer and $a_1 + \cdots + a_q = s$. We write $u_1^{a_1} \ldots u_q^{a_q} >_{lex} u_1^{b_1} \ldots u_q^{b_q}$ if $(a_1, \ldots, a_q) >_{lex} (b_1, \ldots, b_q)$. We say that $M = u_1^{a_1} \ldots u_q^{a_q}$ is the *maximal expression* if $(a_1, \ldots, a_q) >_{lex} (b_1, \ldots, b_q)$ for any other s-fold product $M = u_1^{b_1} \ldots u_q^{b_q}$.

NOTATION 2.6. If $G(I) = \{u_1, \ldots, u_q\}$, then the set of all s-fold products is denoted by $F(I^s) = \{u_{i_1} \ldots u_{i_s} : u_{i_1}, \ldots, u_{i_s} \in G(I)\}$.

We also generalize Definition 2.6 in [6] from second powers to all powers.

DEFINITION 2.7 (**Rooted order/list on powers**). Let $\mathcal{R}(P_n) = u_1, \ldots, u_q$. We define a total order $>_{\mathcal{R}}$ on $F(J(P_n)^s)$ which we call *rooted order* as follows. For $M, N \in F(J(P_n)^s)$ with maximal expressions $M = u_1^{a_1} \ldots u_q^{a_q}$ and $N = u_1^{b_1} \ldots u_q^{b_q}$ we set $M >_{\mathcal{R}} N$ if $(a_1, \ldots, a_q) >_{lex} (b_1, \ldots, b_q)$.

Let $G(J(P_n)^s) = \{U_1, \ldots, U_r\}$. Then we say $U_1, \ldots, U_r$ is a *rooted list* of minimal generators of $J(P_n)^s$ if $U_1 >_{\mathcal{R}} \ldots >_{\mathcal{R}} U_r$. In such case, we denote the rooted list of generators by $\mathcal{R}(J(P_n)^s) = U_1, \ldots, U_r$.

REMARK 2.8. If $n = 1$, then $\mathcal{R}(J(P_n)^s) = 1$ for every s.

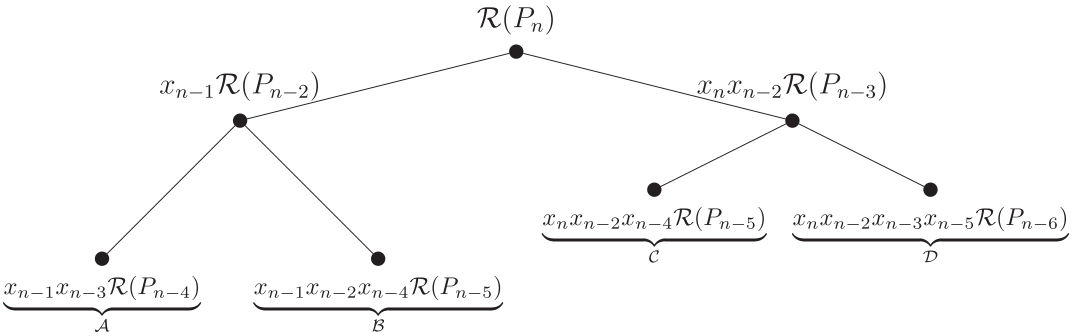

Fig. 1. $\mathcal{R}(P_n)$ in 2 steps

REMARK 2.9. If $n = 6$, then Figure 1 is still valid if we make the convention $\mathcal{R}(P_0) = 1$. In this case, the lists $\mathcal{B}, \mathcal{C}, \mathcal{D}$ each has only one term:

$$\mathcal{B} = x_{n-1}x_{n-2}x_{n-4},\ \mathcal{C} = x_n x_{n-2} x_{n-4},\ \mathcal{D} = x_n x_{n-2} x_{n-3} x_{n-5}.$$

3. Properties of rooted order and $G(J(P_n)^s)$

In this section, we will establish some properties of rooted order and minimal generating set of $J(P_n)^s$ which will be useful in the sequel.

REMARK 3.1. Observe that if $n \geq 2$, then every minimal vertex cover of P_n contains either x_n or x_{n-1}, but not both. Therefore if $U, V \in F(J(P_n)^s)$ such that $U|V$, then the highest power of x_n (respectively x_{n-1}) dividing U is the same as that of x_n (respectively x_{n-1}) dividing V.

Lemma 3.2 ([6, Lemma 3.5]). *Let $n \geq 3$ and let $u \in G(J(P_n))$ such that $x_n|u$. Then there exists $v \in G(J(P_{n-2}))$ such that v divides u/x_n.*

REMARK 3.3. Let $n \geq 4$ and let $\mathcal{R}(P_{n-2}) = u_1, \ldots, u_m$. Observe that by definition of rooted order the expression $u_{i_1} \ldots u_{i_s}$ is maximal with $i_1 \leq \cdots \leq i_s$ if and only if $(x_{n-1}u_{i_1}) \ldots (x_{n-1}u_{i_s})$ is the maximal expression with $x_{n-1}u_{i_1} \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} x_{n-1}u_{i_s}$ in $\mathcal{R}(P_n)$.

REMARK 3.4. Let $n \geq 5$ and let $\mathcal{R}(P_{n-3}) = u_1, \ldots, u_m$. Observe that by definition of rooted order the expression $u_{i_1} \ldots u_{i_s}$ is maximal with $i_1 \leq \cdots \leq i_s$ if and only if $(x_n x_{n-2} u_{i_1}) \ldots (x_n x_{n-2} u_{i_s})$ is the maximal expression with $x_n x_{n-2} u_{i_1} \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} x_n x_{n-2} u_{i_s}$ in $\mathcal{R}(P_n)$.

According to recursive definition of rooted list, for $n \geq 4$ each factor of an s-fold product of minimal generators of $J(P_n)$ belongs to either $x_{n-1}\mathcal{R}(P_{n-2})$ or $x_n x_{n-2}\mathcal{R}(P_{n-3})$. If all of the factors are from $x_{n-1}\mathcal{R}(P_{n-2})$ or all of the factors are from $x_n x_{n-2}\mathcal{R}(P_{n-3})$, then the s-fold product is *pure*. Otherwise s-fold product is *mixed*. Now, we make some observations on pure and mixed s-fold products.

Lemma 3.5 (**Pure s-fold product divisible by $\mathbf{x_{n-1}^s}$**). *Let $n \geq 3$. Then $U, V \in F(J(P_{n-2})^s)$ if and only if $x_{n-1}^s U, x_{n-1}^s V \in F(J(P_n)^s)$. Moreover, in such case the following statements hold.*

(1) *$U >_{\mathcal{R}} V$ if and only if $x_{n-1}^s U >_{\mathcal{R}} x_{n-1}^s V$.*
(2) *$U \in G(J(P_{n-2})^s)$ if and only if $x_{n-1}^s U \in G(J(P_n)^s)$.*

Proof. The fist statement is clear from the definition of rooted list and Lemma 2.2. To see (1) let $\mathcal{R}(P_{n-2}) = u_1, \ldots, u_m$. Suppose that $U = u_{i_1} \ldots u_{i_s}$ and $V = u_{j_1} \ldots u_{j_s}$ are maximal expressions with $i_1 \leq \cdots \leq i_s$ and $j_1 \leq \cdots \leq j_s$. Then by Remark 3.3 the expressions $(x_{n-1}u_{i_1}) \ldots (x_{n-1}u_{i_s})$ and $(x_{n-1}u_{j_1}) \ldots (x_{n-1}u_{j_s})$ are maximal as well. Suppose $U \neq V$ and let t be the smallest index such that $i_t \neq j_t$. Then

$$U >_{\mathcal{R}} V \Longleftrightarrow i_t < j_t \Longleftrightarrow x_{n-1}^s U >_{\mathcal{R}} x_{n-1}^s V$$

as desired. For proof of (2), the direction ($\Leftarrow$) is straightforward and the direction ($\Rightarrow$) follows from Remark 3.1. □

Lemma 3.6 (**Pure s-fold product divisible by $\mathbf{x_n^s}$**). *Let $n \geq 4$. Then $U, V \in F(J(P_{n-3})^s)$ if and only if both $x_n^s x_{n-2}^s U$ and $x_n^s x_{n-2}^s V$ belong to $F(J(P_n)^s)$. Moreover, in such case the following statements hold.*

(1) *$U >_{\mathcal{R}} V$ if and only if $x_n^s x_{n-2}^s U >_{\mathcal{R}} x_n^s x_{n-2}^s V$.*
(2) *$U \in G(J(P_{n-3})^s)$ if and only if $x_n^s x_{n-2}^s U \in G(J(P_n)^s)$.*

Proof. Similar to proof of Lemma 3.5 using Remark 3.4. □

Lemma 3.7 (**Mixed s-fold product**). *Let $\mathcal{R}(P_{n-2}) = u_1, \ldots, u_a$ and let $\mathcal{R}(P_{n-3}) = v_1, \ldots, v_b$ for some $n \geq 4$. Let $U = u_{i_1} \ldots u_{i_q}$, $V = v_{j_1} \ldots v_{j_k}$ and $W = x_{n-1}^q x_n^k x_{n-2}^k UV$.*

(1) *If $W = (x_{n-1}u_{i_1}) \ldots (x_{n-1}u_{i_q})(x_n x_{n-2} v_{j_1}) \ldots (x_n x_{n-2} v_{j_k})$ is the maximal expression in $F(J(P_n)^{k+q})$ with $x_{n-1}u_{i_1} \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} x_{n-1}u_{i_q} >_{\mathcal{R}} x_n x_{n-2} v_{j_1} \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} x_n x_{n-2} v_{j_k}$, then the expression $U = u_{i_1} \ldots u_{i_q}$ is maximal in $F(J(P_{n-2})^q)$ with $i_1 \leq \cdots \leq i_q$ and the expression $V = v_{j_1} \ldots v_{j_k}$ is maximal in $F(J(P_{n-3})^k)$ with $j_1 \leq \cdots \leq j_k$.*
(2) *If $W \in G(J(P_n)^{q+k})$, then $U \in G(J(P_{n-2})^q)$ and $V \in G(J(P_{n-3})^k)$.*

Proof. Proof is straightforward and left to the reader. □

Note that in the previous lemma, the converses of (1) and (2) are not true.

- Consider $q = k = 1$ and $n = 7$ with $\mathcal{R}(P_7) = u_1, \dots, u_7$. Then $u_4 \in x_6\mathcal{R}(P_5)$ and $u_5 \in x_7x_5\mathcal{R}(P_4)$ but u_4u_5 is not the maximal expression as $u_3u_6 = u_4u_5$. Thus the converse of (1) is not true.
- Consider $q = k = 1$ and $n = 5$. Then $U = x_1x_3 \in G(J(P_3))$ and $V = x_2 \in G(J(P_2))$ but $(x_4U)(x_3x_5V) \notin G(J(P_5)^2)$. Thus the converse of (2) is not true.

3.1. Reduction to second powers. In this section we will reduce the problem of describing minimal generating set of $J(P_n)^s$ to the case when $s = 2$. To this end, first we will explicitly describe $G(J(P_n)^s)$ for some small values of n. These results will then form the basis step of inductive proof of Theorem 3.11 which will be our next goal.

Lemma 3.8. *If $2 \leq n \leq 4$, then $G(J(P_n)^s) = F(J(P_n)^s)$ for all s. Moreover, in that case every $U \in F(J(P_n)^s)$ has a unique expression as an s-fold product of minimal generators of $J(P_n)$.*

Proof. Case 1: Suppose $n = 2$ or $n = 3$. Then $\mathcal{R}(P_n) = u_1, u_2$ where x_{n-1} divides u_1 and x_n divides u_2. Let $V = u_1^{\alpha}u_2^{\beta}$ be an s-fold product which divisible by another s-fold product $U = u_1^a u_2^b$. Since the exponents of x_n in U and V are respectively b and β it follows from Remark 3.1 that $b = \beta$. Similarly, since the exponents of x_{n-1} are equal we get $a = \alpha$ and $U = V$.

Case 2: Suppose $n = 4$. Then $\mathcal{R}(P_4) = u_1, u_2, u_3$ where $u_1 = x_1x_3, u_2 = x_2x_3, u_3 = x_2x_4$. Let $U = u_1^a u_2^b u_3^c$ and $V = u_1^{\alpha}u_2^{\beta}u_3^{\gamma}$ be s-fold products such that U divides V. Remark 3.1 implies that $c = \gamma$ and $a + b = \alpha + \beta$. Since the exponents of x_1 in U and V are respectively a and α it follows that $a \leq \alpha$. Similarly, comparing exponents of x_2 we get $b \leq \beta$. Thus $a = \alpha$, $b = \beta$ and $U = V$. □

Lemma 3.9. *Let $\mathcal{R}(P_5) = u_1, u_2, u_3, u_4$. If $U = u_1^{\alpha}u_2^{\beta}u_3^{\gamma}u_4^{\delta} \in F(J(P_5)^s) \setminus G(J(P_5)^s)$, then $\beta, \delta > 0$.*

Proof. Let $u_1 = x_2x_4$, $u_2 = x_1x_3x_4$, $u_3 = x_1x_3x_5$, $u_4 = x_2x_3x_5$. Let $V = u_1^a u_2^b u_3^c u_4^d \in G(J(P_5)^s)$ such that $V|U$. First note that by Remark 3.1 we have

$$a + b = \alpha + \beta \text{ and } c + d = \gamma + \delta. \tag{3.1}$$

Moreover, since the degree of V is less than degree of U we have

$$2a + 3b + 3c + 3d < 2\alpha + 3\beta + 3\gamma + 3\delta. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain $b < \beta$ and thus $\beta > 0$. Then we get $a > \alpha$. Comparing the exponents of x_2 in U and V we get $a + d \leq \alpha + \delta$ and thus $\delta > 0$. □

Lemma 3.10. *Let $\mathcal{R}(P_n) = u_1, \dots, u_r$ with $n \geq 2$ and let $s \geq 2$.*

1. *If $u_{i_1} \dots u_{i_s} \in G(J(P_n)^s)$, then $u_pu_q \in G(J(P_n)^2)$ for all $p, q \in \{i_1, \dots, i_s\}$.*
2. *If $u_{i_1} \dots u_{i_s}$ is the maximal expression for some $i_1 \leq \dots \leq i_s$, then for all $p, q \in \{i_1, \dots, i_s\}$ with $p < q$ the expression u_pu_q is maximal.*

Proof. To see (1) assume for a contradiction $u_{i_1} \dots u_{i_s} \in G(J(P_n)^s)$ but there exist $p, q \in \{i_1, \dots, i_s\}$ such that $u_pu_q \notin G(J(P_n)^2)$. Then there exists $u_{p'}u_{q'} \in G(J(P_n)^2)$ which strictly divides $u_p u_q$. Then $u_{i_1} \dots u_{i_s} u_{p'} u_{q'} / (u_p u_q)$ is an s-fold product and it strictly divides $u_{i_1} \dots u_{i_s}$, contradicting our initial assumption. Proof of (2) is similar. □

Theorem 3.11. *Let $G(J(P_n)) = \{u_1, \dots, u_r\}$ with $n \geq 2$ and $s \geq 2$. Let $U = u_1^{a_1} \dots u_r^{a_r}$ be an s-fold product in $F(J(P_n)^s)$. If $U \notin G(J(P_n)^s)$, then there exist p and q with $a_p, a_q > 0$ such that $u_p u_q \notin G(J(P_n)^2)$.*

Proof. We use induction on n. Suppose that $U \notin G(J(P_n)^s)$. If $n \leq 4$, then the statement is vacuously true by Lemma 3.8. If $n = 5$, then $u_1 u_3$ strictly divides $u_2 u_4$ and the statement is true by Lemma 3.9. Therefore let us assume that $n \geq 6$.

Keeping Figure 1 in mind, observe that if x_{n-1}^s divides U, then the result follows from Lemma 3.5 and the induction assumption on P_{n-2}. Similarly, if x_n^s divides U, then the result follows from Lemma 3.6 and the induction assumption on P_{n-3}. Therefore, let us assume that U is divisible by $x_n x_{n-1}$.

If there exist p and q with $a_p, a_q > 0$ such that $x_{n-4} x_{n-1} | u_p$ and $x_{n-3} x_n | u_q$, then the result follows from [6, Lemma 4.1]. Therefore, it suffices to consider the following cases:

Case 1: Suppose that U is product of factors from $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in Figure 1 such that at least one factor from $\mathcal{A}$ or $\mathcal{B}$ is divisible by x_{n-4}. Then we can write

$$U = (x_{n-1} x_{n-3})^{\alpha} V (x_{n-1} x_{n-2} x_{n-4})^{\beta} W (x_n x_{n-2} x_{n-4})^{\gamma} Y$$

for some $V \in F(J(P_{n-4})^{\alpha})$, $W \in F(J(P_{n-5})^{\beta})$, $Y \in F(J(P_{n-5})^{\gamma})$. Let $U' \in G(J(P_n)^s)$ such that U' strictly divides U. Keeping Remark 2.9 in mind, suppose that

$$U' = (x_{n-1} x_{n-3})^{\alpha'} V' (x_{n-1} x_{n-2} x_{n-4})^{\beta'} W' (x_n x_{n-2} x_{n-4})^{\gamma'} Y' (x_n x_{n-2} x_{n-3} x_{n-5})^{\delta'} Z'$$

for some $V' \in F(J(P_{n-4})^{\alpha'})$, $W' \in F(J(P_{n-5})^{\beta'})$, $Y' \in F(J(P_{n-5})^{\gamma'})$, $Z' \in F(J(P_{n-6})^{\delta'})$. We claim that

$$(\alpha, \beta, \gamma, 0) = (\alpha', \beta', \gamma', \delta').$$

By Remark 3.1 we have $\alpha + \beta = \alpha' + \beta'$ and $\gamma = \gamma' + \delta'$. Since the exponent of x_{n-2} in U' is less than or equal to that of U we have

$$\beta + \gamma \geq \beta' + \gamma' + \delta'.$$

Similarly, since the exponent of x_{n-3} in U' is less than or equal to that of U we have

$$\alpha \geq \alpha' + \delta'.$$

Then adding up the inequalities we get $\delta' = 0$. Then $\gamma = \gamma' + \delta'$ implies $\gamma = \gamma'$. Therefore $\alpha = \alpha'$ and $\beta = \beta'$ as desired.

Therefore, $V'W'Y'$ strictly divides VWY. By recursive definition of $\mathcal{R}(P_{n-2})$ (see Figure 2) observe that

$$U^* = x_{n-3}^{\alpha} V (x_{n-2} x_{n-4})^{\beta} W (x_{n-2} x_{n-4})^{\gamma} Y \in F(J(P_{n-2})^s) \setminus G(J(P_{n-2})^s).$$

Then by induction assumption on P_{n-2}, one of V, W or Y contains a non-minimal 2-fold product. By adding the suitable variables, one can see that U satisfies the desired condition.

Case 2: Suppose that U is product of factors from $\mathcal{A}, \mathcal{C}, \mathcal{D}$ such that no factor from $\mathcal{A}$ is divisible by x_{n-4}. Then we can write

$$U = (x_{n-1}x_{n-3})^{\mu} V (x_n x_{n-2})^{\nu} X$$

for some $V \in F(J(P_{n-4})^{\mu})$, $X \in F(J(P_{n-3})^{\nu})$, where $\mu, \nu > 0$ and $\mu + \nu = s$. We claim that U is divisible by some $U' \in G(J(P_n)^s)$ of the same form. Indeed, if

$$U' = (x_{n-1}x_{n-3})^{\mu'} V' (x_{n-1}x_{n-2}x_{n-4})^{\beta'} W' (x_n x_{n-2})^{\nu'} X'$$

for some $V' \in F(J(P_{n-4})^{\mu'})$, $W' \in F(J(P_{n-5})^{\beta'})$ and $X' \in F(J(P_{n-3})^{\nu'})$, then we must have $\mu' + \beta' = \mu$ and $\nu = \nu'$ by Remark 3.1. Then comparing the exponents of x_{n-2} in U and U' we see that $\beta' = 0$.

Therefore, $V'X'$ strictly divides VX. Then by recursive definition of $\mathcal{R}(P_{n-1})$ observe that

$$U^* = x_{n-2}^{\nu} X (x_{n-1}x_{n-3})^{\mu} V \in F(J(P_{n-1})^s) \setminus G(J(P_{n-1})^s).$$

Then by induction assumption on P_{n-1}, either V or X contains a non-minimal 2-fold product. By adding the suitable variables, one can see that U satisfies the desired condition. □

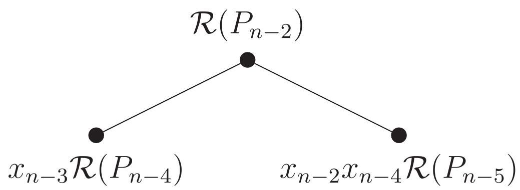

Fig. 2. Recursive definition of rooted list of P_{n-2}

As a consequence of Theorem 3.11 we characterize minimal generating set of $J(P_n)^s$ for $s \geq 3$ in terms of minimal generating set of second power of $J(P_n)$.

Corollary 3.12. *Let $G(J(P_n)) = \{u_1, \dots, u_r\}$ and let $s \geq 2$. The following statements are equivalent.*

(1) $u_{i_1} \dots u_{i_s} \in G(J(P_n)^s)$.
(2) $u_p u_q \in G(J(P_n)^2)$ *for all* $p, q \in \{i_1, \dots, i_s\}$.

Proof. Immediate from Lemma 3.10 and Theorem 3.11. □

Given a monomial ideal I, let $\mu(I)$ denote the cardinality of $G(I)$. If $G(I) = \{u_1, \dots, u_q\}$, then by counting the number of s-element multi-subsets of $[q] = \{1, \dots, q\}$ one can see that $\mu(I^s) \leq \binom{q+s-1}{q-1}$. This upper bound may not be achieved in general for two reasons. Firstly, a product of the form $u_{i_1} \dots u_{i_s}$ may be equal to another product $u_{j_1} \dots u_{j_s}$ with $\{i_1, \dots, i_s\} \neq \{j_1, \dots, j_s\}$ as multi-sets. Secondly, $u_{i_1} \dots u_{i_s}$ may be strictly divisible by another product $u_{j_1} \dots u_{j_s}$. In fact, when I is generated by monomials of the same degrees, the latter cannot happen. Therefore, although the computation of $\mu(I^s)$ is a challenging problem, one can describe the set $G(I^s)$ explicitly when I is generated in the same degree. On the other hand, when I is not generated in the same degree, description of $G(I^s)$ remains a difficult problem as well as computation of $\mu(I^s)$.

It is well-known ([18]) that the function $g(s) = \mu(I^s)$ is a polynomial in s for $s \gg 0$. In [4], the authors addressed the question of how small $\mu(I^2)$ can be in terms of $\mu(I)$ when I is a monomial ideal in polynomial ring with $n = 2$ variables. Behaviour of $\mu(I^s)$ was considered in some other articles, see for example [1, 10, 16, 17]. Recently, Drabkin and Guerrieri [3] studied Freiman cover ideals. Given a cover ideal $J(G)$, it is a demanding task to find the minimal generating set of $J(G)^s$ or $\mu(J(G)^s)$. Therefore, Corollary 3.12 might be of interest in computation of $\mu(J(P_n)^s)$.

We will next see how Theorem 3.11 will be useful to extend the following result to all powers of $J(P_n)$.

Lemma 3.13 ([6, Lemma 4.5])**.** *Let $U \in F(J(P_n)^2) \setminus G(J(P_n)^2)$. Then there exists $V \in G(J(P_n)^2)$ such that $V >_{\mathcal{R}} U$ and $V|U$.*

Lemma 3.14. *Let $U \in F(J(P_n)^s) \setminus G(J(P_n)^s)$. Then there exists $V \in G(J(P_n)^s)$ such that $V >_{\mathcal{R}} U$ and $V|U$.*

Proof. Let $\mathcal{R}(P_n) = u_1, \dots, u_p$ with $n \geq 2$. Let $U = u_1^{\alpha_1} \dots u_p^{\alpha_p}$ be the maximal expression. By Theorem 3.11 there exists $u_i u_j$ with $\alpha_i, \alpha_j \neq 0$ and $u_i u_j \notin G(J(P_n)^2)$. Without loss of generality assume that $i \leq j$. Note that $u_i u_j$ is the maximal expression by Lemma 3.10. Then by Lemma 3.13 there exists $v \in G(J(P_n)^2)$ such that v strictly divides $u_i u_j$ and $v >_{\mathcal{R}} u_i u_j$. Let $v = u_k u_\ell$ be the maximal expression with $k \leq \ell$. Consider the s-fold product $V = (U u_k u_\ell)/(u_i u_j)$. Observe that $V >_{\mathcal{R}} U$ and V strictly divides U. If V is a minimal generator, then we are done, otherwise this process can be repeated. □

4. Linear quotients of $J(P_n)^s$ with respect to rooted order

In this section we will show that $J(P_n)^s$ has linear quotients with respect to rooted order. Before that, we prove the following result which will be crucial in the last case of proof of Theorem 4.3.

Proposition 4.1. *Let $\mathcal{R}(P_n) = u_1, \dots, u_q$ and let $\mathcal{R}(J(P_n)^s) = Y_1, \dots, Y_p$. Suppose that $Y_r = u_{i_1} \dots u_{i_s}$ is the maximal expression for some $2 \leq r \leq p$ with $i_1 \leq \cdots \leq i_s$.*

(1) *For each $1 \leq t \leq s$ with $2 \leq i_t$ we have*

$$(u_1, u_2, \dots, u_{i_t - 1}) : (u_{i_t}) \subseteq (Y_1, \dots, Y_{r-1}) : (Y_r).$$

(2) *If $x_n | Y_r$, then $x_{n-1} \in (Y_1, \dots, Y_{r-1}) : (Y_r)$.*

Proof. (1): First note that by Lemma 2.2 the ideal $(u_1, u_2, \dots, u_{i_t - 1}) : (u_{i_t})$ is generated by variables. Let $\ell < i_t$ with $u_\ell : u_{i_t} = x_z$ for some variable x_z. Consider the s-fold product $M = Y_r u_\ell / u_{i_t}$. Then $M : Y_r = x_z$ and $M >_{\mathcal{R}} Y_r$. If M is a minimal generator, nothing is left to show. Otherwise, by Lemma 3.14 there exists $M' \in G(J(P_n)^s)$ such that $M' >_{\mathcal{R}} M$ and $M'|M$. Then since $M' \neq Y_r$ and $M' : Y_r$ divides $M : Y_r$ it follows that $M' : Y_r = x_z$ and $x_z \in (Y_1, \dots, Y_{r-1}) : (Y_r)$.

(2): Suppose that x_n divides u_{i_k} for some $k \in \{1, \dots, s\}$. Then by definition of rooted list $i_k \geq 2$. By part (1) it suffices to show that

$$x_{n-1} \in (u_1, u_2, \dots, u_{i_k - 1}) : (u_{i_k})$$

which is immediate from [6, Lemma 3.6]. □

We will also need the following result from [6].

Proposition 4.2 ([6, Proposition 4.2])**.** *Let $\mathcal{R}(P_n) = u_1, \dots, u_k$ where $n \geq 2$. Let $1 < i < j \leq k$. Suppose u_j contains a variable from $(u_1, \dots, u_{i-1}) : (u_i)$. Then either $u_i u_j$ is not a minimal generator of $J(P_n)^2$ or $u_i u_j$ is not the maximal 2-fold expression.*

We can now prove the main result of this section.

Theorem 4.3. *Let $\mathcal{R}(J(P_n)^s) = Y_1, \dots, Y_p$. Then $J(P_n)^s$ has linear quotients with respect to $Y_1, \dots, Y_p$.*

Proof. We will proceed by induction on $n + s$. We will show that $(Y_1, \dots, Y_{r-1}) : (Y_r)$ is generated by variables, for all $r \geq 2$.

Basis step (n ≤ 3 or s = 1): The case when $s = 1$ is Lemma 2.2. If $n = 2$ or $n = 3$ with $\mathcal{R}(P_n) = u_1, u_2$, then by Lemma 3.8 we have $\mathcal{R}(J(P_n)^s) = u_1^s, u_1^{s-1}u_2, \dots, u_2^s$ and it is straightforward to show that $(Y_1, \dots, Y_{r-1}) : (Y_r) = (u_1) : (u_2) = (x_{n-1})$ holds for every $r \geq 2$.

Induction step: Let us assume that $n \geq 4$ and $s \geq 2$. We set some notation for the following rooted lists.

- $\mathcal{R}(P_{n-2}) = u_1, \dots, u_a$
- $\mathcal{R}(P_{n-3}) = v_1, \dots, v_b$
- $\mathcal{R}(J(P_{n-2})^s) = U_1, \dots, U_A$
- $\mathcal{R}(J(P_{n-3})^s) = V_1, \dots, V_B$.

Case 1: Suppose that x_n^s divides Y_r. Assume that Y_r has the maximal expression

$$Y_r = (x_n x_{n-2} v_{i_1})(x_n x_{n-2} v_{i_2}) \dots (x_n x_{n-2} v_{i_s})$$

for some $i_1 \leq \cdots \leq i_s$. From Proposition 4.1 (2) we know that x_{n-1} is a generator of $(Y_1, \dots, Y_{r-1}) : (Y_r)$. From Lemma 3.6 we can set $V_t = v_{i_1} v_{i_2} \dots v_{i_s}$ for some $t \in \{1, \dots, B\}$. If $t = 1$, then by definition of rooted order we have

$$(Y_1, \dots, Y_{r-1}) : (Y_r) = (x_{n-1})$$

and nothing is left to show. Therefore let us assume that $t > 1$. Observe that because of induction assumption on P_{n-3} it suffices to show the equality

$$(x_{n-1}) + (V_1, V_2, \dots, V_{t-1}) : (V_t) = (Y_1, \dots, Y_{r-1}) : (Y_r).$$

Because of Lemma 3.6 we already have the inclusion

$$(x_{n-1}) + (V_1, V_2, \dots, V_{t-1}) : (V_t) \subseteq (Y_1, \dots, Y_{r-1}) : (Y_r).$$

We now prove the reverse containment. For any $\ell \leq r - 1$, if $x_{n-1} | Y_\ell$, then it is clear that $Y_\ell : Y_r \in (x_{n-1})$. Otherwise, $(x_{n-2}x_n)^s | Y_\ell$ and by Lemma 3.6, we have $Y_\ell / (x_{n-2}x_n)^s = V_k$ for some k. Moreover, since $Y_\ell >_{\mathcal{R}} Y_r$ Lemma 3.6 implies that $V_k >_{\mathcal{R}} V_t$. Hence $Y_\ell : Y_r = V_k : V_t$, proving the reverse containment.

Case 2: Suppose that x_{n-1}^s divides Y_r. Let

$$Y_r = (x_{n-1} u_{i_1})(x_{n-1} u_{i_2}) \dots (x_{n-1} u_{i_s})$$

be the maximal expression for some $i_1 \leq \cdots \leq i_s$. Then the expression $u_{i_1} \dots u_{i_s}$ is also maximal by Remark 3.3. By Lemma 3.5 we can set $U_t = u_{i_1} \dots u_{i_s}$ for some $t > 1$ as $r > 1$. By induction assumption on P_{n-2} it suffices to show that

$$(U_1, \dots, U_{t-1}) : (U_t) = (Y_1, \dots, Y_{r-1}) : (Y_r).$$

By Lemma 3.5 the inclusion $\subseteq$ is clear. To see the reverse, let $\ell \leq r-1$. By definition of rooted list, Y_ℓ is divisible by either x_{n-1}^s or $x_{n-1}x_n$. Because of Lemma 3.5 we may assume that Y_ℓ is divisible by $x_{n-1}x_n$. Let Y_ℓ have the maximal expression

$$Y_\ell = (x_{n-1}u_{j_1}) \dots (x_{n-1}u_{j_c})(x_n x_{n-2} v_{k_1}) \dots (x_n x_{n-2} v_{k_d})$$

for some $1 \leq j_1 \leq \dots \leq j_c \leq a$ and $1 \leq k_1 \leq \dots \leq k_d \leq b$.

By Lemma 3.2 we can form the s-fold product

$$P = (x_{n-1}u_{j_1}) \dots (x_{n-1}u_{j_c})(x_{n-1}u_{k'_1}) \dots (x_{n-1}u_{k'_d})$$

where $u_{k'_i}$ divides $x_{n-2}v_{k_i}$ for each $i = 1, \dots, d$. By definition of $>_{\mathcal{R}}$ we now have

$$P >_{\mathcal{R}} Y_\ell >_{\mathcal{R}} Y_r \text{ in } F(J(P_n)^s).$$

Lemma 3.5 implies that

$$u_{j_1} \dots u_{j_c} u_{k'_1} \dots u_{k'_d} >_{\mathcal{R}} u_{i_1} \dots u_{i_s} \text{ in } F(J(P_{n-2})^s).$$

Observe that

$$\begin{aligned} P \in G(J(P_n)^s) &\Longrightarrow u_{j_1} \dots u_{j_c} u_{k'_1} \dots u_{k'_d} = U_{t'} \text{ for some } t' < t \text{ by Lemma 3.5} \\ &\Longrightarrow P : Y_r \in (U_1, \dots, U_{t-1}) : (U_t) \text{ as } P : Y_r = U_{t'} : U_t \\ &\Longrightarrow Y_\ell : Y_r \in (U_1, \dots, U_{t-1}) : (U_t) \text{ as } P : Y_r \text{ divides } Y_\ell : Y_r \end{aligned}$$

as desired. On the other hand, if $P \notin G(J(P_n)^s)$, then by Lemma 3.14, there exists $Y_\alpha \in G(J(P_n)^s)$ such that $Y_\alpha | P$ and $Y_\alpha >_{\mathcal{R}} P$. Since $Y_\alpha | P$ it follows from Remark 3.1 that $x_{n-1}^s | Y_\alpha$. Since $Y_\alpha > Y_r$ by Lemma 3.5 we get $Y_\alpha : Y_r \in (U_1, \dots, U_{t-1}) : (U_t)$. Since $Y_\alpha : Y_r$ divides $P : Y_r$ and $P : Y_r$ divides $Y_\ell : Y_r$, we have $Y_\alpha : Y_r$ divides $Y_\ell : Y_r$ and $Y_\ell : Y_r \in (U_1, \dots, U_{t-1}) : (U_t)$ as desired.

Case 3: Suppose that Y_r is divisible by $x_n x_{n-1}$ and it has the maximal expression

$$Y_r = (x_{n-1}u_{i_1}) \dots (x_{n-1}u_{i_q})(x_n x_{n-2} v_{j_1}) \dots (x_n x_{n-2} v_{j_k})$$

for some $1 \leq i_1 \leq \dots \leq i_q \leq a$ and $1 \leq j_1 \leq \dots \leq j_k \leq b$. First note that from Proposition 4.1 we have

$$x_{n-1} \in (Y_1, \dots, Y_{r-1}) : (Y_r). \tag{4.1}$$

Let $t < r$. Since $Y_t >_{\mathcal{R}} Y_r$ and because of (4.1) we may assume that Y_t has the maximal expression

$$Y_t = (x_{n-1}u_{\alpha_1}) \dots (x_{n-1}u_{\alpha_{q'}})(x_n x_{n-2} v_{\beta_1}) \dots (x_n x_{n-2} v_{\beta_{k'}})$$

for some $1 \leq \alpha_1 \leq \dots \leq \alpha_{q'} \leq a$ and $1 \leq \beta_1 \leq \dots \leq \beta_{k'} \leq b$ with $q' \leq q$. We will now consider the following cases.

Case 3.1: Suppose that $i_\ell = \alpha_\ell$ for all $\ell = 1, \dots, q'$. Then $q' = q$ since $Y_t >_{\mathcal{R}} Y_r$. This implies $k = k'$. By Lemma 3.7 we get $v_{\beta_1} \dots v_{\beta_k} >_{\mathcal{R}} v_{j_1} \dots v_{j_k}$ in $\mathcal{R}(J(P_{n-3})^k)$. Observe that

$$Y_t : Y_r = v_{\beta_1} \dots v_{\beta_k} : v_{j_1} \dots v_{j_k}.$$

By the induction assumption on $J(P_{n-3})^k$ there exists a variable x_z such that

$$x_z \text{ divides } v_{\beta_1} \dots v_{\beta_k} : v_{j_1} \dots v_{j_k} \text{ and } x_z = v_{\gamma_1} \dots v_{\gamma_k} : v_{j_1} \dots v_{j_k}$$

for some $v_{\gamma_1} \dots v_{\gamma_k} >_{\mathcal{R}} v_{j_1} \dots v_{j_k}$ in $\mathcal{R}(J(P_{n-3})^k)$. Therefore it suffices to show that

$$x_z \in (Y_1, \dots, Y_{r-1}) : (Y_r).$$

Consider the s-fold product $P = (x_{n-1}u_{i_1}) \dots (x_{n-1}u_{i_q})(x_n x_{n-2} v_{\gamma_1}) \dots (x_n x_{n-2} v_{\gamma_k})$. By definition of rooted order $P >_{\mathcal{R}} Y_r$. Clearly $P : Y_r = x_z$. If $P \in G(J(P_n)^s)$ nothing is left to show. Otherwise, the result follows from Lemma 3.14.

Case 3.2: Suppose that there is a smallest index ℓ among $1, \dots, q'$ such that $i_\ell \neq \alpha_\ell$. Since $Y_t >_{\mathcal{R}} Y_r$ we have $i_\ell > \alpha_\ell$. Then according to Lemma 2.2 there exists a variable in $(u_1, \dots, u_{i_\ell - 1}) : (u_{i_\ell})$, say x_z, which divides $u_{\alpha_\ell} : u_{i_\ell}$. Note that $x_z \neq x_{n-2}$ because of recursive definition of $\mathcal{R}(P_{n-2})$. Also, it is clear that $x_z \neq x_n, x_{n-1}$ because x_z is a vertex of P_{n-2}. From Proposition 4.1 we see that

$$x_z \in (x_{n-1}u_1, \dots, x_{n-1}u_{i_\ell - 1}) : (x_{n-1}u_{i_\ell}) \subseteq (Y_1, \dots, Y_{r-1}) : (Y_r)$$

and thus it suffices to show that x_z divides $Y_t : Y_r$. From Lemma 3.10 and Proposition 4.2 we see that

$$x_z \nmid (x_{n-1}u_{i_\ell})(x_{n-1}u_{i_{\ell+1}}) \dots (x_{n-1}u_{i_q})(x_n x_{n-2} v_{j_1}) \dots (x_n x_{n-2} v_{j_k}).$$

By the choice of ℓ since $u_{i_1} \dots u_{i_{\ell-1}} = u_{\alpha_1} \dots u_{\alpha_{\ell-1}}$ the result follows. □

Using Theorem 4.3 one can obtain an exact formula for the regularity of powers of $J(P_n)$ as in the next corollary.

Corollary 4.4. *For any $n \geq 2$ and $s \geq 1$*

$$\operatorname{reg}(J(P_n)^s) = \begin{cases} 2ks & \text{if } n = 3k+1 \text{ or } n = 3k \\ 2ks + s & \text{if } n = 3k+2. \end{cases}$$

Proof. Similar to proof of [6, Corollary 5.3]. □

5. Rooted order for chordal graphs

In this section we will see how to generalize the concept of rooted list to chordal graphs. To simplify the notation we will use a set A of vertices of G interchangeably with the squarefree monomial $\prod_{x_i \in A} x_i$.

NOTATION 5.1. For each $i = 1, \dots, r$ let L_i be the list $L_i = a^i_1, \dots, a^i_{k_i}$. Then by $L = L_1, L_2, \dots, L_r$ we denote a new list L which is obtained by joining the lists in the given order. More precisely,

$$L = a^1_1, \dots, a^1_{k_1}, a^2_1, \dots, a^2_{k_2}, \cdots, a^r_1, \dots, a^r_{k_r}.$$

DEFINITION 5.2 (**ROOTED LIST FOR CHORDAL GRAPHS**). Suppose that G is a chordal graph with a simplicial vertex x_1 such that $N[x_1] = \{x_1, \dots, x_m\}$ for some $m \geq 2$. We say $\mathcal{R}(G)$ is a *rooted list* of G if it can be written in the form

$$\mathcal{R}(H_1)N(x_1), \mathcal{R}(H_2)N(x_2), \dots, \mathcal{R}(H_m)N(x_m)$$

where the list $\mathcal{R}(H_i)$ is a rooted list of the subgraph $H_i = G \setminus N[x_i]$ for each $i = 1, \ldots, m$. If G has no edges, then we set $\mathcal{R}(G) = 1$.

Remark 5.3. Observe that one can construct rooted lists in different ways as they depend on the choice of simplicial vertex. In Definition 2.1 we always picked the last vertex x_n of P_n as a simplicial vertex.

Lemma 5.4. *Let G be a chordal graph with a rooted list $\mathcal{R}(G) = u_1, \ldots, u_q$. Then*
(1) $G(J(G)) = \{u_1, \ldots, u_q\}$
(2) *$J(G)$ has linear quotients with respect to $u_1, \ldots, u_q$.*

Proof. Proof follows from [5, Theorem 3.1] and [22, Theorem 2.13]. □

Definition 5.5 (**Rooted order/list for powers**). Let G be a chordal graph with a rooted list $\mathcal{R}(G) = u_1, \ldots, u_q$. We define a total order $>_{\mathcal{R}}$ on $F(J(G)^s)$ which we call *rooted order* as follows. For $M, N \in F(J(G)^s)$ with maximal expressions $M = u_1^{a_1} \ldots u_q^{a_q}$ and $N = u_1^{b_1} \ldots u_q^{b_q}$ we set $M >_{\mathcal{R}} N$ if $(a_1, \ldots, a_q) >_{lex} (b_1, \ldots, b_q)$.

Let $G(J(G)^s) = \{U_1, \ldots, U_r\}$. Then we say $U_1, \ldots, U_r$ is a *rooted list* of minimal generators of $J(G)^s$ if $U_1 >_{\mathcal{R}} \ldots >_{\mathcal{R}} U_r$. In such case, we denote the rooted list of generators by $\mathcal{R}(J(G)^s) = U_1, \ldots, U_r$.

The following lemma is a version of Proposition 4.1.

Lemma 5.6. *Let G be a chordal graph with $F(J(G)^s) = G(J(G)^s)$. Let $\mathcal{R}(G) = u_1, \ldots, u_q$ and let $\mathcal{R}(J(G)^s) = Y_1, \ldots, Y_p$. Suppose that $Y_r = u_{j_1} \ldots u_{j_s}$ is the maximal expression for some $2 \le r \le p$ with $j_1 \le \cdots \le j_s$. For each $1 \le t \le s$ with $2 \le j_t$ we have*

$$(u_1, u_2, \ldots, u_{j_t - 1}) : (u_{j_t}) \subseteq (Y_1, \ldots, Y_{r-1}) : (Y_r).$$

Proof. The ideal $(u_1, u_2, \ldots, u_{j_t-1}) : (u_{j_t})$ is generated by variables since the rooted order gives linear quotients by Lemma 5.4. Let $\ell < j_t$ with $u_\ell : u_{j_t} = x_z$ for some variable x_z. Consider the s-fold product $M = Y_r u_\ell / u_{j_t}$. Then $M : Y_r = x_z$ and $M >_{\mathcal{R}} Y_r$. By assumption M is a minimal generator of $J(G)^s$ and the proof follows. □

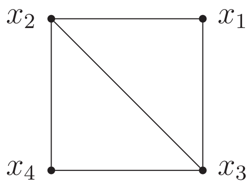

Fig. 3. Diamond graph

In the next chordal example, we construct a rooted list $\mathcal{R}(G)$ such that rooted order $>_{\mathcal{R}}$ on the generators of $J(G)^s$ yield linear quotients for all $s \ge 1$.

Example 5.7. Let G be the chordal graph in Figure 3. As in notation of Definition 5.2 the vertex x_1 is a simplicial vertex and $N(x_1) = \{x_2, x_3\}$. Observe that H_1 is the graph consisting of the isolated vertex x_4. Also, H_2 and H_3 are empty graphs. Therefore we take $\mathcal{R}(H_1) = \mathcal{R}(H_2) = \mathcal{R}(H_3) = 1$. Then the rooted list of G is $\mathcal{R}(G) = u_1, u_2, u_3$ where

$$u_1 = x_2x_3, u_2 = x_1x_3x_4, u_3 = x_1x_2x_4.$$

It is not hard to see that

$$F(J(G)^s) = G(J(G)^s)$$

and every s-fold product has a unique expression. Let $\mathcal{R}(J(G)^s) = Y_1, \dots, Y_p$. Now we will show that $J(G)^s$ has linear quotients with respect to the order $Y_1, \dots, Y_p$. Suppose that $Y_r = u_1^{\alpha}u_2^{\beta}u_3^{\gamma}$ with $r \geq 2$. Consider the ideal I defined by

$$I = \begin{cases} (x_2, x_3) & \text{if } \beta \neq 0 \text{ and } \gamma \neq 0 \\ (x_3) & \text{if } \beta = 0 \text{ and } \gamma \neq 0 \\ (x_2) & \text{if } \beta \neq 0 \text{ and } \gamma = 0 \end{cases}$$

Since $r \geq 2$, we claim that $I = (Y_1, \dots, Y_{r-1}) : (Y_r)$. It is clear from Lemma 5.6 that $I \subseteq (Y_1, \dots, Y_{r-1}) : (Y_r)$ because $(u_1) : (u_2) = (x_2)$ and $(u_1, u_2) : (u_3) = (x_3)$. To see the reverse, assume for a contradiction there exists $\ell < r$ such that no variable in I divides $Y_\ell : Y_r$. Let $Y_\ell = u_1^{\alpha'}u_2^{\beta'}u_3^{\gamma'}$.

Case 1: Suppose $\beta \neq 0$ and $\gamma \neq 0$. Comparing exponents of x_2 and x_3 in Y_ℓ and Y_r we see that $\alpha' + \gamma' \leq \alpha + \gamma$ and $\alpha' + \beta' \leq \alpha + \beta$. Since both Y_ℓ and Y_r are s-fold products we have $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma'$ and thus $\alpha' \leq \alpha$. Since $Y_\ell >_{\mathcal{R}} Y_r$ by definition of rooted order we get $\alpha' = \alpha$. This implies $\beta = \beta'$ and $\gamma = \gamma'$ and $\ell = r$, contradiction.

Case 2: Suppose $\beta = 0$ and $\gamma \neq 0$. Comparing exponents of x_3 in Y_ℓ and Y_r we see that $\alpha' + \beta' \leq \alpha$. In particular $\alpha' \leq \alpha$. By definition of rooted order $\alpha' = \alpha$ must hold. This implies $\beta' = 0$ and $\gamma' = \gamma$. Therefore $\ell = r$, contradiction.

Case 3: Suppose $\beta \neq 0$ and $\gamma = 0$. Comparing exponents of x_2 in Y_ℓ and Y_r we see that $\alpha' + \gamma' \leq \alpha$. In particular $\alpha' \leq \alpha$. By definition of rooted order $\alpha' = \alpha$ must hold. This implies $\gamma' = 0$ and $\beta' = \beta$. Therefore $\ell = r$, contradiction.

We do not know any example of a power of a chordal graph which does not give linear quotients with respect to a rooted order. Therefore this led us to the following question.

Question 5.8. Given a chordal graph G, does there exist a rooted list $\mathcal{R}(G)$ such that the rooted order $>_{\mathcal{R}}$ on the minimal generating set of $J(G)^s$ yields linear quotients for every $s \geq 1$?

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