Yang, X.D.
Osaka J. Math.
58 (2021), 711–729

INVARIANT MANIFOLDS FOR NONAUTONOMOUS STOCHASTIC EVOLUTION EQUATION

Xiangdong YANG

(Received October 23, 2019, revised May 28, 2020)

Abstract

New results pertaining to the invariant manifolds of stochastic partial differential equations are presented. We prove the existence of local and global invariant manifolds for a non-autonomous stochastic evolution equation. These manifolds are constituted by trajectories of the solutions belonging to particular function spaces and the theory of Ornstein-Uhlenbeck process.

1. Introduction.

Consider the non-autonomous stochastic evolution equation

$$dx(t) = [A(t)x(t) + F(t, x(t))]\,dt + x(t)\,dW(t),$$

where $A(t)$ is in general an unbounded linear operator on a Banach space X for every fixed t and $x(t)\,dW(t)$ is a noise satisfies suitable conditions which will be established later, $F : \mathbb{R}_+ \times X \to \mathbb{R}$ is a nonlinear function.

In recent years, for autonomous stochastic evolution equations, existence, uniqueness, stability, invariant measures, invariant manifolds and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by many authors. It is well known that these topics have been developed mainly by semigroup approach.

In [9], the following nonlinear stochastic evolution equation is considered:

$$dx(t) = [Ax(t) + F(x(t))]\,dt + x(t)\,dW(t),$$

where A is a generator of a C_0-semigroup e^{At} satisfying an exponential dichotomy condition. The existence and smoothness of Lipschitz continuous stable and unstable manifolds are proved by the Lyapunov-Perron's method. Similar problems are investigate in [8], [10] and [11]. Although many researchers paid attention to study the non-autonomous stochastic evolution equations which have been widely used to describe abrupt changes such as shocks, harvesting, and natural disasters, as far as we know there is very few attention paid to invariant manifolds for such equations (see [6], [3], [4], [15], [16] and [17] for example).

Instead of using the smallness of Lipschitz constants in classical sense, the concept of admissible spaces is used to construct the invariant manifolds for

$$dx(t) = [A(t)x(t) + F(t, x(t))]\,dt,$$

2020 Mathematics Subject Classification. Primary 60H15; Secondary 37L55, 37L25, 37D10.

Supported by National Natural Science Foundation of China (Grant No. 11261024).

where $A(t)$ is in general an unbounded linear operator on a Banach space X for every fixed t and $F(t, x(t))$ is a nonlinear operator (see [13] and [14]). These invariant manifolds are constituted by trajectories of the solutions belonging to admissible function spaces. However, to the best of our knowledge, the invariant manifolds problem for non-autonomous stochastic system (1) has not been investigated yet. Motivated by this consideration, in this paper we will study the existence of local and global invariant manifolds for a non-autonomous stochastic evolution equation. Specifically, we study the invariant manifolds for a non-autonomous stochastic evolution equation (1) under the assumption that the family of operators $(U(t, s))_{t\geq s\geq 0}$ on a Banach space X associated with $A(t)$ is said to be a (strongly continuous, exponential bounded) evolution family and the nonlinear term $F(t, x)$ is Lipschitz continuous. In fact, the results in this paper are motivated by the recent work of [8], [9], [10], [11] and the invariant manifolds discussed in [13], [14]. The main tools used in this paper are stochastic analysis techniques, theory of Ornstein-Uhlenbeck process and characterization of the exponential dichotomy of evolution equations in particular spaces of functions defined on the half-line. Note that the Lyapunov-Perron's method which is frequently used in [8], [9], [10], [11] can not be applied to (1). We will use the method to construct invariant manifold in [13], [14], where the work is actually done for nonlinear operators.

In Section 2, we recall some basic concepts and results for stochastic partial differential equations, evolution family and random dynamical systems. The existence of local stable manifold for (1) is proved in Section 3. In Section 4, we prove the existence of stable manifold for (1).

2. Preliminaries.

In this section, we recall some basic background knowledge on stochastic non-autonomous partial differential equations, theory of Ornstein-Uhlenbeck process and characterization of the exponential dichotomy of evolution equation.

2.1. Stochastic non-autonomous partial differential equations. Denote by H an infinite dimensional separable Hilbert space with norm $\|\cdot\|$. Consider the stochastic non-autonomous partial differential equation

$$dx(t) = \left[A(t)x(t) + F(t, x(t))\right] dt + x(t)\, dW(t), \tag{1}$$

where $x \in H$, $W(t)$ is the standard $\mathbb{R}$-valued Wiener process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, furthermore, $x(t)\, dW(t)$ is interpreted as a Stratonovich stochastic differential (see [9]), $A(t)$ is in general an unbounded linear operator on a Banach space X for every fixed t, $F : \mathbb{R}_+ \times X \to \mathbb{R}$ is a nonlinear function.

In the case of unbounded $A(t)$, we may use the evolution family $(U(t, s))_{t\geq s\geq 0}$ arising in well-posed homogeneous Cauchy problems. Let us recall the definition of an evolution family (see [1]).

DEFINITION 2.1. A family of operators $(U(t, s))_{t\geq s\geq 0}$ on a Banach space X associated with $A(t)$ is said to be a (strongly continuous, exponential bounded) evolution family on the half-line if the following conditions hold:
(a) $U(s, s) = I, U(t, s) = U(t, \tau)U(\tau, s)$ for $t \geq \tau \geq s \geq 0$.

(b)the map $(t, s) \to U(t, s)x$ is continuous for every $x \in X$.
(c)there are constants $K, c \geq 0$ such that $\|U(t, s)\| \leq Ke^{c(t-s)}$ for $t \geq s \geq 0$.

We recall the notion of exponential dichotomy in the following definition.

DEFINITION 2.2. An evolution family $(U(t, s))_{t\geq s\geq 0}$ on a Banach space X is said to have exponential dichotomy if there are projections $P(t), t \in \mathbb{R}_+$, uniformly bounded and strongly continuous in t, and constants $M, \delta > 0$ such that
(a) $U(t, s)P(s) = P(t)U(t, s)$ for all $t \geq s \geq 0$;
(b) the restriction $U(t, s)_| : \ker P(s) \to \ker P(t)$ is an isomorphism for all $t \geq s \geq 0$ (and we denote its inverse by $U(s, t)_| : \ker P(t) \to \ker P(s)$;
(c) $\|U(t, s)P(s)\| \leq Me^{-\delta(t-s)}$ and $\|U_Q(s, t)Q(t)\| \leq Me^{-\delta(t-s)}$ for all $t \geq s \geq 0$.

Here and below $Q := I - P$. If $P(t) = I$ for $t \in \mathbb{R}$, then $(U(t, s))_{t\geq s\geq 0}$ is exponentially stable. We also denote by $X_0(t) := P(t)$ and $X_1(t) := Q(t) = I - P(t)$.

DEFINITION 2.3. If U is a hyperbolic evolution family, then

$$\Gamma(t, s) := \begin{cases} U(t, s)P(s) & \text{if } t \geq s, t, s \in \mathbb{R}, \\ -U_Q(t, s)Q(s) & \text{if } t < s, t, s \in \mathbb{R}, \end{cases}$$

is called Greens function corresponding to U and $P(\cdot)$.

Also $\Gamma(t, s)$ satisfies the estimate

$$\|\Gamma(t, s)\| \leq Me^{-\delta|t-s|} \tag{2}$$

for $t \neq s \geq 0$.

From [16] and [17], we can conclude that (1) has a uniqueness mild solution which is given by

Lemma 2.1. *Suppose that $W(t)$ is the standard $\mathbb{R}$-valued Wiener process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. If F is strongly measurable, adapted and assumed and locally Lipschitz continuous on H*

$$\|F(t, x) - F(t, y)\| \leq L\|x - y\|, \ L > 0, \tag{3}$$

and

$$\|F(t, x)\| \leq C(1 + \|x\|), \tag{4}$$

then (1) *has a unique solution which can be written as follows in a mild sense*

$$x(t) = U(t, 0)x(0) + \int_0^t U(t, \tau)F(\tau, x(\tau))d\tau + \int_0^t U(t, \tau)x(\tau)dW. \tag{5}$$

2.2. Conjugated random PDEs. Following [2], [8], [9] and [5], we also recall some basic concepts in random dynamical systems. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. A flow θ of mappings $\{\theta_t\}_{t\in\mathbb{R}}$ is defined on the sample space Ω such that

$$\theta : \mathbb{R} \times \Omega \to \Omega, \ \theta_0 = id, \ \theta_{t_1}\theta_{t_2} = \theta_{t_1+t_2},$$

for $t_1, t_2 \in \mathbb{R}$. We call $(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}, \theta)$ a metric dynamical system if the flow θ is supposed to be $(\mathfrak{B}(\mathbb{R}) \bigotimes \mathfrak{F}, \mathfrak{F})$- measurable, where $\mathfrak{B}(\mathbb{R})$ is the σ-algebra of Borel sets on the real line $\mathbb{R}$, in addition, the measure $\mathbb{P}$ is assumed to be ergodic with respect to $\{\theta_t\}_{t\in\mathbb{R}}$.

Let $W(t)$ be a two-sided Wiener process with trajectories in the space $C_0(\mathbb{R}, \mathbb{R})$ of real continuous functions defined on $\mathbb{R}$, taking zero value at $t = 0$. This set is equipped with the compact open topology. On this set we consider the measurable flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$, defined by $\theta_t\omega = \omega(\cdot + t) - \omega(t)$. The distribution of this process generates a measure on $\mathfrak{B}(C_0(\mathbb{R}, \mathbb{R}))$ which is called the Wiener measure. Note that this measure is ergodic with respect to the above flow. We shall consider, instead of the whole $C_0(\mathbb{R}, \mathbb{R})$, a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant $\Omega \subset C_0(\mathbb{R}, \mathbb{R})$ of $\mathbb{P}$ measure one and the trace σ-algebra $\mathfrak{F}$ of $\mathfrak{B}(C_0(\mathbb{R}, \mathbb{R}))$ with respect to Ω. A set Ω is called $\{\theta_t\}_{t\in\mathbb{R}}$-invariant if $\theta_t\Omega = \Omega$ for $t \in \mathbb{R}$. On $\mathfrak{F}$ we consider the restriction of the Wiener measure also denoted by $\mathbb{P}$.

Consider the following linear stochastic differential equation

$$dz + zdt = dW. \tag{6}$$

We call a solution of this equation as Ornstein-Uhlenbeck process. In [8], the following result is proved.

Lemma 2.2. i) *There exists a* $\{\theta_t\}_{t\in\mathbb{R}}$*-invariant* $\Omega \in \mathfrak{B}(C_0(\mathbb{R}, \mathbb{R}))$ *of full measure with sublinear growth:*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{|t|} = 0, \ \omega \in \Omega$$

of $\mathbb{P}$*-measure one.*
ii) *For* $\omega \in \Omega$ *the random variable*

$$z(\omega) = -\int_{-\infty}^{0} e^{\tau}\omega(\tau)d\tau$$

exists and generates a unique stationary solution of (6) *given by*

$$\Omega \times \mathbb{R} \ni (\omega, t) \to z(\theta_t\omega) = -\int_{-\infty}^{0} e^{\tau}\theta_t\omega(\tau)d\tau = -\int_{-\infty}^{0} e^{\tau}\omega(\tau + t)d\tau + \omega(t)$$

The mapping $t \to z(\theta_t\omega)$ *is continuous.*
iii) *In particular, we have*

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0, \ for\ \omega \in \Omega. \tag{7}$$

iv) *In addition,*

$$\lim_{t\to\pm\infty} \frac{\int_0^t z(\theta_\tau\omega)d\tau}{t} = 0, \ for\ \omega \in \Omega.$$

Let us recall the following transformations

$$T(\omega, x) = xe^{-z(\omega)} \tag{8}$$

and its inverse transform

$$T^{-1}(\omega, x) = xe^{z(\omega)} \tag{9}$$

for $z \in H$ and $\omega \in \Omega$.

From Lemma 2.2 in [9], we may use the transformations (8) and (9) to convert (1) into a random differential equation.

Lemma 2.3. *Suppose that x is the solution of*

$$\frac{dx(t)}{dt} = A(t)x(t) + z(\theta_t\omega)x(t) + G(t, \theta_\tau\omega, x(t))) \tag{10}$$

where $G(t, \omega, x(t))) = e^{z(\omega)}F(t, e^{-z(\omega)}x(t))$ *and z is the solution of* (6)*, then for any* $x \in H$

$$\hat{x}(t, \omega, \nu_0) := T^{-1}(\theta_t\omega, x(t, \omega, T(\omega, \nu_0))) \tag{11}$$

is a solution to (1).

3. Local-stable manifolds.

In this section we shall prove the existence of local-stable manifolds for solutions of (1). We shall give some definitions. For the nonlinear term we need some locally Lipschitz properties.

DEFINITION 3.1. Let X be a Banach space and B_ρ be the ball with radius ρ centered at the origin in X, i.e., $B_\rho := \{f \in B : \|f\| \le \rho\}$. A function $f : [0, +\infty) \times X \to \mathbb{R}$ is said to have (C, ρ) properties for some positive constants C, if (3) and (4) are satisfied for $x \in B_\rho$ and a.e. $t \in \mathbb{R}_+$.

Fix some positive number $\delta_0 < \delta$ where δ is the constant defined in (2). We denote by

$$L_{\delta_0} = \{f \in L(\mathbb{R}_+) \,|\, \|f\|_{\delta_0} = \sup_{t \ge 0} |f(t)|e^{-\delta_0 t} < \infty\},$$

which is Banach space.

Recall a multifunction $S = \{S(\omega)\}_{\omega\in\Omega}$ of nonempty closed sets $S(\omega), \omega \in \Omega$, contained in a complete separable metric space (H, d_H) is called a *random set* (see [9]) if

$$\omega \to \inf_{y \in S(\omega)} d_H(x, y)$$

is a random variable for any $x \in H$.

Next we give the definition of local-stable manifolds for the solutions to (1) (See [13]).

DEFINITION 3.2. A random set $\mathbf{S}(\omega) \subset \mathbb{R}_+ \times X$ is said to be a local-stable manifold of L_{δ_0} class for the solutions of (1) if for every $t \in \mathbb{R}_+$ the phase space X splits into a direct sum $X = X_0(t) \bigoplus X_1(t)$ such that $\inf_t \inf\{\|x_0 + x_1\|\} > 0$ for $x_j \in X_j(t), \|x_j\| = 1,\ j = 0, 1$ and if there exist positive constants ρ, ρ_0, ρ_1 and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_0} \cap X_0(t) \to B_{\rho_1} \cap X_1(t)$$

with Lipschitz constants independent of t such that

i) $\mathbf{S}(\omega) = \{(t, x + g_t(x, \omega)) \in \mathbb{R}_+ \times (X_0(t) \bigoplus X_1(t)) \,|\, t \in \mathbb{R}_+, x \in B_{\rho_0} \cap X_0(t)\}$;

ii) Denote by $\mathbf{S_t}(\omega) := \{x + g_t(x, \omega) : (t, x + g_t(x, \omega)) \in \mathbf{S}(\omega)\}$. Then $\mathbf{S_t}(\omega)$ is homeomorphic to $B_{\rho_0} \cap X_0(t)$:=$\{x \in X_0(t) : \|x\| \le \rho_0\}$ for all $t \ge 0$.

iii) to each $x_0 \in \mathbf{S}_{t_0}(\omega)$ there corresponds one and only one solution $x(t)$ of (1) on $[t_0, +\infty)$

satisfying conditions $x(t_0) = x_0$ and the function

$$y(t) = \begin{cases} x(t) & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$$

belongs to the ball with radius ρ in L_{δ_0}(i.e., the ball $\mathfrak{B}_\rho := \{g \in L_{\delta_0} : \|g\|_{L_{\delta_0}} \leq \rho\}$.

Following [14], for each $t_0 \geq 0$ the space $X_0(t_0) = P(t_0)X$ can be characterized as

$$X_0(t_0) = \{x \in X : \text{ the function } y(t) = \begin{cases} U(t, t_0)x & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases} \quad \text{belongs to } B\},$$

where B is a Banach space. Concretely, for L_{δ_0}, we have that

$$X_0(t_0) = \Big\{x \in X : \sup_{t\in[0,\infty)} \|U(t, t_0)x\| < \infty\Big\}.$$

The following lemma gives the solution of (10), which belongs to L_{δ_0}.

Lemma 3.1. *Let the evolution family $(U(t, s))_{t\geq s\geq 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geq 0}$ and dichotomy constants $M, \eta > 0$. Let $F : \mathbb{R}_+ \times B_\rho \to X$ belong to class (C, ρ). Let $x(t)$ be a solution of* (1) *such that for fixed t_0 the function*

$$y(t) = \begin{cases} x(t) & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$$

belongs to $\mathfrak{B}_\rho := \{g \in L_{\delta_0} : \|g\|_{L_{\delta_0}} \leq \rho\}$. Then, for $t \geq t_0$, $x(t)$ can be written in the form

$$x(t) = U(t, t_0)\upsilon_0 + \int_{t_0}^{\infty} \Gamma(t, \tau)(z(\theta_\tau\omega)x(\tau) + G(\tau, \theta_\tau\omega, x(\tau)))d\tau, \tag{12}$$

where $\Gamma(t, \tau)$ is the Greens function defined by equality (2).

Proof. Denote

$$w(t) = \int_{t_0}^{\infty} \Gamma(t, \tau)(z(\theta_\tau\omega)x(\tau) + G(\tau, \theta_\tau\omega, x(\tau)))d\tau \tag{13}$$

for $t \geq t_0$ and $w(t) = 0$ for $t < t_0$. Using (2) and (4) we obtain

$$\begin{aligned} &\|w(t)\| \\ &\leq M \int_{t_0}^{\infty} e^{-\delta|t-\tau|+\delta_0\tau} e^{-\delta_0\tau}(\|z(\theta_\tau\omega)\|\|x\| \\ &+C\|e^{-z(\theta_\tau\omega)}\|(1 + \|e^{z(\theta_\tau\omega)}\|\|x\|))d\tau \\ &\leq MC_{1,\omega} \int_{t_0}^{\infty} e^{-\delta|t-\tau|+\delta_0\tau} d\tau, \end{aligned}$$

where

$$C_{1,\omega} = \sup_{\tau\geq t_0}(\|z(\theta_\tau\omega)\|e^{-\delta_0\tau}\rho) + C(\sup_{\tau\geq t_0}(\|e^{-z(\theta_\tau\omega)}\|e^{-\delta_0\tau}) + \rho). \tag{14}$$

Using the decomposition

$$\int_{t_0}^{\infty} e^{-\delta|t-\tau|+\delta_0\tau}d\tau = \int_{t_0}^{t} e^{-\delta(t-\tau)+\delta_0\tau}d\tau + \int_{t}^{\infty} e^{-\delta(\tau-t)+\delta_0\tau}d\tau$$

yields

$$\|w(t)\| \leq \frac{2\delta MC_{1,\omega}e^{\delta_0 t}}{\delta^2-\delta_0^2}.$$

Thus,

$$\|w(t)\|_{\delta_0} \leq \frac{2\delta MC_{1,\omega}}{\delta^2-\delta_0^2}.$$

It is straightforward to verify that $w(\cdot)$ satisfies the equation

$$w(t) = U(t,t_0)w_0 + \int_{t_0}^{t} U(t,\tau)(z(\theta_\tau\omega)x(\tau) + G(\tau,\theta_\tau\omega,x(\tau)))d\tau,$$

for $t \geq t_0$.

Since $x(t)$ is a solution of (10) we obtain that

$$x(t) - w(t) = U(t,t_0)(x(t_0) - w(t_0)) = y(t) - w(t)$$

for $t \geq t_0$. Denote by $v_0 = w(t_0) - x(t_0)$, since $y(t)$ and $w(t)$ are in L_{δ_0}, we can conclude that $v_0 \in X_0(t_0)$. The conclusion follows from the equality $x(t) = U(t,t_0)v_0 + w(t)$. □

In order to compare solutions on the manifolds, we should recall the cone inequality theorem.(See p.7-8 of [13])
A closed subset C of a Banach space X is called a *cone* if it has the following properties:
(i) $x_0 \in C$ implies $\lambda x_0 \in C$ for all $\lambda \geq 0$;
(ii) $x_1, x_2 \in C$ implies $x_1 + x_2 \in C$;
(iii) $\pm x_0 \in C$ implies $x_0 = 0$.
Fix a cone C in a Banach space X, for $x, y \in X$ we will use the notation $x \leq y$ if $x - y \in C$.
If the cone C is invariant under a linear operator A, then it is easy to see that A preserves the inequality, i.e.,$x \leq y$ implies $\mathrm{A}x \leq \mathrm{A}y$.
The following cone inequality theorem which can be found in Theorem I.9.3 in [7] will be used later as a lemma.

Lemma 3.2. *Let C be a cone given in a Banach space X such that C is invariant under a bounded linear operator* A *having spectral radius $r_A < 1$. If a vector $x \in X$ satisfies the inequality*

$$x \leq \mathrm{A}x + \eta$$

for some given $\eta \in X$, then it also satisfies the estimate $x \leq y$, where $y \in X$ is the solution of the equation $y = \mathrm{A}y + \eta$.

Now we may construct the structure of certain solutions of (10) in the following theorem.

Theorem 3.1. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geq 0}$ and dichotomy constants $M, \delta > 0$. Let $F : \mathbb{R}_+ \times B_\rho \to X$ belong to class (C,ρ). If*

$$\frac{2\delta MC_{1,\omega}}{\delta^2-\delta_0^2} < \min\left\{1, \frac{\rho}{2}\right\} \textit{ and } \frac{2\delta(ML+C_{z,\omega})}{\delta^2-\delta_0^2} < 1,$$

where $C_{1,\omega}$ *is defined in* (14) *and* $C_{z,\omega} = \sup_{\tau\geq t_0}(\|z(\theta_\tau\omega)\|e^{-\delta_0\tau})$, *then for* $r = \frac{\rho}{2M}$ *and* $t_0 \geq 0$ *there corresponds to each* $\upsilon_0 \in B_r \cap X_0(t_0)$ *one and only one solution* $x(t)$ *of* (10) *on* $[t_0, +\infty)$ *satisfying the conditions that* $P_0x(t_0) = \upsilon_0$ *and the the function*

$$y(t) = \begin{cases} x(t) \text{ for } t \geq t_0 \\ 0 \quad \text{ for } t < t_0 \end{cases}$$

belongs to the ball $\mathfrak{B}_\rho$ *in* L_{δ_0}. *Moreover, the following estimate is valid for any two solutions* $x_1(t)$ *and* $x_2(t)$ *be two solutions of* (10) *corresponding to different values* $\upsilon_1, \upsilon_2 \in B_r \cap X_0(t_0)$

$$\|x_1(t) - x_2(t)\| \leq C_\mu e^{-\mu(t-t_0)}\|\upsilon_1 - \upsilon_2\| \tag{15}$$

for $t \geq t_0$, *where* $\mu < \delta$, $\frac{2(\delta-\mu)ML}{(\delta-\mu)^2-\delta_0^2} < 1$ *and* $C_\mu = \frac{M}{1-\frac{2(\delta-\mu)ML}{(\delta-\mu)^2-\delta_0^2}}$.

Proof. We shall show that the following transformation T defined by

$$(Tx)(t) = \begin{cases} U(t,t_0)\upsilon_0 + \int_{t_0}^\infty \Gamma(t,\tau)(z(\theta_\tau\omega)x(\tau) + G(\tau,\theta_\tau\omega,x(\tau)))d\tau & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$$

acts from $\mathfrak{B}_\rho$ into $\mathfrak{B}_\rho$ and is a contraction for $\upsilon_0 \in B_r \cap X_0(t_0)$. Note that $\|F(t,x)\| \leq C(1+\|x\|)$ for $x(\cdot) \in \mathfrak{B}_\rho$, therefore, putting

$$y(t) = \begin{cases} U(t,t_0)\upsilon_0 + \int_{t_0}^\infty \Gamma(t,\tau)(z(\theta_\tau\omega)x(\tau) + G(\tau,\theta_\tau\omega,x(\tau)))d\tau & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$$

by proof of Lemma 3.1, we have

$$\|y(t)\| \leq Me^{-\delta(t-t_0)}\|\upsilon_0\| + \frac{2\delta MC_{1,\omega}e^{\delta_0 t}}{\delta^2-\delta_0^2}$$

where $C_{1,\omega}$ is defined in (14).

Thus,

$$\|y(t)\|_{L_{\delta_0}} \leq M\|\upsilon_0\|_{L_{\delta_0}} + \frac{2\delta MC_{1,\omega}}{\delta^2-\delta_0^2}.$$

Now the fact $\|\upsilon_0\|_{L_{\delta_0}} \leq \frac{\rho}{2M}$ and $\frac{2\delta MC_{1,\omega}}{\delta^2-\delta_0^2} < \frac{\rho}{2}$ yields $\|y(t)\|_{L_{\delta_0}} \leq \rho$.

Therefore, the transformation T acts from $\mathfrak{B}_\rho$ into $\mathfrak{B}_\rho$. We now estimate

$$\begin{aligned}
&\|(Tx)(t) - (Tw)(t)\| \\
&\leq \int_{t_0}^\infty \|\Gamma(t,\tau)\|(\|G(\tau,\theta_\tau\omega,x(\tau)) - G(\tau,\theta_\tau\omega,w(\tau))\| \\
&+\|z(\theta_\tau\omega)\|\|x(\tau) - w(\tau)\|)d\tau \\
&\leq (ML + C_{z,\omega})\int_{t_0}^\infty e^{-\delta|t-\tau|}\|x(\cdot) - w(\cdot)\|d\tau.
\end{aligned}$$

Therefore,

$$\|(Tx)(t)-(Tw)(t)\|_{L_{\delta_0}} \leq \frac{2\delta(ML+C_{z,\omega})}{\delta^2-\delta_0^2}\|x(\cdot)-w(\cdot)\|_{L_{\delta_0}}.$$

It follows from assumptions that $\frac{2\delta(ML+C_{z,\omega})}{\delta^2-\delta_0^2} < 1$. Hence, $T : \mathfrak{B}_\rho$ into $\mathfrak{B}_\rho$ is a contraction. Thus, there exists a unique $x \in \mathfrak{B}_\rho$ such that $Tx = x$.

By Lemma 3.1 we know that $x(t)$ is the unique solution in $\mathfrak{B}_\rho$ of (10) for $t \geq t_0$.

Denote by $x_1(t)$ and $x_2(t)$ be two solutions of (10) corresponding to different values $\upsilon_1, \upsilon_2 \in B_r \cap X_0(t_0)$. Then,

$$\begin{aligned}
&x_1(t)-x_2(t)\\
&= U(t,t_0)(\upsilon_1-\upsilon_2)\\
&+\int_{t_0}^{\infty}\Gamma(t,\tau)(G(\tau,\theta_\tau\omega,x_1(\tau))-G(\tau,\theta_\tau\omega,x_2(\tau))\\
&+z(\theta_\tau\omega)(x_1(\tau)-x_2(\tau)))d\tau,
\end{aligned}$$

for $t \geq t_0$. It follows that,

$$\begin{aligned}
&\|x_1(t)-x_2(t)\|\\
&\leq Me^{-\delta(t-t_0)}\|\upsilon_1-\upsilon_2\|\\
&+(ML+C_{z,\omega})\int_{t_0}^{\infty} e^{-\delta|t-\tau|}\|x_1(\tau)-x_2(\tau)\|d\tau,
\end{aligned}$$

for $t \geq t_0$.

Denote by $\psi(t) = \|x_1(t)-x_2(t)\|$, then $\operatorname{ess\,sup}_{t\geq t_0}\psi(t) < \infty$ and

$$\psi(t) \leq Me^{-\delta(t-t_0)}\|\upsilon_1-\upsilon_2\| + (ML+C_{z,\omega})\int_{t_0}^{\infty} e^{-\delta|t-\tau|}\psi(\tau)d\tau, \tag{16}$$

for $t \geq t_0$. We will apply the cone inequality theorem to Banach space $L_{\delta_0}[t_0,\infty)$ which is the space of real-valued functions defined and essentially bounded on $[t_0,\infty)$ with the cone C being the set of all (a.e.) nonnegative functions. Consider the linear operator A defined for $x \in L_{\delta_0}[t_0,\infty)$ by

$$(\mathrm{A}x)(t) = (ML+C_{z,\omega})\int_{t_0}^{\infty} e^{-\delta|t-\tau|}x(\tau)d\tau.$$

Then,

$$\begin{aligned}
&\sup_{t\geq t_0}(\mathrm{A}x)(t)\\
&\leq (ML+C_{z,\omega})\int_{t_0}^{\infty} e^{-\delta|t-\tau|+\delta_0\tau}e^{-\delta_0\tau}x(\tau)d\tau\\
&\leq (ML+C_{z,\omega})\int_{t_0}^{\infty} e^{-\delta|t-\tau|+\delta_0\tau}d\tau\cdot\|x\|_{\delta_0}\\
&\leq \frac{2\delta(ML+C_{z,\omega})}{\delta^2-\delta_0^2}\|x\|_{\delta_0}.
\end{aligned}$$

Thus, we have proved that A is a bounded linear operator with $\|\mathrm{A}\| < 1$, leaving the cone C invariant. We may rewrite the inequality (16) as

$$\psi \leq \mathrm{A}\psi + \eta(t) \text{ for } \eta(t) = Me^{-\delta(t-t_0)}\|\upsilon_1 - \upsilon_2\|.$$

Hence, by cone inequality Lemma 3.2, we obtain that $\psi \leq \varphi$, where φ is a solution of the equation $\varphi = \mathrm{A}\varphi + \eta$ which can be rewritten as

$$\varphi(t) = Me^{-\delta(t-t_0)}\|\upsilon_1 - \upsilon_2\| + (ML + C_{z,\omega})\int_{t_0}^{\infty} e^{-\delta|t-\tau|}\varphi(\tau)d\tau, \tag{17}$$

for $t \geq t_0$.

In order to estimate φ, we set $\phi(t) = e^{\mu(t-t_0)}\varphi(t)$ for $\mu < \delta$ and $\frac{2(\delta-\mu)(ML+C_{z,\omega}))}{(\delta-\mu)^2-\delta_0^2} < 1$. By (17), we have

$$\phi(t) = Me^{-(\delta-\mu)(t-t_0)}\|\upsilon_1 - \upsilon_2\| + (ML + C_{z,\omega})\int_{t_0}^{\infty} e^{-\delta|t-\tau|+\mu(t-\tau)}\phi(\tau)d\tau, \tag{18}$$

for $t \geq t_0$. Consider the linear operator D defined for $x \in L_{\delta_0}[t_0, \infty)$ by

$$(\mathrm{D}x)(t) = (ML + C_{z,\omega})\int_{t_0}^{\infty} e^{-\delta|t-\tau|+\mu(t-\tau)}x(\tau)d\tau.$$

Then,

$$\begin{aligned}
&\sup_{t\geq t_0}(\mathrm{D}x)(t)\\
&\leq (ML + C_{z,\omega})\int_{t_0}^{\infty} e^{-(\delta-\mu)|t-\tau|}x(\tau)d\tau\\
&\leq (ML + C_{z,\omega})\int_{t_0}^{\infty} e^{-(\delta-\mu)|t-\tau|+\delta_0\tau}d\tau \cdot \|x\|_{\delta_0}\\
&\leq \frac{2(\delta-\mu)(ML + C_{z,\omega})}{(\delta-\mu)^2 - \delta_0^2}\|x\|_{\delta_0}.
\end{aligned}$$

Thus, we have proved that D is a bounded linear operator with $\|\mathrm{D}\| < 1$. Then (18) can be rewritten as

$$\phi = \mathrm{D}\phi + \eta(t) \text{ for } \eta(t) = Me^{-(\delta-\mu)(t-t_0)}\|\upsilon_1 - \upsilon_2\|.$$

Thus, $\phi = (\mathrm{I}-\mathrm{D})^{-1}\eta$ uniquely solves the equation $\phi = \mathrm{D}\phi + \eta(t)$ in $L_{\delta_0}[t_0, \infty)$. Furthermore,

$$\begin{aligned}
&\|\phi\|_{\delta_0}\\
&= \|(\mathrm{I}-\mathrm{D})^{-1}\eta\|_{\delta_0}\\
&\leq \|(\mathrm{I}-\mathrm{D})^{-1}\|\|\eta\|_{\delta_0}\\
&\leq \frac{M}{1-\|\mathrm{D}\|}\|\upsilon_1 - \upsilon_2\|\\
&\leq \frac{M}{1-\frac{2(\delta-\mu)(ML+C_{z,\omega})}{(\delta-\mu)^2-\delta_0^2}}\|\upsilon_1 - \upsilon_2\| := C_\mu\|\upsilon_1 - \upsilon_2\|,
\end{aligned}$$

which yields

$$\phi(t) \leq C_\mu\|\upsilon_1 - \upsilon_2\|$$

for $t \geq t_0$. Hence $\varphi(t) = e^{-\mu(t-t_0)}\phi(t) \leq e^{-\mu(t-t_0)}C_\mu\|\upsilon_1 - \upsilon_2\|$. Recall the definition of $\psi(t) = \|x_1(t) - x_2(t)\| \leq \varphi(t)$, we get the conclusion

$$\|x_1(t) - x_2(t)\| \le e^{-\mu(t-t_0)} C_\mu \|v_1 - v_2\|$$

for $t \ge t_0$. □

We now prove our result on local stable manifold.

Theorem 3.2. *Let the evolution family $(U(t,s))_{t\ge s\ge 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\ge 0}$ and dichotomy constants $M, \delta > 0$. Let $F : \mathbb{R}_+ \times B_\rho \to X$ belong to class (C,ρ). If*

$$\frac{2\delta M C_{1,\omega}}{\delta^2 - \delta_0^2} < \min\left\{1, \frac{\rho}{2}\right\} \text{ and } \frac{2\delta(ML + C_{z,\omega})}{\delta^2 - \delta_0^2} < 1,$$

where $C_{1,\omega}$ is defined in (14) *and $C_{z,\omega}$ is defined in* Theorem 3.1*, then there exists a local-stable manifold $\mathbf{S}(\omega)$ of L_{δ_0} class for the solutions of* (10)*. Moreover, every two solutions $x_1(t)$ and $x_2(t)$ on the manifold $\mathbf{S}(\omega)$ attract each other exponentially in the sense that there exist positive constants μ and C_μ independent of $t_0 \ge 0$ such that*

$$\|x_1(t) - x_2(t)\| \le C_\mu e^{-\mu(t-t_0)} \|P(t_0)v_1(t_0) - P(t_0)v_2(t_0)\|. \tag{19}$$

Proof. Since the evolution family $(U(t,s))_{t\ge s\ge 0}$ have exponential dichotomy, we know that for each $t \ge 0$ there are projections $P(t), t \in \mathbb{R}_+$, uniformly bounded and strongly continuous in t, such that the phase space X splits into the direct sum $X = X_0(t) \oplus X_1(t)$, where $X_0(t) = P(t)X$ and $X_1(t) = kerP(t)$. Furthermore, we can conclude that $\inf_t \inf\{\|x_0 + x_1\|\} > 0$ for $x_j \in X_j(t), \|x_j\| = 1,\ j = 0, 1$ from $\sup_{t\ge 0} \|P(t)\| < \infty$. We should construct the family of Lipschitz continuous mapping $(g_t)_{t\ge 0}$ which satisfied the conditions of Definition 3.1. Defined

$$g_{t_0}(y,\omega) = \int_{t_0}^{\infty} (\Gamma(t_0,\tau)G(\tau,\theta_\tau\omega,x(\tau)) + z(\theta_\tau\omega)x(\tau))d\tau,$$

where $y \in B_r \cap X_0(t_0)$ with $r = \frac{\rho}{2M}$, x is the solution in B_ρ of (10) on $[t_0,\infty)$ which satisfies $P(t_0)(x_0) = y$ and $x(t) = 0,\ t < t_0$. We can conclude by definition of Green's function that $g_{t_0}(y,\omega) \in X_1(t_0)$. Since

$$\begin{aligned}
&\|g_{t_0}(y)\| \\
&\le \int_{t_0}^{\infty} \|\Gamma(t_0,\tau)\|(\|G(\tau,\theta_\tau\omega,x(\tau))\| + \|z(\theta_\tau\omega)\|\|x\|)d\tau \\
&\le MC_{1,\omega}\int_{t_0}^{\infty} e^{-\delta|t-\tau|+\delta_0\tau}d\tau \\
&\le \frac{2\delta MC_{1,\omega}}{\delta^2-\delta_0^2} < \frac{\rho}{2}
\end{aligned}$$

where $C_{1,\omega}$ is defined in (14). Hence $g_{t_0}(y,\omega)$ is a mapping from $B_r \cap X_0(t_0)$ to $B_\rho \cap X_1(t_0)$. For y_1 and y_2 belonging to $B_r \cap X_0(t_0)$ we have

$$\begin{aligned}
&\|g_{t_0}(y_1,\omega) - g_{t_0}(y_2,\omega)\| \\
&\le \int_0^{\infty} \|\Gamma(t_0,\tau)\|(\|G(\tau,\theta_\tau\omega,x_1(\tau)) - G(\tau,\theta_\tau\omega,x_2(\tau))\| \\
&+\|z(\theta_\tau\omega)\|\|x_1(\tau) - x_2(\tau)\|)d\tau
\end{aligned}$$

$$\leq (ML + C_{z,\omega}) \int_0^\infty e^{-\delta|t_0-\tau|+\delta_0\tau} \|x_1(\cdot) - x_2(\cdot)\|_{\delta_0} d\tau \cdot$$
$$\leq \frac{2\delta(ML + C_{z,\omega})}{\delta^2 - \delta_0^2} \|x_1(\cdot) - x_2(\cdot)\|_{\delta_0}.$$

Since $x_i(\cdot)$ is the unique solution in B_ρ of (10) on $[t_0, \infty)$ satisfying $P(t_0)x_i(t_0) = y_i,\ i = 1, 2,$, respectively, we have that

$$\begin{aligned}
&\|x_1(t) - x_2(t)\| \\
&\leq \|U(t, t_0)\|\|y_1 - y_2\| \\
&+ \int_{t_0}^\infty \|\Gamma(t, \tau)\|(\|G(\tau, \theta_\tau\omega, x_1(\tau)) - G(\tau, \theta_\tau\omega, x_2(\tau))\| \\
&+\|z(\theta_\tau\omega)\|\|x_1(\tau) - x_2(\tau)\|)d\tau \\
&\leq M\|y_1 - y_2\| + \int_0^\infty e^{-\delta|t-\tau|+\delta_0\tau} d\tau \cdot (ML + C_{z,\omega})\|x_1(\cdot) - x_2(\cdot)\|_{\delta_0} \\
&\leq M\|y_1 - y_2\| + \frac{2\delta(ML + C_{z,\omega})}{\delta^2 - \delta_0^2}\|x_1(\cdot) - x_2(\cdot)\|_{\delta_0}.
\end{aligned}$$

Denote by $\beta = \frac{2\delta(ML+C_{z,\omega})}{\delta^2-\delta_0^2} < 1$, we obtain that

$$\|x_1(t) - x_2(t)\|_{\delta_0} \leq \frac{M}{1-\beta}\|y_1 - y_2\|.$$

Hence we have proven that g_{t_0} is Lipschitz continuous with Lipschitz constant independent of t_0,

$$\|g_{t_0}(y_1, \omega) - g_{t_0}(y_2, \omega)\| \leq \frac{M\beta}{2(1-\beta)}\|y_1 - y_2\|.$$

Denote by $\rho_0 := r = \frac{\rho}{2M}$ and $\rho_1 := \rho/2$ we obtain that the family of mappings $(g_t)_{t\geq 0}$ $(g_t : B_{\rho_0} \cap X_0(t) \to B_{\rho_1} \cap X_1(t))$ are Lipschitz continuous with the Lipschitz constant $\frac{M\beta}{2(1-\beta)}$ independent of t.

Define the transformation $Zy := y + g_{t_0}(y, \omega)$ for all $y \in B_r \cap X_0(t_0)$, applying the Implicit Function Theorem for Lipschitz continuous mapping (see [12]), we have that, if Lipschitz constant $\frac{M\beta}{2(1-\beta)}$ of g_{t_0} satisfies $\frac{M\beta}{2(1-\beta)} < 1$, then Z is a homeomorphism. Put $\mathbf{S}(\omega) = \Big\{(t, x + g_t(x, \omega)) \in \mathbb{R}_+\times(X_0(t) \bigoplus X_1(t)) \ \Big|\ t \in \mathbb{R}_+, x \in B_r\cap X_0(t)\Big\}$, then for each $t_0 \geq 0$ we have proven that $\mathbf{S}_{\mathbf{t_0}}(\omega) = \{(x + g_{t_0}(x, \omega)) : (t_0, x + g_{t_0}(x, \omega)) \in \mathbf{S}(\omega)\}$ is homeomorphic to $B_r \cap X_0(t_0)$. Therefore, the condition (ii) in Definition 3.2 follows. The condition (iii) of Definition 3.2 now follows from Theorem 3.1. Finally, the inequality (19) follows from inequality (15) in Theorem 3.1. □

Theorem 3.3. *Let* $\mathbf{S}(\omega) = \{(t, x + g_t(x, \omega)) \in \mathbb{R}_+ \times (X_0(t) \bigoplus X_1(t))| t \in \mathbb{R}_+, x \in B_{\rho_0} \cap X_0(t)\}$ *be the local-stable manifold* $\mathbf{S}(\omega)$ *of* L_{δ_0} *class for the solutions of* (10)*, which is obtained in* Theorem 3.2*. Then* $\hat{\mathbf{S}}(\omega) := \Big\{(t, \hat{x} + g_t(\hat{x})) \in \mathbb{R}_+ \times (X_0(t) \bigoplus X_1(t)) \ \Big|\ t \in \mathbb{R}_+, x \in B_{\rho_0} \cap X_0(t)\Big\}$ *is a local-stable manifold of* L_{δ_0} *class for the solutions of* (5)*.*

Proof. Let $x(t, \omega, v_0)$ be the solution of (10) and $\hat{x}(t, \omega, v_0)$ be the solution of (5). From Lemma 2.3

$$\hat{x}(t,\omega,\hat{\mathbf{S}}) = T^{-1}(\theta_t\omega, x(t, T(\omega,\hat{\mathbf{S}}))) = T^{-1}(\theta_t\omega, x(t,\omega,\mathbf{S})) \subset T^{-1}(\theta_t\omega, \mathbf{S}(\theta_t\omega)) = \hat{\mathbf{S}}(\theta_t\omega).$$

Thus, $\hat{\mathbf{S}}$ is an invariant set. Notice that

$$\begin{aligned}
&\hat{\mathbf{S}}(\omega)\\
&= T(\omega, \mathbf{S}(\omega))\\
&= \{\nu_0 = T^{-1}(\omega, x + g_t(x,\omega)))\,|\,x \in B_{\rho_0} \cap X_0(t)\}\\
&= \{\nu_0 = e^{z(\theta_\tau\omega)}(x + g_t(x,\omega)))\,|\,x \in B_{\rho_0} \cap X_0(t)\}\\
&= \{\nu_0 = (x + g_t(e^{-z(\theta_\tau\omega)}x,\omega)))\,|\,x \in B_{\rho_0} \cap X_0(t)\}
\end{aligned}$$

which implies that $\hat{\mathbf{S}}(\omega)$ is a Lipschitz stable manifold. □

4. Global stable manifolds.

The existence of invariant (global) manifolds will be proved in this section. As in the previous section, for the linear part we need the fact that the evolution family has an exponential dichotomy. Then, we impose some kind of global Lipschitz properties on the nonlinear term $F(t, x)$. Precisely, we have the following definition.

Definition 4.1. Let X be a Banach space. A function $f : [0, +\infty) \times X \to \mathbb{R}$ is said to have Lipschitz properties for some positive constants C, if (3) and (4) are satisfied for $x \in X$ and a.e. $t \in \mathbb{R}_+$.

The definition of stable manifolds for the solutions to (1) is as follows.(See [13])

Definition 4.2. A random set $\mathbf{S}(\omega) \subset \mathbb{R}_+ \times X$ is said to be a stable manifold for the solutions of (1) if for every $t \in \mathbb{R}_+$ the phase space X splits into a direct sum $X = X_0(t) \bigoplus X_1(t)$ such that $\inf_t \inf\{\|x_0 + x_1\|\} > 0$ for $x_j \in X_j(t), \|x_j\| = 1,\ j = 0, 1$ and if there exist a family of Lipschitz continuous mappings

$$g_t : X_0(t) \to X_1(t)$$

with Lipschitz constants independent of $t \in \mathbb{R}_+$ such that
i) $\mathbf{S}(\omega) = \{(t, x + g_t(x,\omega)) \in \mathbb{R}_+ \times (X_0(t) \bigoplus X_1(t))|t \in \mathbb{R}_+, x \in X_0(t)\}$;
ii) Denote by $\mathbf{S_t}(\omega) := \{x + g_t(x,\omega) : (t, x + g_t(x)) \in \mathbf{S}\}$. Then $\mathbf{S_t}(\omega)$ is homeomorphic to $X_0(t)$ for all $t \geq 0$.
iii) to each $x_0 \in \mathbf{S}_{t_0}(\omega)$ there corresponds one and only one solution $x(t)$ of (1) on $[t_0, +\infty)$ satisfying conditions $x(t_0) = x_0$ and the function

$$y(t) = \begin{cases} x(t) & \text{for } t \geq t_0\\ 0 & \text{for } t < t_0 \end{cases}$$

belongs to L_{δ_0}.
iv) $\mathbf{S}(\omega)$ is invariant under (1) in the sense that, if $x(\cdot)$ is a solution of (1) with $x(t_0) = x_0 \in \mathbf{S}_{t_0}(\omega)$ then $x(s) \in \mathbf{S}_s(\omega)$ for all $s \geq t_0$.

The following lemma gives the form of solution of (10), which belongs to L_{δ_0}.

Lemma 4.1. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geq 0}$ and dichotomy constants $M,\delta > 0$. Let $F : \mathbb{R}_+ \times B_\rho \to X$ satisfy Lipschitz properties* (3) *and* (4)*. Let $x(t)$ be a solution of* (1) *such that for fixed t_0 the function*

$$y(t) = \begin{cases} x(t) & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$$

belongs to L_{δ_0}. Then, for $t \geq t_0$, $x(t)$ can be written in the form

$$x(t) = U(t,t_0)\upsilon_0 + \int_{t_0}^{\infty} \Gamma(t,\tau)(z(\theta_\tau\omega)x(\tau) + G(\tau,\theta_\tau\omega,x(\tau)))d\tau, \tag{20}$$

where $\Gamma(t,\tau)$ is the Green's function defined by equality (2).

Proof. Denote

$$w(t) = \int_{t_0}^{\infty} \Gamma(t,\tau)(z(\theta_\tau\omega)x(\tau) + G(\tau,\theta_\tau\omega,x(\tau)))d\tau \tag{21}$$

for $t \geq t_0$ and $w(t) = 0$ for $t < t_0$. Using the proof of Lemma 3.1 we obtain

$$\begin{aligned} &||w(t)||_{\delta_0} \\ &\leq M \int_{t_0}^{\infty} e^{-\delta|t-\tau|+\delta_0\tau} e^{-\delta_0\tau}(||z(\theta_\tau\omega)||||x|| + C(||e^{-z(\theta_\tau\omega)}|| + ||x||))d\tau \\ &\leq \frac{2\delta M C_{2,\omega}}{\delta^2 - \delta_0^2}, \end{aligned}$$

where

$$C_{2,\omega} = \sup_{\tau\geq t_0} e^{-\delta_0\tau}(||z(\theta_\tau\omega)||||x|| + C(||e^{-z(\theta_\tau\omega)}|| + ||x||)). \tag{22}$$

It is straightforward to verify that $w(\cdot)$ satisfies the equation

$$w(t) = U(t,t_0)w_0 + \int_{t_0}^{t} U(t,\tau)(z(\theta_\tau\omega)x(\tau) + G(\tau,\theta_\tau\omega,x(\tau)))d\tau,$$

for $t \geq t_0$.

Since $x(t)$ is a solution of (10) we obtain that

$$x(t) - w(t) = U(t,t_0)(x(t_0) - w(t_0)) = y(t) - w(t)$$

for $t \geq t_0$. Denote by $\upsilon_0 = w(t_0) - x(t_0)$, since $y(t)$ and $w(t)$ are in L_{δ_0}, we can conclude that $\upsilon_0 \in X_0(t_0)$. The conclusion follows from the equality $x(t) = U(t,t_0)\upsilon_0 + w(t)$. □

Theorem 4.1. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geq 0}$ and dichotomy constants $M,\delta > 0$. Let $F : \mathbb{R}_+ \times B_\rho \to X$ satisfy Lipschitz properties* (3) *and* (4)*, then there corresponds to each $\upsilon_0 \in X_0(t_0)$ one and only one solution $x(t)$ of* (1) *on $[t_0,+\infty)$ satisfying the conditions that $P_0x(t_0) = \upsilon_0$ and the the function*

$$y(t) = \begin{cases} x(t) & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$$

belongs to L_{δ_0}*. Moreover, the following estimate is valid for any two solutions* $x_1(t)$ *and* $x_2(t)$ *be two solutions of* (10) *corresponding to different values* $\upsilon_1, \upsilon_2 \in X_0(t_0)$

(23) $$\|x_1(t) - x_2(t)\| \leq C_\mu e^{-\mu(t-t_0)}\|\upsilon_1 - \upsilon_2\|.$$

for $t \geq t_0$*, where* $\mu < \delta$, $\frac{2(\delta-\mu)(ML+C_{z,\omega})}{(\delta-\mu)^2-\delta_0^2} < 1$ *and* $C_\mu = \frac{M}{1-\frac{2(\delta-\mu)(ML+C_{z,\omega})}{(\delta-\mu)^2-\delta_0^2}}$.

Proof. We shall show that the following transformation T defined by

$$(Tx)(t) = \begin{cases} U(t,t_0)\upsilon_0 + \int_{t_0}^{\infty} \Gamma(t,\tau)(z(\theta_\tau\omega)x(\tau) + G(\tau,\theta_\tau\omega,x(\tau)))d\tau & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$$

acts from X into X is a contraction for $\upsilon_0 \in X_0(t_0)$. Note that $\|F(t,x)\| \leq C(1+\|x\|)$ for $x(\cdot) \in X$, therefore, putting

$$y(t) = \begin{cases} U(t,t_0)\upsilon_0 + \int_{t_0}^{\infty} \Gamma(t,\tau)(z(\theta_\tau\omega)x(\tau) + G(\tau,\theta_\tau\omega,x(\tau)))d\tau & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0. \end{cases}$$

Using the proof of Lemma 3.1 we obtain

$$\begin{aligned} &\|y(t)\| \\ &\leq Me^{-\delta(t-t_0)}\|\nu_0\| \\ &+M\int_{t_0}^{\infty} e^{-\delta|t-\tau|+\delta_0\tau}e^{-\delta_0\tau}(\|z(\theta_\tau\omega)\|\|x(\tau)\| + C(\|e^{-z(\theta_\tau\omega)}\| + \|x(\tau)\|))d\tau \end{aligned}$$

thus

$$\|y(t)\|_{\delta_0} \leq M\|\nu_0\|_{\delta_0} + \frac{2\delta MC_{2,\omega}}{\delta^2-\delta_0^2}$$

where $C_{2,\omega}$ is defined in (22).

Therefore, the transformation T acts L_{δ_0} into L_{δ_0}. We now estimate

$$\begin{aligned} &\|(Tx)(t) - (Tw)(t)\| \\ &\leq \int_0^{\infty} \|\Gamma(t,\tau)\|(\|G(\tau,\theta_\tau\omega,x(\tau)) - G(\tau,\theta_\tau\omega,w(\tau))\| \\ &+\|z(\theta_\tau\omega)\|\|x(\tau) - w(\tau)\|)d\tau \end{aligned}$$

$$\leq (ML + C_{z,\omega})\int_0^{\infty} e^{-\delta|t-\tau|+\delta_0\tau}e^{-\delta_0\tau}\|x(\tau) - w(\tau)\|d\tau.$$

Therefore,

$$\|(Tx)(t) - (Tw)(t)\|_{\delta_0} \leq \frac{2\delta(ML + C_{z,\omega})}{\delta^2-\delta_0^2}\|x(\cdot) - w(\cdot)\|_{\delta_0}.$$

It follows from assumptions that $\frac{2\delta(ML+C_{z,\omega})}{\delta^2-\delta_0^2} < 1$. Hence, T acts L_{δ_0} into L_{δ_0} is a contraction. Thus, there exists a unique $x \in L_{\delta_0}$ such that $Tx = x$.

By Lemma 4.1 we know that $x(t)$ is the unique solution of (10) for $t \geq t_0$.

Finally, the last inequality can now be proved by the same way as the proof of inequality in Theorem 3.1. □

We now prove our result on global stable manifold.

Theorem 4.2. *Let the evolution family* $(U(t,s))_{t\geq s\geq 0}$ *have an exponential dichotomy with the corresponding dichotomy projections* $(P(t))_{t\geq 0}$ *and dichotomy constants* $M, \delta > 0$. *Let* $F : \mathbb{R}_+ \times B_\rho \to X$ *satisfy Lipschitz properties* (3) *and* (4).

$$\frac{2\delta M C_{2,\omega}}{\delta^2 - \delta_0^2} < \min\left\{1, \frac{\rho}{2}\right\} \text{ and } \frac{2\delta(ML + C_{z,\omega})}{\delta^2 - \delta_0^2} < 1,$$

where $C_{2,\omega}$ *is defined in* (22) *and* $C_{z,\omega}$ *is defined in* Theorem 3.1, *then there exists a global stable manifold* $\mathbf{S}(\omega)$ *of* L_{δ_0} *class for the solutions of* (10). *Moreover, every two solutions* $x_1(t)$ *and* $x_2(t)$ *on the manifold* $\mathbf{S}(\omega)$ *attract each other exponentially in the sense that there exist positive constants* μ *and* C_μ *independent of* $t_0 \geq 0$ *such that*

$$\|x_1(t) - x_2(t)\| \leq C_\mu e^{-\mu(t-t_0)} \|P(t_0)v_1(t_0) - P(t_0)v_2(t_0)\|.$$

Proof. Since the evolution family $(U(t,s))_{t\geq s\geq 0}$ have exponential dichotomy, we know that for each $t \geq 0$ there are projections $P(t), t \in \mathbb{R}_+$, uniformly bounded and strongly continuous in t, such that the phase space X splits into the direct sum $X = X_0(t) \oplus X_1(t)$, where $X_0(t) = P(t)X$ and $X_1(t) = kerP(t)$. Furthermore, we can conclude that $\inf_t \inf\{\|x_0 + x_1\|\} > 0$ for $x_j \in X_j(t), \|x_j\| = 1,\ j = 0, 1$ from $\sup_{t\geq 0} \|P(t)\| < \infty$. We should construct the family of Lipschitz continuous mapping $(g_t)_{t\geq 0}$ which satisfied the conditions of Definition 4.2. Define

$$g_{t_0}(y,\omega) = \int_{t_0}^{\infty} (\Gamma(t_0,\tau)G(\tau,\theta_\tau\omega, x(\tau)) + z(\theta_\tau\omega)x(\tau))d\tau,$$

where $y \in X_0(t_0)$, x is the solution of (10) on $[t_0, \infty)$ which satisfies $P(t_0)(x_0) = y$ and $x(t) = 0,\ t < t_0$ (note that the existence and uniqueness of $x(t)$ is obtained in Theorem 4.1). For y_1 and y_2 belonging to $X_0(t_0)$ we have

$$\begin{aligned}
&\|g_{t_0}(y_1,\omega) - g_{t_0}(y_2,\omega)\| \\
&\leq \int_0^{\infty} \|\Gamma(t_0,\tau)\|(\|G(\tau,\theta_\tau\omega, x_1(\tau)) - G(\tau,\theta_\tau\omega, x_2(\tau))\| \\
&+ \|z(\theta_\tau\omega)\|\|x_1(\tau) - x_2(\tau)\|)d\tau \\
&\leq (ML + C_{z,\omega}) \int_0^{\infty} e^{-\delta|t-\tau|+\delta_0\tau} e^{-\delta_0\tau} \|x_1(\tau) - x_2(\tau)\| d\tau \\
&\leq \frac{2\delta(ML + C_{z,\omega})}{\delta^2 - \delta_0^2} \|x_1(\tau) - x_2(\tau)\|_{\delta_0}.
\end{aligned}$$

Since $x_i(\cdot)$ is the unique solution of (10) in L_{δ_0} on $[t_0, \infty)$ satisfying $P(t_0)x_i(t_0) = y_i,\ i = 1, 2$, respectively, we have that

$$\begin{aligned}
&\|x_1(t) - x_2(t)\| \\
&\leq \|U(t,t_0)\|\|y_1 - y_2\| + \\
&\int_{t_0}^{\infty} \|\Gamma(t,\tau)\|(\|G(\tau,\theta_\tau\omega, x_1(\tau)) - G(\tau,\theta_\tau\omega, x_2(\tau))\|
\end{aligned}$$

$$+\|z(\theta_\tau\omega)\|\|x_1(\tau)-x_2(\tau)\|)d\tau$$
$$\leq M\|y_1-y_2\|+(ML+C_{z,\omega})\int_0^\infty e^{-\delta|t-\tau|+\delta_0\tau}e^{-\delta_0\tau}\|x_1(\tau)-x_2(\tau)\|d\tau$$
$$\leq M\|y_1-y_2\|+\frac{2\delta(ML+C_{z,\omega})}{\delta^2-\delta_0^2}\|x_1(\tau)-x_2(\tau)\|_{\delta_0}.$$

Denote by $\beta=\frac{2\delta(ML+C_{z,\omega})}{\delta^2-\delta_0^2}<1$, we obtain that

$$\|x_1(t)-x_2(t)\|_{\delta_0}\leq\frac{M}{1-\beta}\|y_1-y_2\|.$$

Hence we have proven that g_{t_0} is Lipschitz continuous with Lipschitz constant independent of t_0,

$$\|g_{t_0}(y_1,\omega)-g_{t_0}(y_2,\omega)\|\leq\frac{M\beta}{2(1-\beta)}\|y_1-y_2\|.$$

Thus, we have obtained that the family of mappings $(g_t)_{t\geq 0}$ are Lipschitz continuous with the Lipschitz constant $\frac{M\beta}{2(1-\beta)}$ independent of t.

Define the transformation $Zy := y+g_{t_0}(y,\omega)$ for all $y\in X_0(t_0)$, applying the Implicit Function Theorem for Lipschitz continuous mapping (see [12]), we have that, if Lipschitz constant $\frac{M\beta}{2(1-\beta)}$ of g_{t_0} satisfies $\frac{M\beta}{2(1-\beta)}<1$, then Z is a homeomorphism. Put $\mathbf{S}(\omega)=\Big\{(t,x+g_t(x,\omega))\in\mathbb{R}_+\times(X_0(t)\bigoplus X_1(t))\ \Big|\ t\in\mathbb{R}_+, x\in X_0(t)\Big\}$, then for each $t_0\geq 0$ we have proven that $\mathbf{S}_{\mathbf{t_0}}(\omega)=\{(x+g_{t_0}(x,\omega)):(t_0,x+g_{t_0}(x,\omega))\in\mathbf{S}\}$ is homeomorphic to $X_0(t_0)$. Therefore, the condition (ii) in Definition 4.2 follows. The condition (iii) of Definition 4.2. now follows from Theorem 4.1. We should prove that the condition (iv) of Definition 4.2 is satisfied. By Lemma 4.1 we have that, for $s\geq t_0$ the solution $u(s)$ can be rewritten in the form

$$x(s)=U(s,t_0)\upsilon_0+\int_{t_0}^\infty\Gamma(s,\tau)(z(\theta_\tau\omega)x(\tau)+G(\tau,\theta_\tau\omega,x(\tau))d\tau,$$

for some $\upsilon_0\in X_0(t_0)=P(t_0)X$ where $\Gamma(t,\tau)$ is the Greens function defined by equality (2).

Denote by

$$W(s):=U(s,t_0)\upsilon_0+\int_{t_0}^s\Gamma(s,\tau)(z(\theta_\tau\omega)x(\tau)+G(\tau,\theta_\tau\omega,x(\tau)))d\tau,$$

we obtain that $W(s)\in P(s)X$ and

$$x(s)=W(s)+\int_s^\infty\Gamma(s,\tau)(z(\theta_\tau\omega)x(\tau)+G(\tau,\theta_\tau\omega,x(\tau)))d\tau. \tag{24}$$

For $t\geq s$, by (12), straightforward computation yields

$$x(t)=U(t,s)W(s)+\int_s^\infty\Gamma(t,\tau)(z(\theta_\tau\omega)x(\tau)+G(\tau,\theta_\tau\omega,x(\tau)))d\tau. \tag{25}$$

Thus, combing (24) and (25) with definition of g_t we obtain that $x(s)=W(s)+g_sW(s)$ yielding that $x(s)\in\mathbf{S}_s(\omega)$ for all $s\geq t_0$. Finally, the last inequality follows from the last inequality in Theorem 4.1. □

Theorem 4.3. *Let* $\mathbf{S}(\omega)=\Big\{(t,x+g_t(x,\omega))\in\mathbb{R}_+\times(X_0(t)\bigoplus X_1(t))\ \Big|\ t\in\mathbb{R}_+, x\in X_0(t)\Big\}$ *be the global stable manifold* $\mathbf{S}(\omega)$ *of* L_{δ_0} *class for the solutions of* (10)*, which is obtained* *in* Theorem 4.2. *Then the manifold* $\hat{\mathbf{S}}(\omega) := \{(t, \hat{x} + g_t(\hat{x}, \omega)) \in \mathbb{R}_+ \times (X_0(t) \bigoplus X_1(t)) \,|\, t \in \mathbb{R}_+, x \in X_0(t)\}$ *is a global stable manifold of* L_{δ_0} *class for the solutions of* (5).

Proof. Let $x(t, \omega, \nu_0)$ be the solution of (10) and $\hat{x}(t, \omega, \nu_0)$ be the solution of (5). From Lemma 2.3,

$$\hat{x}(t, \omega, \hat{\mathbf{S}}) = T^{-1}(\theta_t\omega, x(t, T(\omega, \hat{\mathbf{S}}))) = T^{-1}(\theta_t\omega, x(t, \omega, \mathbf{S})) \subset T^{-1}(\theta_t\omega, \mathbf{S}(\theta_t\omega)) = \hat{\mathbf{S}}(\theta_t\omega).$$

Thus, $\hat{\mathbf{S}}$ is an invariant set. Notice that

$$\begin{aligned}
&\hat{\mathbf{S}}(\omega)\\
&= T(\omega, \mathbf{S}(\omega))\\
&= \{\nu_0 = T^{-1}(\omega, x + g_t(x, \omega))) \,|\, x \in X_0(t)\}\\
&= \{\nu_0 = e^{z(\theta_\tau\omega)}(x + g_t(x, \omega))) \,|\, x \in X_0(t)\}\\
&= \{\nu_0 = (x + g_t(e^{-z(\theta_\tau\omega)}x, \omega))) \,|\, x \in X_0(t)\}
\end{aligned}$$

which implies that $\hat{\mathbf{S}}(\omega)$ is a Lipschitz stable manifold. □

References

[1] H. Amann: Linear and Quasilinear Parabolic Problems, Vol. 1. Birkhauser, Boston, 1995.

[2] L. Arnold: Random Dynamical Systems, Springer, New York, 1998.

[3] V. Barbu, Z. Brzeźniak, E. Hausenblas and L. Tubaro: *Existence and convergence results for infinite dimensional nonlinear stochastic equations with multiplicative noise*, Stochastic Process. Appl. **123** (2013) 934–951.

[4] J. Blot, C. Buse and P. Cieutat: *Local attractivity in nonautonomous semilinear evolution equations*, Nonauton. Dyn. Syst. **1** (2014), 72–82.

[5] T. Caraballo, J. Duan, K. Lu and B. Schmalfuß: *Invariant manifolds for random and stochastic partial differential equations*, Adva. Nonlinear Stud. **10** (2010), 23–52.

[6] T.E. Duncan, B. Maslowski and B. Pasik-Duncan: *Stochastic equations in Hilbert space with a multiplicative fractional Gaussian noise*, Stochastic Process. Appl. **115** (2005), 1357–1383.

[7] Ju.L. Daleckii and M.G. Krein: Stability of Solutions of Differential Equations in Banach Spaces, Amer. Math. Soc., Providence, RI, 1974.

[8] J. Duan, K. Lu and B. Schmalfuß: *Invariant manifolds for stochastic partial differential equations*, Ann. Prob. **31** (2003), 2109–2135.

[9] J. Duan, K. Lu and B. Schmalfuß: *Smooth stable and unstable manifolds for stochastic evolutionary equations*, J. Dynam. Differential Equations **16** (2004), 949–972.

[10] M. Garrido-Atienza and K. Lu: *Random dynamical systems for stochastic partial differential equations driven by a fractional Brownian motion*, Discrete Contin. Dyn. Syst. Ser. B **14** (2010), 473–493.

[11] M. Garrido-Atienza, K. Lu and B. Schmalfuß: *Unstable invariant manifolds for stochastic PDEs driven by a fractional Brownian motion*, J. Differ. Equations **248** (2010), 1637–1667.

[12] R.H. Martin: Nonlinear Operators and Differential Equations in Banach Spaces, Wiley-Interscience, New York, 1976.

[13] N.T. Huy: *Invariant manifolds of admissible classes for semi-linear evolution equations*, J. Differ. Equations **246** (2009), 1820–1844.

[14] N.T. Huy: *Inertial manifolds for semi-linear parabolic equations in admissible spaces*, J. Math. Anal. Appl. **386** (2012), 894–909.

[15] R. Jahanipur: *Stochastic functional evolution equations with monotone nonlinearity: Existence and stability of the mild solutions*, J. Differ. Equations **248** (2010), 1230–1255.

[16] M. Pronk and M. Veraar: *A new approach to stochastic evolution equations with adapted drift*, J. Differ. Equations **256** (2014), 3634–3683.
[17] MC. Veraar: *Non-autonomous stochastic evolution equations and applications to stochastic partial differential equations*, J. Evol. Equ. **10** (2010), 85–127.

Department of Mathematics
KunMing University of Science and Technology
650093 KunMing, YunNan Province
China
e-mail: yangsddp@126.com