

ON SYMMETRIC EQUIVALENCE OF SYMMETRIC UNION DIAGRAMS

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(Received February 5, 2020)

Abstract

Eisermann and Lamm introduced a notion of symmetric equivalence among symmetric union diagrams and studied it using a refined form of the Jones polynomial. We introduced invariants of symmetric equivalence via refined versions of topological spin models and provided a partial answer to a question left open by Eisermann and Lamm. In this paper we adopt a new approach to the symmetric equivalence problem and give a complete answer to the original question left open by Eisermann and Lamm.

1. Introduction

Eisermann and Lamm introduced a notion of symmetric equivalence among symmetric union diagrams and defined a Laurent polynomial invariant under symmetric equivalence [2]. The authors of the present paper tackled the problem of symmetric equivalence by considering a stronger version of symmetric equivalence and using topological spin models to define invariants for both types of equivalence [1]. Here we introduce a different approach to study symmetric equivalence and, as an application, we resolve a question left open in both [2] and [1]. In Subsections 1.1, 1.2 and 1.3 we collect the necessary background material and in Subsection 1.4 we state our results.

1.1. Symmetric diagrams and symmetric equivalences. The involution of the real two-plane $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $\rho(x, y) = (-x, y)$ fixes the axis $\ell = \{0\} \times \mathbb{R} \subset \mathbb{R}^2$ pointwise. We declare two diagrams $D, D' \subset \mathbb{R}^2$ to be identical if one is sent to the other by an orientation-preserving diffeomorphism $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $h \circ \rho = \rho \circ h$ and $h(D) = D'$. An oriented link diagram $D \subset \mathbb{R}^2$ is *symmetric* if $\rho(D)$ is obtained from D by changing the orientation and switching all the crossings on the axis. A symmetric diagram D is a *symmetric union* if ρ sends each component \widehat{D} of D to itself in an orientation-reversing fashion, implying that \widehat{D} crosses the axis perpendicularly in exactly two non-crossing points. Figure 1 illustrates the symmetric union diagrams D_4 and D'_4 , first considered by Eisermann and Lamm [2]. Following [2], we define a *symmetric Reidemeister move off the axis* as an ordinary Reidemeister move on a symmetric diagram carried out, away from the axis ℓ , together with its mirror-symmetric counterpart with respect to ℓ . A *symmetric Reidemeister move on the axis* is one of the moves illustrated in Figure 2. Eisermann and Lamm consider also two extra moves $S1(\pm)$ and $S2(v)$, some of which are illustrated in Figure 3. It is understood

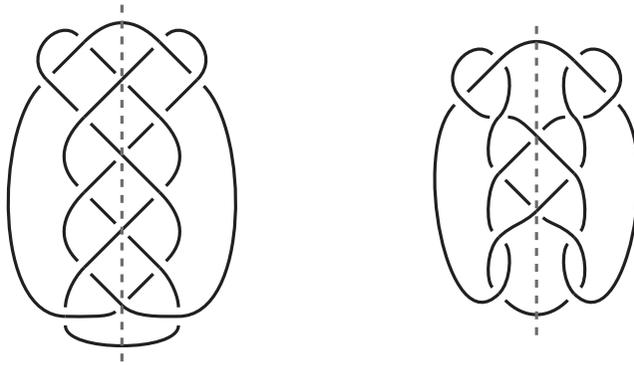


Fig. 1. The symmetric union diagrams D_4 (left) and D'_4 (right)

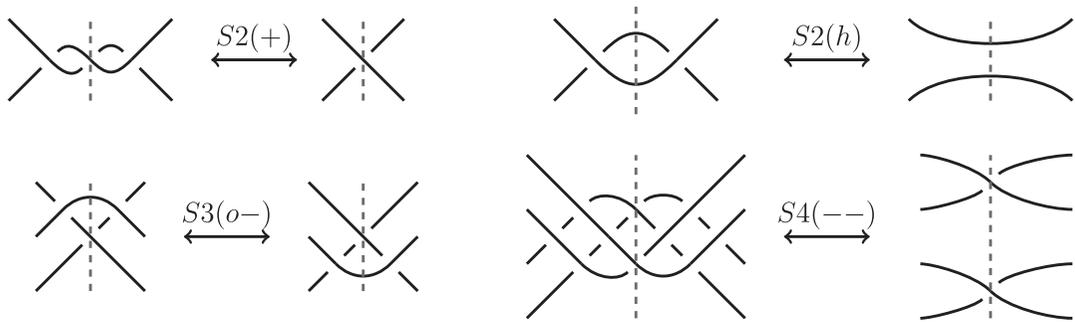


Fig. 2. Symmetric Reidemeister moves on the axis

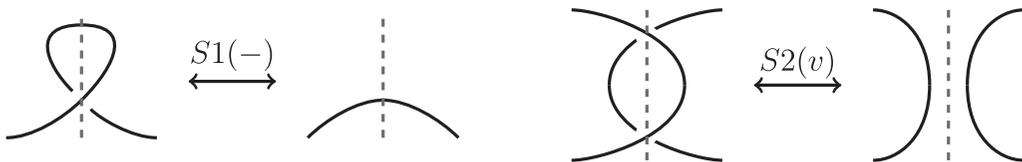


Fig. 3. Moves $S1(-)$ and $S2(v)$

that all of these moves admit variants obtained by turning the corresponding pictures upside down, mirroring, rotating them around the axis and, for the $S4$ -moves, switching one of the crossings on the axis (cf. [2, §2.3]).

DEFINITION. Two oriented, symmetric diagrams which can be obtained from each other via a finite sequence of symmetric Reidemeister moves on and off the axis (sR -moves) and $S1$ -moves will be called *symmetrically equivalent*. If they can be obtained from each other using sR -moves, $S1$ - and $S2(v)$ -moves, we will say that the diagrams are *weakly symmetrically equivalent*.

1.2. Eisermann and Lamm’s results. Eisermann and Lamm [2] showed that there exists an infinite family of pairs (D_n, D'_n) of symmetric union 2-bridge knot diagrams such that D_n and D'_n are Reidemeister equivalent but not weakly symmetrically equivalent for $n = 3$ and $n \geq 5$. D_4 and D'_4 are the diagrams of Figure 1. Eisermann and Lamm established their result

using an invariant of weak symmetric equivalence defined as follows. Let $\vec{\mathcal{D}}$ denote the set of oriented planar link diagrams $D \subset \mathbb{R}^2$ transverse to the axis $\ell = \{0\} \times \mathbb{R}$. Let $\mathbb{Z}(s^{1/2}, t^{1/2})$ be the quotient field of the ring of Laurent polynomials with integer coefficients in the variables $s^{1/2}$ and $t^{1/2}$. Eisermann and Lamm [2] define a map $W : \vec{\mathcal{D}} \rightarrow \mathbb{Z}(s^{1/2}, t^{1/2})$ such that $W(D) = W(D')$ if D and D' are weakly symmetrically equivalent. By [2, Proposition 5.6], if $D \in \vec{\mathcal{D}}$ represents an n -component link L and has no crossings on the axis then

$$(1.1) \quad W(D) = \left(\frac{s^{1/2} + s^{-1/2}}{t^{1/2} + t^{-1/2}} \right)^{n-1} V_L(t),$$

where $V_L(t)$ is the Jones-polynomial of the link L , normalized so that on the n -component unlink it takes the value $(-t^{1/2} - t^{-1/2})^{n-1}$. Moreover, if D has crossings on the axis, then the following skein-like recursion formulas hold:

$$(1.2) \quad W\left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \vdots \\ \vdots \end{array}\right) = -s^{-1/2} W\left(\begin{array}{c} \diagup \diagdown \\ \vdots \\ \vdots \end{array}\right) - s^{-1} W\left(\begin{array}{c} \vdots \\ \vdots \end{array}\right)$$

$$(1.3) \quad W\left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \vdots \\ \vdots \end{array}\right) = -s^{1/2} W\left(\begin{array}{c} \diagdown \diagup \\ \vdots \\ \vdots \end{array}\right) - s W\left(\begin{array}{c} \vdots \\ \vdots \end{array}\right)$$

It turns out [2, Proposition 1.8] that when D is a symmetric union knot diagram, then $W(D)$ is an honest Laurent polynomial that we shall call the *refined Jones polynomial*. The diagrams D_4 and D'_4 have the same refined Jones polynomial, so the question of their weak symmetric equivalence was left unanswered in [2].

1.3. Invariants from topological spin models. The theory of topological spin models for links in S^3 was introduced in [3]. Here we follow the reformulation used in [1], to which we refer the reader for further details. Fix an integer $n \geq 2$, denote by $\text{Mat}_n(\mathbb{C})$ the space of square $n \times n$ complex matrices, and let $d \in \{\pm\sqrt{n}\}$. Given a symmetric, complex matrix $W^+ \in \text{Mat}_n(\mathbb{C})$ with non-zero entries, let $W^- \in \text{Mat}_n(\mathbb{C})$ be the matrix uniquely determined by the equation

$$(1.4) \quad W^+ \circ W^- = J,$$

where \circ is the Hadamard, i.e. entry-wise, product and J is the all-1 matrix. Define, for each matrix $X \in \text{Mat}_n(\mathbb{C})$ with non-zero entries and $a, b \in \{1, \dots, n\}$, the vector $Y_{ab}^X \in \mathbb{C}^n$ by setting

$$Y_{ab}^X(x) := \frac{X(x, a)}{X(x, b)} \in \mathbb{C}, \quad x \in \{1, \dots, n\}.$$

Then, the pair $M = (W^+, d)$ is a *spin model* if the following equations hold:

$$(1.5) \quad W^+ Y_{ab}^{W^+} = d W^- (a, b) Y_{ab}^{W^+} \quad \text{for every } a, b \in \{1, \dots, n\}.$$

The following definition was introduced in [1, Remark 1.3].

DEFINITION. A *Potts-refined spin model* is a triple $\widehat{M} = (W^+, V^+, d)$ such that:

- $M = (W^+, d)$ is a spin model;
- $V^+ = (-\xi^{-3})I + \xi(J - I)$, where ξ is one of the four complex numbers such that $d = -\xi^2 - \xi^{-2}$.

Let $\widehat{M} = (W^+, V^+, d)$ be a Potts-refined spin model, D a symmetric union diagram and c a chequerboard colouring of $\mathbb{R}^2 \setminus D$. Let Γ_D be the planar, signed medial graph associated

to the black regions of c . Let Γ_D^0, Γ_D^1 be the sets of vertices, respectively edges of Γ_D and let $N = |\Gamma_D^0|$. Given $e \in \Gamma_D^1$, we denote by v_e and w_e (in any order) the vertices of e . The set Γ_D^1 contains the set Γ_ℓ^1 of edges corresponding to crossings on the axis. Let $V^- = (-\xi^3)I + \xi^{-1}(J - I)$, and define the *partition function* $Z_{\widehat{M}}(D, c)$ by the formula

$$Z_{\widehat{M}}(D, c) := d^{-N} \sum_{\sigma: \Gamma_D^0 \rightarrow \{1, \dots, n\}} \prod_{e \in \Gamma_\ell^1} V^{s(e)}(\sigma(v_e), \sigma(w_e)) \prod_{e \in \Gamma_D^1 \setminus \Gamma_\ell^1} W^{s(e)}(\sigma(v_e), \sigma(w_e)),$$

where $s(e) \in \{+, -\}$ is the sign of the edge e , and the *normalized partition function* $I_{\widehat{M}}(D, c)$ by

$$I_{\widehat{M}}(D, c) := (-\xi)^{p_\ell(D) - n_\ell(D)} Z_{\widehat{M}}(D, c),$$

where $p_\ell(D)$ and $n_\ell(D)$ denote, respectively, the numbers of positive and negative crossings on the axis. When D is not connected $Z_{\widehat{M}}(D, c)$ and $I_{\widehat{M}}(D, c)$ are defined as the product of the values of $Z_{\widehat{M}}$ and, respectively, $I_{\widehat{M}}$ on its connected components with the induced colourings. It turns out [1] that the complex number $I_{\widehat{M}}(D, c)$ is independent of the choice of c , so we can write more simply $I_{\widehat{M}}(D)$. Moreover, by a special case of [1, Theorem 1.3], if D and D' are oriented, weakly symmetrically equivalent symmetric union diagrams, then

$$(1.6) \quad I_{\widehat{M}}(D) = I_{\widehat{M}}(D').$$

In [1, Subsection 4.2] we showed that, for a suitable choice of M , the invariant $I_{\widehat{M}}$ defined above can distinguish, up to weak symmetric equivalence, infinitely many Reidemeister equivalent symmetric union diagrams. We also showed [1, Subsection 4.2] that more general invariants can distinguish the diagrams D_4 and D'_4 up to symmetric equivalence, but we were unable to use invariants coming from spin models to rule out that the diagrams D_4 and D'_4 of Figure 1 are *weakly* symmetrically equivalent.

1.4. Statements of results. Given a symmetric union diagram D and an integer $h \in \mathbb{Z}$, define a new symmetric union diagram $D(h)$ by replacing each crossing on the axis with $|h|$ consecutive crossings, having the same or opposite type depending on the sign of h . The precise convention is specified in Figure 4, where a number $m = \pm h$ inside a box denotes a sequence of $|m|$ consecutive half-twists on the axis, each of sign equal to $\text{sgn}(m)$, the sign of m . Note that, in particular, $D = D(1)$ for every diagram D .

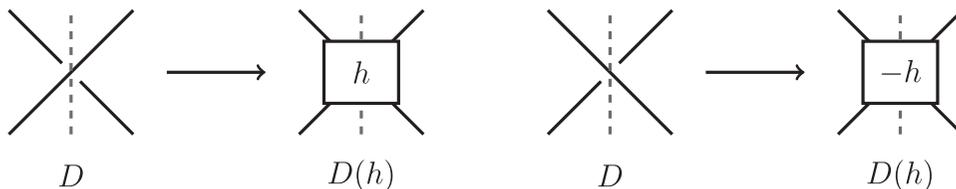


Fig.4. Definition of $D(h)$, $h \in \mathbb{Z}$

The following theorem is established in Section 2.

Theorem 1. *Let D and D' be two symmetric diagrams. If D and D' are (weakly) symmetrically equivalent, then $D(h)$ and $D'(h)$ are (weakly) symmetrically equivalent for each $h \in \mathbb{Z}$.*

Clearly, if for any integer $h \in \mathbb{Z}$ the diagrams $D(h)$ and $D'(h)$ can be shown to be (weakly) symmetrically inequivalent, it follows from Theorem 1 that D and D' cannot be (weakly) symmetrically equivalent. It is therefore natural to ask whether the weak symmetric equivalence of D_4 and D'_4 could be decided by showing that $D_4(h)$ and $D'_4(h)$ have different refined Jones polynomials or different Potts-refined spin model invariants. It turns out that this is impossible: in Section 3 we show that, for any $h \in \mathbb{Z}$, the diagrams $D_4(h)$ and $D'_4(h)$ have the same refined Jones polynomial and Potts-refined spin model invariants. Nevertheless, in Section 4 we use Theorem 1 to prove the following.

Theorem 2. *The Reidemeister equivalent symmetric union diagrams D_4 and D'_4 are not weakly symmetrically equivalent.*

Notice that Theorem 2 resolves the question left open in [2, 1] about the weak symmetric equivalence of the diagrams D_4 and D'_4 . The proof of Theorem 2 is based on the simple fact that if two symmetric union diagrams D and D' are symmetrically equivalent then they are, in particular, Reidemeister equivalent and therefore represent the same link in S^3 . Thus, to prove Theorem 2 it suffices to show that the knots K and K' , represented respectively by the diagrams $D_4(2)$ and $D'_4(2)$, are distinct. This can be accomplished in a number of ways. We sketch a few, and provide the details of a computation showing that K and K' have different torsion numbers.

The paper is organized as follows. In Section 2 we prove Theorem 1. In Section 3 we show that $D_4(h)$ and $D'_4(h)$ have the same refined Jones polynomial and Potts-refined spin model invariants. In Section 4 we prove Theorem 2.

2. Proof of Theorem 1

The following Lemmas 1 and 2 deal with generalizations of, respectively, the $S4$ -move and the $S2$ -move. The lemmas play a key rôle in the proof of Theorem 1.

Lemma 1. *Suppose that the symmetric diagrams D and D' differ by the $S4(m, n)$ -move defined in Figure 5. Then, D and D' are connected by a sequence of symmetric Reidemeister moves off the axis and $S4$ -moves. In particular, D and D' are symmetrically equivalent.*

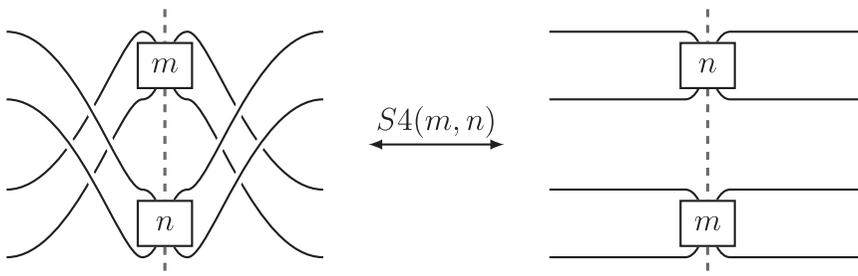


Fig.5. Definition of the $S4(m, n)$ -move, where m and n are integers denoting the number of crossings on the axis with the appropriate signs.

Proof. Since when $n = 0$ or $m = 0$ an $S4(m, n)$ -move reduces to a symmetric pair of second Reidemeister moves off the axis, we may assume without loss of generality that $mn \neq 0$. We suppose first that m and n are both positive and we establish the statement by

induction on m and n . The basis of the induction holds because an $S4(1, 1)$ -move is just an ordinary $S4$ -move. Assume that the statement holds for $S4(h, k)$ -moves with $1 \leq h < m$ and $1 \leq k < n$. The inductive step is established by proving the statement for $S4(m, k)$ -moves and $S4(h, n)$ -moves. Figure 6 shows that an $S4(m, k)$ -move can be decomposed into a sequence of symmetric Reidemeister moves and $S4(h, k)$ -moves with $1 \leq h < m$. More precisely, to

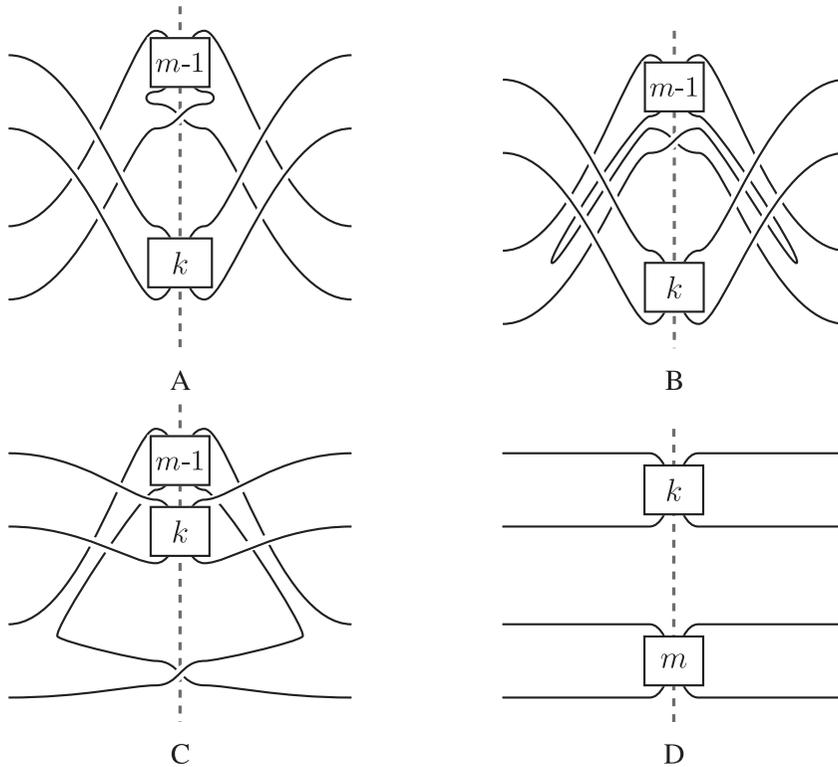


Fig.6. Decomposition of an $S4(m, k)$ -move.

go from Figure 6A to Figure 6B we use two symmetric second Reidemeister moves off the axis, to go from Figure 6B to Figure 6C one $S4(1, k)$ -move and to go from Figure 6C to Figure 6D one $S4(m - 1, k)$ -move. A similar sequence of moves can be used to prove the inductive step for an $S4(h, n)$ -move.

For the other choices of signs of m and n the argument is essentially the same, except that one needs to perform the double induction on $|m|$ and $|n|$ and modify accordingly Figure 6 and its analogue for the $S4(h, n)$ -move. The obvious details are left to the reader. \square

Lemma 2. *Suppose that the symmetric diagrams D and D' differ by the $S2(\pm, n)$ -move defined in Figure 7. Then, D and D' are connected by a sequence of symmetric Reidemeister moves off the axis, $S4$ -moves and $|n|$ $S2$ -moves. In particular, D and D' are symmetrically equivalent.*

Proof. Note that the statement is obvious for $n = 0, 1, -1$. We describe the proof for the $S(-, n)$ -move with $n < 0$ because the other cases can be proved similarly. We are going to argue by induction on n , so we start assuming that the statement is true for $S(-, k)$ -moves with $n < k \leq -1$. Performing a symmetric $R2$ -move on the left-hand side tangle of Fig-

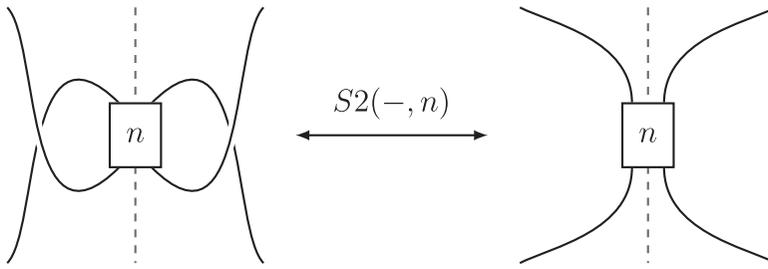


Fig. 7. Definition of the $S2(-, n)$ -move. The $S2(+, n)$ -move is obtained by switching the crossings off the axis in the above picture.

ure 7 we obtain the tangle of Figure 8A. After an $S4(-1, n + 1)$ -move and some symmetric Reidemeister moves off the axis, the tangle of Figure 8A can be modified into the tangle in Figure 8B. By Lemma 1 this means that the tangles of Figures 8A and 8B are obtained from each other via a sequence of $S4$ -moves and Reidemeister moves off the axis. By a third Reidemeister move off the axis followed by a second Reidemeister move off the axis, the tangle of Figure 8B can be turned into the tangle in Figure 8C. Now we make use of the inductive hypothesis and perform an $S(-, n + 1)$ -move to get the tangle of Figure 8D. Finally, a single $S2$ -move leads us from Figure 8D to the right-hand side of Figure 7, concluding the proof. \square

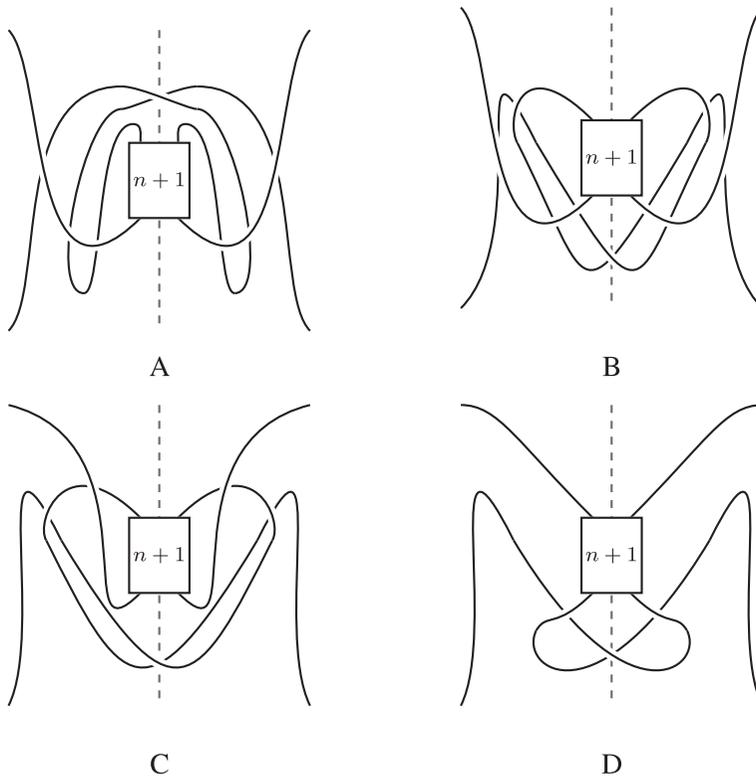


Fig. 8. Decomposition of an $S2(\pm, n)$ -move.

Proof of Theorem 1. We will argue that, whenever the diagrams D and D' differ by a symmetric Reidemeister move M , then $D(n)$ and $D'(n)$ are weakly symmetrically equivalent if M is of type $S2(v)$, and symmetrically equivalent otherwise. This is clear for symmetric Reidemeister moves off the axis and $S2(h)$ -moves because they do not involve crossings on the axis. Suppose that D' is obtained from D by applying an $S1$ -move. Then, it is immediate that $D'(n)$ is obtained from $D(n)$ by applying $|n|$ $S1$ -moves. A similar reasoning applies to $S2(v)$ -moves and $S3$ -moves, and since all the verifications are very simple we leave them to the reader. If D' is obtained from D by an $S4(\varepsilon_1, \varepsilon_2)$ -move with $\varepsilon_i \in \{\pm\}$, then $D'(n)$ is obtained from $D(n)$ via an $S4(\varepsilon_1 n, \varepsilon_2 n)$ -move and by Lemma 1 $D(n)$ and $D'(n)$ are symmetrically equivalent. Finally, if D' is obtained from D by an $S2(\pm)$ -move, then $D'(n)$ is obtained from $D(n)$ via a $S2(\pm, n)$ -move and by Lemma 2 $D(n)$ and $D'(n)$ are symmetrically equivalent. \square

3. Negative results for W and the Potts-refined spin model invariants

Our aim in this section is to show that the diagrams $D_4(h)$ and $D'_4(h)$ cannot be distinguished up to any symmetric equivalence using either Eisermann and Lamm’s refined Jones polynomial or any invariant coming from a Potts-refined topological spin model. This will be established in Corollary 1. We start with the following Proposition 1, which will be used in the proof of Corollary 1. Recall that $\vec{\mathcal{D}}$ denotes the set of oriented planar link diagrams $D \subset \mathbb{R}^2$ transverse to the axis $\ell = \{0\} \times \mathbb{R}$. Let U^m , for $m \geq 1$, denote any crossingless, symmetric union diagram of the m -component unlink.

Proposition 1. *Let R be a ring and $\Psi : \vec{\mathcal{D}} \rightarrow R$ a map such that*

- (1) $\Psi(D) = \Psi(D')$ if D and D' are weakly symmetrically equivalent;
- (2) $\Psi\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = a_+ \Psi\left(\begin{array}{c} \frown \\ \smile \end{array}\right) + b_+ \Psi\left(\begin{array}{c} \vdots \\ \vdots \end{array}\right) \left(\begin{array}{c} \circ \\ \circ \end{array}\right)$ and $\Psi\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right) = a_- \Psi\left(\begin{array}{c} \frown \\ \smile \end{array}\right) + b_- \Psi\left(\begin{array}{c} \vdots \\ \vdots \end{array}\right) \left(\begin{array}{c} \circ \\ \circ \end{array}\right)$, with
- (3) $b_+ b_- = 1$ and $(a_+ b_- + a_- b_+) \Psi(U^2) + a_+ a_- \Psi(U^3) = 0$.

Then, $\Psi(D_4(h)) = \Psi(D'_4(h))$ for each $h \in \mathbb{Z}$.

Proof. Clearly $D_4(0) = D'_4(0)$. We only prove the statement for $h > 0$ because the proof for $h < 0$ is essentially the same. Let $D_4(t, h, b)$, $t, b \geq 0$, $h \geq 1$, be the diagram obtained from D_4 by replacing the top (respectively bottom) crossing on the axis with t (respectively b) consecutive crossings on the axis of the same sign, and each of the other crossings on the axis with h consecutive crossings on the axis of the same sign. Observe that $D_4(h, h, h) = D_4(h)$ and $D_4(0, h, 0) = D'_4(h)$ for each $h \geq 1$. Therefore, it suffices to prove that $\Psi(D_4(k, h, k)) = \Psi(D_4(h))$ for each $h \geq k \geq 0$ with $h \geq 1$. This follows by an easy downward induction on k starting from $k = h \geq 1$ once we show that the equality $\Psi(D_4(k, h, k)) = \Psi(D_4(k - 1, h, k - 1))$ holds. It will be convenient to use the following terminology and notation. We call a horizontal resolution of a crossing on the axis a *0-resolution* and a vertical resolution of such a crossing a *1-resolution*. We denote by D_{xy} , with $x, y \in \{0, 1\}$, any symmetric union diagram obtained from $D_4(k, h, k)$ by an x -resolution of any of its k top crossings on the axis and a y -resolution of any of its k bottom crossings on the axis. It is easy to check that D_{01} and D_{10} are weakly symmetrically equivalent to U^2 , D_{00} is weakly symmetrically equivalent to U^3 and $D_{11} = D_4(k - 1, h, k - 1)$. A simple calculation using Assumption (2) yields

$$\Psi(D_4(k, h, k)) = b_+b_- \Psi(D_4(k - 1, h, k - 1)) + (a_+b_- + a_-b_+) \Psi(U^2) + a_+a_- \Psi(U^3),$$

which by (3) gives the claimed equality $\Psi(D_4(k, h, k)) = \Psi(D_4(k - 1, h, k - 1))$. □

Corollary 1. *For any $h \in \mathbb{Z}$, the refined Jones polynomial and any Potts-refined spin model invariant take the same values on $D_4(h)$ and $D'_4(h)$.*

Proof. Let W be the refined Jones polynomial from [2]. We recalled in Subsection 1.2 that W satisfies Assumption (1) of Proposition 1. By Equation (1.1), since $V_{U^m}(t) = (-t^{1/2} - t^{-1/2})^{m-1}$ we obtain $W(U^m) = (-s^{1/2} - s^{-1/2})^{m-1}$. Together with Equations (1.2) and (1.3) this immediately implies that W satisfies Assumptions (2) and (3) of Proposition 1 and therefore $W(D_4(h)) = W(D'_4(h))$ for every $h \in \mathbb{Z}$.

Let $\widehat{M} = (W^+, V^+, d)$ be a Potts-refined spin model. By Equation (1.6) we know that $I_{\widehat{M}}$ satisfies Assumption (1) of Proposition 1, and it follows immediately from the definition that $I_{\widehat{M}}(U^m) = d^m$. Using that $V^+ = (-\xi^{-3} - \xi)I + \xi J = d\xi^{-1}I + \xi J$, $V^- = -\xi^3I + \xi^{-1}(J - I) = (-\xi^3 - \xi^{-1})I + \xi^{-1}J = d\xi I + \xi^{-1}J$ and $d = -\xi^{-2} - \xi^2$, it is straightforward to check that

$$I_{\widehat{M}}\left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}\right) = -\xi^{-2}I_{\widehat{M}}\left(\begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array}\right) - \xi^{-4}I_{\widehat{M}}\left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}\right) \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}\right) \text{ and } I_{\widehat{M}}\left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}\right) = -\xi^2I_{\widehat{M}}\left(\begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array}\right) - \xi^4I_{\widehat{M}}\left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}\right) \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}\right).$$

Thus, $I_{\widehat{M}}$ satisfies Assumptions (2) and (3) of Proposition 1 and $I_{\widehat{M}}(D_4(h)) = I_{\widehat{M}}(D'_4(h))$ for every $h \in \mathbb{Z}$. □

4. Proof of Theorem 2

In this section we show that the diagrams $D_4(2)$ and $D'_4(2)$ represent distinct knots K and K' . Since this implies that $D_4(2)$ and $D'_4(2)$ are not Reidemeister equivalent, combining this fact with Theorem 1 yields Theorem 2.

As one referee pointed out to us, to show that K and K' are distinct it is possible to use both the Kauffmann polynomial and the colored Jones polynomial. Another possibility is to show that K and K' have different second Alexander ideals. In a previous version of this paper ¹ we worked out the details of the computation of the second Alexander ideals for the infinite families of knots $\{K_s\}$ and $\{K'_s\}$ given by the diagrams $D_{4s}(2)$ and $D'_{4s}(2)$, $s \geq 1$. As it turns out, the second Alexander ideals distinguish K_s from K'_s for each $s \geq 1$. In this paper we just prove that K and K' are distinct using their third cyclic branched covers.

Lemma 3. *Let $\Sigma = \Sigma_3(K)$ and $\Sigma' = \Sigma_3(K')$ denote the three-fold branched covers of K and K' , respectively. Then, we have the following isomorphisms of Abelian groups:*

$$H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \quad \text{and} \quad H_1(\Sigma'; \mathbb{Z}) \cong \mathbb{Z}/49\mathbb{Z} \oplus \mathbb{Z}/49\mathbb{Z}.$$

Proof. We start by computing Seifert matrices for K and K' . Consider the Seifert surface Σ for $D_4(2)$ and the basis for its first homology group shown in Figure 9. The generators are divided into two groups, each of which is shown separately in Figure 9 to maximize readability. It is straightforward to check that the associated Seifert matrix is

¹<https://arxiv.org/abs/1901.10270v2>

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

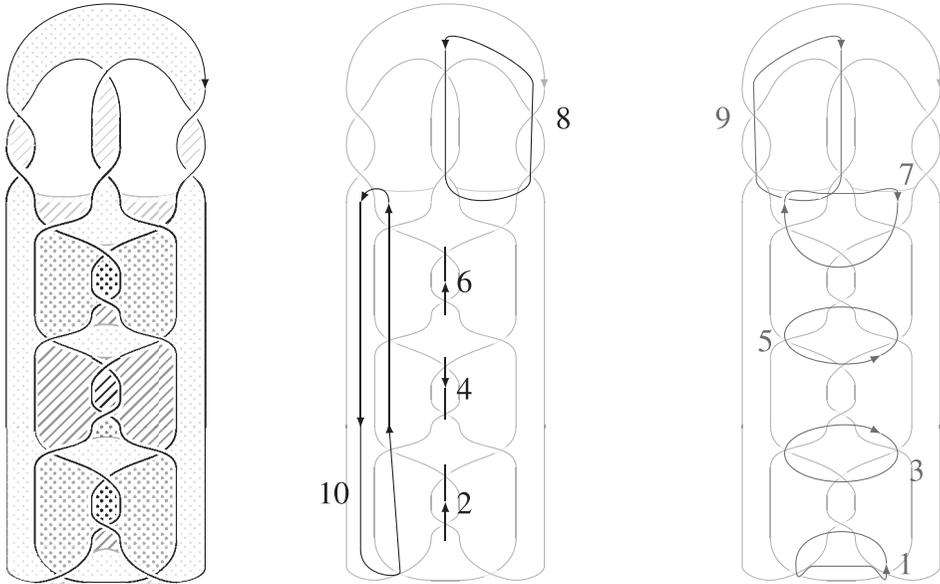


Fig. 9. A Seifert surface Σ for $D_4(2)$ and a basis of $H_1(\Sigma; \mathbb{Z})$

Next, we consider a Seifert surface Σ' for $D'_4(2)$ and the basis for $H_1(\Sigma', \mathbb{Z})$ illustrated in Figure 10. As before, the generators are divided into two groups, which are shown separately. It is easy to check that the corresponding Seifert matrix V' is the following:

$$V' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}$$

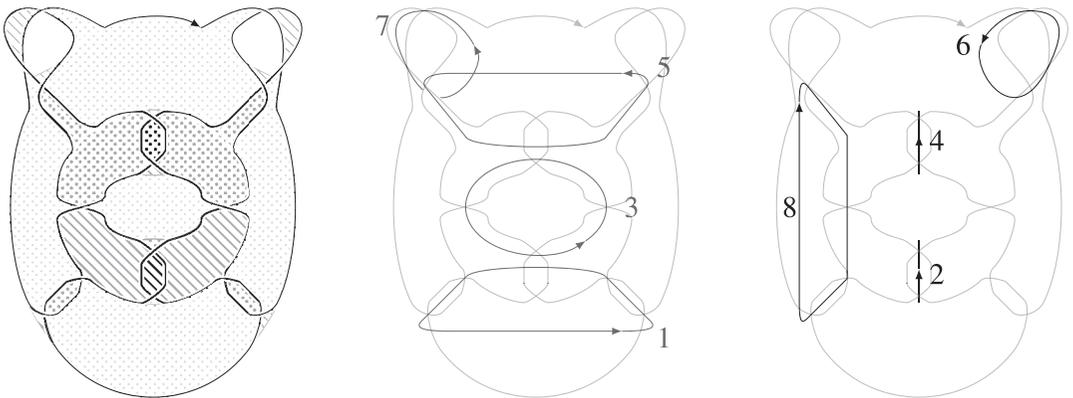


Fig. 10. A Seifert surface Σ' for $D'_4(2)$ and a basis of $H_1(\Sigma', \mathbb{Z})$

Recall [4, Satz I] (see also [5, Theorem 3]) that, given a knot K with Seifert matrix V , a presentation matrix for $H_1(\Sigma_3(K); \mathbb{Z})$ is $\Gamma_V^3 - (\Gamma_V - I)^3$, where $\Gamma_V = -V(tV - V)^{-1}$ and I is the identity matrix. The statement follows by computing the elementary divisors of the matrices Γ_V and $\Gamma_{V'}$. \square

Proof of Theorem 2. The statement is an immediate consequence of Lemma 3 and Theorem 1. \square

References

- [1] C. Collari and P. Lisca: *Symmetric union diagrams and refined spin models*, *Canad. Math. Bull.* **62** (2019), 451–468.
- [2] M. Eisermann and C. Lamm: *A refined Jones polynomial for symmetric unions*, *Osaka J. Math.* **48** (2011), 333–370.
- [3] V. Jones: *On knot invariants related to some statistical mechanical models*, *Pacific J. Math.* **137** (1989), 311–334.
- [4] H. Seifert: *Über das Geschlecht von Knoten*, *Math. Ann.* **110** (1935), 571–592.
- [5] H. Trotter: *Homology of group systems with applications to knot theory*, *Ann. of Math.* **76** (1962), 464–498.

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