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QUANDLE TWISTED ALEXANDER INVARIANTS

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Abstract

We establish a quandle version of the twisted Alexander polynomial. We also develop a theory that reduces the size of a twisted Alexander matrix with column relations. The reduced matrix can be used to refine invariants derived from the twisted Alexander matrix.

1. Introduction

The twisted Alexander polynomial [9, 11] is a twisted version of the Alexander polynomial, which is twisted by a group representation. The twisted Alexander polynomial is an invariant for a pair of a (knot) group and its group representation. Behavior of the twisted Alexander polynomial for topological properties of knots such as the genus and fiberedness has been studied (e.g. [1, 4, 7, 8]). A quandle [6, 10] is an algebra whose axioms correspond to the Reidemeister moves on link diagrams. A knot quandle is known as a complete knot invariant, although it is not easy to distinguish two knot quandles. In this paper, we introduce a quandle version of the twisted Alexander polynomial, which is an invariant for a pair of a (knot) quandle and its quandle representation. It can be used to extract information from knot quandles.

The usual (twisted) Alexander polynomial is defined through a reduced (twisted) Alexander matrix, which is obtained by using one relation between columns of the (twisted) Alexander matrix. In this paper, we also develop a theory that reduces the size of a quandle twisted Alexander matrix with column relations, where the quandle twisted Alexander matrix is a matrix obtained by using the derivative with an Alexander pair introduced in [5]. We emphasize that our theory covers multiple relations between columns of a matrix. We introduce a notion of a column relation map, which controls a relation between columns. We then construct an invariant through the reduced quandle twisted Alexander matrix.

This paper is organized as follows. In Section 2, we introduce a column relation matrix of a matrix and define an equivalence relation on pairs of matrices and their column relation matrices. We introduce two invariants for the equivalence classes. In Section 3, we recall quandle presentations and Tietze transformations on them. In Section 4, we recall a quandle derivative and introduce a column relation map, which yields a column relation matrix. In Section 5, we see that an Alexander pair and a column relation map give an invariant of (link) quandles, whose invariance is proven in Section 7. We also see that the twisted Alexander polynomial is recoverable as an invariant in our framework. In Section 6, we give calculation examples of our invariant. In Section 8, we introduce the notion of cohomologous

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for Alexander pairs and column relation maps and show that cohomologous Alexander pairs and column relation maps induce the same invariant.

2. Relation matrices

In this section, we introduce a column relation matrix of a matrix and define an equivalence relation on pairs of matrices and their column relation matrices. We introduce two invariants for the equivalence classes.

Let R be a ring. We denote by $M(m,n;R)$ the set of $m\times n$ matrices over R. We say that two matrices A_1 and A_2 over R are *equivalent* ($A_1 \sim A_2$) if they are related by a finite sequence of the following transformations:

- $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_i,\dots,\boldsymbol{a}_j,\dots,\boldsymbol{a}_n) \leftrightarrow (\boldsymbol{a}_1,\dots,\boldsymbol{a}_i+\boldsymbol{a}_j r,\dots,\boldsymbol{a}_j,\dots,\boldsymbol{a}_n)$ $(r\in R)$,
- $\begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_i\\ \vdots\\ \boldsymbol{a}_j\\ \vdots\\ \boldsymbol{a}_n\end{pmatrix} \leftrightarrow \begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_i+r\boldsymbol{a}_j\\ \vdots\\ \boldsymbol{a}_j\\ \vdots\\ \boldsymbol{a}_n\end{pmatrix}$ $(r\in R)$,
- $A \leftrightarrow \begin{pmatrix}A\\ \boldsymbol{0}\end{pmatrix}$,
- $A \leftrightarrow \begin{pmatrix}A & \boldsymbol{0}\\ \boldsymbol{0} & 1\end{pmatrix}$.

We denote the $n\times n$ identity matrix by E_n or simply E. Let $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ be the standard unit column vectors in $\mathbb{R}^n$, that is, $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n)=E_n$. We set

$$E_{ij}(r) := (\boldsymbol{e}_1,\dots,\boldsymbol{e}_{j-1},\boldsymbol{e}_j+r\boldsymbol{e}_i,\boldsymbol{e}_{j+1},\dots,\boldsymbol{e}_n),$$

whose (i,j)-entry is r. Then, the first and second transformations are written as $A\leftrightarrow AE_{ji}(r)$ and $A\leftrightarrow E_{ij}(r)A$, respectively. We also set

$$P_{ij} := (\boldsymbol{e}_1,\dots,\boldsymbol{e}_j,\dots,\boldsymbol{e}_i,\dots,\boldsymbol{e}_n),$$

which is a permutation matrix. We denote by $R^\times$ the group of units of R.

Proposition 2.1 (c.f. [5]). *We have the following equivalences:*

(1) $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_i,\dots,\boldsymbol{a}_j,\dots,\boldsymbol{a}_n) \sim (\boldsymbol{a}_1,\dots,\boldsymbol{a}_j,\dots,-\boldsymbol{a}_i,\dots,\boldsymbol{a}_n)$,

(2) $(\boldsymbol{a}_1,\dots,\boldsymbol{a}_i,\dots,\boldsymbol{a}_n) \sim (\boldsymbol{a}_1,\dots,\boldsymbol{a}_i u,\dots,\boldsymbol{a}_n)$ $(u\in R^\times)$,

(3) $\begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_i\\ \vdots\\ \boldsymbol{a}_j\\ \vdots\\ \boldsymbol{a}_n\end{pmatrix} \sim \begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_j\\ \vdots\\ -\boldsymbol{a}_i\\ \vdots\\ \boldsymbol{a}_n\end{pmatrix}$,

(4) $\begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_i\\ \vdots\\ \boldsymbol{a}_n\end{pmatrix} \sim \begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ u\boldsymbol{a}_i\\ \vdots\\ \boldsymbol{a}_n\end{pmatrix}$ $(u\in R^\times)$.

Let R be a commutative ring, and let $A\in M(m,n;R)$. A k-minor of A is the determinant of a $k\times k$ submatrix of A. The (0th) *elementary ideal* $E(A)$ of A is the ideal of R generated by all n-minors of A if $n\le m$; otherwise $E(A)=0$. Suppose that R is a GCD domain. Then, the (0th) *Alexander invariant* $\Delta(A)$ of A is the greatest common divisor of all n-

minors of A if $n \leq m$; otherwise $\Delta(A) = 0$. We remark that $\Delta(A)$ coincides with the greatest common divisor of generators of $E(A)$ and is determined up to unit multiple. If $A \sim B$, then $E(A) = E(B)$ and $\Delta(A) \doteq \Delta(B)$, where the symbol $\doteq$ indicates equality up to a unit factor. See [3] for more details.

REMARK 2.2. We may regard a matrix in $M(m, n; M(k, k; R))$ as a matrix in $M(km, kn; R)$. We call such matrices *flat matrices*, and emphasize that equivalent matrices are equivalent as flat matrices. The twisted Alexander polynomial is defined through this process.

Lemma 2.3. *Let R be a commutative ring. For $A \in M(m, n; R)$ and $B \in M(n, n; R)$, we have $E(AB) = (\det B)E(A)$. Let R be a GCD domain. For $A \in M(m, n; R)$ and $B \in M(n, n; R)$, we have $\Delta(AB) \doteq (\det B)\Delta(A)$.*

Proof. If $n \leq m$, we have

$$\begin{aligned} E(AB) &= I(\{\det A'B \mid A' \text{ is an } n \times n \text{ submatrix of } A\}) \\ &= (\det B)I(\{\det A' \mid A' \text{ is an } n \times n \text{ submatrix of } A\}) \\ &= (\det B)E(A); \end{aligned}$$

otherwise $E(AB) = 0 = (\det B)E(A)$, where $I(S)$ denotes the ideal generated by a set S. Then $\Delta(AB) \doteq (\det B)\Delta(A)$ follows from $E(AB) = (\det B)E(A)$. □

Let R be a ring. For $A = (a_{ij}) \in M(m, n; R)$, $\boldsymbol{i} = (i_1, \ldots, i_s)$ and $\boldsymbol{j} = (j_1, \ldots, j_t)$, we define

$$A_{\boldsymbol{i},\boldsymbol{j}} := \begin{pmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_t} \\ a_{i_2 j_1} & a_{i_2 j_2} & \cdots & a_{i_2 j_t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_s j_1} & a_{i_s j_2} & \cdots & a_{i_s j_t} \end{pmatrix}.$$

For example,

$$A_{(3,2),(1,4)} = \begin{pmatrix} a_{31} & a_{34} \\ a_{21} & a_{24} \end{pmatrix}$$

for $A = (a_{ij}) \in M(4, 4; R)$. We further note that

$$A_{\boldsymbol{i},\boldsymbol{j}} = {}^t(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_s})A(\boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_t}).$$

Put $\bar{n} := (1, \ldots, n)$. For $l \leq n$, we set $S_n(l) := \{(\sigma(1), \ldots, \sigma(l)) \mid \sigma \in S_n\}$. For $\boldsymbol{j} = (j_1, \ldots, j_l) \in S_n(l)$, we denote by $\boldsymbol{j}^c$ the vector obtained by removing $j_1, \ldots, j_l$ from $\bar{n}$.

DEFINITION 2.4. Let $A \in M(m, n; R)$. We call $B \in M(n, l; R)$ a *column relation matrix* of A if $AB = O$. A column relation matrix $B \in M(n, l; R)$ is *regular* if $\det B_{\boldsymbol{j},\bar{l}} \neq 0$ for some $\boldsymbol{j} \in S_n(l)$.

Let R be an integral domain. For $a, b \in R \setminus \{0\}$ and ideals I, J of R, we write $I/a = J/b$ if $bI = aJ$, where $aI := \{ax \mid x \in I\}$. For $a, b \in R \setminus \{0\}$ and $x, y \in R$, we write $x/a \doteq y/b$ if $bx \doteq ay$. We remark that these are equivalence relations.

Definition 2.5. Let $A \in M(m,n;R)$. Let $B \in M(n,l;R)$ be a regular column relation matrix of A. We choose $\boldsymbol{j} \in S_n(l)$ so that $\det B_{\boldsymbol{j},\bar{l}} \neq 0$. When R is an integral domain, we define

$$E(A,B) := E(A_{\bar{m},\boldsymbol{j}^c}) / \det B_{\boldsymbol{j},\bar{l}}.$$

When R is a GCD domain, we define

$$\Delta(A,B) := \Delta(A_{\bar{m},\boldsymbol{j}^c}) / \det B_{\boldsymbol{j},\bar{l}}.$$

When we consider $E(A,B)$, we implicitly assume that the base ring is an integral domain. When we consider $\Delta(A,B)$, we implicitly assume that the base ring is a GCD domain. The following proposition implies that $E(A,B)$ and $\Delta(A,B)$ are independent of the choices of $\boldsymbol{j}$.

Proposition 2.6. *Let $A \in M(m,n;R)$. Let $B \in M(n,l;R)$ be a regular column relation matrix of A. We choose $\boldsymbol{j},\boldsymbol{k} \in S_n(l)$ so that $\det B_{\boldsymbol{j},\bar{l}} \neq 0$ and $\det B_{\boldsymbol{k},\bar{l}} \neq 0$. When R is an integral domain, we have*

$$E(A_{\bar{m},\boldsymbol{j}^c}) / \det B_{\boldsymbol{j},\bar{l}} = E(A_{\bar{m},\boldsymbol{k}^c}) / \det B_{\boldsymbol{k},\bar{l}}.$$

When R is a GCD domain, we have

$$\Delta(A_{\bar{m},\boldsymbol{j}^c}) / \det B_{\boldsymbol{j},\bar{l}} \doteq \Delta(A_{\bar{m},\boldsymbol{k}^c}) / \det B_{\boldsymbol{k},\bar{l}}.$$

Proof. By permutating rows and columns, we may assume that

$$A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \end{pmatrix}, \qquad A_{\bar{m},\boldsymbol{j}^c} = \begin{pmatrix} A_3 & A_4 \end{pmatrix}, \qquad A_{\bar{m},\boldsymbol{k}^c} = \begin{pmatrix} A_2 & A_4 \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix}, \qquad B_{\boldsymbol{j},\bar{l}} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad B_{\boldsymbol{k},\bar{l}} = \begin{pmatrix} B_1 \\ B_3 \end{pmatrix}.$$

Since we have $A_1B_1 + \cdots + A_4B_4 = O$, we then have

$$\begin{aligned}
\begin{pmatrix} E_{n_1} & O & O \\ O & A_3 & A_4 \end{pmatrix} \begin{pmatrix} B_1 & O \\ B_3 & O \\ O & E_{n_4} \end{pmatrix} &= \begin{pmatrix} B_1 & O \\ A_3B_3 & A_4 \end{pmatrix} \\
&= \begin{pmatrix} B_1 & O \\ -A_1B_1 - A_2B_2 - A_4B_4 & A_4 \end{pmatrix} \\
&\sim \begin{pmatrix} B_1 & O \\ -A_2B_2 & A_4 \end{pmatrix} \\
&\sim \begin{pmatrix} B_1 & O \\ A_2B_2 & A_4 \end{pmatrix} \\
&= \begin{pmatrix} E_{n_1} & O & O \\ O & A_2 & A_4 \end{pmatrix} \begin{pmatrix} B_1 & O \\ B_2 & O \\ O & E_{n_4} \end{pmatrix},
\end{aligned}$$

where n_i is the number of columns of A_i, which coincides with that of rows of B_i. Then we have

$$E\left(\begin{pmatrix} E_{n_1} & O \\ O & A_{\bar{m},\boldsymbol{j}^c} \end{pmatrix}\begin{pmatrix} B_{\boldsymbol{k},\bar{l}} & O \\ O & E_{n_4} \end{pmatrix}\right) = E\left(\begin{pmatrix} E_{n_1} & O \\ O & A_{\bar{m},\boldsymbol{k}^c} \end{pmatrix}\begin{pmatrix} B_{\boldsymbol{j},\bar{l}} & O \\ O & E_{n_4} \end{pmatrix}\right).$$

By Lemma 2.3, we have $(\det B_{\boldsymbol{k},\bar{l}})E(A_{\bar{m},\boldsymbol{j}^c}) = (\det B_{\boldsymbol{j},\bar{l}})E(A_{\bar{m},\boldsymbol{k}^c})$, which implies that $E(A_{\bar{m},\boldsymbol{j}^c})/\det B_{\boldsymbol{j},\bar{l}} = E(A_{\bar{m},\boldsymbol{k}^c})/\det B_{\boldsymbol{k},\bar{l}}$. From this equality, we have $\Delta(A_{\bar{m},\boldsymbol{j}^c})/\det B_{\boldsymbol{j},\bar{l}} \doteq \Delta(A_{\bar{m},\boldsymbol{k}^c})/\det B_{\boldsymbol{k},\bar{l}}$. □

DEFINITION 2.7. For matrices A_1, A_2 over a ring R and their column relation matrices B_1, B_2, we write $(A_1, B_1) \sim_T (A_2, B_2)$ if they are related by a finite sequence of the following transformations:

$$(A, B) \leftrightarrow (AP_{ij}, P_{ij}B), \qquad (A, B) \leftrightarrow (E_{ij}(r)A, B),$$

$$(A, B) \leftrightarrow \left(\begin{pmatrix} A \\ \boldsymbol{0} \end{pmatrix}, B\right), \qquad (A, B) \leftrightarrow \left(\begin{pmatrix} A & \boldsymbol{0} \\ \boldsymbol{a} & 1 \end{pmatrix}, \begin{pmatrix} B \\ -\boldsymbol{a}B \end{pmatrix}\right).$$

We note that $(A_1, B_1) \sim_T (A_2, B_2)$ implies $A_1 \sim A_2$.

Suppose that R is an integral domain. Then B_1 is regular if and only if B_2 is regular, since their ranks coincide when we regard B_1, B_2 as matrices over the field of fractions of R.

Proposition 2.8. *Let B_1, B_2 be regular column relation matrices of matrices A_1, A_2, respectively. If $(A_1, B_1) \sim_T (A_2, B_2)$, then $E(A_1, B_1) = E(A_2, B_2)$ and $\Delta(A_1, B_1) \doteq \Delta(A_2, B_2)$.*

Proof. It is sufficient to show $E(A_1, B_1) = E(A_2, B_2)$ for each transformation in Definition 2.7. Let $B \in M(n, l; R)$ be a regular column relation matrix of $A \in M(m, n; R)$. It is easy to see that $E(A, B) = E(AP_{ij}, P_{ij}B)$. Hence, by permutating rows and columns, we may assume that $\det B_{\bar{l},\bar{l}} \neq 0$. Then, the desired equalities follow from

$$A_{\bar{m},\bar{l}^c} \sim E_{ij}(r)A_{\bar{m},\bar{l}^c}, \qquad A_{\bar{m},\bar{l}^c} \sim \begin{pmatrix} A_{\bar{m},\bar{l}^c} \\ \boldsymbol{0} \end{pmatrix}, \qquad A_{\bar{m},\bar{l}^c} \sim \begin{pmatrix} A_{\bar{m},\bar{l}^c} & \boldsymbol{0} \\ \boldsymbol{a}_{(1),\bar{l}^c} & 1 \end{pmatrix},$$

where $\bar{l}^c = (l+1, \ldots, n)$. □

REMARK 2.9. Let A_1, A_2 be matrices over $M(k, k; R)$, and let B_1, B_2 be column relation matrices of A_1, A_2, respectively. Here, we denote by $\overline{A}$ the flat matrix of a matrix A. If $(A_1, B_1) \sim_T (A_2, B_2)$, then $(\overline{A_1}, \overline{B_1}) \sim_T (\overline{A_2}, \overline{B_2})$, which implies $E(\overline{A_1}, \overline{B_1}) = E(\overline{A_2}, \overline{B_2})$ and $\Delta(\overline{A_1}, \overline{B_1}) \doteq \Delta(\overline{A_2}, \overline{B_2})$.

3. Quandles and their presentations

A *quandle* [6, 10] is a set Q equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

(Q1) For any $a \in Q$, $a \triangleleft a = a$.
(Q2) For any $a \in Q$, the map $\triangleleft a : Q \to Q$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.
(Q3) For any $a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote the map $(\triangleleft a)^n : Q \to Q$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$.

For quandles $(X_1, \triangleleft_1)$ and $(X_2, \triangleleft_2)$, a *quandle homomorphism* $f : X_1 \to X_2$ is a map satisfying $f(a \triangleleft_1 b) = f(a) \triangleleft_2 f(b)$ for any $a, b \in X_1$. We call a bijective quandle homomorphism

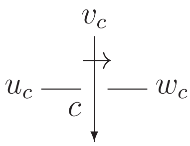

Fig. 1.

a *quandle isomorphism*. A quandle homomorphism $\rho : X \to Q$ is also called a *quandle representation* of X to Q. A quandle representation is *trivial* if it is a constant map. Let $\rho_1 : X_1 \to Q$ and $\rho_2 : X_2 \to Q$ be quandle representations. We say that (X_1, ρ_1) and (X_2, ρ_2) are *isomorphic* if there exists a quandle isomorphism $f : X_1 \to X_2$ such that $\rho_1 = \rho_2 \circ f$.

For a group G, the *n-fold conjugation quandle*, denoted by $\operatorname{Conj}_n G$, is the quandle $(G, \triangleleft)$ defined by $a \triangleleft b = b^{-n}ab^n$. The quandle $\operatorname{Conj}_1 G$ is called the *conjugation quandle* and denoted by $\operatorname{Conj} G$. The *dihedral quandle*, denoted by R_n, is the quandle $(\mathbb{Z}_n, \triangleleft)$ defined by $a \triangleleft b = 2b - a$, where $\mathbb{Z}_n$ stands for $\mathbb{Z}/n\mathbb{Z}$. For a group G, the *core quandle*, denoted by $\operatorname{Core} G$, is the quandle $(G, \triangleleft)$ defined by $a \triangleleft b = ba^{-1}b$. Let $R[t^{\pm 1}]$ be the Laurent polynomial ring over a commutative ring R and M an $R[t^{\pm 1}]$-module. The *Alexander quandle* $(M, \triangleleft)$ is defined by $a \triangleleft b = ta + (1 - t)b$.

We denote by $F_{\mathsf{Qnd}}(S)$ the free quandle on a set S. A presentation $\langle S \,|\, R \rangle$ of a quandle can be used to represent a quandle, where $R \subset F_{\mathsf{Qnd}}(S) \times F_{\mathsf{Qnd}}(S)$. We call the elements of S the *generators* of $\langle S \,|\, R \rangle$ and call the elements of R the *relators* of $\langle S \,|\, R \rangle$. A relator (a, b) is also written as $a = b$. A presentation $\langle S \,|\, R \rangle$ is *finite* if S and R are finite. For a finitely presented quandle, we often write

$$\langle x_1, \dots, x_n \,|\, r_1, \dots, r_m \rangle := \langle \{x_1, \dots, x_n\} \,|\, \{r_1, \dots, r_m\} \rangle.$$

See [2] for details of a presentation of a quandle.

Let L be an oriented link represented by a diagram D. A normal orientation is often used to represent an orientation of a link on its diagram. The normal orientation is obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. We denote by $C(D)$ and $\mathcal{A}(D)$ the sets of crossings and arcs of D, respectively. For a crossing c of D, we denote the relator $(u_c \triangleleft v_c, w_c)$ by r_c, where v_c is the over-arc of c and u_c, w_c are the under-arcs of c such that the normal orientation of v_c points from u_c to w_c (see Figure 1). The fundamental quandle $Q(L)$ of L is the quandle whose presentation given by

$$\langle x \ (x \in \mathcal{A}(D)) \,|\, r_c \ (c \in C(D)) \rangle. \tag{1}$$

This is called the *Wirtinger presentation* of $Q(L)$ with respect to D. We denote by $E(L)$ the exterior of L. We remark that we obtain a presentation of the fundamental group $G(L) := \pi_1(E(L))$ by replacing r_c by $v_c^{-1}u_c v_c w_c^{-1}$ in (1), which is the Wirtinger presentation of $G(L)$ with respect to D.

Let L_i be an oriented link and $\rho_i : Q(L_i) \to Q$ a quandle representation for $i \in \{1, 2\}$. We say that (L_1, ρ_1) and (L_2, ρ_2) are *isomorphic* if there exists an orientation-preserving homeomorphism $f : S^3 \to S^3$ such that $f(L_1) = L_2$ and $\rho_1 = \rho_2 \circ f_*$, where $f_* : Q(L_1) \to Q(L_2)$ is the induced isomorphism.

Let $\langle S_1 \,|\, R_1 \rangle$ and $\langle S_2 \,|\, R_2 \rangle$ be finite presentations of quandles. Let $\rho_1 : \langle S_1 \,|\, R_1 \rangle \to Q$ and $\rho_2 : \langle S_2 \,|\, R_2 \rangle \to Q$ be quandle representations. Then $(\langle S_1 \,|\, R_1 \rangle, \rho_1)$ and $(\langle S_2 \,|\, R_2 \rangle, \rho_2)$ are isomorphic if and only if they can be transformed into each other by a finite sequence of the following transformations:

(T1-1) $(\langle S \mid R\rangle, \rho) \leftrightarrow (\langle S \mid R \cup \{(x,x)\}\rangle, \rho)$ $(x \in F_{\mathsf{Qnd}}(S))$,
(T1-2) $(\langle S \mid R \cup \{(a,b)\}\rangle, \rho) \leftrightarrow (\langle S \mid R \cup \{(a,b),(b,a)\}\rangle, \rho)$,
(T1-3) $(\langle S \mid R \cup \{(a,b),(b,c)\}\rangle, \rho) \leftrightarrow (\langle S \mid R \cup \{(a,b),(b,c),(a,c)\}\rangle, \rho)$,
(T1-4) $(\langle S \mid R \cup \{(a_1,a_2),(b_1,b_2)\}\rangle, \rho)$
$\leftrightarrow (\langle S \mid R \cup \{(a_1,a_2),(b_1,b_2),(a_1 \triangleleft b_1, a_2 \triangleleft b_2)\}\rangle, \rho)$,
(T1-5) $(\langle S \mid R \cup \{(a_1,a_2),(b_1,b_2)\}\rangle, \rho)$
$\leftrightarrow (\langle S \mid R \cup \{(a_1,a_2),(b_1,b_2),(a_1 \triangleleft^{-1} b_1, a_2 \triangleleft^{-1} b_2)\}\rangle, \rho)$,
(T2) $(\langle S \mid R\rangle, \rho) \leftrightarrow (\langle S \cup \{y\} \mid R \cup \{(y, w_y)\}\rangle, \rho)$ $(y \notin F_{\mathsf{Qnd}}(S),\ w_y \in F_{\mathsf{Qnd}}(S))$,

where we use the same symbol ρ to represent quandle representations which coincide on S. See [5, Lemma 3.3] for more details.

4. Derivatives and column relation maps

In [5], we introduced the notion of a derivative with an Alexander pair and defined a quandle twisted Alexander matrix, which yields an Alexander type invariant. In this section, we recall the definition of the derivative with an Alexander pair and introduce the notion of a column relation map, which will be used to define a column relation matrix of the quandle twisted Alexander matrix.

DEFINITION 4.1. Let $(Q, \triangleleft)$ be a quandle. Let R be a ring. The pair (f_1, f_2) of maps $f_1, f_2 : Q \times Q \to R$ is an *Alexander pair* if f_1 and f_2 satisfy the following conditions:

- For any $a \in Q$, $f_1(a,a) + f_2(a,a) = 1$.
- For any $a, b \in Q$, $f_1(a,b)$ is invertible.
- For any $a, b, c \in Q$,

$$\begin{aligned} f_1(a \triangleleft b, c) f_1(a,b) &= f_1(a \triangleleft c, b \triangleleft c) f_1(a,c), \\ f_1(a \triangleleft b, c) f_2(a,b) &= f_2(a \triangleleft c, b \triangleleft c) f_1(b,c), \text{ and} \\ f_2(a \triangleleft b, c) &= f_1(a \triangleleft c, b \triangleleft c) f_2(a,c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b,c). \end{aligned}$$

Let $Q = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle$ be a finitely presented quandle. Put $S := \{x_1, \ldots, x_n\}$. Let $\mathrm{pr} : F_{\mathsf{Qnd}}(S) \to Q$ be the canonical projection. We often omit "pr" to represent $\mathrm{pr}(a)$ as a. Let $f = (f_1, f_2)$ be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$. The *$f$-derivative with respect to x_j* is the unique map $\frac{\partial_f}{\partial x_j} : F_{\mathsf{Qnd}}(S) \to R$ satisfying

$$\frac{\partial_f}{\partial x_j}(a \triangleleft b) = f_1(a,b) \frac{\partial_f}{\partial x_j}(a) + f_2(a,b) \frac{\partial_f}{\partial x_j}(b), \qquad \frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij}$$

for any $a, b \in F_{\mathsf{Qnd}}(S)$ and $i \in \{1, \ldots, n\}$, where δ_{ij} is the Kronecker delta. For a relator $r = (r_1, r_2)$, we define

$$\frac{\partial_f}{\partial x_j}(r) := \frac{\partial_f}{\partial x_j}(r_1) - \frac{\partial_f}{\partial x_j}(r_2).$$

DEFINITION 4.2. Let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$. A *column relation map* $f_{\mathrm{col}} : Q \to R$ is a map satisfying

$$f_{\mathrm{col}}(a \triangleleft b) = f_1(a,b) f_{\mathrm{col}}(a) + f_2(a,b) f_{\mathrm{col}}(b)$$

for any $a, b \in Q$.

Proposition 4.3. *For each $c \in Q$, the map $f_{\mathrm{col}} : Q \to R$ defined by $f_{\mathrm{col}}(x) = f_2(x \triangleleft^{-1} c, c)$ is a column relation map.*

Proof. As we have

$$\begin{aligned} f_{\mathrm{col}}(a \triangleleft b) &= f_2((a \triangleleft b) \triangleleft^{-1} c, c) \\ &= f_2((a \triangleleft^{-1} c) \triangleleft (b \triangleleft^{-1} c), c) \\ &= f_1(a,b) f_2(a \triangleleft^{-1} c, c) + f_2(a,b) f_2(b \triangleleft^{-1} c, c) \\ &= f_1(a,b) f_{\mathrm{col}}(a) + f_2(a,b) f_{\mathrm{col}}(b), \end{aligned}$$

the map f_{col} is a column relation map. □

Lemma 4.4. *Let $Q = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$ be a finitely presented quandle. Put $S := \{x_1, \ldots, x_n\}$. Let $f = (f_1, f_2)$ be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$. Let $f_{\mathrm{col}} : Q \to R$ be a column relation map. For $w \in F_{\mathsf{Qnd}}(S)$, we have*

$$f_{\mathrm{col}}(w) = \sum_{j=1}^{n} \frac{\partial_f}{\partial x_j}(w) f_{\mathrm{col}}(x_j).$$

Proof. It is sufficient to show that

$$\begin{aligned} &f_{\mathrm{col}}((\cdots((x_{i_0} \triangleleft^{\varepsilon_1} x_{i_1}) \triangleleft^{\varepsilon_2} x_{i_2}) \cdots) \triangleleft^{\varepsilon_k} x_{i_k}) \\ &= \sum_{j=1}^{n} \frac{\partial_f}{\partial x_j}((\cdots((x_{i_0} \triangleleft^{\varepsilon_1} x_{i_1}) \triangleleft^{\varepsilon_2} x_{i_2}) \cdots) \triangleleft^{\varepsilon_k} x_{i_k}) f_{\mathrm{col}}(x_j) \end{aligned}$$

for any $i_0, \ldots, i_k \in \{1, \ldots, n\}$ and $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$. We show this equality by induction on the length k. When $k = 0$, we have

$$f_{\mathrm{col}}(x_i) = \sum_{j=1}^{n} \delta_{ij} f_{\mathrm{col}}(x_j) = \sum_{j=1}^{n} \frac{\partial_f}{\partial x_j}(x_i) f_{\mathrm{col}}(x_j).$$

We suppose that the equality holds for any length less than k. Put $w := (\cdots((x_{i_0} \triangleleft^{\varepsilon_1} x_{i_1}) \triangleleft^{\varepsilon_2} x_{i_2}) \cdots) \triangleleft^{\varepsilon_{k-1}} x_{i_{k-1}}$. We then have

$$\begin{aligned} &\sum_{j=1}^{n} \frac{\partial_f}{\partial x_j}(w \triangleleft x_i) f_{\mathrm{col}}(x_j) \\ &= \sum_{j=1}^{n} \left(f_1(w, x_i) \frac{\partial_f}{\partial x_j}(w) + f_2(w, x_i) \frac{\partial_f}{\partial x_j}(x_i) \right) f_{\mathrm{col}}(x_j) \\ &= f_1(w, x_i) \sum_{j=1}^{n} \frac{\partial_f}{\partial x_j}(w) f_{\mathrm{col}}(x_j) + f_2(w, x_i) \sum_{j=1}^{n} \delta_{ij} f_{\mathrm{col}}(x_j) \\ &= f_1(w, x_i) f_{\mathrm{col}}(w) + f_2(w, x_i) f_{\mathrm{col}}(x_i) \\ &= f_{\mathrm{col}}(w \triangleleft x_i). \end{aligned}$$

In a similar manner, by using

$$\frac{\partial_f}{\partial x_j}(a \triangleleft^{-1} b) = f_1(a \triangleleft^{-1} b, b)^{-1}\frac{\partial_f}{\partial x_j}(a) - f_1(a \triangleleft^{-1} b, b)^{-1} f_2(a \triangleleft^{-1} b, b)\frac{\partial_f}{\partial x_j}(b)$$

and

$$f_{\mathrm{col}}(a \triangleleft^{-1} b) = f_1(a \triangleleft^{-1} b, b)^{-1} f_{\mathrm{col}}(a) - f_1(a \triangleleft^{-1} b, b)^{-1} f_2(a \triangleleft^{-1} b, b) f_{\mathrm{col}}(b),$$

we have

$$\sum_{j=1}^{n} \frac{\partial_f}{\partial x_j}(w \triangleleft^{-1} x_i) f_{\mathrm{col}}(x_j) = f_{\mathrm{col}}(w \triangleleft^{-1} x_i),$$

which completes the proof. □

We give examples of Alexander pairs and column relation maps.

EXAMPLE 4.5. Let Q be a quandle, R a ring, and $f : Q \to \operatorname{Conj} R^{\times}$ a quandle homomorphism.

(1) The maps $f_1, f_2 : Q \times Q \to R$ defined by $f_1(a, b) = f(b)^{-1}$ and $f_2(a, b) = f(b)^{-1} f(a) - f(b)^{-1}$ form an Alexander pair, and the map $f_{\mathrm{col}} : Q \to R$ defined by $f_{\mathrm{col}}(x) = f(x) - 1$ is a column relation map.

(2) The maps $f_1, f_2 : Q \times Q \to R$ defined by $f_1(a, b) = f(b)^{-1}$ and $f_2(a, b) = 1 - f(b)^{-1}$ form an Alexander pair, and the map $f_{\mathrm{col}} : Q \to R$ defined by $f_{\mathrm{col}}(x) = 1$ is a column relation map.

By setting $f(x) = t^{-1}x^n$, we have the following:

EXAMPLE 4.6. Let G be a group, and R a commutative ring. Let $R[t^{\pm 1}][G]$ be the group ring of G over the Laurent polynomial ring $R[t^{\pm 1}]$. Let $Q := \operatorname{Conj}_n G$.

(1) The maps $f_1, f_2 : Q \times Q \to R[t^{\pm 1}][G]$ defined by $f_1(a, b) = tb^{-n}$ and $f_2(a, b) = b^{-n}a^n - tb^{-n}$ form an Alexander pair, and the map $f_{\mathrm{col}} : Q \to R[t^{\pm 1}][G]$ defined by $f_{\mathrm{col}}(x) = t^{-1}x^n - 1$ is a column relation map.

(2) The maps $f_1, f_2 : Q \times Q \to R[t^{\pm 1}][G]$ defined by $f_1(a, b) = tb^{-n}$ and $f_2(a, b) = 1 - tb^{-n}$ form an Alexander pair, and the map $f_{\mathrm{col}} : Q \to R[t^{\pm 1}][G]$ defined by $f_{\mathrm{col}}(x) = 1$ is a column relation map.

EXAMPLE 4.7. Let G be a group, and $R[G]$ the group ring of G over a commutative ring R. Let $Q := \operatorname{Core} G$. The maps $f_1, f_2 : Q \times Q \to R[G]$ defined by $f_1(a, b) = -ba^{-1}$ and $f_2(a, b) = 1 + ba^{-1}$ form an Alexander pair, and the maps $f_{\mathrm{col},1}, f_{\mathrm{col},2} : Q \to R[G]$ defined by $f_{\mathrm{col},1}(x) = 1$ and $f_{\mathrm{col},2}(x) = x$ are column relation maps.

EXAMPLE 4.8. Let R be a commutative ring with $t \in R^{\times}$. Let Q be the Alexander quandle R with $a \triangleleft b = ta + (1 - t)b$. The maps $f_1, f_2 : Q \times Q \to R$ defined by $f_1(a, b) = t$ and $f_2(a, b) = 1 - t$ form an Alexander pair, and the maps $f_{\mathrm{col},1}, f_{\mathrm{col},2} : Q \to R$ defined by $f_{\mathrm{col},1}(x) = 1$ and $f_{\mathrm{col},2}(x) = x$ are column relation maps.

For $n \in \mathbb{Z}$, we define $P_n \in \mathbb{Z}[t]$ by

Table 1. P_n and Q_n

n	P_n	Q_n
0	2	0
1	t	$t-2$
2	t^2-2	t^2-4
3	t^3-3t	t^2-t-2
4	t^4-4t^2+2	t^3-4t
5	t^5-5t^3+5t	t^3-t^2-3t+2
6	$t^6-6t^4+9t^2-2$	t^4-5t^2+4
7	$t^7-7t^5+14t^3-7t$	$t^4-t^3-4t^2+3t+2$
8	$t^8-8t^6+20t^4-16t^2+2$	t^5-6t^3+8t
9	$t^9-9t^7+27t^5-30t^3+9t$	$t^5-t^4-5t^3+4t^2+5t-2$

$$P_n = \frac{(t+\sqrt{t^2-4})^n}{2^n} + \frac{2^n}{(t+\sqrt{t^2-4})^n}.$$

We then have $P_n = P_{-n}$ and

$$P_{n+1} - tP_n + P_{n-1} = 0 \tag{2}$$

for any $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$, we define $Q_n \in \mathbb{Z}[t]$ by

$$Q_{2n+1} = P_{n+1} - P_n \qquad \text{and} \qquad Q_{2n} = P_{n+1} - P_{n-1}.$$

In Table 1, we list P_n and Q_n for $0 \le n \le 9$.

Lemma 4.9. *We have $P_{k+n} = P_k$ in $\mathbb{Z}[t]/(Q_n)$ for any $k \in \mathbb{Z}$.*

Proof. We write $x \equiv y$ if $x - y = zQ_n$ for some $z \in \mathbb{Z}[t]$. It is sufficient to show that $P_n \equiv P_0$ and $P_{1+n} \equiv P_1$, since we have $P_{k+n} \equiv P_k$ by using

$$\begin{aligned} P_{i+n} - P_i &= tP_{i+n-1} - P_{i+n-2} - tP_{i-1} + P_{i-2} \\ &= t(P_{(i-1)+n} - P_{i-1}) - (P_{(i-2)+n} - P_{i-2}), \text{ or} \\ P_{i+n} - P_i &= tP_{i+n+1} - P_{i+n+2} - tP_{i+1} + P_{i+2} \\ &= t(P_{(i+1)+n} - P_{i+1}) - (P_{(i+2)+n} - P_{i+2}) \end{aligned}$$

inductively, where the first and third equalities follow from (2).

Suppose $n = 2m+1$. We show that $P_{m+j} \equiv P_{m+1-j}$ for any $j \ge 0$. By the definition of Q_{2m+1}, we have $P_m - P_{m+1} = -Q_n$ and $P_{m+1} - P_m = Q_n$ for $j = 0, 1$. We have $P_{m+j} \equiv P_{m+1-j}$ by using

$$\begin{aligned} P_{m+i} - P_{m+1-i} &= tP_{m+i-1} - P_{m+i-2} - tP_{m+2-i} + P_{m+3-i} \\ &= t(P_{m+(i-1)} - P_{m+1-(i-1)}) - (P_{m+(i-2)} - P_{m+1-(i-2)}) \end{aligned}$$

inductively, where the first equality follows from (2). Putting $j = m+1, m+2$, we have $P_n = P_{2m+1} \equiv P_0$ and $P_{1+n} = P_{2m+2} \equiv P_{-1} = P_1$.

Suppose $n = 2m$. We show that $P_{m+j} \equiv P_{m-j}$ for any $j \ge 0$. By the definition of Q_{2m}, we have $P_m - P_m = 0$ and $P_{m+1} - P_{m-1} = Q_n$ for $j = 0, 1$. We have $P_{m+j} \equiv P_{m-j}$ by using

$$\begin{aligned}P_{m+i} - P_{m-i} &= tP_{m+i-1} - P_{m+i-2} - tP_{m-i+1} + P_{m-i+2}\\ &= t(P_{m+(i-1)} - P_{m-(i-1)}) - (P_{m+(i-2)} - P_{m-(i-2)})\end{aligned}$$

inductively, where the first equality follows from (2). Putting $j = m, m+1$, we have $P_n = P_{2m} \equiv P_0$ and $P_{1+n} = P_{2m+1} \equiv P_{-1} = P_1$. □

Proposition 4.10. *Let Q be the dihedral quandle R_n of order n. The maps $f_1, f_2 : Q \times Q \to \mathbb{Z}[t]/(Q_n)$ defined by*

$$f_1(a,b) = -1 \qquad \text{and} \qquad f_2(a,b) = P_{a-b}$$

form an Alexander pair.

We remark that P_{a-b} is well-defined for $a, b \in R_n$ by Lemma 4.9.

Proof. Since $f_1(a,b) = -1$ and $f_2(a,a) = 2$, it is sufficient to show

$$\begin{aligned}f_2(a,b) &= f_2(a \triangleleft c, b \triangleleft c),\\ f_2(a \triangleleft b, c) &= -f_2(a,c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b,c)\end{aligned}$$

for $a, b, c \in R_n$. The first equality follows from $P_{a-b} = P_{b-a}$. The second equality follows from

$$P_{2b-a-c} = -P_{a-c} + P_{b-a}P_{b-c},$$

which can be obtained by direct calculation. □

Since we have

$$\begin{aligned}Q_{2n+3} - tQ_{2n+1} + Q_{2n-1} &= 0,\\ Q_{2n+2} - tQ_{2n} + Q_{2n-2} &= 0,\end{aligned}$$

it is easy to see that Q_n is divisible by $t-2$ for any $n \in \mathbb{Z}$. From Proposition 4.10, we have the following corollary.

Corollary 4.11. *Let Q be the dihedral quandle R_3 of order 3. The maps $f_1, f_2 : Q \times Q \to \mathbb{Z}$ defined by $f_1(a,b) = -1$ and $f_2(a,b) = 3\delta_{ab} - 1$ form an Alexander pair, and the map $f_{\mathrm{col},c} : Q \to \mathbb{Z}$ defined by $f_{\mathrm{col},c}(x) = 3\delta_{xc} - 1$ is a column relation map for $c \in Q$.*

5. Quandle twisted Alexander invariants

Let $X = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$ be a finitely presented quandle. Let $\rho : X \to Q$ be a quandle representation. Let $f = (f_1, f_2)$ be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$. Then $f \circ \rho^2 = (f_1 \circ \rho^2, f_2 \circ \rho^2)$ is an Alexander pair of maps $f_1 \circ \rho^2, f_2 \circ \rho^2 : X \times X \to R$. The *$f$-twisted Alexander matrix* of (X, ρ) is

$$A(X,\rho; f_1, f_2) = \begin{pmatrix} \frac{\partial_{f\circ\rho^2}}{\partial x_1}(r_1) & \cdots & \frac{\partial_{f\circ\rho^2}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f\circ\rho^2}}{\partial x_1}(r_m) & \cdots & \frac{\partial_{f\circ\rho^2}}{\partial x_n}(r_m) \end{pmatrix}.$$

Let $f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},l} : Q \to R$ be column relation maps. Then $f_{\mathrm{col},1} \circ \rho, \ldots, f_{\mathrm{col},l} \circ \rho : X \to R$

are column relation maps. We define

$$R_{\mathrm{col}}(X,\rho;f_{\mathrm{col},1},\dots,f_{\mathrm{col},l}) := \begin{pmatrix} (f_{\mathrm{col},1}\circ\rho)(x_1) & \cdots & (f_{\mathrm{col},l}\circ\rho)(x_1) \\ \vdots & \ddots & \vdots \\ (f_{\mathrm{col},1}\circ\rho)(x_n) & \cdots & (f_{\mathrm{col},l}\circ\rho)(x_n) \end{pmatrix}.$$

We denote $R_{\mathrm{col}}(X,\rho;f_{\mathrm{col},1},\dots,f_{\mathrm{col},l})$ by $R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})$ for short, where $\boldsymbol{f}_{\mathbf{col}}$ indicates $(f_{\mathrm{col},1},\dots,f_{\mathrm{col},l})$.

Proposition 5.1. *The matrix $R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})$ is a column relation matrix of $A(X,\rho;f_1,f_2)$.*

Proof. We may assume that $\boldsymbol{f}_{\mathbf{col}}=(f_{\mathrm{col}})$, since $AB_1=O$ and $AB_2=O$ imply $A\begin{pmatrix}B_1 & B_2\end{pmatrix}=O$. For a relator $r=(r_1,r_2)$, we have

$$\begin{aligned}
&\sum_{j=1}^{n}\frac{\partial_{f\circ\rho^2}}{\partial x_j}(r)(f_{\mathrm{col}}\circ\rho)(x_j)\\
&=\sum_{j=1}^{n}\frac{\partial_{f\circ\rho^2}}{\partial x_j}(r_1)(f_{\mathrm{col}}\circ\rho)(x_j)-\sum_{j=1}^{n}\frac{\partial_{f\circ\rho^2}}{\partial x_j}(r_2)(f_{\mathrm{col}}\circ\rho)(x_j)\\
&=(f_{\mathrm{col}}\circ\rho)(r_1)-(f_{\mathrm{col}}\circ\rho)(r_2)=0,
\end{aligned}$$

where the second equality follows from Lemma 4.4. This completes the proof. □

When R is an integral domain, we define

$$\begin{aligned}
E(X,\rho;f_1,f_2;\boldsymbol{f}_{\mathbf{col}}) &:= E(A(X,\rho;f_1,f_2),R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})),\\
\Delta(X,\rho;f_1,f_2;\boldsymbol{f}_{\mathbf{col}}) &:= \Delta(A(X,\rho;f_1,f_2),R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})).
\end{aligned}$$

When R is a matrix ring consisting of $k\times k$ matrices over an integral domain, we define

$$\begin{aligned}
E(X,\rho;f_1,f_2;\boldsymbol{f}_{\mathbf{col}}) &:= E(\overline{A(X,\rho;f_1,f_2)},\overline{R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})}),\\
\Delta(X,\rho;f_1,f_2;\boldsymbol{f}_{\mathbf{col}}) &:= \Delta(\overline{A(X,\rho;f_1,f_2)},\overline{R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})}).
\end{aligned}$$

The following theorem shows that they are invariants.

Theorem 5.2. *Let $X=\langle \boldsymbol{x}\,|\,\boldsymbol{r}\rangle$ and $X'=\langle \boldsymbol{x}'\,|\,\boldsymbol{r}'\rangle$ be finitely presented quandles, and let $\rho: X\to Q$ and $\rho': X'\to Q$ be quandle representations. Let (f_1,f_2) be an Alexander pair of maps $f_1,f_2: Q\times Q\to R$. Let $f_{\mathrm{col},1},\dots,f_{\mathrm{col},l}: Q\to R$ be column relation maps. If $(X,\rho)\cong(X',\rho')$, then we have*

$$(A(X,\rho;f_1,f_2),R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}}))\sim_T(A(X',\rho';f_1,f_2),R_{\mathrm{col}}(X',\rho';\boldsymbol{f}_{\mathbf{col}})).$$

Furthermore, we have the following.

- *If R is an integral domain and $R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})$ is regular, then we have*

$$\begin{aligned}
E(A(X,\rho;f_1,f_2),R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})) &= E(A(X',\rho';f_1,f_2),R_{\mathrm{col}}(X',\rho';\boldsymbol{f}_{\mathbf{col}})),\\
\Delta(A(X,\rho;f_1,f_2),R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})) &\doteq \Delta(A(X',\rho';f_1,f_2),R_{\mathrm{col}}(X',\rho';\boldsymbol{f}_{\mathbf{col}})).
\end{aligned}$$

- *If R is a matrix ring consisting of $k\times k$ matrices over an integral domain and $\overline{R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})}$ is regular, then we have*

$$E(\overline{A(X,\rho;f_1,f_2)},\overline{R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})}) = E(\overline{A(X',\rho';f_1,f_2)},\overline{R_{\mathrm{col}}(X',\rho';\boldsymbol{f}_{\mathbf{col}})}),$$
$$\Delta(\overline{A(X,\rho;f_1,f_2)},\overline{R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})}) \doteq \Delta(\overline{A(X',\rho';f_1,f_2)},\overline{R_{\mathrm{col}}(X',\rho';\boldsymbol{f}_{\mathbf{col}})}).$$

The twisted Alexander polynomial [9, 11] can be realized as the invariant $\Delta(X,\rho;f_1,f_2;f_{\mathrm{col}})$ for some Alexander pair (f_1,f_2) and column relation map f_{col}.

Let L be an oriented link, and D a diagram of L. Let

$$Q(L) = \langle x_1,\dots,x_n \mid u_1 \triangleleft v_1 = w_1,\dots,u_m \triangleleft v_m = w_m\rangle,$$
$$G(L) = \langle x_1,\dots,x_n \mid v_1^{-1}u_1v_1w_1^{-1},\dots,v_m^{-1}u_mv_mw_m^{-1}\rangle$$

be the Wirtinger presentations of the fundamental quandle $Q(L)$ and the fundamental group $G(L)$ with respect to D, respectively. See Section 3. Let R be a commutative ring. Set $G := GL(k;R)$. Let $\rho : G(L) \to G$ be a group representation. The *induced quandle representation* of ρ is a quandle homomorphism from $Q(L)$ to $\mathrm{Conj}\,G$ that sends x_i to $\rho(x_i)$, and we denote it by the same symbol $\rho : Q(L) \to \mathrm{Conj}\,G$.

Proposition 5.3. *Let $\Delta_{L,\rho}(t)$ be the twisted Alexander polynomial of (L,ρ) with the abelianization $\alpha : G(L) \to \langle t\rangle$ that sends every meridian to t^{-1}. Let $f_1, f_2 : Q(L)\times Q(L) \to R[t^{\pm1}][G]$ be the maps defined by $f_1(a,b) = tb^{-1}$ and $f_2(a,b) = b^{-1}a - tb^{-1}$. Let $f_{\mathrm{col}} : Q(L) \to R[t^{\pm1}][G]$ be the map defined by $f_{\mathrm{col}}(x) = t^{-1}x - 1$. Then we have*

$$\Delta_{L,\rho}(t) \doteq \Delta(\overline{A(Q(L),\rho;f_1,f_2)},\overline{R_{\mathrm{col}}(Q(L),\rho;f_{\mathrm{col}})}).$$

Proof. We note that (f_1,f_2) and f_{col} are an Alexander pair and column relation map. See Example 4.6 (1) with $n = 1$. In [5], we showed that the twisted Alexander matrix of (L,ρ) coincides with $\overline{A(Q(L),\rho;f_1,f_2)}$. Then, the twisted Alexander polynomial (L,ρ) is defined by

$$\Delta_{L,\rho}(t) \doteq \Delta(A(Q(L),\rho;f_1,f_2)_{\bar{m},(j)^c})/\det(t^{-1}\rho(x_j) - E_k),$$

which coincides with $\Delta(\overline{A(Q(L),\rho;f_1,f_2)},\overline{R_{\mathrm{col}}(Q(L),\rho;f_{\mathrm{col}})})$. □

In a similar manner, we have the following proposition:

Proposition 5.4. *Let $\Delta_L(t)$ be the Alexander polynomial of L with the abelianization $\alpha : G(L) \to \langle t\rangle$ that sends every meridian to t^{-1}. Let $f_1, f_2 : Q(L)\times Q(L) \to R[t^{\pm1}]$ be the maps defined by $f_1(a,b) = t$ and $f_2(a,b) = 1-t$. Let $f_{\mathrm{col}} : Q(L) \to R[t^{\pm1}]$ be the map defined by $f_{\mathrm{col}}(x) = t^{-1} - 1$. Then we have*

$$\frac{\Delta_L(t)}{t^{-1}-1} \doteq \Delta(A(Q(L),\rho;f_1,f_2),R_{\mathrm{col}}(Q(L),\rho;f_{\mathrm{col}})).$$

Furthermore, setting $f_{\mathrm{col}}(x) = 1$, we have

$$\Delta_L(t) \doteq \Delta(A(Q(L),\rho;f_1,f_2),R_{\mathrm{col}}(Q(L),\rho;f_{\mathrm{col}})).$$

REMARK 5.5. We note that the (twisted) Alexander polynomials with the abelianization α that sends every meridian to t can be obtained by setting

$$f_1(a,b) = t^{-1}b^{-1}, \qquad f_2(a,b) = b^{-1}a - t^{-1}b^{-1}, \qquad f_{\mathrm{col}}(x) = tx - 1,$$
$$f_1(a,b) = t^{-1}, \qquad f_2(a,b) = 1 - t^{-1}, \qquad f_{\mathrm{col}}(x) = t - 1$$

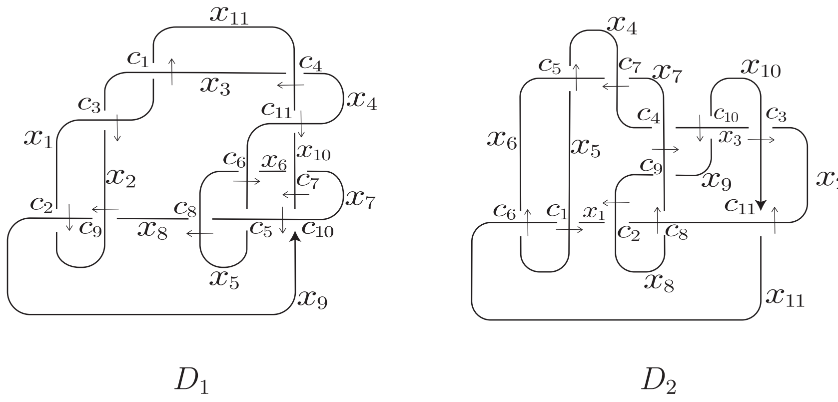

Fig. 2. The knots $11n38$ and $11n102$

in Propositions 5.3 and 5.4, respectively.

6. Examples

In this section, we investigate our invariant with the Alexander pair and the two column relation maps given in Corollary 4.11.

Let Q be the dihedral quandle R_3 of order 3. Let $f_1, f_2 : Q \times Q \to \mathbb{Z}$ be the Alexander pair defined by $f_1(a, b) = -1$ and $f_2(a, b) = 3\delta_{ab} - 1$. Let $f_{\mathrm{col},c} : Q \to \mathbb{Z}$ be the column relation map defined by $f_{\mathrm{col},c}(x) = 3\delta_{xc} - 1$ for $c \in Q$. See Corollary 4.11. Let L be an oriented link. Let $\rho : Q(L) \to Q$ be a quandle representation.

First, we see that, for a trivial representation ρ,

$$\Delta(Q(L), \rho; f_1, f_2; f_{\mathrm{col},c}) \doteq \begin{cases} \operatorname{Det} L/2 & \text{if } \operatorname{Im}\rho = \{c\}, \\ \operatorname{Det} L & \text{if } \operatorname{Im}\rho \neq \{c\}, \end{cases} \tag{3}$$

where $\operatorname{Det} L$ is the determinant of L. We remark that $\operatorname{Det} L = |\Delta_L(-1)|$. Let $g_1, g_2 : Q \times Q \to \mathbb{Z}$ be the Alexander pair defined by $g_1(a, b) = -1$ and $g_2(a, b) = 2$. Let $g_{\mathrm{col}} : Q \to \mathbb{Z}$ be the column relation map defined by $g_{\mathrm{col}}(x) = 1$. By Proposition 5.4, we have

$$\Delta(Q(L), \rho; g_1, g_2; g_{\mathrm{col}}) \doteq \Delta_L(-1).$$

Since $f_1 \circ \rho^2 = g_1 \circ \rho^2$ and $f_2 \circ \rho^2 = g_2 \circ \rho^2$, we have $A(Q(L), \rho; f_1, f_2) = A(Q(L), \rho; g_1, g_2)$. Since $(f_{\mathrm{col},c} \circ \rho)(x) = (3\delta_{xc} - 1)(g_{\mathrm{col}} \circ \rho)(x)$, we have

$$R_{\mathrm{col}}(Q(L), \rho; f_{\mathrm{col},c}) = \begin{cases} 2R_{\mathrm{col}}(Q(L), \rho; g_{\mathrm{col}}) & \text{if } \operatorname{Im}\rho = \{c\}, \\ R_{\mathrm{col}}(Q(L), \rho; g_{\mathrm{col}}) & \text{if } \operatorname{Im}\rho \neq \{c\}. \end{cases}$$

Thus we have (3).

Let K_1 be the knot $11n38$, and let K_2 be the knot $11n102$. Let D_1 and D_2 be their diagrams depicted in Figure 2. Then, we see that

$$\Delta(Q(K_1), \rho_1; f_1, f_2; f_{\mathrm{col},0}, f_{\mathrm{col},1}) \doteq 2/3,$$
$$\Delta(Q(K_2), \rho_2; f_1, f_2; f_{\mathrm{col},0}, f_{\mathrm{col},1}) \doteq 7/3$$

for any nontrivial quandle representation $\rho_i : Q(K_i) \to Q$. We note that both $Q(K_1)$ and

$Q(K_2)$ have 6 nontrivial quandle representations. We also note that $\Delta_{K_1}(t) \doteq \Delta_{K_2}(t)$ and $E_d(K_1) = E_d(K_2)$ for any d, where $E_d(K)$ is the dth Alexander ideal of a knot K.

The Wirtinger presentation of $Q(K_1)$ with respect to D_1 is

$$Q(K_1) = \left\langle x_1, \dots, x_{11} \;\middle|\; \begin{array}{l} x_1 \triangleleft x_3 = x_{11},\ x_1 \triangleleft x_9 = x_2,\ x_3 \triangleleft x_1 = x_2, \\ x_4 \triangleleft x_{11} = x_3,\ x_4 \triangleleft x_7 = x_5,\ x_5 \triangleleft x_4 = x_6, \\ x_7 \triangleleft x_{10} = x_6,\ x_7 \triangleleft x_5 = x_8,\ x_8 \triangleleft x_2 = x_9, \\ x_{10} \triangleleft x_7 = x_9,\ x_{11} \triangleleft x_4 = x_{10} \end{array} \right\rangle.$$

Putting $a = \rho_1(x_1)$, $b = \rho_1(x_2)$ and $c = \rho_1(x_3)$, we have $a \neq b$, $c = 2a + 2b$ and

$$\begin{aligned} &\rho_1(x_4) = \rho_1(x_8) = a, \quad \rho_1(x_5) = \rho_1(x_{11}) = b, \\ &\rho_1(x_6) = \rho_1(x_7) = \rho_1(x_9) = \rho_1(x_{10}) = c. \end{aligned}$$

Then, $A(Q(K_1), \rho_1; f_1, f_2)$ is

$$\begin{pmatrix} f_1^\bullet & 0 & f_2^{\neq} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ f_1^\bullet & -1 & 0 & 0 & 0 & 0 & 0 & 0 & f_2^{\neq} & 0 & 0 \\ f_2^{\neq} & -1 & f_1^\bullet & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & f_1^\bullet & 0 & 0 & 0 & 0 & 0 & 0 & f_2^{\neq} \\ 0 & 0 & 0 & f_1^\bullet & -1 & 0 & f_2^{\neq} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_2^{\neq} & f_1^\bullet & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & f_1^\bullet & 0 & 0 & f_2^{=} & 0 \\ 0 & 0 & 0 & 0 & f_2^{\neq} & 0 & f_1^\bullet & -1 & 0 & 0 & 0 \\ 0 & f_2^{\neq} & 0 & 0 & 0 & 0 & 0 & f_1^\bullet & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f_2^{=} & 0 & -1 & f_1^\bullet & 0 \\ 0 & 0 & 0 & f_2^{\neq} & 0 & 0 & 0 & 0 & 0 & -1 & f_1^\bullet \end{pmatrix},$$

where $f_1^\bullet = -1$, $f_2^{=} = 2$ and $f_2^{\neq} = -1$. The matrix $A(Q(K_1), \rho_1; f_1, f_2)_{\overline{11},(1,2)^c}$ is equivalent to the 1×1 matrix $\begin{pmatrix} 2 \end{pmatrix}$. We have

$$R_{\mathrm{col}}(Q(K_1), \rho_1; f_{\mathrm{col},0}, f_{\mathrm{col},1}) = \begin{pmatrix} f_{\mathrm{col},0}(a) & f_{\mathrm{col},1}(a) \\ f_{\mathrm{col},0}(b) & f_{\mathrm{col},1}(b) \\ f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\ f_{\mathrm{col},0}(a) & f_{\mathrm{col},1}(a) \\ f_{\mathrm{col},0}(b) & f_{\mathrm{col},1}(b) \\ f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\ f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\ f_{\mathrm{col},0}(a) & f_{\mathrm{col},1}(a) \\ f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\ f_{\mathrm{col},0}(c) & f_{\mathrm{col},1}(c) \\ f_{\mathrm{col},0}(b) & f_{\mathrm{col},1}(b) \end{pmatrix} = \begin{pmatrix} 3\delta_{a0} - 1 & 3\delta_{a1} - 1 \\ 3\delta_{b0} - 1 & 3\delta_{b1} - 1 \\ 3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\ 3\delta_{a0} - 1 & 3\delta_{a1} - 1 \\ 3\delta_{b0} - 1 & 3\delta_{b1} - 1 \\ 3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\ 3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\ 3\delta_{a0} - 1 & 3\delta_{a1} - 1 \\ 3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\ 3\delta_{c0} - 1 & 3\delta_{c1} - 1 \\ 3\delta_{b0} - 1 & 3\delta_{b1} - 1 \end{pmatrix}.$$

Since $R_{\mathrm{col}}(Q(K_1), \rho_1; f_{\mathrm{col},0}, f_{\mathrm{col},1})_{(1,2),\bar{2}}$ is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix},$$

we have $\det R_{\mathrm{col}}(Q(K_1), \rho_1; f_{\mathrm{col},0}, f_{\mathrm{col},1})_{(1,2),\bar{2}} \doteq 3$. Thus we have

$$
\begin{aligned}
&\Delta(Q(K_1),\rho_1;f_1,f_2;f_{\mathrm{col},0},f_{\mathrm{col},1})\\
&= \Delta(A(Q(K_1),\rho_1;f_1,f_2)_{\overline{11},(1,2)^c} / \det R_{\mathrm{col}}(Q(K_1),\rho_1;f_{\mathrm{col},0},f_{\mathrm{col},1})_{(1,2),\bar{2}}\\
&\doteq 2/3.
\end{aligned}
$$

The Wirtinger presentation of $Q(K_2)$ with respect to D_2 is

$$
Q(K_2) = \left\langle x_1,\dots,x_{11} \;\middle|\; \begin{array}{l} x_{11} \triangleleft x_5 = x_1,\ x_2 \triangleleft x_8 = x_1,\ x_3 \triangleleft x_{10} = x_2,\\ x_4 \triangleleft x_7 = x_3,\ x_5 \triangleleft x_6 = x_4,\ x_5 \triangleleft x_{11} = x_6,\\ x_7 \triangleleft x_4 = x_6,\ x_8 \triangleleft x_2 = x_7,\ x_8 \triangleleft x_7 = x_9,\\ x_{10} \triangleleft x_3 = x_9,\ x_{11} \triangleleft x_2 = x_{10} \end{array} \right\rangle.
$$

Putting $a = \rho_2(x_1)$, $b = \rho_2(x_2)$ and $c = \rho_2(x_8)$, we have $a \neq b$, $c = 2a + 2b$ and

$$
\begin{aligned}
&\rho_2(x_3) = \rho_2(x_4) = \rho_2(x_5) = \rho_2(x_6) = \rho_2(x_7) = \rho_2(x_{11}) = a,\\
&\rho_2(x_9) = b, \quad \rho_2(x_{10}) = c.
\end{aligned}
$$

Then, $A(Q(K_2),\rho_2;f_1,f_2)$ is

$$
\begin{pmatrix}
-1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1\\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 2\\
0 & 0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1
\end{pmatrix}.
$$

The matrix $A(Q(K_2),\rho_2;f_1,f_2)_{\overline{11},(1,2)^c}$ is equivalent to the 1×1 matrix $\begin{pmatrix}7\end{pmatrix}$. In the same manner as $\det R_{\mathrm{col}}(Q(K_1),\rho_1;f_{\mathrm{col},0},f_{\mathrm{col},1})_{(1,2),\bar{2}}$, we have

$$
\det R_{\mathrm{col}}(Q(K_2),\rho_2;f_{\mathrm{col},0},f_{\mathrm{col},1})_{(1,2),\bar{2}} \doteq 3.
$$

Thus we have

$$
\begin{aligned}
&\Delta(Q(K_2),\rho_2;f_1,f_2;f_{\mathrm{col},0},f_{\mathrm{col},1})\\
&= \Delta(A(Q(K_2),\rho_2;f_1,f_2)_{\overline{11},(1,2)^c} / \det R_{\mathrm{col}}(Q(K_2),\rho_2;f_{\mathrm{col},0},f_{\mathrm{col},1})_{(1,2),\bar{2}}\\
&\doteq 7/3.
\end{aligned}
$$

7. Proof of Theorem 5.2

We show

$$
(A(X,\rho;f_1,f_2), R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})) \sim_T (A(X',\rho';f_1,f_2), R_{\mathrm{col}}(X',\rho';\boldsymbol{f}_{\mathbf{col}})).
$$

It is sufficient to show this equivalence for the transformations (T1-1)–(T1-5) and (T2) in Section 3. We set

$$A := A(\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle, \rho; f_1, f_2), \qquad B := R_{\mathrm{col}}(\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle, \rho; \boldsymbol{f}_{\mathbf{col}}),$$
$$A' := A(\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle, \rho'; f_1, f_2), \qquad B' := R_{\mathrm{col}}(\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle, \rho'; \boldsymbol{f}_{\mathbf{col}}).$$

We denote by $\boldsymbol{a}_i$ the i-th row vector of A and denote by a_{ij} the (i, j) entry of A.

For (T1-1), we suppose

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m, x = x \rangle \ (x \in F_{\mathsf{Qnd}}(\boldsymbol{x})).$$

We then have

$$(A, B) \sim_T \left(\begin{pmatrix} A \\ \mathbf{0} \end{pmatrix}, B \right) = (A', B').$$

For (T1-2), we suppose

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m, a = b \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m, a = b, b = a \rangle.$$

We then have

$$(A, B) = \left(\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \end{pmatrix}, B \right) \sim_T \left(\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \mathbf{0} \end{pmatrix}, B \right) \sim_T \left(\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ -\boldsymbol{a}_{m+1} \end{pmatrix}, B \right) = (A', B').$$

For (T1-3), we suppose

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m, a = b, b = c \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m, a = b, b = c, a = c \rangle.$$

We then have

$$(A, B) = \left(\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \end{pmatrix}, B \right) \sim_T \left(\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \\ \mathbf{0} \end{pmatrix}, B \right) \sim_T \left(\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_{m+1} \\ \boldsymbol{a}_{m+2} \\ \boldsymbol{a}' \end{pmatrix}, B \right) = (A', B'),$$

where $\boldsymbol{a}' = \boldsymbol{a}_{m+1} + \boldsymbol{a}_{m+2}$.

For (T1-4), we suppose

$$\langle \boldsymbol{x} \,|\, \boldsymbol{r} \rangle = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2 \rangle,$$
$$\langle \boldsymbol{x}' \,|\, \boldsymbol{r}' \rangle = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m, a_1 = a_2, b_1 = b_2, a_1 \triangleleft b_1 = a_2 \triangleleft b_2 \rangle.$$

We then have

$$(A,B)=\left(\begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_{m+1}\\ \boldsymbol{a}_{m+2}\end{pmatrix},B\right)\sim_T\left(\begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_{m+1}\\ \boldsymbol{a}_{m+2}\\ \boldsymbol{0}\end{pmatrix},B\right)\sim_T\left(\begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_{m+1}\\ \boldsymbol{a}_{m+2}\\ \boldsymbol{a}'\end{pmatrix},B\right)=(A',B'),$$

where $\boldsymbol{a}'=(f_1\circ\rho^2)(a_1,b_1)\boldsymbol{a}_{m+1}+(f_2\circ\rho^2)(a_1,b_1)\boldsymbol{a}_{m+2}$.

For (T1-5), we suppose

$$\begin{aligned}\langle\boldsymbol{x}\,|\,\boldsymbol{r}\rangle&=\langle x_1,\dots,x_n\,|\,r_1,\dots,r_m,a_1=a_2,b_1=b_2\rangle,\\ \langle\boldsymbol{x}'\,|\,\boldsymbol{r}'\rangle&=\langle x_1,\dots,x_n\,|\,r_1,\dots,r_m,a_1=a_2,b_1=b_2,a_1\triangleleft^{-1}b_1=a_2\triangleleft^{-1}b_2\rangle.\end{aligned}$$

We then have

$$(A,B)=\left(\begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_{m+1}\\ \boldsymbol{a}_{m+2}\end{pmatrix},B\right)\sim_T\left(\begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_{m+1}\\ \boldsymbol{a}_{m+2}\\ \boldsymbol{0}\end{pmatrix},B\right)\sim_T\left(\begin{pmatrix}\boldsymbol{a}_1\\ \vdots\\ \boldsymbol{a}_{m+1}\\ \boldsymbol{a}_{m+2}\\ \boldsymbol{a}'\end{pmatrix},B\right)=(A',B'),$$

where $\boldsymbol{a}'=(f_1\circ\rho^2)(a_1\triangleleft^{-1}b_1,b_1)^{-1}\boldsymbol{a}_{m+1}-(f_1\circ\rho^2)(a_1\triangleleft^{-1}b_1,b_1)^{-1}(f_2\circ\rho^2)(a_1\triangleleft^{-1}b_1,b_1)\boldsymbol{a}_{m+2}$.

For (T2), we suppose

$$\begin{aligned}\langle\boldsymbol{x}\,|\,\boldsymbol{r}\rangle&=\langle x_1,\dots,x_n\,|\,r_1,\dots,r_m\rangle,\\ \langle\boldsymbol{x}'\,|\,\boldsymbol{r}'\rangle&=\langle x_1,\dots,x_n,y\,|\,r_1,\dots,r_m,y=w\rangle\ (y\notin F_{\mathsf{Qnd}}(\boldsymbol{x}),\ w\in F_{\mathsf{Qnd}}(\boldsymbol{x})).\end{aligned}$$

We then have

$$(A,B)=\left(\begin{pmatrix}\frac{\partial_{f\circ\rho^2}}{\partial x_1}(r_1)&\cdots&\frac{\partial_{f\circ\rho^2}}{\partial x_n}(r_1)\\ \vdots&\ddots&\vdots\\ \frac{\partial_{f\circ\rho^2}}{\partial x_1}(r_m)&\cdots&\frac{\partial_{f\circ\rho^2}}{\partial x_n}(r_m)\end{pmatrix},B\right)\sim_T\left(\begin{pmatrix}A&\boldsymbol{0}\\ \boldsymbol{a}'&1\end{pmatrix},\begin{pmatrix}B\\ \boldsymbol{b}'\end{pmatrix}\right)=(A',B'),$$

where

$$\begin{aligned}\boldsymbol{a}'&=\left(-\frac{\partial_{f\circ\rho^2}}{\partial x_1}(w),\dots,-\frac{\partial_{f\circ\rho^2}}{\partial x_n}(w)\right),\\ \boldsymbol{b}'&=(f_{\mathrm{col},1}(\rho(y)),\dots,f_{\mathrm{col},l}(\rho(y)))\\ &=\left(\sum_{i=1}^{n}\frac{\partial_{f\circ\rho^2}}{\partial x_i}(w)f_{\mathrm{col},1}(\rho(x_i)),\dots,\sum_{i=1}^{n}\frac{\partial_{f\circ\rho^2}}{\partial x_i}(w)f_{\mathrm{col},l}(\rho(x_i))\right).\end{aligned}$$

The rest follows from Proposition 2.8 and Remark 2.9.

8. Cohomologous Alexander pairs and column relation maps

Let (f_1,f_2) and (g_1,g_2) be Alexander pairs of maps $f_1,f_2,g_1,g_2 : Q\times Q\to R$. Let $f_{\mathrm{col}}:Q\to R$ and $g_{\mathrm{col}}:Q\to R$ be column relation maps with respect to (f_1,f_2) and (g_1,g_2), respectively. Two triples $(f_1,f_2,f_{\mathrm{col}})$ and $(g_1,g_2,g_{\mathrm{col}})$ are *cohomologous* if there exists a map $h:Q\to R$ satisfying the following conditions:

- For any $a \in Q$, $h(a)$ is invertible in R.
- For any $a, b \in Q$, $h(a \triangleleft b) f_1(a, b) = g_1(a, b) h(a)$.
- For any $a, b \in Q$, $h(a \triangleleft b) f_2(a, b) = g_2(a, b) h(b)$.
- For any $a \in Q$, $h(a) f_{\mathrm{col}}(a) = g_{\mathrm{col}}(a)$.

We then write $(f_1, f_2, f_{\mathrm{col}}) \sim_h (g_1, g_2, g_{\mathrm{col}})$ to specify h. Let $f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},l} : Q \to R$ and $g_{\mathrm{col},1}, \ldots, g_{\mathrm{col},l} : Q \to R$ be column relation maps with respect to (f_1, f_2) and (g_1, g_2), respectively. When $(f_1, f_2, f_{\mathrm{col},i}) \sim_h (g_1, g_2, g_{\mathrm{col},i})$ for any i, we write $(f_1, f_2, \boldsymbol{f}_{\mathbf{col}}) \sim_h (g_1, g_2, \boldsymbol{g}_{\mathbf{col}})$.

EXAMPLE 8.1. For an Alexander pair (f_1, f_2) and $a \in Q$, we define $f_1 \triangleleft a$ and $f_2 \triangleleft a$ by

$$(f_1 \triangleleft a)(x, y) = f_1(x \triangleleft a, y \triangleleft a), \qquad (f_2 \triangleleft a)(x, y) = f_2(x \triangleleft a, y \triangleleft a).$$

For a column relation map f_{col} and $a \in Q$, we define $f_{\mathrm{col}} \triangleleft a$ by

$$(f_{\mathrm{col}} \triangleleft a)(x) = f_1(x, a) f_{\mathrm{col}}(x).$$

Putting $h(x) := f_1(x, a)$, we have

$$(f_1, f_2, f_{\mathrm{col}}) \sim_h (f_1 \triangleleft a, f_2 \triangleleft a, f_{\mathrm{col}} \triangleleft a).$$

Proposition 8.2. *Let $X = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$ be a finitely presented quandle, and let $\rho : X \to Q$ be a quandle representation. Let (f_1, f_2) and (g_1, g_2) be Alexander pairs of maps $f_1, f_2, g_1, g_2 : Q \times Q \to R$. Let $f_{\mathrm{col},1}, \ldots, f_{\mathrm{col},l} : Q \to R$ and $g_{\mathrm{col},1}, \ldots, g_{\mathrm{col},l} : Q \to R$ be column relation maps with respect to (f_1, f_2) and (g_1, g_2), respectively. Suppose $(f_1, f_2, \boldsymbol{f}_{\mathbf{col}}) \sim_h (g_1, g_2, \boldsymbol{g}_{\mathbf{col}})$.*

- *If R is an integral domain and $R_{\mathrm{col}}(X, \rho; \boldsymbol{f}_{\mathbf{col}})$ is regular, then we have*

$$E(A(X, \rho; f_1, f_2), R_{\mathrm{col}}(X, \rho; \boldsymbol{f}_{\mathbf{col}})) = E(A(X, \rho; g_1, g_2), R_{\mathrm{col}}(X, \rho; \boldsymbol{g}_{\mathbf{col}})),$$
$$\Delta(A(X, \rho; f_1, f_2), R_{\mathrm{col}}(X, \rho; \boldsymbol{f}_{\mathbf{col}})) \doteq \Delta(A(X, \rho; g_1, g_2), R_{\mathrm{col}}(X, \rho; \boldsymbol{g}_{\mathbf{col}})).$$

- *If R is a matrix ring consisting of $k \times k$ matrices over an integral domain and $\overline{R_{\mathrm{col}}(X, \rho; \boldsymbol{f}_{\mathbf{col}})}$ is regular, then we have*

$$E(\overline{A(X, \rho; f_1, f_2)}, \overline{R_{\mathrm{col}}(X, \rho; \boldsymbol{f}_{\mathbf{col}})}) = E(\overline{A(X, \rho; g_1, g_2)}, \overline{R_{\mathrm{col}}(X, \rho; \boldsymbol{g}_{\mathbf{col}})}),$$
$$\Delta(\overline{A(X, \rho; f_1, f_2)}, \overline{R_{\mathrm{col}}(X, \rho; \boldsymbol{f}_{\mathbf{col}})}) \doteq \Delta(\overline{A(X, \rho; g_1, g_2)}, \overline{R_{\mathrm{col}}(X, \rho; \boldsymbol{g}_{\mathbf{col}})}).$$

Proof. We assume that

$$X = \langle x_1, \ldots, x_n \,|\, u_1 \triangleleft v_1 = w_1, \ldots, u_m \triangleleft v_m = w_m \rangle$$

for some $u_1, \ldots, u_m, v_1, \ldots, v_m, w_1, \ldots, w_m \in \{x_1, \ldots, x_n\}$, where we note that any finitely presented quandle can be presented in this form. By the proof of Theorem 9.3 in [5], we have

$$\mathrm{diag}(h(\rho(w_1)), \ldots, h(\rho(w_m))) A(X, \rho; f_1, f_2) = A(X, \rho; g_1, g_2)\, \mathrm{diag}(h(\rho(x_1)), \ldots, h(\rho(x_n))).$$

Since $h(\rho(x_i)) f_{\mathrm{col},j}(\rho(x_i)) = g_{\mathrm{col},j}(\rho(x_i))$, we have

$$\mathrm{diag}(h(\rho(x_1)), \ldots, h(\rho(x_n))) R_{\mathrm{col}}(X, \rho; \boldsymbol{f}_{\mathbf{col}}) = R_{\mathrm{col}}(X, \rho; \boldsymbol{g}_{\mathbf{col}}).$$

We choose $\boldsymbol{j} \in S_n(l)$ so that $\det R_{\mathrm{col}}(X, \rho; \boldsymbol{f}_{\mathbf{col}})_{\boldsymbol{j}, \bar{l}} \neq 0$. Then we have

$$\begin{aligned}
&\operatorname{diag}(h(\rho(w_1)),\dots,h(\rho(w_m)))A(X,\rho;f_1,f_2)_{\bar{m},\boldsymbol{j}^c}\\
&= A(X,\rho;g_1,g_2)_{\bar{m},\boldsymbol{j}^c}\operatorname{diag}(h(\rho(x_1)),\dots,h(\rho(x_n)))_{\boldsymbol{j}^c,\boldsymbol{j}^c},\\
&\operatorname{diag}(h(\rho(x_1)),\dots,h(\rho(x_n)))_{\boldsymbol{j},\boldsymbol{j}}R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})_{\boldsymbol{j},\bar{l}} = R_{\mathrm{col}}(X,\rho;\boldsymbol{g}_{\mathbf{col}})_{\boldsymbol{j},\bar{l}}
\end{aligned}$$

which imply

$$\begin{aligned}
&A(X,\rho;f_1,f_2)_{\bar{m},\boldsymbol{j}^c}\sim A(X,\rho;g_1,g_2)_{\bar{m},\boldsymbol{j}^c},\\
&R_{\mathrm{col}}(X,\rho;\boldsymbol{f}_{\mathbf{col}})_{\boldsymbol{j},\bar{l}}\sim R_{\mathrm{col}}(X,\rho;\boldsymbol{g}_{\mathbf{col}})_{\boldsymbol{j},\bar{l}},
\end{aligned}$$

respectively. The desired equalities follow from these equivalences. □

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