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GENERALIZED SCHRÖDINGER FORMS WITH APPLICATIONS TO MAXIMUM PRINCIPLES

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Abstract

We study criticalities of generalized Schrödinger operators in terms of the Schrödinger forms induced by generalized Feynman-Kac perturbations of symmetric Markov processes, which extend earlier work due to Takeda [26]. The related functional inequalities and analytic characterizations of criticalities of Schrödinger forms are given. As applications, we establish some maximum principles via the analytic characterization.

1. Introduction and main results

In this paper, we study subcriticality, criticality and supercriticality of the Schrödinger forms induced by generalized Feynman-Kac perturbations and characterize their properties in terms of the bottom of the spectrum relative to the Schrödinger form, with applications to some maximum principles. The notion of criticalities for Schrödinger operators has been extensively studied over the recent decades (see [21], [22], [23], [25], [27], [31] and the references therein) which is closely related to the existence of harmonic functions in the sense of Schrödinger operators.

Let E be a locally compact separable metric space and $\mathfrak{m}$ a positive Radon measure on E with full topological support. In [26], Takeda introduced a new way to define subcriticality, criticality and supercriticality of the Schrödinger form with a local perturbation

$$\mathcal{E}^{\mu}(f,f)=\mathcal{E}(f,f)-\int_E f^2\,\mathrm{d}\mu,\qquad f\in\mathcal{D}(\mathcal{E}^{\mu})(=\mathcal{D}(\mathcal{E})).$$

through Doob's h-transform, where $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a symmetric regular Dirichlet form on $L^2(E;\mathfrak{m})$ and $\mu=\mu^+-\mu^-$ is a signed Radon measure of Kato class of the symmetric Hunt process $\mathbf{X}$ associated with $(\mathcal{E},\mathcal{D}(\mathcal{E}))$. It was shown in [26] that these definitions of criticalities are well-defined by the existence of superharmonic functions and the recurrence or transience of h-transformed Dirichlet forms. Furthermore, some analytic chracterizations for these definitions were studied in terms of the bottom of the spectrum $\lambda(\mu)$ of the time-changed process of $\mathbf{X}$ by μ^+: the Schrödinger form $(\mathcal{E}^{\mu},\mathcal{D}(\mathcal{E}^{\mu}))$ is subcritical (resp. critical and supercritical) if and only if $\lambda(\mu)>1$ (resp. $=1$ and <1). These characterizations were then extended to more general cases containing non-local perturbations by Li [20]. Global

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property such as recurrence and transience of a symmetric Markov process is closely related to some functional inequalities with respect to the associated Dirichlet form. Takeda established a Poincaré-type inequality for the critical Schrödinger form by using an L^2-version of Oshima's inequality and from which he extended the dichotomy result on diffusion processes due to Pinchover and Tintarev [23] to more general symmetric Markov processes ([26, Theorem 3.2, Corollary 3.4 and Theorem 3.7]).

The purpose of this paper is to extend the previous results in [26] to the Schrödinger form induced by the so-called *generalized Feynman-Kac semigroup*, with applications to some maximum principles studied in [29], [30]. Before stating our results, let us explain the necessary notations and some known facts. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric regular Dirichlet form on $L^2(E;\mathfrak{m})$ and $\mathbf{X} = (X_t, \mathbf{P}_x, \zeta)$ the associated $\mathfrak{m}$-symmetric Hunt process on E, which is assumed to satisfy **(I)** and **(RSF)** (see Section 2 for the definitions). Denote by (N, H) be the Lévy system of $\mathbf{X}$. We may and do assume throughout the whole exposition that all considering measures and additive functionals are supposed to be in the strict sense. Let μ_1 and μ_2 be positive smooth measures corresponding to positive continuous additive functionals $A_t^{\mu_1}$ and $A_t^{\mu_2}$ of $\mathbf{X}$, respectively. Set $\mu := \mu_1 - \mu_2$. Let F_1 and F_2 be positive symmetric bounded Borel functions on $E \times E$ vanishing on the diagonal. Set $F := F_1 - F_2$. Then $A_t^F := A_t^{F_1} - A_t^{F_2}$ with $A_t^{F_i} := \sum_{0<s\le t} F_i(X_{s-}, X_s)$, $(i = 1, 2)$ is an (discontinuous) additive functional of $\mathbf{X}$ whenever it is summable. For a bounded continuous Borel function u on E locally in $\mathcal{D}(\mathcal{E})$ ($\mathcal{D}_{\rm loc}(\mathcal{E})$ in notation), let N_t^u be the continuous additive functional of zero quadratic variation appearing in the Fukushima decomposition of $u(X_t) - u(X_0)$ (see (2.3) below). Note that N_t^u is not necessarily of bounded variation in general. It is natural to consider the following generalized non-local Feynman-Kac functional by the additive functionals $A_t := N_t^u + A_t^\mu + A_t^F$ of the form

$$e_A(t) := \exp(A_t) \tag{1.1}$$

because the process $\mathbf{X}$ admits many continuous additive functionals which do not have bounded variations, and many discontinuous additive functionals. With (1.1), we define the Feynman-Kac semigroup $\{P_t^A\}_{t\ge 0}$ and resolvent $\{R_\alpha^A\}_{\alpha>0}$ by

$$P_t^A f(x) := \mathbf{E}_x\left[e_A(t) f(X_t)\right], \quad R_\alpha^A f(x) := \int_0^\infty P_t^A f(x)\,\mathrm{d}t \tag{1.2}$$

for any Borel measurable function f. Let $\mathcal{Q}$ be the symmetric quadratic form defined by

$$\mathcal{Q}(f,g) := \mathcal{E}(f,g) + \mathcal{E}(u, fg) - \mathcal{A}(f,g), \tag{1.3}$$

where

$$\mathcal{E}(u, fg) := \frac{1}{2}\int_E f\,\mathrm{d}\mu_{\langle u,g\rangle} + \frac{1}{2}\int_E g\,\mathrm{d}\mu_{\langle u,f\rangle},$$
$$\mathcal{A}(f,g) := \int_E f(x)g(x)\mu(\mathrm{d}x) + \int_E\int_E f(x)g(y)(e^{F(x,y)} - 1)N(x,\mathrm{d}y)\mu_H(\mathrm{d}x).$$

In view of Stollmann-Voigt's inequality (see (2.2) below), we can see that $\mathcal{Q}(f,g)$ is well-defined under some mild conditions on u, μ and F.

Let $\mathbf{X}^*$ be the subprocess of $\mathbf{X}$ killed by $e^{-A_t^{\mu_2} - A_t^{F_2}}$. Note that $\mathbf{X}^*$ is a transient Markov process under **(I)** of $\mathbf{X}$ provided $\mu_2 + N(F_2)\mu_H$ is non-trivial. So we always assume the transience of $\mathbf{X}$ when $\mu_2 + N(F_2)\mu_H = 0$. Let $S^1_{EK}(\mathbf{X})$ (resp. $S^1_K(\mathbf{X})$ and $S^1_D(\mathbf{X})$) denote the class of positive smooth measures of extended Kato class (resp. of Kato class and of Dynkin class) with respect to $\mathbf{X}$. In addition, under the transience of $\mathbf{X}$, we denote by $S^1_{CK_\infty}(\mathbf{X})$ the family of Green-tight measures of Kato class with respect to $\mathbf{X}$ (see Section 2 for the definitions). We make the following condition:

$$(\mathbf{A})^* \qquad \begin{cases} \mu_1 + N(e^{-F_2}(e^{F_1}-1))\mu_H \in S^1_{EK}(\mathbf{X}^*),\ \mu_{\langle u\rangle} \in S^1_K(\mathbf{X}^*), \\ \mu_2 + N(F_2)\mu_H \in S^1_D(\mathbf{X}). \end{cases}$$

Under $(\mathbf{A})^*$, the generalized Feynman-Kac semigroup $\{P^A_t\}_{t\ge0}$ forms a strongly continuous semigroup on $L^2(E;\mathfrak{m})$ associated with the Schrödinger form $(\mathcal{Q},\mathcal{D}(\mathcal{Q}))$ (see [15, Lemma 2.1]).

In Section 3, we introduce the class of positive superharmonic functions of $\{P^A_t\}_{t\ge0}$,

$$\mathcal{H}^A_+ := \left\{h \in \mathcal{D}_{\rm loc}(\mathcal{E}) \cap C(E) \mid h > 0 \text{ and } P^A_t h \le h\right\}.$$

Suppose $\mathcal{H}^A_+ \neq \emptyset$ and take $h \in \mathcal{H}^A_+$. Then the h-transformed semigroup $\{P^{A,h}_t\}_{t\ge0}$ of $\{P^A_t\}_{t\ge0}$ defined by $P^{A,h}_t f(x) := (1/h(x))P^A_t(fh)(x)$ is naturally Markovian and its associated quadratic form $(\mathcal{Q}^h,\mathcal{D}(\mathcal{Q}^h))$ is given by

$$\begin{cases} \mathcal{D}(\mathcal{Q}^h) := \{f \in L^2(E;h^2\mathfrak{m}) \mid fh \in \mathcal{D}(\mathcal{Q})\}, \\ \mathcal{Q}^h(f,g) := \mathcal{Q}(fh,gh), \quad f,g \in \mathcal{D}(\mathcal{Q}^h). \end{cases}$$

Note that $(\mathcal{Q}^h,\mathcal{D}(\mathcal{Q}^h))$ is to be a regular Dirichlet form on $L^2(E;h^2\mathfrak{m})$ (Lemma 3.1). We make the following definition for the subcriticality, criticality and supercriticality of the Schrödinger form $(\mathcal{Q},\mathcal{D}(\mathcal{Q}))$ in terms of the transience or recurrence of $(\mathcal{Q}^h,\mathcal{D}(\mathcal{Q}^h))$, and the emptyness of $\mathcal{H}^A_+$.

Definition 1.1. The Schrödinger form $(\mathcal{Q},\mathcal{D}(\mathcal{Q}))$ is said to be

(1) *subcritical* if $\mathcal{H}^A_+ \neq \emptyset$ and $(\mathcal{Q}^h,\mathcal{D}(\mathcal{Q}^h))$ is transient for some $h \in \mathcal{H}^A_+$.
(2) *critical* if $\mathcal{H}^A_+ \neq \emptyset$ and $(\mathcal{Q}^h,\mathcal{D}(\mathcal{Q}^h))$ is recurrent for some $h \in \mathcal{H}^A_+$.
(3) *supercritical* if $\mathcal{H}^A_+ = \emptyset$.

We show that these classifications for criticalities of $(\mathcal{Q},\mathcal{D}(\mathcal{Q}))$ are well-defined (Proposition 3.1). Furthermore, we will prove the following functional inequalities relative to the subcriticality and criticality for $(\mathcal{Q},\mathcal{D}(\mathcal{Q}))$ which extend the previous results in [23], [26]. Let $\mathcal{D}_e(\mathcal{Q})$ be the extended Schödinger space of $(\mathcal{Q},\mathcal{D}(\mathcal{Q}))$ (see Section 3).

Theorem 1.1. *Suppose that $\mathcal{H}^A_+ \neq \emptyset$ and $(\mathbf{A})^*$. Then the following dichotomy holds:*

(1) *There exists a strictly positive measurable function g on E such that*

$$\int_E f^2 g\,d\mathfrak{m} \le \mathcal{Q}(f,f), \quad f \in \mathcal{D}_e(\mathcal{Q}).$$

(2) *For any $\varphi \in \mathcal{B}_b(E)$ with compact support on E and $h \in \mathcal{H}^A_+$ satisfying $\int_E \varphi h\,d\mathfrak{m} \neq 0$, there exists a strictly postive function $g \in L^1(E;\mathfrak{m})$ and a constant $C > 0$ such that*

$$\frac{1}{C}\int_E f^2 g\,d\mathfrak{m} \le \mathcal{Q}(f,f) + C\left(\int_E f\varphi\,d\mathfrak{m}\right)^2, \quad f \in \mathcal{D}_e(\mathcal{Q}).$$

Define the bottom of the spectrum of the Schrödinger form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ by

$$\lambda^{\mathcal{Q}}(\eta) := \inf\left\{\mathcal{Q}(f,f) \,\middle|\, f \in \mathcal{D}(\mathcal{Q}), \int_E f^2 \mathrm{d}\eta = 1\right\}, \tag{1.4}$$

where $\eta := \eta_{u,\mu,F} = \mu_1 + N(e^{-F_2}(e^{U+F_1} - U - 1))\mu_H + \frac{1}{2}\mu^c_{\langle u\rangle}$ with $U(x,y) := u(x) - u(y)$.

In Section 4, we give analytic characterizations for the criticality and subcriticality of $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ in terms of the bottom of the spectrum (1.4) by proving the existence of the superharmonic functions in each cases. The main result is the following:

Theorem 1.2. *Assume that* $\mu_1 + N(F_1)\mu_H + \mu_{\langle u\rangle} \in S^1_{CK_\infty}(\mathbf{X}^*)$ *and* $\mu_2 + N(F_2)\mu_H \in S^1_K(\mathbf{X})$. *Then the quadratic form* $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ *is*

(1) *subcritical, if and only if* $\lambda^{\mathcal{Q}}(\eta) > 0$.
(2) *critical, if and only if* $\lambda^{\mathcal{Q}}(\eta) = 0$.
(3) *supercritical, if and only if* $\lambda^{\mathcal{Q}}(\eta) < 0$.

In a series of papers [28], [29], [30], the author obtained a necessary and sufficient condition for several maximum principles for Schrödinger operators with local perturbations. In Section 5, we partially extend the results in [29], [30] to the case of generalized Feynman-Kac perturbation by applying the criticalities of generalized Schrödinger forms obtained in the previous section. In particular, we prove that the analytic characterzation for the subcriticality of $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is equivalent to several maximum principles (Theorem 5.2 and Corollary 5.1).

Throughout this paper, we use the following notations: For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \min\{a, b\}$. We denote by $\mathcal{B}_b(E)$ (resp. $C_b(E)$) the space of bounded Borel functions (resp. the space of bounded continuous functions) on E.

2. Preliminaries

Let E be a locally compact separable metric space and $\mathfrak{m}$ a positive Radon measure on E with full topological support. Let ∂ be a point added to E so that $E_\partial := E \cup \{\partial\}$ is the one-point compactification of E. The point ∂ also serves as the cemetery point for E. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric regular Dirichlet form on $L^2(E; \mathfrak{m})$. A function f on E is said to be *locally in* $\mathcal{D}(\mathcal{E})$ (denoted as $f \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E})$) if for any relatively compact open set G of E there exists an element $f_G \in \mathcal{D}(\mathcal{E})$ such that $f = f_G$ $\mathfrak{m}$-a.e. on G. Let $\mathcal{D}_e(\mathcal{E})$ be the family of $\mathfrak{m}$-measurable functions f on E such that $|f| < \infty$ $\mathfrak{m}$-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{f_n\}_{n=1}^\infty$ of functions in $\mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty} f_n = f$ $\mathfrak{m}$-a.e.. We call $\mathcal{D}_e(\mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Let $\mathbf{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbf{P}_x, \zeta)$ be the $\mathfrak{m}$-symmetric Hunt process on E generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Here $\mathcal{F}$ and $\mathcal{F}_t$ are the minimal (augmented) admissible filtration of $\mathbf{X}$ and ζ is the lifetime of $\mathbf{X}$, $\zeta := \inf\{t > 0 \mid X_t = \partial\}$. Here and in the sequel, unless mentioned otherwise, we use the convention that a function defined on E takes the value 0 at ∂. Let denote by $\{P_t\}_{t\geq 0}$ and $\{R_\alpha\}_{\alpha\geq 0}$ the transition semigroup and the resolvent of $\mathbf{X}$ which are defined by

$$P_t f(x) = \mathbf{E}_x[f(X_t)], \qquad R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) \mathrm{d}t.$$

Throughout this paper, we always assume that $\mathbf{X}$ satisfies the following properties:

(I) (Irreducibility): If a Borel set $B \subset E$ is P_t-invariant, that is, $P_t(\mathbf{1}_B f) = \mathbf{1}_B P_t f$ $\mathfrak{m}$-a.e. for any $f \in L^2(E;\mathfrak{m}) \cap \mathcal{B}_b(E)$ and $t > 0$, then B satisfies either $\mathfrak{m}(B) = 0$ or $\mathfrak{m}(B^c) = 0$.

(RSF) (Resolvent Strong Feller Property) : $R_\alpha(\mathcal{B}_b(E)) \subset C_b(E)$ for any/some $\alpha > 0$.

The process $\mathbf{X}$ is said to satisfy the *absolute continuity condition* ((**AC**) in abbreviation) if the transition kernel $p_t(x, \mathrm{d}y)$ of $\mathbf{X}$ is absolutely continuous with respect to $\mathfrak{m}$, that is, $p_t(x, \mathrm{d}y) = p_t(x, y)\mathfrak{m}(\mathrm{d}y)$ for any $x \in E$ and $t > 0$. Under **(AC)**, there exists a non-negative jointly measurable α-order resolvent kernel $R_\alpha(x, y)$ defined for all $x, y \in E$ and $\alpha > 0$. Moreover, $R_\alpha(x, y)$ is α-excessive in $x \in E$ and in $y \in E$ (see Lemma 4.2.4 in [10]). Since $\alpha \mapsto R_\alpha(x, y)$ is decreasing for each $x, y \in E$, one can define 0-order resolvent kernel $R_0(x, y) := \lim_{\alpha\to 0} R_\alpha(x, y)$ provided $\mathbf{X}$ is transient. We simply write $R(x, y)$ for $R_0(x, y)$ and call the Green function of $\mathbf{X}$. For a non-negative Borel measure ν, we write

$$R_\alpha\nu(x) := \int_E R_\alpha(x, y)\nu(\mathrm{d}y)$$

and $R\nu(x) := R_0\nu(x)$. In particular, $R_\alpha f(x) := R_\alpha(f\mathfrak{m})(x)$ for any $f \in \mathcal{B}_b(E)$. We remark that **(RSF)** implies **(AC)**.

A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E;\mathfrak{m})$ is said to be *transient* if there exists a bounded $\mathfrak{m}$-integrable function g strictly positive $\mathfrak{m}$-a.e. on E such that

$$\int_E |f| g \,\mathrm{d}\mathfrak{m} \le \sqrt{\mathcal{E}(f, f)}, \quad \text{for any } f \in \mathcal{D}(\mathcal{E})$$

([10, (1.5.6)]). A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E;\mathfrak{m})$ is said to be *recurrent* if the constant function 1 belongs to $\mathcal{D}_e(\mathcal{E})$ and $\mathcal{E}(1, 1) = 0$ ([10, Theorem 1.6.3(iii)]).

A positive Radon measure ν on E is of *positive-order finite energy integral* if

$$\int_E R_1\nu(x)\nu(\mathrm{d}x) := \iint_{E\times E} R_1(x, y)\nu(\mathrm{d}x)\nu(\mathrm{d}y) < \infty. \tag{2.1}$$

We denote by $S_0(\mathbf{X})$ the set of all positive Radon measure of positive-order finite energy integral ([10, (2.2.1)]). Let $S_{00}(\mathbf{X})$ be the set of $\nu \in S_0(\mathbf{X})$ such that $\nu(E) < \infty$ and $\sup_{x\in E} R_1\nu(x) < \infty$. A positive Radon measure ν on E of 0-*order finite energy integral* ($\nu \in S_0^{(0)}(\mathbf{X})$ in notation) can be similarly defined by the validity of (2.1) with $R(x, y)$ instead of $R_1(x, y)$ under the transience of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Similarly, we also define $S_{00}^{(0)}(\mathbf{X})$ with $R(x, y)$ instead of $R_1(x, y)$.

An increasing sequence $\{K_n\}_{n=1}^\infty$ of compact sets is called a generalized compact nest if for any compact set K, $\lim_{n\to\infty} \mathrm{Cap}(K \setminus K_n) = 0$. We call a non-negative Borel measure ν on E *smooth* if there exists a generalized compact nest $\{K_n\}_{n=1}^\infty$ such that $\nu(E \setminus \bigcup_{n=1}^\infty K_n) = 0$ and $\mathbf{1}_{K_n}\cdot\nu \in S_{00}(\mathbf{X})$ (cf. [10, Theorem 2.2.4]). We denote by $S(\mathbf{X})$ the family of all smooth measures. A non-negative Borel measure ν on E is said to be *smooth in the strict sense* if there exists a sequence $\{E_n\}_{n=1}^\infty$ of Borel sets increasing to E such that $\mathbf{1}_{E_n}\cdot\nu \in S_{00}(\mathbf{X})$ for each n and $\mathbf{P}_x(\lim_{n\to\infty} \sigma_{E\setminus E_n} \ge \zeta) = 1$ for any $x \in E$, where $\sigma_{E\setminus E_n} = \inf\{t > 0 \mid X_t \in E \setminus E_n\}$. We denote by $S_1(\mathbf{X})$ the totality of the smooth measures in the strict sense ([10, Section 5.1]).

A measure $\nu \in S_1(\mathbf{X})$ is said to be of *Dynkin class* (resp. of *Green-bounded*) of $\mathbf{X}$ if for some $\alpha > 0$, $\sup_{x\in E} R_\alpha\nu(x) < \infty$ (resp. $\sup_{x\in E} R\nu(x) < \infty$ under the transience of $\mathbf{X}$). We denote by $S^1_D(\mathbf{X})$ (resp. $S^1_{D_0}(\mathbf{X})$) the family of measures of Dynkin class (resp. of Green-bounded). Note that every measure $\nu \in S^1_D(\mathbf{X})$ is a Radon measure on E on account of the regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and the Stollmann-Voigt inequality ([24, Theorem 3.1]): For $\nu \in S^1_D(\mathbf{X})$ and $\alpha > 0$,

$$\int_E f^2 \mathrm{d}\nu \leq \|R_\alpha \nu\|_\infty \mathcal{E}_\alpha(f, f), \quad f \in \mathcal{D}(\mathcal{E}), \tag{2.2}$$

where $\mathcal{E}_\alpha(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + (\cdot, \cdot)_m$. This inequality also holds for $f \in \mathcal{D}_e(\mathcal{E}), \nu \in S^1_{D_0}(\mathbf{X})$ and $\alpha = 0$.

A measure $\nu \in S_1(\mathbf{X})$ is said to be in the *Kato class* (resp. *extended Kato class*) with respect to $\mathbf{X}$ if $\lim_{\alpha\to\infty} \sup_{x\in E} R_\alpha \nu(x) = 0$ (resp. $\lim_{\alpha\to\infty} \sup_{x\in E} R_\alpha \nu(x) < 1$) and denote by $S^1_K(\mathbf{X})$ (resp. $S^1_{EK}(\mathbf{X})$) the family of measures of Kato class (resp. of extended Kato class). Clearly, $S^1_K(\mathbf{X}) \subset S^1_{EK}(\mathbf{X}) \subset S^1_D(\mathbf{X})$ and $S^1_{D_0}(\mathbf{X}) \subset S^1_D(\mathbf{X})$.

Now we recall the notion of Green-tight measures of Kato class ([2, Definition 2.2]).

DEFINITION 2.1. Suppose that $\mathbf{X}$ is transient. A measure $\nu \in S_1(\mathbf{X})$ is said to be a *Green-tight measure of Kato class* with respect to $\mathbf{X}$ if for any $\varepsilon > 0$ there exists a Borel subset $K = K(\varepsilon)$ of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all ν-measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{x\in E} R(\mathbf{1}_{B\cup K^c}\nu)(x) = \sup_{x\in E} \int_{B\cup K^c} R(x, y)\, \nu(\mathrm{d}y) < \varepsilon.$$

We denote by $S^1_{CK_\infty}(\mathbf{X})$ the family of Green-tight measures of Kato class with respect to $\mathbf{X}$. It is known in [2, Proposition 2.2] that $S^1_{CK_\infty}(\mathbf{X}) \subset S^1_{D_0}(\mathbf{X})$ and $S^1_{CK_\infty}(\mathbf{X}) \subset S^1_K(\mathbf{X})$.

We say that a positive continuous additive functional (PCAF in abbreviation) in the strict sense A^ν_t of $\mathbf{X}$ and a measure $\nu \in S_1(\mathbf{X})$ are in the Revuz correspondence if they satisfy for any $t > 0$, $f \in \mathcal{B}_b(E)$,

$$\int_E f(x)\nu(\mathrm{d}x) =\uparrow \lim_{t\downarrow 0} \frac{1}{t}\mathbf{E}_m\left[\int_0^t f(X_s)\, \mathrm{d}A^\nu_s\right].$$

It is known that the family of equivalence classes of the set of PCAFs in the strict sense and the family of positive measures belonging to $S_1(\mathbf{X})$ are in one to one correspondence under the Revuz correspondence ([10, Theorem 5.1.4]).

Let $(N(x, \mathrm{d}y), H_t)$ be a Lévy system for $\mathbf{X}$, that is, $N(x, \mathrm{d}y)$ is a kernel on $(E, \mathcal{B}(E))$ and H_t is a PCAF with bounded 1-potential such that for any nonnegative Borel function ϕ on $E \times E$ vanishing on the diagonal and any $x \in E$,

$$\mathbf{E}_x\left[\sum_{s\leq t} \phi(X_{s-}, X_s)\right] = \mathbf{E}_x\left[\int_0^t \int_E \phi(X_s, y)N(X_s, \mathrm{d}y)\, \mathrm{d}H_s\right].$$

To simplify notation, we will write

$$N\phi(x) := \int_E \phi(x, y)N(x, \mathrm{d}y).$$

Let μ_H be the Revuz measure of the PCAF H_t. Then the jumping measure J and the killing measure κ of $\mathbf{X}$ are given by

$$J(\mathrm{d}x\mathrm{d}y) = \frac{1}{2}N(x, \mathrm{d}y)\mu_H(\mathrm{d}x) \quad \text{and} \quad \kappa(\mathrm{d}x) = N(x, \{\partial\})\mu_H(\mathrm{d}x).$$

These measures feature in the Beurling-Deny decomposition of $\mathcal{E}$: for $f, g \in \mathcal{D}_e(\mathcal{E})$,

$$\mathcal{E}(f,g) = \mathcal{E}^c(f,g) + \int_{E\times E}(f(x)-f(y))(g(x)-g(y))J(\mathrm{d}x\mathrm{d}y) + \int_E f(x)g(x)\kappa(\mathrm{d}x),$$

where $\mathcal{E}^c$ is the strongly local part of $\mathcal{E}$.

Lemma 2.1. *The condition $\mu_2 + N(F_2)\mu_H \in S^1_D(\mathbf{X})$ always implies $\mu_2 + N(F_2)\mu_H \in S^1_{D_0}(\mathbf{X}^*)$.*

Proof. For notational convenience, let $A_t^{N(F_2)\mu_H} := \int_0^t N(F_2)(X_s)\mathrm{d}H_s$. Note that $M_t^{F_2} := A_t^{F_2} - A_t^{N(F_2)\mu_H}$ is a martingale additive functional under $N(F_2)\mu_H \in S^1_D(\mathbf{X})$. Denote by R^* the 0-order resolvent of $\mathbf{X}^*$. Then

$$\begin{aligned} R^*(\mu_2 + N(F_2)\mu_H)(x) &= \mathbf{E}_x\left[\int_0^\infty e^{-A_t^{\mu_2}-A_t^{F_2}}\mathrm{d}\left(A_t^{\mu_2} + A_t^{N(F_2)\mu_H}\right)\right] \\ &\le \mathbf{E}_x\left[\int_0^\infty e^{-A_t^{\mu_2}}\mathrm{d}A_t^{\mu_2}\right] + \mathbf{E}_x\left[\int_0^\infty e^{-A_t^{F_2}}\mathrm{d}A_t^{N(F_2)\mu_H}\right] \\ &= 1 - \mathbf{E}_x\left[e^{-A_\zeta^{\mu_2}}\right] + \mathbf{E}_x\left[\int_0^\infty e^{-A_t^{F_2}}\mathrm{d}A_t^{N(F_2)\mu_H}\right]. \end{aligned}$$

Since the third term of the last sentence is estimated by

$$\begin{aligned} \mathbf{E}_x\left[\int_0^\infty e^{-A_t^{F_2}}\mathrm{d}A_t^{N(F_2)\mu_H}\right] &= \lim_{t\to\infty}\mathbf{E}_x\left[\int_0^t e^{-A_s^{F_2}}\mathrm{d}A_s^{N(F_2)\mu_H}\right] = \lim_{t\to\infty}\mathbf{E}_x\left[\int_0^t e^{-A_s^{F_2}}\mathrm{d}A_s^{F_2}\right] \\ &= \lim_{t\to\infty}\mathbf{E}_x\left[\sum_{s\le t} e^{-A_s^{F_2}}F_2(X_{s-}, X_s)\right] \\ &\le \lim_{t\to\infty}\mathbf{E}_x\left[\sum_{s\le t} e^{-A_s^{F_2}}\left(e^{F_2(X_{s-},X_s)} - 1\right)\right] \\ &\le \lim_{t\to\infty}\mathbf{E}_x\left[\sum_{s\le t} -\Delta\left(e^{-A_s^{F_2}}\right)\right] \\ &= \lim_{t\to\infty}\mathbf{E}_x\left[1 - e^{-A_t^{F_2}}\right] = 1 - \mathbf{E}_x\left[e^{-A_\zeta^{F_2}}\right] \le 1, \end{aligned}$$

we see that $\sup_{x\in E} R^*(\mu_2 + N(F_2)\mu_H)(x) \le 2 < \infty$. □

Let us consider a bounded Borel function $u \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \cap C(E)$ satisfying $\mu_{\langle u\rangle} \in S^1_D(\mathbf{X})$. In [16, Theorem 6.2(2)], we proved that the additive functional $u(X_t) - u(X_0)$ admits the following strict decomposition:

(2.3) $$u(X_t) - u(X_0) = M_t^u + N_t^u \quad t \in [0, \zeta[\quad \mathbf{P}_x\text{-a.s. for all } x \in E,$$

where M_t^u is a square integrable martingale additive functional in the strict sense on the random time interval $[\![0,\zeta]\!]$ (see [4] for the definition) and N_t^u is a continuous additive functional (CAF in abbreviation) in the strict sense which is locally of zero energy. M_t^u can be decomposed as

$$M_t^u = M_t^{u,c} + M_t^{u,j} + M_t^{u,\kappa}, \tag{2.4}$$

where $M_t^{u,j}$, $M_t^{u,\kappa}$ and $M_t^{u,c}$ are the jumping, killing and continuous part of M_t^u respectively. Those are defined $\mathbf{P}_x$-a.s. for all $x \in E$ by [16, Theorem 6.2(2)]. Let $\mu_{\langle u \rangle}$, $\mu^c_{\langle u \rangle}$, $\mu^j_{\langle u \rangle}$ and $\mu^\kappa_{\langle u \rangle}$ be the smooth Revuz measures associated with the quadratic variational processes (or the sharp bracket PCAFs in the strict sense) $\langle M^u \rangle_t$, $\langle M^{u,c} \rangle_t$, $\langle M^{u,j} \rangle_t$ and $\langle M^{u,\kappa} \rangle_t$ respectively. Then

$$\mu_{\langle u \rangle}(\mathrm{d}x) = \mu^c_{\langle u \rangle}(\mathrm{d}x) + \mu^j_{\langle u \rangle}(\mathrm{d}x) + \mu^\kappa_{\langle u \rangle}(\mathrm{d}x).$$

Note that $\mathcal{E}(f,f) = \frac{1}{2}\nu_{\langle f \rangle}(E)$ with $\nu_{\langle f \rangle} := \mu^c_{\langle f \rangle} + \mu^j_{\langle f \rangle} + 2\mu^\kappa_{\langle f \rangle}$ provided $f \in \mathcal{D}_e(\mathcal{E})$.

3. Schrödinger forms and their related inequalities

Let us introduce a class of superharmonic functions:

$$\mathcal{H}_+^A := \left\{ h \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \cap C(E) \mid h > 0 \text{ and } P_t^A h \le h \right\}.$$

Assume that $\mathcal{H}_+^A \neq \emptyset$. For $h \in \mathcal{H}_+^A$, define the quadratic form $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ on $L^2(E; h^2\mathfrak{m})$ by

$$\begin{cases} \mathcal{D}(\mathcal{Q}^h) := \{ f \in L^2(E; h^2\mathfrak{m}) \mid fh \in \mathcal{D}(\mathcal{Q}) \}, \\ \mathcal{Q}^h(f,g) := \mathcal{Q}(fh, gh), \quad f,g \in \mathcal{D}(\mathcal{Q}^h). \end{cases} \tag{3.1}$$

By a similar way of [26, Lemmas 2.3, 2.4 and 2.5], we can see that $\mathcal{D}(\mathcal{E}) \cap C_0(E) = \mathcal{D}(\mathcal{Q}) \cap C_0(E) = \mathcal{D}(\mathcal{Q}^h) \cap C_0(E)$ provided the condition $(\mathbf{A})^*$. Here $C_0(E)$ denotes the space of continuous functions on E with compact support.

Lemma 3.1. *Assume that* $(\mathbf{A})^*$ *and* $\mathcal{H}_+^A \neq \emptyset$. *For any* $h \in \mathcal{H}_+^A$, *the quadratic form* $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ *is a regular symmetric Dirichlet form on* $L^2(E; h^2\mathfrak{m})$.

Proof. It is enough to show the regularity of $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$. We see by [15, Lemma 2.1] that there exists $C > 0$ such that

$$C^{-1}\mathcal{E}_1(f,f) \le \mathcal{Q}_\alpha(f,f) \le C\mathcal{E}_1(f,f), \quad f \in \mathcal{D}(\mathcal{E})$$

for some large $\alpha > 0$ and the generalized Feynman-Kac semigroup $\{P_t^A\}_{t \ge 0}$ forms a strongly continuous semigroup on $L^2(E; \mathfrak{m})$. From this fact and (3.1), the required regularity can be easily derived under $\mathcal{H}_+^A \neq \emptyset$. □

Let $\mathbf{X}^h$ be the Hunt process generated by $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$. Clearly, $\mathbf{X}^h$ is $h^2\mathfrak{m}$-symmetric and its semigroup $\{P_t^{A,h}\}_{t \ge 0}$ and resolvent $\{R_\alpha^{A,h}\}_{\alpha > 0}$ are given by

$$P_t^{A,h} f(x) := \mathbf{E}_x\left[L_t^{A,h} f(X_t) \right], \quad R_\alpha^{A,h} f(x) = \int_0^\infty P_t^{A,h} f(x)\,\mathrm{d}t,$$

where

$$L_t^{A,h} := \frac{h(X_t)}{h(X_0)} e_A(t) \tag{3.2}$$

and $e_A(t)$ is the generalized Feynman-Kac functional given by (1.1). Note that $\mathbf{X}^h$ also satisfies $(\mathbf{I})$ because of the positivity of $L_t^{A,h}$ and $(\mathbf{I})$ of $\mathbf{X}$.

Let $\mathcal{D}_e(\mathcal{Q}^h)$ be the extended Dirichlet space of the regular Dirichlet form $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$. Using $\mathcal{D}_e(\mathcal{Q}^h)$, we define the *extended Schrödinger space* $\mathcal{D}_e(\mathcal{Q})$ by

$$\mathcal{D}_e(\mathcal{Q}) := \left\{ f \;\middle|\; f/h \in \mathcal{D}_e(\mathcal{Q}^h) \right\}.$$

We also give another definition of the extended Schrödinger space $\widetilde{\mathcal{D}}_e(\mathcal{Q})$, the family of $\mathfrak{m}$-measurable functions f on E such that $|f| < \infty$ $\mathfrak{m}$-a.e. and there exists a $\mathcal{Q}$-Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $\mathcal{D}(\mathcal{Q})$ such that $\lim_{n\to\infty} f_n = f$ $\mathfrak{m}$-a.e.. Note that $\mathcal{Q}$ is a non-negative definite quadratic form on $L^2(E;\mathfrak{m})$ under the (sub)criticality. For $f \in \widetilde{\mathcal{D}}_e(\mathcal{Q})$ and the sequence $\{f_n\}_{n=1}^{\infty}$, define $\widetilde{\mathcal{Q}}(f,f) = \lim_{n\to\infty} \mathcal{Q}(f_n, f_n)$. Then we see that $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{D}}_e(\mathcal{Q}))$ is well-defined. Moreover, by the same way as in the proof of [26, Lemma 2.8], we can show that $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{D}}_e(\mathcal{Q})) = (\mathcal{Q}, \mathcal{D}_e(\mathcal{Q}))$.

Now, the following proposition guarantees that Definition 1.1 is well-defined.

Proposition 3.1. *Assume* $(\mathbf{A})^*$ *and* $\mathcal{H}_+^A \neq \emptyset$. *If* $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ *is transient (resp. recurrent) for some* $h \in \mathcal{H}_+^A$, *then* $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ *is transient (resp. recurrent) for any* $h \in \mathcal{H}_+^A$.

Proof. The proof can be similarly deduced by [26, Lemmas 4.1 and 4.2]. We address here the proof for reader's convenience. Note that since $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ satisfies $(\mathbf{I})$, it is either to be transient or recurrent. For h_1 and h_2 in $\mathcal{H}_+^A$, let us suppose that $(\mathcal{Q}^{h_1}, \mathcal{D}(\mathcal{Q}^{h_1}))$ and $(\mathcal{Q}^{h_2}, \mathcal{D}(\mathcal{Q}^{h_2}))$ are transient and recurrent, respectively. Then, it follows from the transience of $(\mathcal{Q}^{h_1}, \mathcal{D}(\mathcal{Q}^{h_1}))$ and [11, Proposition 2.2] that there exists a bounded measurable function $g > 0$ $\mathfrak{m}$-a.e. such that $\|R^{A,h_1} g\|_\infty \leq 1$ and

$$\int_E f^2 g\, {h_1}^2 \mathrm{d}\mathfrak{m} \leq \|R^{A,h_1} g\|_\infty \mathcal{Q}^{h_1}(f,f) \leq \mathcal{Q}^{h_1}(f,f), \quad \text{for any } f \in \mathcal{D}_e(\mathcal{Q}^{h_1}).$$

By the definition (3.1), this is equivalent to

$$\int_E f^2 g \,\mathrm{d}\mathfrak{m} \leq \mathcal{Q}(f,f), \quad \text{for any } f \in \mathcal{D}_e(\mathcal{Q}). \tag{3.3}$$

On the other hand, the recurrence of $(\mathcal{Q}^{h_2}, \mathcal{D}(\mathcal{Q}^{h_2}))$ implies that $1 \in \mathcal{D}_e(\mathcal{Q}^{h_2})$ and $\mathcal{Q}^{h_2}(1,1) = 0$, equivalently, $h_2 \in \mathcal{D}_e(\mathcal{Q})$ and $\mathcal{Q}(h_2, h_2) = 0$. Applying this fact to (3.3), we can conclude that $h_2 = 0$ $\mathfrak{m}$-a.e., which is contradictory. □

We say that an $\mathfrak{m}$-symmetric Hunt process $\mathbf{X} = (X_t, \mathbf{P}_x)$ is *Harris recurrent* if $\mathfrak{m}(B) > 0$, then $\mathbf{P}_x(\int_0^\infty \mathbf{1}_B(X_t)\,\mathrm{d}t = \infty) = 1$ for any $x \in E$ and $B \in \mathcal{B}(E)$. Denote by $\mathcal{B}_{b,0}(E)$ (resp. $\mathcal{B}_{b,0}^+(E)$) the set of bounded (resp. positive bounded) Borel functions on E with compact support.

Proposition 3.2. *If* $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ *is critical, equivalently,* $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ *is recurrent for some* $h \in \mathcal{H}_+^A$, *then there exist a strictly positive function* $g \in L^1(E;\mathfrak{m})$ *and a constant* $C > 0$ *such that*

$$\frac{1}{C}\int_E f^2 g\,\mathrm{d}\mathfrak{m} \leq \mathcal{Q}(f,f) + C\left(\int_E f\varphi\,\mathrm{d}\mathfrak{m}\right)^2, \quad f \in \mathcal{D}_e(\mathcal{Q}) \tag{3.4}$$

for any $\varphi \in \mathcal{B}_{b,0}^+(E)$ *satisfying* $\int_E \varphi h\,\mathrm{d}\mathfrak{m} \neq 0$.

To prove Proposition 3.2, we need several lemmas:

Lemma 3.2. *For any compact set* $K \subset E$ *and* $\beta > 0$, *we have*

$$\inf_{x,y\in K} R_\beta(x,y) > 0.$$

Proof. First we note that for any $\beta > 0$ and $x, y \in E$, $R_\beta(x,y) > 0$ by [13, Lemma 6.1]. For each fixed $y \in E$, $x \mapsto R_\beta(x,y)$ is β-excessive, hence it is lower semi continuous. Thus for any $y \in E$,

$$f_K(y) := \inf_{x\in K} R_\beta(x,y) = \min_{x\in K} R_\beta(x,y) > 0$$

holds. It is easy to see that for each $z \in E$, $\alpha R_{\alpha+\beta} f_K(z) \le f_K(z)$ so that

$$\overline{f}_K(z) := \lim_{\alpha\to\infty} \alpha R_{\alpha+\beta} f_K(z) \le f_K(z).$$

Then we have $\overline{f}_K(z) = \lim_{\alpha\to\infty} \alpha R_{\alpha+\beta}\overline{f}_K(z)$, that is, $\overline{f}_K$ is β-excessive, hence it is lower semi continuous under **(RSF)** of $\mathbf{X}$. From this, we obtain the conclusion

$$\inf_{z\in K} f_K(z) = \inf_{x,z\in K} R_\beta(x,z) > 0.$$

Indeed, since $\overline{f}_K \le f_K$ on E, $f_K(z) = 0$ for $z \in K$ implies $\overline{f}_K(z) = 0$ so that $R_{\alpha+\beta} f_K(z) = 0$ for any $\alpha > 0$. Thus we get $\int_E f_K(y) R_{\alpha+\beta}(z,y)\mathfrak{m}(dy) = 0$ and hence $f_K(y) = 0$ m-a.e. $y \in E$. This contradicts to $f_K(y) > 0$ for all $y \in E$. □

Lemma 3.3. *Let $\check{R}_1(x,y)$ be the 1-resolvent density of the time changed process of $\mathbf{X}$ by the PCAF $\int_0^t \varphi(X_s)\mathrm{d}s$, $\varphi \in \mathcal{B}^+_{b,0}(E)$. Then*

$$\inf_{x,y\in F} \check{R}_1(x,y) > 0,$$

where F is the fine support of $\varphi \cdot \mathfrak{m}$.

Proof. Let K be the compact support of $\varphi \cdot \mathfrak{m}$. Then $F \subset K$ and the proof is quite similar as in the proof of [26, Lemma 3.3] with Lemma 3.2. □

Lemma 3.4. *Assume* $(\mathbf{A})^*$ *and $\mathcal{H}^A_+ \neq \emptyset$. For any $h \in \mathcal{H}^A_+$, the process $\mathbf{X}^h$ satisfies* **(RSF)**. *In particular, if $\mathbf{X}^h$ (or $(Q^h, \mathcal{D}(Q^h))$) is recurrent, then it is Harris recurrent.*

Proof. By a similar way as in the proof of [5, Lemma 3.2], one can show that

$$\lim_{t\to 0} \sup_{x\in E} \mathbf{E}_x\left[\left|L^{A,h}_t - 1\right|\right] = 0. \tag{3.5}$$

Then for any $f \in \mathcal{B}_b(E)$ and $\alpha, \beta > 0$,

$$\begin{aligned}
&\left\|R^{A,h}_\alpha f - \beta R_\beta R^{A,h}_\alpha f\right\|_\infty \\
&\le \left\|R^{A,h}_\alpha f - \beta R^{A,h}_\beta R^{A,h}_\alpha f\right\|_\infty + \beta\left\|R^{A,h}_\beta R^{A,h}_\alpha f - R_\beta R^{A,h}_\alpha f\right\|_\infty \\
&\le \left\|R^{A,h}_\beta f - \alpha R^{A,h}_\beta R^{A,h}_\alpha f\right\|_\infty + \beta\left\|R^{A,h}_\alpha f\right\|_\infty \int_0^\infty e^{-\beta t}\sup_{x\in E}\mathbf{E}_x\left[\left|L^{A,h}_t - 1\right|\right]\mathrm{d}t \\
&\le \beta^{-1}\|f\|_\infty + \left\|R^{A,h}_\alpha f\right\|_\infty \int_0^\infty e^{-t}\sup_{x\in E}\mathbf{E}_x\left[\left|L^{A,h}_{t/\beta} - 1\right|\right]\mathrm{d}t.
\end{aligned} \tag{3.6}$$

The last term in the right hand side above goes to 0 as $\beta \to \infty$ in view of (3.5) and the dominated convergence theorem. Hence we have $R^{A,h}_\alpha f \in C_b(E)$ because so is $\beta R_\beta R^{A,h}_\alpha f$, which tells us (**RSF**) of $\mathbf{X}^h$. The last assertion follows from the fact that any recurrent symmetric Markov process satisfying (**I**) and (**RSF**) is Harris recurrent ([10, Lemma 4.8.1]). □

Proof of Proposition 3.2. The proof is similar to that of [26, Corollary 3.5] under Lemma 3.3 and Lemma 3.4. Indeed, following to the arguments for proving [10, Theorem 4.8.2(ii)] (or [17, Theorem 2.1])), we can establish the following Poincaré type inequality for the Harris recurrent Dirichlet form $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ for some $h \in \mathcal{H}_+^A$: for any $\phi \in \mathcal{B}_{b,0}(E)$ satisfying $\int_E \phi\, \mathrm{d}\mathfrak{m}_h \neq 0$ ($\mathfrak{m}_h := h^2\mathfrak{m}$), there exists a strictly positive function $g \in L^1(E; \mathfrak{m}_h)$ such that

$$\int_E \left(f - \frac{1}{\int_E \phi\, \mathrm{d}\mathfrak{m}_h} \int_E f\phi\, \mathrm{d}\mathfrak{m}_h \right)^2 g\, \mathrm{d}\mathfrak{m}_h \leq \mathcal{Q}^h(f, f), \quad f \in \mathcal{D}_e(\mathcal{Q}^h),$$

equivalently,

$$\int_E \left(v - \frac{h}{\int_E \varphi h\, \mathrm{d}\mathfrak{m}} \int_E v\varphi\, \mathrm{d}\mathfrak{m} \right)^2 g\, \mathrm{d}\mathfrak{m} \leq \mathcal{Q}(v, v), \quad v \in \mathcal{D}_e(\mathcal{Q})$$

for $v = fh$ and $\varphi = \phi h \in \mathcal{B}_{b,0}(E)$. Then the inequality (3.4) can be induced by the same way as in the proof of [26, Lemma 3.3 and Corollary 3.5]. □

Proof of Theorem 1.1. By Lemma 3.4, the process $\mathbf{X}^h$ satisfies (**I**) and (**RSF**) for any $h \in \mathcal{H}_+^A$. So the Dirichlet form $(\mathcal{Q}^h, \mathcal{D}(\mathcal{Q}^h))$ of $\mathbf{X}^h$ is either transient or Harris recurrent. Hence, the former and the latter imply (1) and (2) of the present theorem in view of (3.3) and Proposition 3.2, respectively. □

4. Analytic characterizations of criticality and subcriticality for Schrödinger forms

4.1. Girsanov and Feynman-Kac transforms. Fix a function $u \in \mathcal{D}_{\mathrm{loc}}(\mathcal{E}) \cap C_b(E)$ with $\mu_{\langle u \rangle} \in S_K^1(\mathbf{X})$. In [16, Theorem 6.2(2)], we proved that the additive functional $u(X_t) - u(X_0)$ admits the following decomposition in the strict sense:

$$u(X_t) - u(X_0) = M_t^{u,c} + M_t^{u,j} + M_t^{u,\kappa} + N_t^u, \quad t \in [0, \zeta[$$

$\mathbf{P}_x$-a.s. for all $x \in E$. Then, there exists a purely discontinuous locally square integrable local martingale additive functional M_t^U on $[\![0, \zeta[\![$ such that $\Delta M_t^U = U(X_{t-}, X_t) = u(X_{t-}) - u(X_t)$, $t \in [0, \zeta[$ $\mathbf{P}_x$-a.s. for all $x \in E$ (see [5, Lemma 3.1]). Moreover, M_t^U is given by

$$M_t^U = M_t^{-u,j} + M_t^{-u,\kappa}.$$

Therefore, there also exists a purely discontinuous locally square integrable local martingale additive functional $M_t^{e^U-1}$ on $[\![0, \zeta[\![$ such that $\Delta M_t^{e^U-1} = e^{U(X_{t-}, X_t)} - 1$, $t \in [0, \zeta[$ $\mathbf{P}_x$-a.s. for all $x \in E$. $M_t^{e^U-1}$ is given by

$$M_t^{e^U-1} = M_t^U + \sum_{0<s\leq t} (e^U - U - 1)(X_{s-}, X_s) - \int_0^t N(e^U - U - 1)(X_s)\mathrm{d}H_s, \quad t < \zeta.$$

Put $M_t := M_t^{e^U-1} + M_t^{-u,c}$ and let $U_t := \mathrm{Exp}(M)_t$ be the Doléans-Dade exponential of M_t, that is, U_t is the unique solution of

$$U_t = 1 + \int_0^t U_{s-}\mathrm{d}M_s, \quad t<\zeta,\ \mathbf{P}_x\text{-a.s.}$$

Note that U_t is positive and a local martingale on $[\![0,\zeta[\![$. Therefore it is a supermartingale on $[\![0,\zeta[\![$. In particular, $U_t\mathbf{1}_{\{t<\zeta\}}$ is a supermartingale with $\mathbf{E}_x[U_t\mathbf{1}_{\{t<\zeta\}}]\le 1$ for all $x\in E$. Moreover, U_t can be represented as

$$U_t = \exp\left(M_t^U + M_t^{-u,c} - \int_0^t N(e^U-U-1)(X_s)\mathrm{d}H_s - \frac{1}{2}\langle M^{u,c}\rangle_t\right),\ \ t<\zeta. \tag{4.1}$$

by a similar way as in the proof of [14, Theorem 3.1]. In addition, we also note that U_t can be defined for $t\in[0,\infty[$ $\mathbf{P}_x$-a.s. for all $x\in E$ provided $\mu_{\langle u\rangle}\in S_D^1(\mathbf{X})$ ([14, Proposition 3.1]).

Let $\mathbf{U}=(X_t,\mathbf{P}_x^U)$ be the transformed process of $\mathbf{X}$ by U_t. Then $\mathbf{U}$ is an $e^{-2u}\mathfrak{m}$-symmetric Hunt process on E (cf. [14, Theorem 3.1] or [6, Lemma 3.2]).

Let $\mathbf{X}^*=(X_t,\mathbf{P}_x^*)$ be the subprocess of $\mathbf{X}$ killed by $e^{-A_t^{\mu_2}-A_t^{F_2}}$. Note that $\mathbf{X}^*$ is a transient and irreducible Markov process and its Lévy system is given by $(e^{-F_2(x,y)}N(x,\mathrm{d}y),H_t)$. We denote by $(\mathcal{E}^*,\mathcal{D}(\mathcal{E}^*))$ the associated Dirichlet form of $\mathbf{X}^*$ on $L^2(E;\mathfrak{m})$.

Let $\mathbf{U}^*=(X_t,\mathbf{P}_x^{U^*})$ be the transformed process of $\mathbf{X}^*$ by the supermartingale multiplicative functional U_t^* defined by

$$U_t^* = \exp\left(M_t^U + M_t^{-u,c} - \int_0^t N(e^{-F_2}(e^U-U-1))(X_s)\mathrm{d}H_s - \frac{1}{2}\langle M^{u,c}\rangle_t\right),\ \ t<\zeta. \tag{4.2}$$

The functional U_t^* plays the same role for getting $\mathbf{U}^*$ from $\mathbf{X}^*$ as U_t does for getting $\mathbf{U}$ from $\mathbf{X}$. Let $(\mathcal{E}^{U^*},\mathcal{D}(\mathcal{E}^{U^*}))$ be the Dirichlet form on $L^2(E;e^{-2u}\mathfrak{m})$ generated by $\mathbf{U}^*$. Then, it follows from [14, Theorem 3.2] that $\mathcal{D}(\mathcal{E}^{U^*})=\mathcal{D}(\mathcal{E}^*)=\mathcal{D}(\mathcal{E})$ and for $f\in\mathcal{D}(\mathcal{E}^{U^*})$

$$\begin{aligned}\mathcal{E}^{U^*}(f,f) &= \frac{1}{2}\int_E e^{-2u(x)}\mu^c_{\langle f\rangle}(\mathrm{d}x)\\ &\quad + \int_{E\times E}(f(x)-f(y))^2 e^{-u(x)-u(y)-F_2(x,y)}N(x,\mathrm{d}y)\mu_H(\mathrm{d}x) + \int_E f(x)^2 e^{-u(x)}\kappa(\mathrm{d}x).\end{aligned}$$

From this expression and the boundedness of u and F, we see that $\mathcal{E}^{U^*}\asymp\mathcal{E}^*$, that is, there exists a constant $C>0$ such that for $f\in\mathcal{D}(\mathcal{E})$

$$C^{-1}\mathcal{E}^*(f,f)\le\mathcal{E}^{U^*}(f,f)\le C\mathcal{E}^*(f,f). \tag{4.3}$$

By [13, Lemma 4.1 and Corollary 5.1(3)], we have the following:

Lemma 4.1. *Assume that $\mu_{\langle u\rangle}\in S_K^1(\mathbf{X}^*)$. Then the following hold.*

(1) *For $\nu\in S_D^1(\mathbf{X}^*)$, $e^{-2u}\nu\in S_D^1(\mathbf{U}^*)$.*
(2) *For $\nu\in S_K^1(\mathbf{X}^*)$, $e^{-2u}\nu\in S_K^1(\mathbf{U}^*)$.*
(3) *For $\nu\in S_{EK}^1(\mathbf{X}^*)$, $e^{-2u}\nu\in S_{EK}^1(\mathbf{U}^*)$.*
(4) *For $\nu\in S_{CK_\infty}^1(\mathbf{X}^*)$, $e^{-2u}\nu\in S_{CK_\infty}^1(\mathbf{U}^*)$ provided $\mu_{\langle u\rangle}\in S_{CK_\infty}^1(\mathbf{X}^*)$.*

Let $\mathbf{Y}^*=(X_t,\mathbf{P}_x^{Y^*})$ be the transformed process of $\mathbf{U}^*$ by the supermartingale multiplicative functional Y_t^* defined by

$$Y_t^* := \exp\left(-\int_0^t N\left(e^{U-F_2}\left(e^{F_1}-1\right)\right)(X_s)\mathrm{d}H_s + A_t^{F_1}\right),\quad t<\zeta. \tag{4.4}$$

Note that $\mathbf{Y}^*$ is also an $e^{-2u}\mathfrak{m}$-symmetric Hunt process on E, because it is the transformed process induced by a pure jump Girsanov-type transform. Let $(\mathcal{E}^{Y^*}, \mathcal{D}(\mathcal{E}^{Y^*}))$ be the Dirichlet form on $L^2(E; e^{-2u}\mathfrak{m})$ generated by $\mathbf{Y}^*$. Then $\mathcal{D}(\mathcal{E}^{Y^*}) = \mathcal{D}(\mathcal{E}^{U^*})$ and $\mathcal{E}^{Y^*}$ can be expressed as

$$\begin{aligned}(4.5)\qquad \mathcal{E}^{Y^*}(f,f) = \mathcal{E}^{U^*}(f,f) &+ \int_E f^2 e^{-2u}\mathrm{d}\left(N\left(e^{U-F_2}\left(e^{F_1}-1\right)\right)\mu_H\right)\\ &- \int_{E\times E} f(x)f(y)e^{-u(x)-u(y)-F_2(x,y)}\left(e^{F_1(x,y)}-1\right)N(x,\mathrm{d}y)\mu_H(\mathrm{d}x)\end{aligned}$$

(cf. [3]). In the sequel, we denote the semigroup and the resolvent of the process $\mathbf{X}^*$ (resp. $\mathbf{U}^*$, $\mathbf{Y}^*$) by $\{P_t^*\}_{t\geq 0}$ (resp. $\{P_t^{U^*}\}_{t\geq 0}$, $\{P_t^{Y^*}\}_{t\geq 0}$) and $\{R_\alpha^*\}_{\alpha>0}$ (resp. $\{R_\alpha^{U^*}\}_{\alpha>0}$, $\{R_\alpha^{Y^*}\}_{\alpha>0}$), respectively.

Consider the generalized non-local Feynman-Kac transforms by the additive functionals $A_t^1 := A_t^{\mu_1} + A_t^{F_1} + N_t^u$ of the form

$$e_{A^1}(t) := \exp(A_t^1), \quad t \geq 0.$$

Then we see that for $\mu_{\langle u\rangle} + N(F_1)\mu_H \in S_D^1(\mathbf{X}^*)$

$$(4.6)\qquad e_{A^1}(t) = e^{u(X_t)-u(x)}U_t^* \exp\left(A_t^{\overline{\nu}_1^*} + A_t^{F_1}\right) = e^{u(X_t)-u(x)}Y_t^* \exp\left(A_t^{\eta}\right),$$

where

$$\overline{\nu}_1^* := \mu_1 + N\left(e^{-F_2}\left(e^U - U - 1\right)\right)\mu_H + \frac{1}{2}\mu_{\langle u\rangle}^c \quad \text{and} \quad \eta = \overline{\nu}_1^* + N\left(e^{U-F_2}\left(e^{F_1}-1\right)\right)\mu_H.$$

Lemma 4.2. *Assume that* $\mu_1 + N(F_1)\mu_H + \mu_{\langle u\rangle} \in S_{CK_\infty}^1(\mathbf{X}^*)$ *and* $\mu_2 + N(F_2)\mu_H \in S_D^1(\mathbf{X})$. *Then* $e^{-2u}\eta \in S_{CK_\infty}^1(\mathbf{Y}^*)$.

Proof. First, we note under the assumptions that $\overline{\nu}_1^* \in S_{CK_\infty}^1(\mathbf{X}^*)$ and $N(e^{U-F_2}(e^{F_1}-1))\mu_H \in S_{CK_\infty}^1(\mathbf{X}^*)$ by the boundedness of u and F. Then we see $\eta \in S_{CK_\infty}^1(\mathbf{X}^*)$, and hence $e^{-2u}\eta \in S_{CK_\infty}^1(\mathbf{U}^*)$ by virtue of Lemma 4.1(4). We prove $S_{CK_\infty}^1(\mathbf{U}^*) \subset S_{CK_\infty}^1(\mathbf{Y}^*)$. Recall that $\mathbf{U}^*$ is obtained from $\mathbf{X}^*$ by the supermartingale multiplicative functional U_t^* under $\mathbf{X}^*$. Applying [13, Corollary 5.1(3)] to the transient process $\mathbf{X}^*$ with $\mu_{\langle u\rangle} \in S_{CK_\infty}^1(\mathbf{X}^*)$, one can get

$$(4.7)\qquad e^{-2u}(\mu_{\langle u\rangle} + N(F_1)\mu_H) \in S_{CK_\infty}^1(\mathbf{U}^*).$$

Now, let us consider the supermartingale mulitiplicative functional Y_t^1 of $\mathbf{X}$ defined by

$$\begin{aligned}Y_t^1 &= \mathrm{Exp}\left(\sum_{s\leq t}(e^{F_1}-1)(X_{s-},X_s) - \int_0^t N(e^{F_1}-1)(X_s)\mathrm{d}H_s\right)\\ &= \exp\left(A_t^{F_1} - \int_0^t N(e^{F_1}-1)(X_s)\mathrm{d}H_s\right).\end{aligned}$$

Then we have $S_{CK_\infty}^1(\mathbf{X}) \subset S_{CK_\infty}^1(\mathbf{Y}^1)$ under $N(F_1)\mu_H \in S_{CK_\infty}^1(\mathbf{X})$ by [14, Corollaries 5.1(3) and 5.2(2)], where $\mathbf{Y}^1$ is the transformed process from $\mathbf{X}$ by Y_t^1. From this observation, we can see the following fact that the supermartingale multiplicative functional Y_t^* under $\mathbf{U}^*$ plays the same role so that $S_{CK_\infty}^1(\mathbf{U}^*) \subset S_{CK_\infty}^1(\mathbf{Y}^*)$ under (4.7) by applying [14, Corollaries 5.1(3) and 5.2(2)] from $\mathbf{U}^*$ to $\mathbf{Y}^*$. The proof is complete. □

For $\nu := \nu_1 - \nu_2 \in S_1(\mathbf{X}) - S_1(\mathbf{X})$, define the semigroup (not necessarily Markovian) $\{P_t^{\nu,F}\}_{t\geq 0}$ and the resolvent $\{R_\alpha^{\nu,F}\}_{\alpha>0}$: for $f \in \mathcal{B}_b(E)$

$$P_t^{\nu,F}f(x) := \mathbf{E}_x\left[e^{A_t^\nu + A_t^F}f(X_t)\right], \quad R_\alpha^{\nu,F}f(x) := \int_0^\infty e^{-\alpha t}P_t^{\nu,F}f(x)\,\mathrm{d}t.$$

Lemma 4.3. *Assume that $\nu_i + N(F_i)\mu_H \in S_K^1(\mathbf{X})$ for $i = 1, 2$. Then there exists $\alpha_0 > 0$ such that for any $\alpha > \alpha_0$ and $f \in \mathcal{B}_b(E)$, $R_\alpha^{\nu,F} f \in C_b(E)$. In particular, if $\nu_1 = F_1 = 0$, then the assertion holds for any $\alpha > 0$.*

Proof. By Khas'minskii's lemma, we see that for sufficiently small $t > 0$,

$$\sup_{x\in E}\mathbf{E}_x\left[\mathrm{Exp}\left(A^{\nu_i} + A^{G_i}\right)_t\right] \le \frac{1}{1 - \sup_{x\in E}\mathbf{E}_x\left[A_t^{\nu_i} + \int_0^t N(G_i)(X_s)\,\mathrm{d}H_s\right]}, \quad (i = 1, 2), \tag{4.8}$$

Here $\mathrm{Exp}(A)_t$ denotes the Stieltjes exponential of a PCAF A and $G_i := e^{F_i} - 1$ $(i = 1, 2)$. Clearly $\mathrm{Exp}\left(A^{\nu_i} + A^{G_i}\right)_t = e^{A_t^{\nu_i} + A_t^{F_i}}$. Then we have

$$\begin{aligned}
\lim_{t\to0}\sup_{x\in E}\mathbf{E}_x\left[\left|e^{A_t^\nu + A_t^F} - 1\right|\right] &\le \lim_{t\to0}\sup_{x\in E}\mathbf{E}_x\left[\left|e^{A_t^{\nu_1} + A_t^{F_1}} - e^{A_t^{\nu_2} + A_t^{F_2}}\right|\right] \\
&\le \lim_{t\to0}\sup_{x\in E}\mathbf{E}_x\left[\left|\mathrm{Exp}\left(A^{\nu_1} + A^{G_1}\right)_t - 1\right|\right] + \lim_{t\to0}\sup_{x\in E}\mathbf{E}_x\left[\left|\mathrm{Exp}\left(A^{\nu_2} + A^{G_2}\right)_t - 1\right|\right] \\
&= 0
\end{aligned}$$

by (4.8). Now the assertion follows from [18, Corollary 5.1]. □

Lemma 4.4. *Assume that $\mu_1 + N(F_1)\mu_H + \mu_{\langle u\rangle} \in S_{CK_\infty}^1(\mathbf{X}^*)$ and $\mu_2 + N(F_2)\mu_H \in S_K^1(\mathbf{X})$. Then, the processes $\mathbf{U}^*$ and $\mathbf{Y}^*$ satisfy* **(RSF)***.*

Proof. In view of Lemma 4.3 of $\mathbf{X}$, the killed process $\mathbf{X}^*$ satisfies (**RSF**) under $\mu_2 + N(F_2)\mu_H \in S_K^1(\mathbf{X})$. Moreover, by a similar way of (3.5) and the estimate (3.6), we can show that

$$\lim_{t\to0}\sup_{x\in E}\mathbf{E}_x^*[|U_t^* - 1|] = 0$$

and for $f \in \mathcal{B}_b(E)$ and $\alpha, \beta > 0$, $\|R_\alpha^{U^*}f - \beta R_\beta^* R_\gamma^{U^*}f\|_\infty \to 0$ as $\beta \to \infty$. From this fact, we see that $\mathbf{U}^*$ satisfies **(RSF)**. On the other hand, both $e^{-2u}N(e^{U-F_2}(e^{F_1} - 1))\mu_H$ and $e^{-2u}N(F_1)\mu_H$ belong to $S_K^1(\mathbf{U}^*)$ by virtue of Lemma 4.1(2) and the boundedness of u and F. Then, by applying $\nu_2 = N(e^{U-F_2}(e^{F_1} - 1))\mu_H$ to Lemma 4.3 with $\mathbf{U}^*$ as the underlying process, we see that there exists $\alpha_0 > 0$ such that for any $\alpha > \alpha_0$ and for $f \in \mathcal{B}_b(E)$, $R_\alpha^{Y^*}f \in C_b(E)$, which implies **(RSF)** of $\mathbf{Y}^*$. □

4.2. Analytic characterzations of criticality and subcriticality. Define the bottom of the spectrum of the quadratic form $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ by

$$\lambda^{\mathcal{Q}}(\eta) := \inf\left\{\mathcal{Q}(f, f)\,\middle|\, f \in \mathcal{D}(\mathcal{Q}), \int_E f^2\mathrm{d}\eta = 1\right\}. \tag{4.9}$$

Let g_A be the gauge function defined by

$$g_A(x) := \mathbf{E}_x^*\left[e_{A^1}(\zeta)\right]. \tag{4.10}$$

It is known by [15, Theorem 1.4] with Lemma 2.1 that g_A is bounded if and only if $\lambda^{\mathcal{Q}}(\overline{\mu}_1^*) > 0$, where $\overline{\mu}_1^* := \mu_1 + N(e^{U+F_1} - U - 1)\mu_H + \frac{1}{2}\mu_{\langle u\rangle}^c$. In addition, the boundedness of g_A is also equivalent to $\lambda^{Q}(\eta) > 0$ because $\overline{\mu}_1^*$ is absolutely continuous with respect to η (cf. [15, Corollary 1.1]).

Proposition 4.1. *Assume that* $\mu_1 + N(F_1)\mu_H + \mu_{\langle u\rangle} \in S^1_{CK_\infty}(\mathbf{X}^*)$ *and* $\mu_2 + N(F_2)\mu_H \in S^1_K(\mathbf{X})$. *If* $\lambda^{Q}(\eta) > 0$, *then the Schrödinger form* $(Q, \mathcal{D}(Q))$ *is subcritical.*

Proof. Let g_A be the gauge function defined in (4.10). First, we prove that $g_A \in \mathcal{H}^A_+$. It is easy to check by the Markov property of $\mathbf{X}^*$ that the function g_A is P^A_t-excessive, that is, $P^A_t g_A(x) \uparrow g_A(x)$ as $t \to 0$ (cf. [26, Lemma 5.2]). Put $\widetilde{g}_A(x) := e^{u(x)}\mathbf{E}^*_x[e_{A^1}(\zeta)e^{-u(X_{\zeta-})}] = \mathbf{E}^{Y^*}_x[e^{A^\eta_\zeta}]$. Then, by a similar way as in the proof of [26, Lemma 5.3], it holds that

$$\widetilde{g}_A(x) = 1 + R^{Y^*}\left(e^{-2u}\widetilde{g}_A\eta\right)(x). \tag{4.11}$$

Note that $R^{Y^*}(e^{-2u}\widetilde{g}_A\eta)$ is bounded under $\lambda^{Q}(\eta) > 0$, because so is $\widetilde{g}_A$ by the boundedness of g_A and u. By using a similar way of (3.5) and the estimate (3.6), we see then $\|R^{Y^*}(e^{-2u}\widetilde{g}_A\eta) - \beta R^{U^*}_\beta R^{Y^*}(e^{-2u}\widetilde{g}_A\eta)\|_\infty \to 0$ as $\beta \to \infty$, which implies that $R^{Y^*}(e^{-2u}\widetilde{g}_A\eta) \in C_b(E)$ because so is $R^{U^*}(R^{Y^*}(e^{-2u}\widetilde{g}_A\eta))$ by **(RSF)** of $\mathbf{U}^*$ proved in Lemma 4.4. Therefore we have $g_A \in C_b(E)$ in view of (4.11) and the continuity of u. Moreover, similarly to the proof of [12, Lemma 4.2], we can show that $R^{Y^*}(e^{-2u}\widetilde{g}_A\eta) \in \mathcal{D}_{\rm loc}(\mathcal{E}^{Y^*}) = \mathcal{D}_{\rm loc}(\mathcal{E})$, As a consequence, $g_A \in \mathcal{D}_{\rm loc}(\mathcal{E})$. Hence we can conclude that $g_A \in \mathcal{D}_{\rm loc}(\mathcal{E}) \cap C_b(E)$.

Next, we prove the transience of $(Q^{g_A}, \mathcal{D}(Q^{g_A}))$. We note that the transformed processes $\mathbf{U}^*$ and $\mathbf{Y}^*$ satisfy (**I**) by the positivities of (4.2) and (4.4), and (**I**) of $\mathbf{X}^*$, respectively. It follows from (4.3) that $\mathcal{D}(\mathcal{E}^{U^*}) = \mathcal{D}(\mathcal{E}^*)$ and $\mathcal{E}^{U^*} \asymp \mathcal{E}^*$. Then we see that $(\mathcal{E}^{U^*}, \mathcal{D}(\mathcal{E}^{U^*}))$ is a transient Dirichlet form on $L^2(E; e^{-u}\mathfrak{m})$. Moreover, since the Dirichlet form $\mathcal{E}^{Y^*}$ in (4.5) can be expressed as

$$\mathcal{E}^{Y^*}(f,f) = \mathcal{E}^{U^*}(f,f) + \frac{1}{2}\int_{E\times E}(f(x)-f(y))^2\, e^{-u(x)-u(y)-F_2(x,y)}\left(e^{F_1(x,y)}-1\right)N(x,\mathrm{d}y)\mu_H(\mathrm{d}x)$$

for $f \in \mathcal{D}(\mathcal{E}^{Y^*})(= \mathcal{D}(\mathcal{E}^*))$, one can also get $\mathcal{E}^{Y^*} \asymp \mathcal{E}^*$ by the boundedness of u and F again. Therefore we see that $(\mathcal{E}^{Y^*}, \mathcal{D}(\mathcal{E}^{Y^*}))$ is also a transient Dirichlet form on $L^2(E; e^{-2u}\mathfrak{m})$. Under this transience, we can take a positive bounded $e^{-2u}\mathfrak{m}$-integrable function g such that

$$\int_E f^2 g\,\mathrm{d}\mathfrak{m} \le \mathcal{E}^{Y^*}(fe^u, fe^u), \quad f \in \mathcal{D}(\mathcal{E}^{Y^*}). \tag{4.12}$$

On the other hand, since

$$\begin{aligned} Q(f,f) &= \mathcal{E}^{U^*}(fe^u, fe^u) - \int_E f^2\mathrm{d}\overline{\nu}_1^* - \int_{E\times E} f(x)f(y)e^{-F_2(x,y)}\left(e^{F_1(x,y)}-1\right)N(x,\mathrm{d}y)\mu_H(\mathrm{d}x) \\ &= \mathcal{E}^{U^*}(fe^u, fe^u) - \int_E f^2\mathrm{d}\left(\overline{\nu}_1^* + N\left(e^{U-F_2}\left(e^{F_1}-1\right)\right)\mu_H\right) \\ &\quad + \frac{1}{2}\int_{E\times E}((fe^u)(x)-(fe^u)(y))^2\, e^{-u(x)-u(y)-F_2(x,y)}\left(e^{F_1(x,y)}-1\right)N(x,\mathrm{d}y)\mu_H(\mathrm{d}x) \end{aligned}$$

by virtue of (4.6), we have

$$Q(f,f) = \mathcal{E}^{Y^*}(fe^u, fe^u) - \int_E f^2\mathrm{d}\eta. \tag{4.13}$$

Then the bottom of the spectrum $\lambda^{Q}(\eta)$ can be written as

$$\lambda^{Q}(\eta) = \inf\left\{ \mathcal{E}^{Y^*}(fe^u, fe^u) - \int_E f^2 \mathrm{d}\eta \;\middle|\; f \in \mathcal{D}(\mathcal{E}^{Y^*}), \int_E f^2 \mathrm{d}\eta = 1 \right\}$$

by applying (4.13) to (4.9), which implies that $\mathcal{E}^{Y^*}(fe^u, fe^u) \geq (\lambda^{Q}(\eta) + 1)\int_E f^2 \mathrm{d}\eta$. From this with (4.12), we have

$$Q(f,f) = \mathcal{E}^{Y^*}(fe^u, fe^u) - \int_E f^2 \mathrm{d}\eta \geq \frac{\lambda^{Q}(\eta)}{\lambda^{Q}(\eta)+1}\int_E f^2 g \,\mathrm{d}\mathfrak{m}.$$

Let

$$\widetilde{g} := \frac{\sqrt{\lambda^{Q}(\eta)} \cdot g}{\sqrt{(\lambda^{Q}(\eta)+1)\int_E g g_A^2 \mathrm{d}\mathfrak{m}}}.$$

Then, we see that for $f \in \mathcal{D}(Q^{g_A})$, equivalently, for $g_A f \in \mathcal{D}(Q)$

$$Q^{g_A}(f,f) = Q(g_A f, g_A f) \geq \frac{\lambda^{Q}(\eta)}{\lambda^{Q}(\eta)+1}\int_E f^2 g\, g_A^2 \mathrm{d}\mathfrak{m} \geq \left(\int_E |f|\widetilde{g}\, g_A^2 \mathrm{d}\mathfrak{m}\right)^2,$$

which implies the transience of $(Q^{g_A}, \mathcal{D}(Q^{g_A}))$. □

Proposition 4.2. *Assume $\mu_1 + N(F_1)\mu_H + \mu_{\langle u \rangle} \in S^1_{CK_\infty}(\mathbf{X}^*)$ and $\mu_2 + N(F_2)\mu_H \in S^1_K(\mathbf{X})$. If $\lambda^{Q}(\eta) = 0$, then the Schrödinger form $(Q, \mathcal{D}(Q))$ is critical.*

Proof. As we have already seen in the proof of Proposition 4.1, the process $\mathbf{Y}^*$ (or $\mathcal{E}^{Y^*}$) is transient and satisfies (**I**). Moreover, $\mathbf{Y}^*$ also satisfies **(RSF)** by Lemma 4.4. In addition, $e^{-2u}\eta \in S^1_{CK_\infty}(\mathbf{Y}^*)$ by Lemma 4.2. Note that $\lambda^{Q}(\eta)$ can be written as

$$\lambda^{Q}(\eta) + 1 = \inf\left\{ \mathcal{E}^{Y^*}(fe^u, fe^u) \;\middle|\; f \in \mathcal{D}_e(\mathcal{E}^{Y^*}), \int_E f^2 \mathrm{d}\eta = 1 \right\}. \tag{4.14}$$

Then, it follows from [25, Theorem 2.1] that there exists a minimizer $h_0 \in \mathcal{D}_e(\mathcal{E}^{Y^*})$ of (4.14) (equivalently, of (4.9)), namely a ground state of $(\mathcal{E}^{Y^*}, \mathcal{D}(\mathcal{E}^{Y^*}))$, that is,

$$\lambda^{Q}(\eta) + 1 = \mathcal{E}^{Y^*}(h_0 e^u, h_0 e^u), \quad \text{and} \quad \int_E h_0^2 \,\mathrm{d}\eta = 1. \tag{4.15}$$

Now, define $h(x) := R^{Y^*}(e^{-2u}h_0\eta)(x)$, a version of the minimizer h_0. Then, by a similar way of Section 5.2 in [26], we can see that $h \in \mathcal{H}^A_+$. The function h belongs to $\mathcal{D}_e(Q)(= \mathcal{D}_e(\mathcal{E}^{Y^*}))$ and $Q(h,h) = \mathcal{E}^{Y^*}(he^u, he^u) - \int_E h^2 \,\mathrm{d}\eta = 0$, equivalently, $1 \in \mathcal{D}_e(Q^h)$ and $Q^h(1,1) = 0$ which implies that $(Q^h, \mathcal{D}(Q^h))$ is recurrent. □

REMARK 4.1. The conditions on measures imposed in Proposition 4.2 is milder than the previous one due to Li [20] even if we restrict ourselves to dealing with only non-local perturbations.

Proof of Theorem 1.2. Note that if $\lambda^{Q}(\eta) < 0$, then $\mathcal{H}^A_+ = \emptyset$. Indeed, if $\mathcal{H}^A_+ \neq \emptyset$, then we can take a function $h \in \mathcal{H}^A_+$ such that $Q(f,f) = Q^h(f/h, f/h) \geq 0$ for any $f \in \mathcal{D}(Q)$, and thus $\lambda^{Q}(\eta) \geq 0$. From this fact with Propositions 4.1 and 4.2, we can finish the proof of the theorem. □

5. Maximum principles related to genenalized Schrödinger forms

In this section, we study some maximum principles related to the generalized Feynman-Kac semigroup by using the criticalities of generalized Schrödinger forms obtained in the previous section. In a series of papers [28], [29], [30], Takeda investigated an analytic condition for several maximum principles for Schrödinger operators with local perturbations. We partially extend his results to the case of generalized Feynman-Kac perturbations.

First, let us consider a condition for the Green kernel to satisfy *generalized Ugaheri's maximum principle* in terms of the spectral function $\lambda^{Q}(\eta)$.

Theorem 5.1. *Assume that* $\mu_1 + N(F_1)\mu_H + \mu_{\langle u\rangle} \in S^1_{CK_\infty}(\mathbf{X}^*)$ *and* $\mu_2 + N(F_2)\mu_H \in S^1_K(\mathbf{X})$. *Let* ν *be a positive Radon measure with compact topological support* S_ν. *Then the following hold:*

(1) *If* $\lambda^{Q}(\eta) > 0$, *then there exists a constant* $C > 0$ *such that*

$$\sup_{x\in E} R^A\nu(x) \le C \sup_{x\in S_\nu} R^A\nu(x).$$

(2) *If* $\lambda^{Q}(\eta) = 0$, *then for any* $\alpha \ge 0$ *there exists a constant* $C > 0$ *such that*

$$\sup_{x\in E} R^A_\alpha\nu(x) \le C \sup_{x\in S_\nu} R^A_\alpha\nu(x).$$

Proof. (1): Let g_A be the gauge function defined in (4.10). As we showed in the proof of Proposition 4.1, $g_A \in \mathcal{H}^A_+$ and $(Q^{g_A}, \mathcal{D}(Q^{g_A}))$ is transient. Moreover, by virtue of [14, Lemma 4.9] and (4.6)

$$0 < e^{-2\|u\|_\infty}\mathbf{E}^{Y^*}_x\left[e^{A^\eta_\zeta}\right] \le e^{-u(x)}\mathbf{E}^{Y^*}_x\left[e^{A^\eta_\zeta}e^{u(X_{\zeta-})}\right] = \mathbf{E}^*_x\left[e_{A^1}(\zeta)\right] = g_A(x) \le \sup_{x\in E} g_A(x) < \infty$$

under the assumptions. Then since $R^A\nu = g_A R^{A,g_A}(\nu/g_A)$, we have

$$\begin{aligned}\sup_{x\in E} R^A\nu(x) &\le \sup_{x\in E} g_A(x)\cdot\sup_{x\in E} R^{A,g_A}(\nu/g_A)(x) = \sup_{x\in E} g_A(x)\cdot\sup_{x\in S_\nu} R^{A,g_A}(\nu/g_A)(x)\\ &= \sup_{x\in E} g_A(x)\cdot\sup_{x\in S_\nu}\left(\frac{1}{g_A(x)}R^A\nu(x)\right) \le \frac{\sup_{x\in E} g_A(x)}{\inf_{x\in S_\nu} g_A(x)}\cdot\sup_{x\in S_\nu} R^A\nu(x)\\ &\le C\sup_{x\in S_\nu} R^A\nu(x).\end{aligned}$$

Here we used in the first equality the Frostman's maximum principle for the resolvent R^{A,g_A} of $\mathbf{X}^{g_A}$, that is, $\sup_{x\in E} R^{A,g_A}\nu(x) = \sup_{x\in S_\nu} R^{A,g_A}\nu(x)$ ([19]).

(2): Let $\alpha > 0$. In this case, one can express that

$$R^A_\alpha\nu(x) = \mathbf{E}^*_x\left[\int_0^\infty e^{-\alpha t + A^1_t}\,\mathrm{d}A^\nu_t\right] = \mathbf{E}^{Y^*}_x\left[\int_0^\infty e^{\widetilde{A}_t}\,\mathrm{d}A^\nu_t\right],$$

where $\widetilde{A}_t := A^\eta_t - \alpha t$. Since $\lambda^{Q}(\eta - \alpha m) > \lambda^{Q}(\eta) = 0$, the gauge function $\mathbf{E}^{Y^*}_\cdot[e^{\widetilde{A}_\zeta}]$ is bounded. Thus, so is $g_{A-\alpha}$ because

$$g_{A-\alpha}(x) := \mathbf{E}^*_x\left[e_{A^1}(\zeta)e^{-\alpha\zeta}\right] = e^{-u(x)}\mathbf{E}^{Y^*}_x\left[e^{\widetilde{A}_\zeta}e^{u(X_{\zeta-})}\right] \le e^{2\|u\|_\infty}\mathbf{E}^{Y^*}_x\left[e^{\widetilde{A}_\zeta}\right]$$

in view of [14, Lemma 4.9] and (4.6) again. Moreover, similar to the proof of Proposition 4.1 we can see that $g_{A-\alpha} \in \mathcal{H}^A_+$. Then, the assertion can be obtained from the proof of (1) with $g_{A-\alpha}$. Now, consider the case $\alpha = 0$. In view of the proof of Proposition 4.2, the assumption $\lambda^{Q}(\eta) = 0$ implies that there exists a strictly positive function $h \in \mathcal{H}^A_+$ such that $\mathbf{X}^h$ (or $(Q^h, \mathcal{D}(Q^h))$) is to be recurrent. Hence $\mathbf{X}^h$ is to be Harris recurrent by Lemma 3.4. Then we have

$$R^A\nu(x) = \mathbf{E}_x\left[\int_0^\infty e^{A_t}\mathrm{d}A^\nu_t\right] = h(x)\mathbf{E}^{A,h}_x\left[\int_0^\infty \frac{1}{h(X_t)}\mathrm{d}A^\nu_t\right] = \infty.$$

for any $x \in E$. □

Next two lemmas are used to show the equivalence between several maximum principles and the analytic condition of subcriticality for genenalized Schrödinger forms.

Lemma 5.1. *Assume* $\mu_1 + N(F_1)\mu_H + \mu_{\langle u\rangle} \in S^1_{CK_\infty}(\mathbf{X}^*)$ *and* $\mu_2 + N(F_2)\mu_H \in S^1_K(\mathbf{X})$*. If* $\lambda^{Q}(\eta) \geq 0$*, then for* $h \in \mathcal{D}_e(Q)$

$$Q(h, \varphi) = \lim_{\beta\to\infty}\beta\left(h - \beta R^A_\beta h, \varphi\right)_{\mathfrak{m}}, \qquad \textit{for any} \quad \varphi \in \mathcal{D}(Q).$$

Proof. We note that for any $f \in \mathcal{D}_e(\mathcal{E})$ and $v \in \mathcal{D}(\mathcal{E})$

$$\mathcal{E}(f, v) = \lim_{\beta\to\infty}\beta\left(f - \beta R_\beta f, v\right)_{\mathfrak{m}} \tag{5.1}$$

([29, Lemma 3.13]). By Definition 1.1 and Theorem 1.2, the condition $\lambda^{Q}(\eta) \geq 0$ is equivalent to $\mathcal{H}^A_+ \neq \emptyset$. Taking $g \in \mathcal{H}^A_+$ and applying (5.1) to $f = h/g$, $v = \varphi/g$ and $\mathcal{E} = Q^g$, we have

$$Q(h, \varphi) = Q^g(h/g, \varphi/g) = \lim_{\beta\to\infty}\beta\left(h/g - \beta R^{A,g}_\beta(h/g), \varphi/g\right)_{g^2\mathfrak{m}} = \lim_{\beta\to\infty}\beta\left(h - \beta R^A_\beta h, \varphi\right)_{\mathfrak{m}}$$

for any $\varphi \in \mathcal{D}(Q)$. □

Lemma 5.2. *Assume* $\mu_1 + N(F_1)\mu_H + \mu_{\langle u\rangle} \in S^1_{CK_\infty}(\mathbf{X}^*)$ *and* $\mu_2 + N(F_2)\mu_H \in S^1_K(\mathbf{X})$*. If* $\lambda^{Q}(\eta) \leq 0$*, then for there exists a positive function* $h \in \mathcal{D}_e(Q) \cap C_b(E)$ *such that*

$$Q(h, \varphi) = 0, \qquad \textit{for any} \quad \varphi \in \mathcal{D}(Q) \cap C^+_0(E).$$

Proof. First, suppose $\lambda^{Q}(\eta) = 0$. In this case, there exists a version of the minimizer $h_1 \in \mathcal{D}_e(Q)$ of (4.14) (or equivalently, (4.9)) which is strictly positive bounded continuous and satisfies $P^A_t h_1 = h_1$ (hence $\beta R^A_\beta h_1 = h_1$ for $\beta > 0$), in view of the proof of Proposition 4.2. Then the assertion of the present lemma follows from Lemma 5.1. Next, suppose that $\lambda^{Q}(\eta) < 0$. Put $\lambda := \lambda^{Q}(\eta) + 1$. Then, by (4.14), one can see

$$\inf\left\{\mathcal{E}^{Y^*}(fe^u, fe^u) \;\middle|\; f \in \mathcal{D}_e(\mathcal{E}^{Y^*}), \int_E f^2\mathrm{d}(\lambda\eta) = 1\right\} = 1 \tag{5.2}$$

which implies that $\lambda^{Q}(\lambda\eta) = 0$. Thus we see from the first case above that there exists a strictly positive bounded continuous function $h_2 \in \mathcal{D}_e(Q)$, a version of the minimizer of (5.2). Note that the relation between h_1 and h_2 is given by $h_1 = \sqrt{\lambda}h_2$. Hence $P^A_t h_2 = h_2$ and the assertion holds in view of Lemma 5.1 again. □

Let us introduce three function spaces on E defined by

$$\mathcal{SH}_1^A = \left\{h \in \mathcal{B}(E) \mid h \text{ is bounded above, } P_t^A h \geq h, \ \forall t \geq 0\right\},$$
$$\mathcal{SH}_2^A = \left\{h \in \mathcal{D}_e(\mathcal{Q}) \cap C(E) \mid h \text{ is bounded above, } P_t^A h \geq h, \ \forall t \geq 0\right\},$$
$$\mathcal{SH}_3^A = \left\{h \in \mathcal{D}_e(\mathcal{Q}) \cap C(E) \mid h \text{ is bounded above, } \mathcal{Q}(h, \varphi) \leq 0, \ \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(E)\right\}.$$

We define the *maximum principle* as follows :

$(\mathbf{MP})_i$ If $h \in \mathcal{SH}_i^A$, then $h(x) \leq 0$ for all $x \in E$ $(i = 1, 2, 3)$.

Remark 5.1. It is easy to see that $(\mathbf{MP})_1$ implies $(\mathbf{MP})_2$ because $\mathcal{SH}_2^A \subset \mathcal{SH}_1^A$. Moreover, if $h \in \mathcal{SH}_2^A$, then $h \leq \beta R_\beta^A h$ $(\beta > 0)$, hence $\mathcal{Q}(h, \varphi) \leq 0$ in view of Lemma 5.1. Therefore we see that $(\mathbf{MP})_3$ implies $(\mathbf{MP})_2$.

Remark 5.2. A function in $\mathcal{SH}_3^A$ is regarded as a weak subsolution to the formal Schrödinger equation $-\mathcal{L}^A u := -(\mathcal{L} + \mathcal{L}u + \mu + \mu_H \mathbf{F})u = 0$, where $\mathbf{F}$ is a non-local linear operator defined by

$$\mathbf{F}f(x) = \int_E (e^{F(x,y)} - 1) f(y) N(x, \mathrm{d}y), \quad x \in E.$$

In this sense, $(\mathbf{MP})_3$ is more closer to the maximum principle analytically defined.

Theorem 5.2. *Suppose that* $\mu_1 + N(F_1)\mu_H + \mu_{\langle u \rangle} \in S^1_{CK_\infty}(\mathbf{X}^*)$ *and* $\mu_2 + N(F_2)\mu_H \in S^1_K(\mathbf{X})$ *holds. Assume further that* $\mathbf{X}^*$ *is almost surely killed, that is,*

$$\mathbf{P}_x^*(\zeta = \infty) = \mathbf{E}_x\left[e^{-A_\zeta^{\mu_2} - A_\zeta^{F_2}}; \zeta = \infty\right] = 0, \quad x \in E.$$

Then the following are equivalent:

(1) $\lambda^{\mathcal{Q}}(\eta) > 0$.

(2) $(\mathbf{MP})_i$ *holds for any* $i = 1, 2, 3$.

Proof. (1) $\Longrightarrow$ (2): In view of Remark 5.1, it suffices to show that $\lambda^{\mathcal{Q}}(\eta) > 0 \Longrightarrow (\mathbf{MP})_1$ and $(\mathbf{MP})_2 \Longrightarrow (\mathbf{MP})_3$. The proof of the former assertion is already made in [13, Theorem 7.1]. To show the latter assertion, we shall prove $\mathcal{SH}_3^A \subset \mathcal{SH}_2^A$. For $h \in \mathcal{SH}_3^A$, let $h_n := he^u \vee (-n)$, $n \in \mathbb{N}$. Define the functional $I_n^{\mathcal{Q}}$ by

$$I_n^{\mathcal{Q}}(\varphi) = -\mathcal{Q}(h_n, \varphi) = -\mathcal{E}^{Y^*}(h_n e^u, \varphi e^u) + \int_E h_n \varphi \, \mathrm{d}\eta, \quad \varphi \in \mathcal{D}(\mathcal{Q}) \cap C_0^+(E). \tag{5.3}$$

Then $I_n^{\mathcal{Q}}$ is a pre-integral for any $n \in \mathbb{N}$, that is, $I^{\mathcal{Q}}(\varphi_\ell) \downarrow 0$ for $\varphi_\ell \in \mathcal{D}(\mathcal{Q}) \cap C_0^+(E)$ such that $\varphi_\ell(x) \downarrow 0$, $x \in E$, as $\ell \to \infty$. Noting that the smallest σ-field generated by $\mathcal{D}(\mathcal{Q}) \cap C_0^+(E)$ is identical with the Borel σ-field by the regularity of $(\mathcal{E}^{Y^*}, \mathcal{D}(\mathcal{E}^{Y^*}))$, one can see that there exists a positive Borel measure $\nu^{(n)}$ such that

$$I_n^{\mathcal{Q}}(\varphi) = \int_E \varphi \, \mathrm{d}\nu^{(n)}, \quad n \in \mathbb{N} \tag{5.4}$$

([9, Theorem 4.5.2]). We prove that $e^{-2u}\nu^{(n)} \in S_1(\mathbf{Y}^*)$ for $n \in \mathbb{N}$. Let K be a compact set of zero capacity. Then for a relatively compact open set D such that $K \subset D$, there exists a sequence $\{\varphi_\ell\} \subset \mathcal{D}(\mathcal{Q}) \cap C_0^+(D)$ such that $\varphi_\ell \geq 1$ on K and $\mathcal{E}^{Y^*}(\varphi_\ell e^u, \varphi_\ell e^u) \to 0$ as $\ell \to \infty$ ([10, Lemma 2.2.7]). By Lemma 4.2, $e^{-2u}\eta \in S^1_{CK_\infty}(\mathbf{Y}^*)$, hence $e^{-2u}\eta \in S^1_{D_0}(\mathbf{Y}^*)$ ([2, Proposition 2.2]). Then by (4.13) and Stollmann-Voigt's inequality (2.2)

$$\mathcal{Q}(\varphi_\ell,\varphi_\ell) \le \left(1+\|R^{Y^*}\left(e^{-2u}\eta\right)\|_\infty\right)\mathcal{E}^{Y^*}(\varphi_\ell e^u,\varphi_\ell e^u) \longrightarrow 0, \quad \text{as} \quad \ell\to\infty$$

and from which with (5.3) and (5.4), we see that $e^{-2u}\nu^{(n)} \in S_0^{(0)}(\mathbf{Y}^*), n\in\mathbb{N}$. This implies that we can take a 0-order potential $U^{Y^*}(e^{-2u}\nu^{(n)}) \in \mathcal{D}_e(\mathcal{E}^{Y^*})$ such that

$$\int_E \varphi\,\mathrm{d}\nu^{(n)} = \mathcal{E}^{Y^*}\left(U^{Y^*}(e^{-2u}\nu^{(n)}),\varphi e^u\right). \tag{5.5}$$

On account of (5.3), (5.4) and (5.5), we see then

$$\mathcal{E}^{Y^*}\left(U^{Y^*}(e^{-2u}\nu^{(n)})+h_ne^u,\varphi e^u\right) = \mathcal{E}^{Y^*}\left(U^{Y^*}(e^{-2u}h_n\eta),\varphi e^u\right)$$

and thus $U^{Y^*}(e^{-2u}\nu^{(n)})+h_ne^u = U^{Y^*}(e^{-2u}h_n\eta)$, $\mathfrak{m}$-a.e. This relation implies $\|U^{Y^*}\nu^{(n)}\|_\infty<\infty$ because $e^{-2u}h_n\eta\in S^1_{D_0}(\mathbf{Y}^*)$. Then for a sequence $\{K_\ell\}$ of relatively compact sets increasing to E, $e^{-2u}\mathbf{1}_{K_\ell}\nu^{(n)}$ is a finite measure in $S_0^{(0)}(\mathbf{Y}^*)$ and $\|U^{Y^*}(e^{-2u}\mathbf{1}_{K_\ell}\nu^{(n)})\|_\infty<\infty$, that is, $e^{-2u}\mathbf{1}_{K_\ell}\nu^{(n)}\in S_{00}^{(0)}(\mathbf{Y}^*)$. Consequently $e^{-2u}\nu^{(n)}\in S_1(\mathbf{Y}^*)$ for $n\in\mathbb{N}$ by virtue of [10, Theorem 5.1.7].

Note that the equation (5.3) leads us to

$$\mathcal{E}^{Y^*}(h_ne^u,\varphi e^u) = \int_E \varphi\,(h_n\mathrm{d}\eta-\mathrm{d}\nu^{(n)}), \quad \varphi\in\mathcal{D}(\mathcal{E}^{Y^*})\cap C_0^+(E).$$

Applying this result to [10, Theorem 5.4.2], we have

$$h_n(X_t)-h_n(X_0) = M_t^{h_n} - \int_0^t h_n(X_s)\,\mathrm{d}A_s^\eta + A_t^{\nu^{(n)}}, \quad t<\zeta,\ \mathbf{P}_x^{Y^*}\text{-a.s. for any } x\in E,$$

where M^{h_n} is a square integrable martingale additive functional. Then by Itô's formula,

$$e^{A_t^\eta}h_n(X_t)-h_n(X_0) = \int_0^t e^{A_s^\eta}\mathrm{d}M_s^{h_n} + \int_0^t e^{A_s^\eta}\mathrm{d}A_s^{\nu^{(n)}}, \quad t<\zeta,\ \mathbf{P}_x^{Y^*}\text{-a.s. for any } x\in E.$$

Put $\tau_k=\inf\{t>0\mid A_t^\eta>k\}$. Since $\eta\in S^1_{D_0}(\mathbf{X}^*)$, $\mathbf{P}_x^*(\lim_{k\to\infty}\tau_k=\infty)=1$. Hence

$$\begin{aligned}\mathbf{E}_x^*\left[e_{A^1}(\tau_k\wedge t)h_n(X_{\tau_k\wedge t})e^{-u(X_{\tau_k\wedge t})}\right] &= e^{-u(x)}\mathbf{E}_x^{Y^*}\left[e^{A^\eta_{\tau_k\wedge t}}h_n(X_{\tau_k\wedge t})\right]\\ &= e^{-u(x)}h_n(x)+e^{-u(x)}\mathbf{E}_x^{Y^*}\left[\int_0^{\tau_k\wedge t}e^{A_s^\eta}\mathrm{d}A_s^{\nu^{(n)}}\right]\\ &\ge e^{-u(x)}h_n(x) \quad \text{for any} \quad x\in E.\end{aligned}$$

Letting $k\to\infty$ and then $n\to\infty$, we see that $P_t^Ah(x)\ge P_t^{A^1}h(x)\ge h(x)$ for any $x\in E$ by the dominated convergence theorem, which implies that $h\in\mathcal{SH}_2^A$.

(2) $\Longrightarrow$ (1): Suppose $\lambda^{\mathcal{Q}}(\eta)\le 0$. Then by Lemma 5.2, there exists a strictly positive bounded continuous function $h\in\mathcal{D}_e(\mathcal{Q})$ such that $\mathcal{Q}(h,\varphi)=0$ for any $\varphi\in\mathcal{D}(\mathcal{Q})\cap C_0^+(E)$, which means that $(\mathbf{MP})_3$ does not hold. □

In the rest of this section, we assume that $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is strongly local, that is, $\mathcal{E}(f,g)=0$ for any $f,g\in\mathcal{D}(\mathcal{E})$ such that f is constant on a neighborhood of the support of g. In this case, the Feynman-Kac semigroup (1.2) is given by

$$P_t^Af(x)=\mathbf{E}_x[e_A(t)f(X_t)] \qquad \text{with} \quad e_A(t)=\exp(N_t^u+A_t^\mu). \tag{5.6}$$

Let $\mathcal{Q}$ be the symmetric quadratic form associated with (5.6). Then by virtue of (1.3)

$$\mathcal{Q}(f,g) := \mathcal{E}(f,g) + \mathcal{E}(u,fg) - \int_E f(x)g(x)\,\mu(\mathrm{d}x).$$

Consider a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ such that $\lim_{n\to\infty} x_n = \partial$ and $\lim_{n\to\infty} \mathbf{P}_{x_n}(\zeta > \varepsilon) = 0$ for any $\varepsilon > 0$. We denote by $\mathcal{S}$ the family of such sequences. Then we see under the condition $\mu_{\langle u\rangle} + \mu_1 \in S^1_K(\mathbf{X})$ that if $h \in \mathcal{SH}^A_1$, then $\overline{\lim}_{n\to\infty} h(x_n) \le 0$ for any $\{x_n\}_{n=1}^{\infty} \in \mathcal{S}$. Indeed, it holds that there exists $p > 1$ such that $\sup_{x\in E} \mathbf{E}_x[e^{p(N^u_t + A^\mu_t)}] < \infty$ for a small $t > 0$ ([8, Lemma 3.1]). Hence, for $\{x_n\}_{n=1}^{\infty} \in \mathcal{S}$,

$$\begin{aligned}\text{(5.7)}\qquad \varlimsup_{n\to\infty} h(x_n) &\le \varlimsup_{n\to\infty} P^A_t h(x_n) \le \|h^+\|_\infty \varlimsup_{n\to\infty} \mathbf{E}_{x_n}\left[e_A(t)\mathbf{1}_{\{t<\zeta\}}\right] \\ &\le \|h^+\|_\infty \sup_{x\in E} \mathbf{E}_x\left[e^{p(N^u_t + A^\mu_t)}\right]^{1/p} \varlimsup_{n\to\infty} \mathbf{P}_{x_n}(\zeta > t)^{(p-1)/p} = 0.\end{aligned}$$

Let us introduce another function spaces on E:

$$\widetilde{\mathcal{SH}}^A_2 = \left\{h \in \mathcal{D}_{\mathrm{loc}}(\mathcal{Q}) \cap C(E) \mid h \text{ is bounded above, } P^A_t h \ge h,\ \forall t \ge 0\right\},$$

$$\widetilde{\mathcal{SH}}^A_3 = \left\{h \in \mathcal{D}_{\mathrm{loc}}(\mathcal{Q}) \cap C(E) \;\middle|\; \begin{array}{l} h \text{ is bounded above, } \mathcal{Q}(h,\varphi) \le 0,\ \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C^+_0(E) \\ \overline{\lim}_{n\to\infty} h(x_n) \le 0 \text{ for any } \{x_n\}_{n=1}^{\infty} \in \mathcal{S} \end{array}\right\}.$$

Now, let us define another maximum principle by

$(\widetilde{\mathbf{MP}})_i$ If $h \in \widetilde{\mathcal{SH}}^A_i$, then $h(x) \le 0$ for all $x \in E$ $\quad (i = 2, 3)$.

In particular, we call $(\widetilde{\mathbf{MP}})_3$ the *refined maximum principle* (cf. [1], [28]).

Corollary 5.1. *Under the same assumptions of* Theorem 5.2*, the following are equivalent:*

(i) $\lambda^{\mathcal{Q}}(\eta) > 0$.
(ii) $(\widetilde{\mathbf{MP}})_i$ *holds for any* $i = 2, 3$.

Proof. By the same argument as in the proof of (1) $\Longrightarrow$ (2) in Theorem 5.2, one can show that $\widetilde{\mathcal{SH}}^A_3 \subset \widetilde{\mathcal{SH}}^A_2$. Moreover, $\mathcal{SH}^A_3 = \widetilde{\mathcal{SH}}^A_3$ in view of (5.7). Then we see that

$$\mathcal{SH}^A_3 = \widetilde{\mathcal{SH}}^A_3 \subset \widetilde{\mathcal{SH}}^A_2 \subset \mathcal{SH}^A_1.$$

Now the assertion is an easy consequence of the inclusions above and Theorem 5.2. □

Remark 5.3. Let $\mathbf{X} = (X_t, \mathbf{P}_x)$ be a Brownian motion on $\mathbb{R}^d$, a typical irreducible strong Feller process of resolvent. The associated Dirichlet form on $L^2(\mathbb{R}^d)$ is $(H^1(\mathbb{R}^d), \frac{1}{2}\mathbf{D})$, where $H^1(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) \mid |\nabla f| \in L^2(\mathbb{R}^d)\}$ and $\mathbf{D}$ is the classical Dirichlet integral on $\mathbb{R}^d$, $\mathbf{D}(f,g) := \int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g(x)\,\mathrm{d}x$. Fix a bounded function $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$.

Let D be a *regular* domain of $\mathbb{R}^d$ with respect to $\mathbf{X}$ (see [7] for the defintion). The absorbing Brownian motion $\mathbf{X}_D = (X_t, \mathbf{P}^D_x, \tau_D)$ (or the part process of $\mathbf{X}$ on D) is defined as the process killed upon leaving D. Here $\tau_D = \inf\{t > 0 \mid X_t \notin D\}$, the first exit time of X_t from D. It is known that $\mathbf{X}_D$ is a transient irreducible strong Feller process of resolvent on D ([7]). Let $R_D(x,y)$ be the Green function of $\mathbf{X}_D$. We say that D is *Green-bounded* if $\sup_{x\in D}\int_D R_D(x,y)\mathrm{d}y = \sup_{x\in D}\mathbf{E}_x[\tau_D] < \infty$, equivalently $\mathfrak{m} \in S^1_{D_0}(\mathbf{X}_D)$, where $\mathfrak{m}$ is the d-dimensional Lebesgue measure on D. Suppose that $\nu(\mathrm{d}x) = \eta(\mathrm{d}x) - \mu_2(\mathrm{d}x) := (\frac{1}{2}|\nabla u|^2 + \mu_1)(\mathrm{d}x) - \mu_2(\mathrm{d}x)$ satisfies the following condition:

$$
(5.8) \qquad \begin{cases} d=1, & D \text{ is bounded and } |\nu|(D)<\infty \text{ or,} \\ d=2, & D \text{ is Green-bounded and } |\nu|\in S^1_K(\mathbf{X}) \text{ with } |\nu|(D)<\infty \text{ or,} \\ d\geq 3, & |\nu|\in S^1_K(\mathbf{X}) \text{ with } |\nu|(D)<\infty. \end{cases}
$$

Then we can see $|\nu| \in S^1_{CK_\infty}(\mathbf{X})$ (see [15, Example 4.1]). Hence, Corollary 5.1 implies the following equivalence:

$$
\lambda^{\mathcal{Q}_D}(\eta) > 0 \iff (\widetilde{\mathbf{MP}})_3 \quad \text{holds,}
$$

where

$$
\lambda^{\mathcal{Q}_D}(\eta) = \inf\left\{ \mathcal{Q}_D(f,f) \;\middle|\; f \in C_0^\infty(D),\ \int_D f(x)^2\left(\frac{1}{2}|\nabla u|^2 + \mu_1\right)(\mathrm{d}x) = 1 \right\}
$$

with $\mathcal{Q}_D(f,f) := \frac{1}{2}\mathbf{D}_D(f,f) + \frac{1}{2}\mathbf{D}_D(f^2,u) - \int_D f^2 \mathrm{d}\mu$ and $\mathbf{D}_D(f,g) := \int_D \nabla f(x)\cdot\nabla g(x)\,\mathrm{d}x$. Moreover, if D is Green-bounded, then the condition $\lambda^{\mathcal{Q}_D}(\eta) > 0$ is equivalent to

$$
\lambda^{\mathcal{Q}_D}(\mathfrak{m}) = \inf\left\{ \mathcal{Q}_D(f,f) \;\middle|\; f \in C_0^\infty(D),\ \int_D f(x)^2 \mathfrak{m}(\mathrm{d}x) = 1 \right\} > 0
$$

in view of [15, Corollary 1.2]. Hence we have

$$
(5.9) \qquad \lambda^{\mathcal{Q}_D}(\mathfrak{m}) > 0 \iff (\widetilde{\mathbf{MP}})_3 \quad \text{holds.}
$$

Note that the result (5.9) can be regarded as an extension of [1, Theorem 1.1] (or [29, Example 4.4]) in which the authors proved (5.9) under $u \equiv 0$ and $\mathfrak{m}(D) < \infty$.

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