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LOWER BOUNDS ON LOEWY LENGTHS OF CENTERS OF BLOCKS

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Abstract

In this short note, we give a lower bound on the Loewy length of the center of a block of a group algebra in terms of a defect group.

Let G be a finite group and k an algebraically closed field of characteristic $p > 0$. For a block B of the group algebra kG, we denote by ZB its center. It is known that the k-dimension of ZB equals the number of irreducible ordinary characters in B and Brauer [1, Problem 20] conjectured that this number is bounded above by $|D|$, where D is a defect group of B. In this note, we consider relations between the Loewy length $LL(ZB)$ of ZB and the structure of D. Some previous works [5, 6, 8, 9] have obtained its upper bounds: for example, Okuyama has proved that $LL(ZB) \leq |D|$.

Motivated by these results, we give a lower bound for $LL(ZB)$ in the following. For the other definitions and terminologies, see [7].

Theorem. *Let p^{ε} be the exponent of the center $Z(D)$ of D. Then*

$$\frac{p^{\varepsilon} + p - 2}{p - 1} \leq LL(ZB).$$

Proof. We may assume $|D| \neq 1$. Let $E = N_G(D, b_D)/DC_G(D)$ be the inertial quotient of B where b_D is a root in $C_G(D)$. Then E can be embedded in the automorphism group of D by the Schur-Zassenhaus Theorem. By a result of Broué [2, Proposition (III) 1.1], there exists an ideal K of ZB such that ZB/K is isomorphic to the algebra $kZ(D)^E$ of fixed points. Hence

$$LL(kZ(D)^E) = LL(ZB/K) \leq LL(ZB).$$

Here we fix an orbit $\mathcal{O}$ of an element in $Z(D)$ of order p^{ε} by the action of E. Remark that $|\mathcal{O}| \neq 0$ in k as it divides $|E|$. We put

$$a = |\mathcal{O}|1 - \sum_{u \in \mathcal{O}} u = \sum_{u \in \mathcal{O}} (1 - u)$$

where 1 is the unit element in $Z(D)$. Since a is contained in the Jacobson radical of $kZ(D)^E$, it suffices to prove that $a^t \neq 0$ where $t = 1 + p + \cdots + p^{\varepsilon - 1}$. For $0 \leq i \leq \varepsilon - 1$,

$$a^{p^i} = |\mathcal{O}|^{p^i} 1 - \sum_{u \in \mathcal{O}} u^{p^i} = |\mathcal{O}|1 - \sum_{u \in \mathcal{O}} u^{p^i}$$

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by Fermat's little theorem. Hence each term of $a^t = a \cdot a^p \cdots a^{p^{\varepsilon-1}}$ has the form

$$(-1)^{|I|}|\mathcal{O}|^{\varepsilon-|I|}\prod_{i\in I}(u_i)^{p^i}$$

where $I \subseteq \{0, 1, \ldots, \varepsilon - 1\}$ and $u_i \in \mathcal{O}$.

Suppose now that $\prod_{i\in I}(u_i)^{p^i} = 1$ for some $I \neq \emptyset$. Since the order of u_i is p^ε for any $i \in I$, we have $|I| \geq 2$. If $r = \min\{I\}$ and $s = \min\{I - \{r\}\}$, we obtain

$$1 = \{\prod_{i\in I}(u_i)^{p^i}\}^{p^{\varepsilon-s}} = \prod_{i\in I}(u_i)^{p^{\varepsilon-s+i}} = (u_r)^{p^{\varepsilon-s+r}} \neq 1,$$

a contradiction.

Thus the coefficient of 1 in a^t is $|\mathcal{O}|^\varepsilon \neq 0$. Therefore $a^t \neq 0$ as claimed. □

We add two remarks.

(1) For a block B with cyclic defect group D,

$$LL(ZB) = \frac{|D| - 1}{|E|} + 1$$

by [3, Corollary 2.8]. In this case our theorem is clear as $|E|$ divides $p - 1$.

We next suppose that B has defect group

$$< x, y \mid x^{p^{d-1}} = y^p = 1,\ y^{-1}xy = x^{1+p^{d-2}} >$$

where $d \geq 4$. As is well known, this p-group has exponent p^{d-1} and has cyclic center of order p^{d-2}. In this case,

$$LL(ZB) = \begin{cases} p^{d-2} & (p = 2) \\ \frac{p^{d-2}-1}{|E|} + 1 & (p \neq 2) \end{cases}$$

by [6, Proposition 10 and its proof]. Hence Theorem does not hold if p^ε is replaced by the exponent of D.

(2) For abelian defect groups, our lower bound refines a result of Külshammer [4, K.Korollar] on the Loewy length $LL(B)$ of B. Namely,

$$p^{\varepsilon-1} < LL(ZB) \leq LL(B)$$

where p^ε is the exponent of D.

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