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ON MARKOVIAN SEMIGROUPS OF LÉVY DRIVEN SDES, SYMBOLS AND PSEUDO–DIFFERENTIAL OPERATORS

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Abstract

We analyse analytic properties of nonlocal transition semigroups associated with a class of stochastic differential equations (SDEs) in $\mathbb{R}^d$ driven by pure jump–type Lévy processes. First, we will show under which conditions the semigroup will be analytic on the Besov space $B^m_{p,q}(\mathbb{R}^d)$ with $1 \leq p, q < \infty$ and $m \in \mathbb{R}$. Secondly, we present some applications by proving the strong Feller property and give weak error estimates for approximating schemes of the SDEs over the Besov space $B^m_{\infty,\infty}(\mathbb{R}^d)$. The choice of Besov spaces is twofold. First, observe that Besov spaces can be defined via the Fourier transform and the partition of unity. Secondly, the space of continuous functions can be characterised by Besov spaces.

1. Introduction

The purpose of the article is to show smoothing properties for the Markovian semigroup generated by stochastic differential equations driven by pure jump–type Lévy processes. To be more precise, let $L = \{L(t) : t \geq 0\}$ be a family of Lévy processes. Let us consider the stochastic differential equations of the form

$$\left\{ \begin{array}{rcl} dX^x(t) & = & b(X^x(t-))\,dt + \sigma(X^x(t-))dL(t), \\ X^x(0) & = & x, \quad x \in \mathbb{R}^d, \end{array} \right. \tag{1.1}$$

where $\sigma : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous. Under this assumption, the existence and uniqueness of a solution to equation (1.1) is well established, see for e.g. [2, p. 367, Theorem 6.2.3]. Let $(\mathcal{P}_t)_{t\geq 0}$ be the Markovian semigroup associated to X defined by

$$(\mathcal{P}_t f)(x) := \mathbb{E}\left[f(X^x(t))\right], \quad t \geq 0, \quad x \in \mathbb{R}^d. \tag{1.2}$$

Then, it is known that $(\mathcal{P}_t)_{t\geq 0}$ is a Feller semigroup (see [2,Theorem 6.7.2]) and its infinitesimal generator is given by

$$Au(x) = \int_{\mathbb{R}^d} e^{ix^T\xi} a(x,\xi)(\mathcal{F}u)(\xi)\,d\xi \quad u \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{F}u$ denotes the Fourier transform of u, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space of infinite often differentiable functions, where all derivatives decrease faster than any power of $|x|$ as $|x|$ tends to infinity, and the symbol a is defined by

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$$(1.3)\qquad a(x,\xi) := -\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}\left[e^{i(X^x(t)-x)^T\xi}-1\right],\quad x\in\mathbb{R}^d,\quad \xi\in\mathbb{R}^d.$$

In [15], the first two named authors investigate the analytic properties of the Markovian semigroup generated by an SDE driven by a Lévy process (see Theorem 2.1 in [15]). These type of results are used to solve several applications which arise in fields related to probability theory such as nonlinear filtering theory [14], or stochastic numerics (see section (6)). In [16], the author introduced a so-called Sobolev index and showed that the evolution problem associated with a Lévy process with Sobolev index α has a unique weak solution in the Sobolev space $H^{\alpha/2}$. In this article, we put a further step and investigate under which constraints the corresponding Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ driven by an SDE with pure jump noise forms an analytic semigroup in the Besov spaces $B^m_{p,q}(\mathbb{R}^d)$ with $1\leq p,q<\infty$ and $m\in\mathbb{R}$. Here, we used Besov-spaces due to two reasons. First, Besov spaces are quite general; one covers on one side the space of continuous functions and on the other side the scale of Hilbert spaces $L^2(\mathbb{R}^d)$ and $H^s_2(\mathbb{R}^d)$, $s\in\mathbb{R}$ (see [49, 2.3.5] or [38, p. 14]). Even, if we exclude in our results the case where $q=\infty$ or $p=\infty$, by embedding Theorems (see [38, p. 30-31]), one gets easily good estimates for $B^s_{\infty,\infty}(\mathbb{R}^d)$, $s\notin\mathbb{N}$, a space which coincides with $C^s_b(\mathbb{R}^d)$. In this way, we can use the analyticity property of the Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ in Besov spaces to obtain the strong Feller property of $(\mathcal{P}_t)_{t\geq 0}$. The strong Feller property of the Markovian semigroup associated with $\mathbb{R}^d$-valued SDEs plays an important role in the long time behaviour or within the proof of the uniqueness of an invariant measure of solution processes. So, our first motivation for this paper was to study the regularity of the Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ (e.g. see Corollary (5.1) and Corollary (5.4)) associated with equation (1.1). In particular, we were interested in getting weak assumptions on the coefficients b and σ. The second motivation was to study the Monte-Carlo error of an approximation of an SDE driven by Lévy noise. To be more precise, it enables us to obtain an explicit estimate of the distance between the semigroup associated with the original problem (1.1) and the semigroup associated with certain approximations of the original problem.

In [22, Theorem 2.2] and [36], the authors derive some estimates on the density of the solution of an SDE driven by a Lévy process. These estimates are uniform in space and are related to our results, see Corollary 5.2. In [28], the authors consider the non-symmetric jump processes and construct the heat kernel. For this heat kernel, the authors deduce some upper bound as well estimates for its fractional derivative and estimates of its gradient. In [5], the authors represent their main result as the propagation of the regularity of the Markovian semigroups induced by the solution process of an SDE driven by a Brownian motion and a Lévy process. In particular, they show that for all $k\in\mathbb{N}$ there exists a constant $C>0$ depending on the operator a and $T>0$ such that

$$\sup_{0<t\leq T}\|\mathcal{P}_t f\|_{W^k_\infty}\leq C\|f\|_{W^k_\infty},$$

for all $f\in C^k_b(\mathbb{R}^d)$. Here $\|f\|_{k,\infty}$ is the supremum norm of f and its first k derivatives. In the case of $k=0$, this means that the semigroup $(\mathcal{P}_t)_{t\geq 0}$ is a Feller semigroup. Kühn, [33], investigates the Feller property of the Markovian semigroup for unbounded diffusion coefficients, see also [40] and [41] for related works by the authors. In [31], the analyticity of the Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ is proven for SDEs with only additive noise; the noise has to have a very special form. Notice also that in [12], the authors derive a Bismuth-Elworthy-

Li type formula for Lévy processes in Hilbert spaces. In [23], the authors get nice estimates on the density of solution processes driven by pure Lévy processes. They investigated the case in $\mathbb{R}^d$, $d > 1$, where the Lévy measure is only supported by the axes and have different exponents. Also, they give some short time estimates for the density.

The paper is organised as follows. In section 2, we give a short review of the symbols associated with the SDEs driven by Lévy processes and introduce some notations. In section 3, we give a short introduction to pseudodifferential operators and fix the notation. In section 4, we study under which conditions on the symbol, the semigroup $(\mathcal{P}_t)_{t\geq 0}$ is an analytic semigroup in a general Besov space $B^m_{p,q}(\mathbb{R}^d)$, $1 \leq p, q < \infty$. The motivation for our main results, i.e. the two applications to solution processes of stochastic differential equations, are presented in section 5 and 6. As the first application, we verify under which constrains the semigroup $(\mathcal{P}_t)_{t\geq 0}$ is strong Feller. As a second application, we calculate the rate of convergence for the Monte Carlo error for SDEs driven by a pure Lévy process; the theoretical result is also verified by some numerical experiments. Finally, in section A, we give a short overview of pseudo-differential operators and investigate under which condition the operator of a symbol is invertible.

Notation 1.1. For a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. For an element $\xi \in \mathbb{R}^n$, let ξ^α be defined by $\xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$. Moreover for a function $f : \mathbb{R}^d \to \mathbb{C}$ we write $\partial_x^\alpha f(x)$ for

$$\frac{\partial^\alpha}{\partial x_1 \partial x_2 \cdots \partial x_d} f(x).$$

In addition, let us define the brackets $\langle \cdot \rangle : \mathbb{R} \ni \xi \mapsto \langle \xi \rangle^\rho := (1 + |\xi|^2)^{\frac{\rho}{2}} \in \mathbb{R}$. Following inequality, also called *Peetres inequality*, is used on several places

$$\langle x + y \rangle^s \leq c_s \langle x \rangle^s \langle y \rangle^{|s|}, \quad x, y \in \mathbb{R}^d,\ s \in \mathbb{R}.$$

Let X be a non empty set and $f, g : X \to [0, \infty)$. We set $f(x) \lesssim g(x)$, $x \in X$, iff there exists a $C > 0$ such that $f(x) \leq C g(x)$ for all $x \in X$. Moreover, if f and g depend on a further variable $z \in Z$, the statement for all $z \in Z$, $f(x,z) \lesssim g(x,z)$, $x \in X$ means that for every $z \in Z$ there exists a real number $C_z > 0$ such that $f(x,z) \leq C_z g(x,z)$ for every $x \in X$. Also we set $f(x) \asymp g(x)$, $x \in X$, iff $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$ for all $x \in X$. Finally, we say $f(x) \gtrsim g(x)$, $x \in X$, iff $g(x) \lesssim f(x)$, $x \in X$. Similarly as above, we handle the case if the functions depend on a further variable.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of infinite often differentiable functions where all derivatives decrease faster than any power of $|x|$, as $|x|$ tends to infinity. Let $\mathcal{S}'(\mathbb{R}^d)$ be the dual of $\mathcal{S}(\mathbb{R}^d)$.

If $m \in \mathbb{N}$ we define

$$C_b^m(\mathbb{R}^d) := \left\{ f \in C_b^0(\mathbb{R}^d) : D^\alpha f \in C_b^0(\mathbb{R}^d), |\alpha| \leq m \right\}$$

endowed with the norm

$$|f|_{C_b^m} := \sum_{|\alpha|\leq m} |D^\alpha f|_{C_b^0}.$$

Let $s \in \mathbb{R} \setminus \mathbb{N}$, then we put $s = [s] + \{s\}$, where $[s]$ is an integer and $0 \leq \{s\} < 1$. Then

$$C_b^s(\mathbb{R}^d) := \left\{ f \in C_b^{[s]}(\mathbb{R}^d) : \sum_{|\alpha|=[s]} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{s\}}} < \infty \right\}$$

equipped with the norm

$$|f|_{C_b^s} := |f|_{C_b^{[s]}} + \sum_{|\alpha|=[s]} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{s\}}}.$$

In order to define Besov spaces as given in [38, Definition 2, pp. 7-8] (compare also to [48]) let us choose first a function $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $0 \leq \psi(x) \leq 1$, $x \in \mathbb{R}^d$ and

$$\psi(x) = \begin{cases} 1, & \text{if} \quad |x| \leq 1, \\ 0 & \text{if} \quad |x| \geq \frac{3}{2}. \end{cases}$$

Then, let us put

$$\begin{cases} \phi_0(x) &=& \psi(x), \ x \in \mathbb{R}^d, \\ \phi_1(x) &=& \psi(\frac{x}{2}) - \psi(x), \ x \in \mathbb{R}^d, \\ \phi_j(x) &=& \phi_1(2^{-j+1}x), \ x \in \mathbb{R}^d, \quad j = 2, 3, \ldots. \end{cases}$$

Since we need it later on let

$$\mathcal{U}_1 \quad = \quad \operatorname{supp}(\phi_1). \tag{1.4}$$

We will use the definition of the Fourier transform $\mathcal{F} = \mathcal{F}^{+1}$ and its inverse $\mathcal{F}^{-1}$ as [38, p. 6]. In particular, with $\langle \cdot, \cdot \rangle$ being the scalar product in $\mathbb{R}^d$, we put

$$(\mathcal{F}^{\pm 1} f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{\mp i \langle x, \xi \rangle} f(x)\, dx, \quad f \in \mathcal{S}(\mathbb{R}^d), \xi \in \mathbb{R}^d.$$

With the choice of $\phi = \{\phi_j\}_{j=0}^\infty$ as above and $\mathcal{F}$ and $\mathcal{F}^{-1}$ being the Fourier and the inverse Fourier transformations (acting on the space $\mathcal{S}'(\mathbb{R}^d)$ of Schwartz distributions) we have the following definition.

Definition 1.1. Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. If $0 < q < \infty$ we put

$$|f|_{B^s_{p,q}} \quad = \quad \left(\sum_{j=0}^\infty 2^{sjq} \left| \mathcal{F}^{-1} \left[\phi_j \mathcal{F} f \right] \right|_{L^p}^q \right)^{\frac{1}{q}} = \left\| \left(2^{sj} \left| \mathcal{F}^{-1} \left[\phi_j \mathcal{F} f \right] \right|_{L^p} \right)_{j \in \mathbb{N}} \right\|_{l^q}.$$

2. Symbols, their definitions and properties

In this section, we give a short review of symbols coming up as Hoh's and Lévy's symbols while dealing with processes generated by Lévy processes. Besides, we introduce some notations. Throughout the remaining article, let $L = \{L^x(t) : t \geq 0, x \in \mathbb{R}^d\}$ be a family of Lévy processes L^x, where L^x is a Lévy process starting at $x \in \mathbb{R}^d$. Then L generates a Markovian semigroup $(\mathcal{P}_t)_{t \geq 0}$ on $C_b(\mathbb{R}^d)$ by

$$\mathcal{P}_t f(x) := \mathbb{E} f(L^x(t)), \quad f \in C_b(\mathbb{R}^d).$$

Let A be the infinitesimal generator of $(\mathcal{P}_t)_{t \geq 0}$ acting on $C_b^2(\mathbb{R}^d)$ defined by

$$A f := \lim_{h\to 0} \frac{1}{h}\left(\mathcal{P}_h - \mathcal{P}_0\right) f, \quad f \in C_b^2(\mathbb{R}^d). \tag{2.1}$$

Another way of defining A is done by Lévy symbols (see [20, 16]). In particular, let

$$\psi(\xi) = \frac{1}{t}\ln(\mathbb{E}e^{i\langle \xi, L(t)\rangle}), \quad \xi \in \mathbb{R}^d.$$

Observe that we have (see e.g. [2, p. 42] and [39])

$$\psi(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle\xi,z\rangle} - 1 - i\langle\xi, x\rangle \mathbb{1}_{\{|z|\le 1\}}\right) \nu(dz), \quad \xi \in \mathbb{R}^d.$$

If L is a pure jump process with symbol ψ, then the infinitesimal generator defined by (2.1) can also be written as

$$(Af)(x) \;=\; -\int_{\mathbb{R}^d} e^{i\langle\xi,x\rangle}\psi(\xi)(\mathcal{F}f)(\xi)\, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{2.2}$$

The operator A, usually denoted in the literature by $\psi = \psi(D)$, is well defined in $C_b^2(\mathbb{R}^d)$, has values in $\mathcal{B}_b(\mathbb{R}^d)$ (bounded Borel functions in $\mathbb{R}^d$) and satisfies the positive maximum principle (see e.g. [24, Theorem 4.5.13]). Therefore, A generates a Feller semigroup on $C_b^\infty(\mathbb{R}^d)$ and a sub–Markovian semigroup on $L^2(\mathbb{R}^d)$ (see e.g. [25, Theorem 2.6.9 and Theorem 2.6.10]). To characterise the symbol, we introduce the generalised Blumenthal–Getoor index (see [7]).

Definition 2.1. Let L be a Lévy process with symbol ψ and $\psi \in C_b^k(\mathbb{R}^d \setminus \{0\})$ for some $k \in \mathbb{N}_0$. Then the Blumenthal–Getoor index of order k is defined by

$$s := \inf_{\substack{\lambda>0\\ |\alpha|\le k}} \left\{\lambda : \lim_{\xi\to\infty} \frac{|\partial_\xi^\alpha \psi(\xi)|}{|\xi|^{\lambda-|\alpha|}} = 0\right\}.$$

Here α denotes a multi-index. If $k = \infty$ then Blumenthal–Getoor index of infinity order is defined by

$$s := \inf_{\substack{\lambda>0\\ \alpha \text{ is a multi–index}}} \left\{\lambda : \lim_{|\xi|\to\infty} \frac{|\partial_\xi^\alpha \psi(\xi)|}{|\xi|^{\lambda-|\alpha|}} = 0\right\}.$$

Remark 2.1. For a function $\psi : \mathbb{R}^d \to \mathbb{R}$, the limit $\lim_{\xi\to\infty}\psi(\xi)$ is a sloppy formulation and means actual

$$\sup_{\xi\in\mathcal{U}_1} \lim_{\lambda\to\infty} \psi(\lambda\xi),$$

where $\mathcal{U}_1$ defined in (1.4). This can be easily seen by analysing, e.g. the proof of the boundedness of the corresponding operator and realizing that the estimate comes up in analysing the summands after decomposing the operator in its dyadic partition of the unity.

Remark 2.2. The Blumenthal–Getoor index of order infinity is defined for the sake of completeness. We are interested in weakening the assumption on the symbol, i.e., reducing the order k.

To analyse properties of the Markovian semigroup $(\mathcal{P}_t)_{t\ge 0}$ and to define the resolvent of the associated operator $\psi(D)$, the range of the symbol is of importance.

Definition 2.2. Let $\mathfrak{Rg}(\psi)$ be the essential range of ψ, i.e.

$$\mathfrak{Rg}(\psi) := \{y \in \mathbb{C} \mid \text{Leb}(\{s \in \mathbb{R}^d : |\psi(s) - y| < \varepsilon\}) > 0 \text{ for each } \varepsilon > 0\}^1.$$

Finally, to characterize the spectrum of the associated operator, one can introduce the type of a symbol.

Definition 2.3. We call a symbol ψ is of type (ω, θ), $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$, iff

$$-\mathfrak{Rg}(\psi) \subset \mathbb{C} \setminus \{\omega\} + \Sigma_{\theta+\frac{\pi}{2}}, \quad \text{where} \quad \Sigma_r := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \delta\}, \quad \delta \in [0, \pi].$$

Remark 2.3. If a symbol ψ is of type $(0, \theta)$, then there exists a constant $c > 0$ such that

$$|\Im(\psi(\xi))| \leq c\, \Re\psi(\xi), \quad \xi \in \mathbb{R}^d.$$

The condition above is often called sector condition of the symbol ψ.

Before presenting a typical example, we introduce stable processes, compare [39, Chapter 3].

Definition 2.4. A probability measure μ on $\mathbb{R}^d$ is infinitely divisible, if for any positive integer $n \in \mathbb{N}$, there exists a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^{(n)*}$.[2]

Observe, due to the independent increments of a Lévy process, the distribution function $L(t)$, $t > 0$, for any Lévy process is an infinitely divisible probability measure.

Definition 2.5 (see Sato [39, Chapter 3]). An infinite divisible probability measure μ is stable, if for any $a > 0$, there exist numbers $b > 0$ and $c \in \mathbb{R}^d$ such that

$$\hat{\mu}(z)^a = \hat{\mu}(bz)\, e^{i\langle c,z\rangle}, \quad z \in \mathbb{R}^d.$$

Here, $\hat{\mu}$ denotes the characteristic function of the probability measure μ, i.e. $\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\mu(dx)$, $z \in \mathbb{R}^d$. The measure μ is called strictly stable, if for any $a > 0$ there exists a number $b > 0$ such that

$$\hat{\mu}(z)^a = \hat{\mu}(bz), \quad z \in \mathbb{R}^d.$$

Definition 2.6. Let $\{X(t) : t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$. It is called a stable or strictly stable process, if the distribution for $X(1)$ is a stable, respectively, a strictly stable infinite divisible measure.

Example 2.1. Let L be a one dimensional strictly α–stable process. In particular, L be a real-valued Lévy process with initial value $L_0 = 0$ that satisfies the self-similarity property

$$L_t / t^{\frac{1}{\alpha}} \stackrel{d}{=} L_1, \quad \forall t > 0.$$

Then, its symbol is given by $\psi(\xi) = c\,|\xi|^\alpha$, the parameter α is called the exponent of the process (see [39, Section 14, p. 77]). Let σ and b be two Lipschitz continuous functions on $\mathbb{R}$. Then, for $\alpha > 1$, the symbol

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[1] Here, Leb denotes the Lebesgue measure.

[2] The symbol $*$ denotes the convolution of two probability measures.

$$a(x,\xi) := |\sigma(x)\xi|^\alpha + ib(x)\xi$$

is of type $(0,\theta)$. If σ is bounded away from zero, then the generalized Blumenthal–Getoor index is α.

Example 2.2. Let $\alpha \in (0,2)$ and L be a symmetric α–stable process without drift. The symbol ψ of L is given by

$$\psi(\xi) = |\xi|^\alpha,$$

the upper and lower index is α, and ψ is of type $(0,\delta)$ for any $\delta > 0$.

Remark 2.4. Let H be a Hilbert space. For $\lambda \in \mathbb{C} \setminus \mathfrak{Rg}(\psi) := \{\zeta \in \mathbb{C} : \exists \xi \text{ with } \psi(\xi) = \zeta\}$ we have (see Theorem 1.4.2 of [18])

$$|R(\lambda, \psi(D))|_{L(H,H)} \le \frac{1}{\operatorname{dist}(\mathfrak{Rg}(\psi), \lambda)}.$$

Moreover, the set $\mathfrak{Rg}(\psi)$ equals the spectrum of the generator A.

For different examples of Lévy processes and their symbols, we refer to [6], [8], [15], [32], or [44]. In case there is no dependence on the space variable, one can derive properties of the Markovian semigroup directly using the range of the symbol. Given a solution process of an SDE, usually, the associated infinitesimal generator of the Markovian semigroup depends on the space variable x. In particular, for $x \in \mathbb{R}^d$ let $X = \{X^x(t) : t \geq 0\}$ be a $\mathbb{R}^d$-valued solution of the SDE given in (1.1) and, as before let $(\mathcal{P}_t)_{t\geq 0}$ be the associated Markovian semigroup defined in (1.2). Let ψ be the Lévy symbol of the Lévy process $L = \{L(t) : t \geq 0\}$. Then, one can show (see Theorem 3.1 [43]), that the infinitesimal generator of the Markovian semigroup associated to X^x has the symbol $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ given by

$$a(x,\xi) \quad = \quad \psi(\sigma^T(x)\xi), \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.3}$$

Let $a_1(x,\xi)$ and $a_2(x,\xi)$ be two given symbols. Due to the dependence on x, the corresponding operators $a_1(x,D)$ and $a_2(x,D)$ do not necessarily commute. Therefore, many techniques working for operators induced by symbols being independent of the space variable x do not work for operators induced by symbols depending on the space variable x. Especially, tricks relying on the Bony's paraproduct gets much more demanding.

In our main result Theorem 4.2 we show under which conditions on the symbol ψ and on the coefficients σ and b the Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ is an analytic semigroup in general Besov spaces $B^s_{p,q}(\mathbb{R}^d)$. To be more precise, we show if σ is bounded away from zero, σ and b are smooth enough, and ψ is of type $(0,\theta)$, $\theta < \frac{\pi}{2}$, sufficiently smooth, and having Blumenthal-Geetor index $\delta \in (1,2)$ of sufficiently high order, then the Markovian semigroup is analytic on $B^s_{p,q}(\mathbb{R}^d)$ for $p, q \in [1,\infty)$.

The choice of Besov spaces is twofold. First, observe that Besov spaces can be defined via the Fourier transform and the partition of the unity (see the paragraph notation or [38, Definition 2, pp. 7-8]). Now, since the operator associated with the symbol $a(x,\xi)$ can be represented by a kernel of the form

$$a(x,D)f(x) = \int_{\mathbb{R}^d} k(x, x-y) f(y)\, dy, \quad x \in \mathbb{R}^d,$$

where the kernel is given by the inverse Fourier transform[3]

$$k(x,z) = \mathcal{F}_{\xi\to z}\left[a(x,\xi)\right](z),$$

Besov spaces come up naturally. Secondly, the strong Feller property is defined via the space of continuous functions $C^s_b(\mathbb{R}^d)$, which is related to the Besov space $B^s_{\infty,\infty}(\mathbb{R}^d)$ for $s \neq 0$. So, it suggests by itself to use Besov spaces and embedding Theorems to prove the strong Feller property for $(\mathcal{P}_t)_{t\geq 0}$.

3. A short introduction to pseudo–differential operators

In this section, we shortly introduce the main definition of pseudo–differential operators and their symbols. Also, we present the definitions and Theorems which are necessary for our purpose. For a detailed introduction on pseudo–differential operators and their symbols in the context of partial differential equations we recommend the books [1, 34, 45, 51, 47, 17], or the monograph of Kumano-go [30], in the context of Markov processes we recommend the books [24, 25, 26] or the survey [9, 29]. Here, we closely follow the book of Abels [1].

Definition 3.1. Let ρ, δ be two real numbers such that $0 \leq \rho \leq 1$ and $0 \leq \delta \leq 1$. Let $S^m_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ be the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where

- $a(x,\xi)$ is infinitely often differentiable, i.e. $a \in C^\infty_b(\mathbb{R}^d \times \mathbb{R}^d)$;
- for any two multi-indices α and β there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\left|\partial^\alpha_{\xi'}\partial^\beta_x a(x,\xi)\mid_{\xi'=\xi\gamma}\right| \leq C_{\alpha,\beta}\langle|\gamma\xi|\rangle^{m-\rho|\alpha|}\langle|x|\rangle^{\delta|\beta|}, \quad x \in \mathbb{R}^d, \xi \in \mathcal{U}_1, \gamma \geq 1.$$

We call any function $a(x,\xi)$ belonging to $\cup_{m\in\mathbb{R}} S^m_{0,0}(\mathbb{R}^d,\mathbb{R}^d)$ a *symbol*. For many estimates, one does not need that the function is infinitely often differentiable. It is often only necessary to know the estimates with respect to ξ and x up to a particular order. For this reason, we introduce the following classes.

Definition 3.2 (compare [51, p. 28]). Let $m \in \mathbb{R}$. Let $\mathcal{A}^m_{k_1,k_2;\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d)$ be the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where

- $a(x,\xi)$ is k_1–times differentiable in ξ and k_2 times differentiable in x;
- for any two multi-indices α and β with $|\alpha| \leq k_1$ and $|\beta| \leq k_2$, there exists a constant $C_{\alpha,\beta} > 0$ depending only on α and β such that

$$\left|\partial^\alpha_{\xi'}\partial^\beta_x a(x,\xi)\mid_{\xi'=\xi\gamma}\right| \leq C_{\alpha,\beta}\langle|\gamma\xi|\rangle^{m-\rho|\alpha|}\langle|x|\rangle^{\delta|\beta|}, \quad x \in \mathbb{R}^d, \xi \in \mathcal{U}_1, \gamma \geq 1.$$

Moreover, we introduce a semi–norm in $\mathcal{A}^m_{k_1,k_2;\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d)$ by

$$\begin{aligned}&\|a\|_{\mathcal{A}^m_{k_1,k_2;\rho,\delta}} \\ &= \sup_{|\alpha|\leq k_1, |\beta|\leq k_2}\ \sup_{(x,\xi)\in\mathbb{R}^d\times\mathcal{U}_1\times\mathbb{R}} \left|\partial^\alpha_\xi\partial^\beta_x a(x,\xi)\mid_{\xi=\xi'\gamma}\right| \langle|\gamma\xi|\rangle^{\rho|\alpha|-m}\langle|x|\rangle^{-\delta|\beta|}, \quad a \in \mathcal{A}^m_{k_1,k_2;\rho,\delta}(\mathbb{R}^d\times\mathbb{R}^d).\end{aligned}$$

[3] $\mathcal{F}_{\xi\to z}[a(x,\xi)](z) = \int_{\mathbb{R}^d} e^{-2\pi i\xi z} a(x,\xi)\, d\xi$.

We have seen in the introduction that, given a symbol, one can define an operator. In case the symbol ψ is a Lévy symbol, the operator defined by (2.2) is an infinitesimal generator of a semigroup of a Lévy process. In case one has an arbitrary symbol, the corresponding operator can be defined similarly.

DEFINITION 3.3 (COMPARE [51, p.28, Def. 4.2]). Let $a(x,\xi)$ be a symbol. Then, $a(x,\xi)$ corresponds to an operator $a(x,D)$ being defined by

$$(a(x,D)\,u)\,(x) := \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} a(x,\xi)\,\hat{u}(\xi)\,d\xi, \quad x \in \mathbb{R}^d,\ u \in \mathcal{S}(\mathbb{R}^d)$$

and being called pseudo–differential operator.

In most applications, one is interested in inverting the operator $a(x,D)$. Here, the symbol has to be elliptic, a terminus being the subject of the next definition.

DEFINITION 3.4 (COMPARE [34, p. 35]). A symbol $a \in S^m_{\rho,\delta}(\mathbb{R}^d\times\mathbb{R}^d)$ is called globally *elliptic*, if there exists a number $r > 0$,

$$\langle|\gamma\xi|\rangle^m \lesssim |a(x,\gamma\xi)|, \quad \gamma \geq r,\ \xi \in \mathcal{U}_1,\ x \in \mathbb{R}^d.$$

In the appendix, we will see that we need upper estimates not only for the symbol itself but also for its derivatives. Therefore, we have to introduce a more sophisticated definition of ellipticity.

DEFINITION 3.5 (COMPARE [34, p. 35]). Let m,ρ,δ be real numbers with $0 \leq \delta < \rho \leq 1$. The class $\mathrm{Hyp}^m_{k_1,k_2,\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ consists of all functions $a(x,\xi)$ such that

- $a(x,\xi)$ is k_1–times differentiable in ξ and k_2 times differentiable in x;
- there exists some $r > 0$ such that

  $$\langle|\gamma\xi|\rangle^m \lesssim |a(x,\gamma\xi)|, \quad \gamma \geq r,\ \xi \in \mathcal{U}_1,\ x \in \mathbb{R}^d,$$

  and for an arbitrary multi-indices α and β there exists a constant $C_{\alpha,\beta} > 0$ with

  $$\left|\partial^\alpha_{\xi'}\partial^\beta_x a(x,\xi')\Big|_{\xi'=\gamma\xi}\right| \leq C_{\alpha,\beta}\langle|\gamma\xi|\rangle^{m-\rho|\alpha|}\langle|x|\rangle^{\delta|\beta|},$$

  for $x \in \mathbb{R}^d, \xi \in \mathcal{U}_1, \gamma \geq r$.

In addition, for $k_1,k_2 \in \mathbb{N}_0$, we define the following semi–norm given by

$$\|a\|_{\mathrm{Hyp}^m_{k_1,k_2;\rho,\delta}} = \sup_{|\alpha|\leq k_1,|\beta|\leq k_2}\ \sup_{x\in\mathbb{R}^d}\ \limsup_{\xi\in\mathcal{U}_1,\gamma\to\infty}\left|\partial^\alpha_{\xi'}\partial^\beta_x\left[\frac{1}{a(x,\xi')}\Big|_{\xi'=\gamma\xi}\right]\right|\langle|\gamma\xi|\rangle^{m+\rho|\alpha|}\langle|x|\rangle^{\delta|\beta|}.$$

In appendix A, we present some theorems and corollaries being necessary for the proof.

4. Analyticity of the Markovian semigroup in general Besov spaces

Given a function space $\mathbb{X}$ over $\mathbb{R}^d$ we are interested under which conditions on the coefficients σ, b and the symbol ψ, the Markovian semigroup $(\mathcal{P}_t)_{t\geq0}$ generates an analytic semigroup on $\mathbb{X}$. Here, one has first to verify that $(\mathcal{P}_t)_{t\geq0}$ generates a strongly continuous semigroup. The Hille–Yosida Theorem gives the necessary and sufficient conditions which have to be satisfied by a semigroup to be strongly continuous. Let us assume that $\mathbb{X}$ is a Banach space. For an operator A, let $\rho(A)$ represent the resolvent set, i.e. $\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)$ is invertible $\}$ and $\sigma(A) = \mathbb{C} \setminus \rho(A)$. Now, if $(A, D(A))$ is closed, densely defined, and for any $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$ one has $\lambda \in \rho(A)$ (compare [13, Theorem 3.5, p. 73], or [35, Theorem 1.5.2]) and

$$\|R(\lambda, A)\|_{L(\mathbb{X},\mathbb{X})} \leq \frac{1}{\Re\lambda}, \tag{4.1}$$

then A generates a strongly continuous semigroup on $\mathbb{X}$. Secondly, to show that this strongly continuous semigroup is analytic, one has to show either that (compare [13, Theorem 4.6, p. 101])

$$M := \sup_{t>0} \|tA\mathcal{P}_t\|_{L(\mathbb{X},\mathbb{X})} < \infty, \tag{4.2}$$

or that there exists a constant $C > 0$ such that

$$\|R(\vartheta + i\tau : A)\|_{L(\mathbb{X},\mathbb{X})} \quad \leq \quad \frac{C}{|\tau|}, \quad \vartheta > 0, \vartheta, \tau \in \mathbb{R}. \tag{4.3}$$

Let $S(A) = \{\langle x^*, Ax\rangle : x \in D(A), x \in \mathbb{X}^*, |x| = 1, |x^*| = 1, \langle x^*, x\rangle = 1\}$ be the numerical range of an operator A. If $\mathbb{X}$ is a Hilbert space and σ constant, $S(A)$ can be characterized by the $\Re\mathfrak{g}(\psi) := \overline{\{a(x,\xi) \in \mathbb{C} : x, \xi \in \mathbb{R}^d\}}$, where $a(x,\xi) := \psi(\sigma^T(x)\xi)$. Since the range of ψ contains the numerical range $S(A)$ of A, we have (see Remark 2.4)

$$\|R(\lambda, A)\|_{L(\mathbb{X},\mathbb{X})} \leq \frac{1}{\operatorname{dist}(\lambda, S(A))}. \tag{4.4}$$

Hence, for $\mathbb{X} = H_2^m(\mathbb{R}^d)$ and $\sigma(x) = \sigma_0$, one can show by analysing the numerical range, which is here given by

$$S(a(x,D)) = \left\{\langle x, a(x,D)x\rangle : x \in \ \operatorname{dom}(a(x,D)), |x|_{H_2^m(\mathbb{R}^d)} = 1, \langle x, x\rangle_{H_2^m} = 1\right\},$$

and some purely geometric considerations, the analyticity of the semigroup $(\mathcal{P}_t)_{t\geq 0}$ in $\mathbb{X}$. Here $\langle\ ,\ \rangle$ represents the inner product in $H_2^m(\mathbb{R}^d)$. In fact, choosing a complex number $\lambda = \vartheta + i\tau$ with $\vartheta > 0$ and $\tau \in \mathbb{R}$, and using that the symbol ψ is of type $(0,\theta)$, we obtain by the following series of computations (see Theorem 3.9 [35, Chapter I])

$$\begin{aligned}
\|R(\lambda, a(x,D))\|_{L(H_2^m(\mathbb{R}^d),H_2^m(\mathbb{R}^d))} &= \|R(\vartheta + i\tau, a(x,D))\|_{L(H_2^m(\mathbb{R}^d),H_2^m(\mathbb{R}^d))} \\
\leq \frac{1}{\operatorname{dist}(\lambda, \bar{S}(a(x,D)))} \leq \frac{1}{\operatorname{dist}(\lambda, \rho(a(x,D)))} &\leq \frac{1}{\operatorname{dist}(\lambda, \rho(a(x,D)))} \\
= \frac{1}{\operatorname{dist}(\vartheta + i\tau, \rho(a(x,D)))} &\leq \frac{\cos\theta}{|\tau|} = \frac{C}{|\tau|},
\end{aligned}$$

where $C = \cos\theta$. These calculations imply that $(\mathcal{P}_t)_{t\geq 0}$ in $\mathbb{X}$ is an analytic semigroup in $\mathbb{X}$.

This result can be generalised to arbitrary Besov spaces. The motivation to analyse the analyticity of the Markovian semigroup in Besov spaces comes from the aim to investigate the strong Feller property of the Markovian semigroup. Since one has the embedding $C^s(\mathbb{R}^d) \subset B^s_{\infty,\infty}(\mathbb{R}^d)$ $(s \neq 0)$, it is natural to switch to Besov spaces. The disadvantage is, abandoning the Hilbert space setting, the numerical range gets more complicated, and it is better to use other methods.

Remark 4.1. Using the abstract theory of classic books such as, e.g. [35, 13, 48], we can prove the following two Theorems for the pseudo–differential operator induced by a simple Lévy process by proving that all assumptions of the corresponding theorems (e.g. Theorem 5.2 [35, Chapter II], Theorem 3.9 [35, Chapter I], Theorem 2.3.3 [48, p.48], etc.) are satisfied. In our case, the underlying stochastic process is not a simple Lévy process, but a solution of stochastic differential equations. Therefore, we apply the same method to prove Theorem 4.1. In particular, we prove that all assumptions of the corresponding theorems (Theorem 5.2 [35, Chapter II], Theorem 3.9 [35, Chapter I], Theorem 2.3.3 [48, p.48], etc.) are satisfied.

Theorem 4.1. *Let us assume that*

- *the symbol* $a(x,\xi)$ *belongs to* $\mathcal{A}^{\delta}_{2d+4,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$, *where* $1<\delta<2$,
- *the symbol* $a(x,\xi)$ *belongs to* $Hyp^{\delta}_{2d+4,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$,
- *and is of type* $(0,\theta)$, $0\le\theta<\frac{\pi}{2}$.

Then, for all $1\le p,q<\infty$ *and* $m\in\mathbb{R}$, *the operator* $a(x,D)$ *generates an analytic semigroup* $(\mathcal{P}_t)_{t\ge0}$ *in* $B^m_{p,q}(\mathbb{R}^d)$.

Let $L=\{L(t):t\ge0\}$ be a family of Lévy processes and let us consider the stochastic differential equations of the form

$$\begin{cases} dX^x(t) &= b(X^x(t-))\,dt+\sigma(X^x(t-))dL(t),\\ X^x(0) &= x,\quad x\in\mathbb{R}^d,\end{cases}$$

where $\sigma:\mathbb{R}^d\to L(\mathbb{R}^d,\mathbb{R}^d)$ and $b:\mathbb{R}^d\to\mathbb{R}^d$ are Lipschitz continuous. Let $(\mathcal{P}_t)_{t\ge0}$ be the Markovian semigroup of X defined in (1.2). Applying Theorem 4.1 to the infinitesimal generator of $(\mathcal{P}_t)_{t\ge0}$ gives following Theorem.

Theorem 4.2. *Let us assume that the symbol* ψ *is of type* $(0,\theta)$, $0\le\theta<\frac{\pi}{2}$, *and*

$$\psi\in\mathcal{A}^{\delta}_{2d+4,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d)\cap Hyp^{\delta}_{2d+4,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d),$$

where $1<\delta<2$ *is the Blumenthal–Getoor index of order* $2d+4$ *of L. In addition, let us assume that*

- $\sigma\in C^{d+3}_b(\mathbb{R}^d)$,
- *and* $b\in C^{d+3}_b(\mathbb{R}^d)$,
- *and that there exists a number* $c>0$ *such that*

$$\inf_{x\in\mathbb{R}^d}\sigma(x)\ge cI.$$

Then, for all $1\le p,q<\infty$ *and* $m\in\mathbb{R}$, *the Markovian semigroup* $(\mathcal{P}_t)_{t\ge0}$ *defined in* (1.2) *is analytic in* $B^m_{p,q}(\mathbb{R}^d)$.

Remark 4.2. The restriction that p has to be strictly smaller than infinity comes from the fact that the space of Schwarz functions $\mathcal{S}(\mathbb{R}^d)$ is not dense in $B^m_{\infty,\infty}(\mathbb{R}^d)$.

Proof of Theorem 4.1:. For simplicity, let us denote $B^m_{p,q}(\mathbb{R}^d)$ by $\mathbb{X}$. Let us assume that the symbol ψ and the coefficients σ and b are infinitely often differentiable. We first show that the operator $(\mathcal{P}_t)_{t\ge0}$ generates a strongly continuous semigroup on $\mathbb{X}$ by proving the required conditions in the Hille–Yosida Theorem. Theorem 2.3.3, p.48 in [48], gives us that the Schwarz space $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathbb{X}$. In addition, it is straightforward to show that $\mathcal{S}(\mathbb{R}^d) \subset \operatorname{dom}(a(x,D))$. This immediately gives that $\operatorname{dom}(a(x,D))$ is dense in $\mathbb{X}$.

Before starting, let us split the operator $a(x,D)$ into two operators in the same way as it is done in Theorem 7.1. Let $R \in \mathbb{N}$ sufficiently large such that

$$R \geq 6 \times \|a\|_{\tilde{\mathcal{A}}^{1,1}_{2d+4,d+3;1,0}}$$

and $\langle|\xi|\rangle^\delta \lesssim |a(x,\xi)|$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ with $|\xi| \geq R$. In addition, let $\chi \in C_b^\infty(\mathbb{R}_0^+)$ such that

$$\chi(\xi) = \begin{cases} 0 & \text{if } |\xi| \leq 1, \\ 1 & \text{if } |\xi| \geq 2, \end{cases}$$

and put $b(x,\xi) := a(x,\xi)(1-\chi(\xi/R))$ and $\tilde{a}(x,\xi) := a(x,\xi)\chi(\xi/R)$. We will show that $\tilde{A} = \tilde{a}(x,D)$ generates an analytic semigroup on $\mathbb{X}$. Due to Theorem 2.1 [35, Chapter 3.2, p. 80] and since $B = b(x,D)$ is bounded on $\mathbb{X}$, it follows that $A = \tilde{A} + B$ generates an analytic semigroup on $\mathbb{X}$.

First, we will show that $(\tilde{a}(x,D), \operatorname{dom}(\tilde{a}(x,D)))$ is closed in $\mathbb{X}$. Let $\{v_n : n \in \mathbb{N}\} \subset \operatorname{dom}(\tilde{a}(x,D))$ be a sequence such that $\lim_{n\to\infty} v_n = v$ in $\operatorname{dom}(\tilde{a}(x,D))$ and $\lim_{n\to\infty} \tilde{a}(x,D)v_n = w$ in $\mathbb{X}$. To show that $(\tilde{a}(x,D), \operatorname{dom}(\tilde{a}(x,D)))$ is closed in $\mathbb{X}$, we have to show that $\tilde{a}(x,D)v = w$. Suppose that $|\tilde{a}(x,D)v - w|_{\mathbb{X}} \neq 0$. In particular, there exists a constant $\hat{C} > 0$ such that $|\tilde{a}(x,D)v - w|_{\mathbb{X}} \geq \hat{C}$. There exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$|v - v_n|_{\operatorname{dom}(\tilde{a}(x,D))} < \frac{\hat{C}}{4\|\tilde{a}(x,D)\|_{L(\operatorname{dom}(\tilde{a}(x,D)),\mathbb{X})}}$$

and

$$|\tilde{a}(x,D)v_n - w|_{\mathbb{X}} < \frac{\hat{C}}{4}.$$

Since $\tilde{a}(x,D)$ is a linear and bounded operator on $\operatorname{dom}(\tilde{a}(x,D))$, we have

$$\begin{aligned} |\tilde{a}(x,D)v - w|_{\mathbb{X}} &\leq |\tilde{a}(x,D)v - \tilde{a}(x,D)v_n|_{\mathbb{X}} + |\tilde{a}(x,D)v_n - w|_{\mathbb{X}} \\ &\leq \|\tilde{a}(x,D)\|_{L(\operatorname{dom}(\tilde{a}(x,D)),\mathbb{X})} |v - v_n|_{\operatorname{dom}(\tilde{a}(x,D))} + |\tilde{a}(x,D)v_n - w|_{\mathbb{X}} < \frac{\hat{C}}{2}. \end{aligned}$$

Since this is a contradiction, we conclude that $w = \tilde{a}(x,D)v$. Next, we show that there exists a constant $C > 0$ such that

$$\|R(\lambda, \tilde{a}(x,D))\|_{L(\mathbb{X},\mathbb{X})} \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\theta+\frac{\pi}{2}}. \tag{4.5}$$

Here, we will apply Theorem 7.1 to get the estimate. In order to do this, first, note that the norm of $\lambda + \tilde{a}(x,\xi)$ in $\tilde{\mathcal{A}}^{\kappa,1}_{2d+4,d+3,;1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ does not depend on λ. Hence,

$$\|\lambda + \tilde{a}\|_{\tilde{\mathcal{A}}^{\kappa,1}_{2d+4,d+3;1,0}} = \|\tilde{a}\|_{\tilde{\mathcal{A}}^{\kappa,1}_{2d+4,d+3;1,0}}, \quad \lambda \in \Sigma_{\theta+\frac{\pi}{2}}.$$

Next, we have to estimate the norm of the operator $\lambda + \tilde{a}(x,\xi)$ in $\operatorname{Hyp}^\delta_{2d+4,d+3;1,0}(\mathbb{R}^d \times \mathbb{R}^d)$. That means, for any multi-indices $|\alpha| \leq 2d+4$ and $|\beta| \leq d+3$ we have to estimate

$$\sup_{\lambda \in \Sigma_{\theta+\frac{\pi}{2}}} |\lambda| \left| \partial_\xi^\alpha \partial_x^\beta \left[\frac{1}{\lambda + \tilde{a}(x,\xi)} \right] \right|.$$

By straightforward calculations it can be shown that this entity is bounded. Here, it is essential that $a(x,D)$ satisfies the sectorial condition, i.e. that there exists a $c>0$ such that $|\Im(\tilde{a}(x,\xi))| \le c|\Re(\tilde{a}(x,\xi))|$. We will consider the case where $|\alpha| = |\beta| = 0$. Separating the real and imaginary part we set $\lambda = \lambda_1 + i\lambda_2$ and $\tilde{a}(x,\xi) = \psi_1(x,\xi) + i\psi(x,\xi)$. Now we have

$$\frac{1}{\lambda + \tilde{a}(x,\xi)} = \frac{\lambda_1 + \psi_1(x,\xi)}{(\lambda_1 + \psi_1(x,\xi))^2 + (\lambda_2 + \psi_2(x,\xi))^2} - i\frac{\lambda_2 + \psi_2(x,\xi)}{(\lambda_1 + \psi_1(x,\xi))^2 + (\lambda_2 + \psi_2(x,\xi))^2}.$$

In particular, simple calculations give

$$\left|\frac{1}{\lambda + \tilde{a}(x,\xi)}\right| \le \frac{\sqrt{(\lambda_1 + \psi_1(x,\xi))^2 + (\lambda_2 + \psi_2(x,\xi))^2}}{(\lambda_1 + \psi_1(x,\xi))^2 + (\lambda_2 + \psi_2(x,\xi))^2} \le \frac{1}{\lambda_2},$$

for $\lambda_1 \ge 1$. Next, we will consider the case where $|\alpha| = |\beta| = 1$, that is let $\alpha = k$ and $\beta = l$ with $k, l \in \{1, \dots, d\}$. Then,

$$\partial_{x_l}\partial_{\xi_k}\left[\frac{1}{\lambda + \tilde{a}(x,\xi)}\right] = -\frac{\partial^2_{x_l\xi_k}\tilde{a}(x,\xi)}{(\lambda + a(x,\xi))^2} + \frac{2\partial_{x_l}a(x,\xi)\partial_{\xi_k}\tilde{a}(x,\xi)}{(\lambda + \tilde{a}(x,\xi))^3}.$$

For simplicity, we will not separate the real and imaginary part. In this way we get

$$|\lambda|\left|\partial_{x_l}\partial_{\xi_k}\left[\frac{1}{\lambda + \tilde{a}(x,\xi)}\right]\right| \;\le\; |\lambda|\left\{\frac{r^{-1}}{|\lambda+1|^2} + \frac{r^{-1}}{|\lambda+1|^3}\right\} \le C(r),$$

where in the Definition 3.5, it is only necessary that there exists some $r > 0$ such that $\langle|\xi|\rangle^m \lesssim |a(x,\xi)|$ for $\xi \in \mathbb{R}^d$ with $|\xi| \ge r$. Similarly, we could get the bound for the general case where for multi-indices satisfying only $|\alpha| \le 2d+4$ and $|\beta| \le d+3$. By an application of Theorem 7.1 we know that (4.5) is satisfied. In particular, that there exists a constant $C > 0$ such that

$$\|R(\lambda, \tilde{a}(x,D))\|_{L(\mathbb{X},\mathbb{X})} \le \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\theta+\frac{\pi}{2}}. \tag{4.6}$$

Finally it remains to show that the semigroup $(\mathcal{P}_t)_{t\ge0}$ is analytic over $\mathbb{X}$. Now pick $\lambda = \vartheta + i\tau \in \Sigma_{\theta+\frac{\pi}{2}}$ such that $\vartheta > 0$ and $\tau \in \mathbb{R}$. From the estimate (4.6), we easily see that,

$$\|R(\vartheta + i\tau, \tilde{a}(x,D))\|_{L(\mathbb{X},\mathbb{X})} \le \frac{C_{d,s}}{|\tau|}, \quad \lambda \in \Sigma_{\theta+\frac{\pi}{2}}.$$

Then by applying the Theorem 5.2 [35, Chapter II] we could conclude that the Markovian semigroup is an analytic semigroup over $\mathbb{X}$. □

By analysing the proof of [46, p. 58], it can be seen that the condition of the differentiability at the origin can be relaxed. Here, it is essential to mention that the proof relies on the Theorem 2.5 [21, p. 120] (see also Theorem 4.23 in [1]), from which we can see that the extension of the Theorem 9.7 of [51] to symbols, whose derivatives have a singularity at $\{0\}$ is possible. Moreover, analysing line by line of the proof of Theorem 9.7 in [51], one can give an estimate of the norm of the operator.

5. The first application: the strong Feller property

Let $L = \{L(t) : t \ge 0\}$ be a family of Lévy processes and let us consider the stochastic differential equations of the following form

$$
\left\{ \begin{array}{rcl} dX^x(t) & = & b(X^x(t-))\,dt + \sigma(X^x(t-))dL(t), \\ X^x(0) & = & x, \quad x \in \mathbb{R}^d, \end{array} \right. \tag{5.1}
$$

where $\sigma : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous. By $C_b^0(\mathbb{R}^d)$ we denote the set of all real valued and uniformly continuous functions on $\mathbb{R}^d$ equipped with the supremum–norm. A Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ of a process is Feller, iff $\mathcal{P}_t u \in C_b^0(\mathbb{R}^d)$ for all $u \in C_b^0(\mathbb{R}^d)$ and $t > 0$, and is strongly continuous in zero, i.e. $\lim_{t\downarrow 0} |\mathcal{P}_t u - u|_{C_b^0} = 0$ for every $u \in C_b^0(\mathbb{R}^d)$. The Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ of a process is called strong Feller, iff for all $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $t > 0$, $\mathcal{P}_t f \in C_b^0(\mathbb{R}^d)$. In this section we will prove under certain assumptions the strong Feller property of the Markovian semigroup, see e.g. [42, p. 30]. Now, we can state our first result.

Theorem 5.1. *Let L be a square integrable Lévy process with Blumenthal–Getoor index $\delta \in (1, 2)$ of order $2d + 4$ and bounded moments of all order. Let $\sigma \in C_b^{d+3}(\mathbb{R}^d)$, such that σ is bounded away from zero (see Theorem 4.1) and $b \in C_b^{d+3}(\mathbb{R}^d)$. Then, for any $\gamma \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q < \infty$, there exists a constant $C > 0$ such that we have*

$$
|\mathcal{P}_t u|_{B^\gamma_{p,q}} \leq \frac{C}{t} \, |u|_{B^{\gamma-\delta}_{p,q}}, \quad t > 0. \tag{5.2}
$$

The estimate (5.2) can be used to prove the strong Feller property of $(\mathcal{P}_t)_{t\geq 0}$.

Corollary 5.1. *Let us assume that L is a square integrable Lévy process with Blumenthal–Getoor index δ, $\delta \in (1, 2)$, of order $2d + 4$. Let $\sigma \in C_b^{d+3}(\mathbb{R}^d)$ is bounded away from zero and $b \in C_b^{d+3}(\mathbb{R}^d)$. Then, the process defined by* (5.1) *is strong Feller. In particular, for all $\gamma \geq 0$ and $n = [\frac{\gamma}{\delta}] + 1$, we have*

$$
|\mathcal{P}_t u|_{C_b^\gamma(\mathbb{R}^d)} \leq \frac{(nC)^n}{t^n} \, |u|_{L^\infty(\mathbb{R}^d)}, \quad t > 0.
$$

Before presenting the proof of Corollary 5.1, we want to illustrate its applicability. Let us define the density $p : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ for the process X by

$$
\mathbb{P}(X^x(t) \in A) = \int_A p_t(x, y)\, dy, \quad A \in \mathcal{F}(\mathbb{R}^d),\ t > 0, \text{ and } x \in \mathbb{R}^d.
$$

Observe, for any $x, y \in \mathbb{R}^d$, we have

$$
p_t(x, y) = (\mathcal{P}_t \delta_x)(y).
$$

By Corollary 5.1, we get also estimates for the density p of X.

Corollary 5.2. *Let us assume that L is a square integrable Lévy process with Blumenthal–Getoor index $1 < \delta < 2$ of order $2d + 4$, $\sigma \in C_b^{d+3}(\mathbb{R}^d)$ is bounded away from zero, and $b \in C_b^{d+3}(\mathbb{R}^d)$. Then, the density of the process is arbitrary often differentiable. In particular, for any $\theta \in \mathbb{N}$ there exists a number $n = [\frac{\theta+d}{\delta}] + 1$ such that we have for any multi-index α of length θ*

$$
\left| \frac{\partial^\alpha}{\partial_y^\alpha} \, p_t(x, y) \right| \leq \frac{C(n, d)}{t^n}.
$$

Proof of Corollary 5.1. Fix $n \in \mathbb{N}$ and $p \in [1, \infty)$ such that $\gamma < n\delta - \frac{d}{p}$. Fix $1 \leq q < \infty$ arbitrary. Then, we know for $\gamma \notin \mathbb{N}_0$ (see [38, p. 14])

$$C_b^\gamma(\mathbb{R}^d) = B^\gamma_{\infty,\infty}(\mathbb{R}^d).$$

Secondly, we apply the embedding $B^{\gamma+\frac{d}{p}}_{p,q}(\mathbb{R}^d) \hookrightarrow B^\gamma_{\infty,\infty}(\mathbb{R}^d)$ (see [38, Chapter 2.2.3], [1, Section 6.4]), and, finally, we apply Theorem 5.1 n times to get,

$$|\mathcal{P}_t u|_{C_b^\gamma} \le |\mathcal{P}_t u|_{B^\gamma_{\infty,\infty}} \le |\mathcal{P}_t u|_{B^{\gamma+\frac{d}{p}}_{p,q}} = \left|(\mathcal{P}_{\frac{t}{n}})^n u\right|_{B^{\gamma+\frac{d}{p}}_{p,q}} \le \frac{(nC)^n}{t^n}\, |u|_{B^{\gamma+\frac{d}{p}-n\delta}_{p,q}}.$$

By means of [1, Excercise 6.25, Corollary 6.14] we get for $\kappa = \frac{d}{p} - n\delta + \gamma$

$$L^\infty(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow B^0_{p,p}(\mathbb{R}^d) \hookrightarrow B^\kappa_{p,1}(\mathbb{R}^d),$$

By fixing $q = 1$ we obtain

$$|\mathcal{P}_t u|_{C_b^\gamma} \le \frac{(nC)^n}{t^n}\, |u|_{B^{\gamma+\frac{d}{p}-n\delta}_{p,1}} \le \frac{(nC)^n}{t^n}\, |u|_{B^{\gamma+\frac{d}{p}-n\delta}_{p,1}} \le \frac{(nC)^n}{t^n}\, |u|_{L^\infty}.$$

The last line gives the assertion. □

Proof of Corollary 5.2. Fix $p \in (1,\infty)$. We know $\delta_x \in B^{-\frac{d}{p'}}_{p,\infty}(\mathbb{R}^d)$, where p' is the conjugate of p (see [10, Formula B.2]). Let $\theta \in \mathbb{N}_0$. A function u is θ times continuous differentiable, if $u \in C_b^\theta(\mathbb{R}^d)$. Since $B^{\gamma_1}_{p,q}(\mathbb{R}^d) \hookrightarrow C_b^\theta(\mathbb{R}^d)$ for $\gamma_1 = \theta + \frac{d}{p}$, we have to estimate $|\mathcal{P}_t\delta_x|_{B^{\gamma_1}_{p,q}}$. Let $n \in \mathbb{N}$ that large that $n\delta > \theta + d$. Then $\gamma_1 - n\delta < -(d - \frac{d}{p})$. Now, we have

$$|\mathcal{P}_t\delta_x|_{B^{\gamma_1}_{p,q}} \le \left(\frac{C}{t}\right)^n |\delta_x|_{B^{\gamma_1-n\delta}_{p,q}} \le \left(\frac{C}{t}\right)^n |\delta_x|_{B^{-\gamma_2}_{p,1}},$$

where $\gamma_2 < -(d - \frac{d}{p})$. Since $\delta_x \in B^{-\frac{d}{p'}}_{p,\infty}(\mathbb{R}^d)$, the right-hand side is bounded. □

Proof of Theorem 5.1. First, note that by Proposition 2.1. [20, p. 793], the symbol ψ of the Lévy process is infinitely often differentiable. If the coefficient σ is independent from the space variable x, then it is possible to write the symbol of the semigroup $(\mathcal{P}_t)_{t\ge0}$ directly as $(e^{t\psi(\xi)})_{t\ge0}$. If σ depends on the space variable x, such a nice representation of the symbol of the semigroup does not exists. We overcome this obstacle by using the representation of the semigroup $(\mathcal{P}_t)_{t\ge0}$ in terms of the contour integral, since we have it already successfully applied in [11] and [19]. Let $\theta' \in (0,\theta)$, $\rho \in (0,\infty)$, and let

$$\Gamma_{\theta'}(\rho, M) = \Gamma^{(1,M)}_{\theta',\rho} + \Gamma^{(2,M)}_{\theta',\rho} + \Gamma^{(3)}_{\theta',\rho},$$

where $\Gamma^{(1)}_{\theta',\rho}$ and $\Gamma^{(2)}_{\theta',\rho}$ are the rays $re^{i(\frac{\pi}{2}+\theta')}$ and $re^{-i(\frac{\pi}{2}+\theta')}$, $\rho \le r \le M < \infty$, and $\Gamma^{(3)}_{\theta',\rho} = \rho^{-1}e^{i\alpha}$, $\alpha \in [-\frac{\pi}{2} - \theta', \frac{\pi}{2} + \theta']$. It follows from [35, Theorem 1.7.7] and Fubini's Theorem that for $t > 0$ and $v \in B^\gamma_{p,q}(\mathbb{R}^d)$ we have

$$\mathcal{P}_t u = \lim_{M\to\infty} \frac{1}{2\pi i} \int_{\Gamma_{\theta'}(\rho,M)} e^{\lambda t} R(\lambda : a(x,D)) v\, d\lambda,$$

where $R(\lambda : a(x,D))$ denotes the inverse of $a(x,\lambda,D) := \lambda I + a(x,D)$. Due to Theorem 4.2 and the assumption of Theorem 5.1, we know that $(\mathcal{P}_t)_{t\ge0}$ is an analytic semigroup in $B^\gamma_{p,q}(\mathbb{R}^d)$. Therefore, for any element $v \in B^\gamma_{p,q}(\mathbb{R}^d)$, the limit exists and is well defined. Let $u \in B^{\gamma-\delta}_{p,q}(\mathbb{R}^d)$ and $\{v_n : n \in \mathbb{N}\}$ be a sequence such that $v_n \in B^\gamma_{p,q}(\mathbb{R}^d)$ and $v_n \to u$ in $B^{\gamma-\delta}_{p,q}(\mathbb{R}^d)$. By a change of variables, we obtain

$$
\begin{aligned}
&\lim_{M\to\infty}\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta'}(\rho,M)} e^{\lambda t}R(\lambda : a(x,D))v_n d\lambda\right|_{B^\gamma_{p,q}}\\
&\le \lim_{M\to\infty}\left|\frac{1}{2\pi it}\int_\rho^M e^{re^{-i(\frac{\pi}{2}+\theta')}}\, R(\frac{r}{t}e^{-i(\frac{\pi}{2}+\theta')}, a(x,D))\, v_n e^{i(\frac{\pi}{2}+\theta')}dr\right|_{B^\gamma_{p,q}}\\
&+ \lim_{M\to\infty}\left|\frac{1}{2\pi it}\int_\rho^M e^{re^{i(\frac{\pi}{2}+\theta')}}\, R(\frac{r}{s}e^{i(\frac{\pi}{2}+\theta')}, a(x,D))\, v_n e^{-i(\frac{\pi}{2}+\theta')}dr\right|_{B^\gamma_{p,q}}\\
&+ \left|\frac{1}{2\pi it}\int_{-\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho e^{i\beta}}\, R(\frac{\rho}{s}e^{i\beta}, a(x,D))\, v_n\rho^{-1}e^{i\beta}d\beta\right|_{B^\gamma_{p,q}}.
\end{aligned}
$$

The Minkowski inequality gives

$$
\begin{aligned}
\dots \le\ & \frac{1}{2t\pi}\int_\rho^\infty e^{-r\sin\theta'}\left|R(\frac{r}{t}e^{-i(\frac{\pi}{2}+\theta')}, a(x,D))v_n\right|_{B^\gamma_{p,q}} dr\\
+\ & \frac{1}{2t\pi}\int_\rho^\infty e^{-r\sin\theta'}\left|R(\frac{r}{t}e^{i(\frac{\pi}{2}+\theta')}, a(x,D))v_n\right|_{B^\gamma_{p,q}} dr\\
+\ & \frac{\rho^{-1}}{2t\pi}\int_{-\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho\cos\beta}\left|R(\frac{\rho}{t}e^{i\beta}, a(x,D))v_n\right|_{B^\gamma_{p,q}} d\beta.
\end{aligned}
\tag{5.3}
$$

We analyse the right-hand side of the estimate above by analysing the operator $R(\frac{\rho}{t}e^{i\beta}, a(x,D))$ and applying Theorem 7.1. Before doing that, we have to calculate the seminorm of $\lambda + a(x,\xi)$ in the space of hypoelliptic operators. In this way, we require the following estimate. Similar to p. 11 in [15], we can see that for $\lambda\in\Sigma_{\theta+\frac{\pi}{2}}$,

$$\langle|\lambda|^{\frac{1}{\delta}} + |\xi|\rangle^\delta \lesssim |\lambda + a(x,\xi)|.$$

The above result is due to the fact that $a\in \mathrm{Hyp}^\delta_{d+1,0;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$, the idenitity (2.3), and since σ is bounded away from zero. Therefore, there exists a number $r>0$ such that we know

$$|\lambda + a(x,\xi)|^{-1} \lesssim \langle|\lambda|^{\frac{1}{\delta}} + |\xi|\rangle^{-\delta} \lesssim \langle|\xi|\rangle^{-\delta},$$

for all $\xi\in\mathbb{R}^d$ with $r\le|\xi|$. In this way we obtain

$$\left|\frac{1}{\lambda + a(x,\xi)}\right| \le \left|\frac{1}{\lambda + \psi(\sigma(x)^T\xi)}\right| \le \left|\frac{1}{\lambda + \langle\sigma(x)^T\xi\rangle^\delta}\right| \le C(\sigma,\delta)\,\langle|\xi|\rangle^{-\delta}.$$

Let $k\in\{1,\dots,d\}$. Then

$$\left|\partial_{\xi_k}\left[\frac{1}{\lambda + a(x,\xi)}\right]\right| = \left|\frac{\partial_{\xi_k}a(x,\xi)}{(\lambda + a(x,\xi))^2}\right| \le \left|\frac{\langle|\xi|\rangle^{\delta-1}}{(\lambda + \langle|\xi|\rangle^\delta)^2}\right| \le C(\sigma,\delta)\,\langle|\xi|\rangle^{-\delta-1}.$$

Next, let $k,l\in\{1,\dots,d\}$. Then,

$$\partial_{\xi_l}\partial_{\xi_k}\left[\frac{1}{\lambda + a(x,\xi)}\right] = -\frac{\partial^2_{\xi_l\xi_k}a(x,\xi)}{(\lambda + a(x,\xi))^2} + \frac{2\partial_{\xi_l}a(x,\xi)\partial_{\xi_k}a(x,\xi)}{(\lambda + a(x,\xi))^3}.$$

Hence, we have

$$\left|\partial_{\xi_l}\partial_{\xi_k}\left[\frac{1}{\lambda + a(x,\xi)}\right]\right| \le C(\sigma,\delta)\left\{\frac{\langle|\xi|\rangle^{\delta-2}}{(\lambda + \langle|\xi|\rangle^\delta)^2} + \frac{\langle|\xi|\rangle^{\delta-2}}{(\lambda + \langle|\xi|\rangle^\delta)^2}\right\} \le C(\sigma,\delta)\langle|\xi|\rangle^{-\delta-2}.$$

Let $\alpha = (\alpha_1,\cdots,\alpha_k)$ be a multi-index. By observing the pattern of the above derivative we can identify the general derivative $\partial_\xi^\alpha[\frac{1}{\lambda+a(x,\xi)}]$ and get the following estimate. There exist $C_1, C_2, \cdots C_{|\alpha|} > 0$ depending on σ and δ such that

$$\left|\partial_\xi^\alpha\left[\frac{1}{\lambda+a(x,\xi)}\right]\right| \le C_1|\lambda+a|^{-|\alpha|-1}\langle\xi\rangle^{\delta|\alpha|-|\alpha|}$$
$$+ C_2|\lambda+a|^{-|\alpha|}\langle\xi\rangle^{\delta(|\alpha|-1)-|\alpha|} + C_3|\lambda+a|^{-|\alpha|+1}\langle\xi\rangle^{\delta(|\alpha|-2)-|\alpha|} + \cdots + C_{|\alpha|}|\lambda+a|^{-2}\langle\xi\rangle^{\delta-|\alpha|}.$$

Therefore

$$\left|\partial_\xi^\alpha\left[\frac{1}{\lambda+a(x,\xi)}\right]\right|\langle\xi\rangle^{-\delta+|\alpha|} \le C_1|\lambda+a|^{-|\alpha|-1}\langle\xi\rangle^{\delta|\alpha|-\delta}$$
$$+ C_2|\lambda+a|^{-|\alpha|}\langle\xi\rangle^{\delta|\alpha|-2\delta} + C_3|\lambda+a|^{-|\alpha|+1}\langle\xi\rangle^{\delta|\alpha|-3\delta} + \cdots + C_{|\alpha|}|\lambda+a|^{-2}.$$

Using the fact that there exists some $r > 0$ such that we have for all $x \in \mathbb{R}^d$ with $|\xi| \ge r$

$$|\lambda + a(x,\xi)|^{-1} \lesssim \langle|\lambda|^{\frac{1}{\delta}} + |\xi|\rangle^{-\delta} \lesssim \langle|\xi|\rangle^{-\delta},$$

we obtain

$$\left|\partial_\xi^\alpha\left[\frac{1}{\lambda+a(x,\xi)}\right]\right|\langle\xi\rangle^{-\delta+|\alpha|} \le (C_1 + C_2 + \ldots C_{|\alpha|})\langle|\xi|\rangle^{-2\delta} \lesssim \langle|\xi|\rangle^{-2\delta} \le C(\sigma,\delta)R^{-2\delta}.$$

The last line shows that $\lambda + a(x,\xi) \in \mathrm{Hyp}^\delta_{d+1,0;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$.

It remains to estimate the norm of the symbol $\lambda + a(x,\xi)$ in $\tilde{\mathcal{A}}^{1,1}_{k_1,k_2;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$ with $k_1 = 2d+4, k_2 = d+3$. Due to the fact that one has to take at least once the derivative with respect to ξ, the constant λ has no influence on the norm in $\tilde{\mathcal{A}}^{1,1}_{k_1,k_2;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$. Since, we have for $a \in \mathcal{A}^\delta_{k_1,k_2;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$ and

$$\|\lambda+a\|_{\tilde{\mathcal{A}}^{\delta,1}_{k_1,k_2;1,0}} = \sup_{1\le|\alpha|\le k_1, |\beta|\le k_2}\ \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d}\left|\partial_x^\alpha\partial_\xi^\beta(\lambda+a(x,\xi))\right|\langle|\xi|\rangle^{|\beta|-\delta},$$

where $k_1 = 2d+4, k_2 = d+3$, we can conclude that $\lambda + a(x,\xi) \in \tilde{\mathcal{A}}^{1,1}_{2d+4,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$.

Going back to (5.3) we can conclude by our discussion before by

$$\begin{aligned}
&\lim_{M\to\infty}\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta'}(\rho,M)} e^{\lambda t}R(\lambda : a(x,D))v_n d\lambda\right|_{B^\gamma_{p,q}} \\
&\le \frac{C(\sigma,\delta)}{2t\pi}\int_\rho^\infty e^{-r\sin\theta'}\,|v_n|_{B^{\gamma-\delta}_{p,q}}dr + \frac{C(\sigma,\delta)}{2t\pi}\int_\rho^\infty e^{-r\sin\theta'}\,|v_n|_{B^{\gamma-\delta}_{p,q}}dr \\
&\quad + \frac{C(\sigma,\delta)\rho^{-1}}{2t\pi}\int_{-\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho\cos\beta}\,|v_n|_{B^{\gamma-\delta}_{p,q}}\,d\beta \\
&\le \frac{C(\sigma,\delta)}{2t\pi}\,|v_n|_{B^{\gamma-\delta}_{p,q}}.
\end{aligned}$$

Taking the limit $n \to \infty$, we get

$$\lim_{M\to\infty}\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta'}(\rho,M)} e^{\lambda t}R(\lambda : a(x,D))u d\lambda\right|_{B^\gamma_{p,q}} \le \frac{C(\sigma,\delta)}{2t\pi}\,|u|_{B^{\gamma-\delta}_{p,q}},$$

which is the assertion. □

The following Corollary is a consequence of Theorem 5.1.

Corollary 5.3. *Let L be a square integrable Lévy process with Blumenthal–Getoor index $\delta \in (1,2)$ of order $2d+4$. Let $\sigma \in C_b^{d+3}(\mathbb{R}^d)$ be bounded away from zero and $b \in C_b^{d+3}(\mathbb{R}^d)$. Let $m(D)$ be a pseudo–differential operator such that $m(\xi) \in S^{\kappa}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ with $0 \le \kappa \le 1$. Then, there exists a constant $C>0$ such that for any $0<\gamma<\frac{\delta-\kappa}{4}$, $\gamma \notin \mathbb{N}$, and $t>0$ we have*

$$|\mathcal{P}_t\, m(D)\, u|_{C_b^{\gamma}(\mathbb{R}^d)} \le \frac{C}{t}\; |u|_{L^\infty(\mathbb{R}^d)}\, .$$

Proof. The proof is a combination of the proof of Theorem 5.1 and Corollary 5.1. Due to this reason, we include only the essential steps of the proof. We have already shown that

$$\lambda + a(x,\xi) \in \mathrm{Hyp}^{\delta}_{d+1,0;1,0}(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{A}^{-1}_{2d+4,d+3;1,0}(\mathbb{R}^d \times \mathbb{R}^d).$$

As already observed in the proof of Theorem 5.1, we have the following representation of the semigroup

$$|\mathcal{P}_t m(D) u|_{B^{\gamma+\frac{d}{p}}_{p,q}} = \lim_{M\to\infty} \left| \frac{1}{2\pi i} \int_{\Gamma_{\theta'}(\rho,M)} e^{\lambda t} R(\lambda : a(x,D)) m(D) u d\lambda \right|_{B^{\gamma+\frac{d}{p}}_{p,q}} .$$

Similarly as in the proof of Theorem 5.1 we can write

$$\begin{aligned}
&\lim_{M\to\infty} \left| \frac{1}{2\pi i} \int_{\Gamma_{\theta'}(\rho,M)} e^{\lambda t} R(\lambda : a(x,D))\, m(D)\, u\, d\lambda \right|_{B^{\gamma+\frac{d}{p}}_{p,q}} \\
&\le \lim_{M\to\infty} \left| \frac{1}{2\pi i t} \int_{\rho}^{M} e^{r e^{-i(\frac{\pi}{2}+\theta')}}\, R(\frac{r}{t} e^{-i(\frac{\pi}{2}+\theta')}, a(x,D))\, m(D)\, u\, e^{i(\frac{\pi}{2}+\theta')} dr \right|_{B^{\gamma+\frac{d}{p}}_{p,q}} \\
&+ \lim_{M\to\infty} \left| \frac{1}{2\pi i t} \int_{\rho}^{M} e^{r e^{i(\frac{\pi}{2}+\theta')}}\, R(\frac{r}{s} e^{i(\frac{\pi}{2}+\theta')}, a(x,D))\, m(D)\, u\, e^{-i(\frac{\pi}{2}+\theta')} dr \right|_{B^{\gamma+\frac{d}{p}}_{p,q}} \\
&+ \left| \frac{1}{2\pi i t} \int_{-\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho e^{i\beta}}\, R(\frac{\rho}{s} e^{i\beta}, a(x,D))\, m(D)\, u\, \rho^{-1} e^{i\beta} d\beta \right|_{B^{\gamma+\frac{d}{p}}_{p,q}} \\
&\le \frac{1}{2t\pi} \int_{\rho}^{\infty} e^{-r\sin\theta'} \left| R(\frac{r}{t} e^{-i(\frac{\pi}{2}+\theta')}, a(x,D))\, m(D)\, u \right|_{B^{\gamma+\frac{d}{p}}_{p,q}} dr \\
&\quad + \frac{1}{2t\pi} \int_{\rho}^{\infty} e^{-r\sin\theta'} \left| R(\frac{r}{t} e^{i(\frac{\pi}{2}+\theta')}, a(x,D))\, m(D)\, u \right|_{B^{\gamma+\frac{d}{p}}_{p,q}} dr \\
&\quad + \frac{\rho^{-1}}{2t\pi} \int_{-\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho\cos\beta} \left| R(\frac{\rho}{t} e^{i\beta}, a(x,D))\, m(D)\, u \right|_{B^{\gamma+\frac{d}{p}}_{p,q}} d\beta.
\end{aligned}$$

Note again that the semi-norms

$$\|\lambda + a\|_{\mathrm{Hyp}^{\delta}_{d+1,0;1,0}(\mathbb{R}^d\times\mathbb{R}^d)} \quad \text{and} \quad \|\lambda + a\|_{\mathcal{A}^{-1}_{2d+4,d+3;1,0}}$$

do not depend on λ. In this way, by separating $m(D)$ and $R(\frac{r}{t} e^{i(\frac{\pi}{2}+\theta')}, a(x,D))$ and applying Theorem (7.1) we get

$$\begin{aligned}
\cdots &\lesssim \frac{1}{2t\pi} \int_{\rho}^{\infty} e^{-r\sin\theta'}\, |m(D)\, u|_{B^{\gamma+\frac{d}{p}-\delta}_{p,q}}\, dr \\
&+ \frac{1}{2t\pi} \int_{\rho}^{\infty} e^{-r\sin\theta'}\, |m(D)\, u|_{B^{\gamma+\frac{d}{p}-\delta}_{p,q}}\, dr + \frac{\rho^{-1}}{2t\pi} \int_{-\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho\cos\beta}\, |m(D)\, u|_{B^{\gamma+\frac{d}{p}-\delta}_{p,q}}\, d\beta,
\end{aligned}$$

$$\lesssim \left[\frac{1}{t\pi}\int_\rho^\infty e^{-r\sin\theta'}\,dr + \frac{1}{2t\pi}\int_{-\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho\cos\beta}\,d\beta\right] |m(D)\,u|_{B_{p,q}^{\gamma+\frac{d}{p}-\delta}} \le \frac{C}{t}\,|m(D)\,u|_{B_{p,q}^{\gamma+\frac{d}{p}-\delta}}.$$

Since $m(\xi) \in S^\kappa_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ we get

$$\begin{aligned}\frac{C}{t}\,|m(D)\,u|_{B_{p,q}^{\gamma+\frac{d}{p}-\delta}} &\le \frac{C}{t}\,\|m\|_{S^\kappa_{1,0}}\,|u|_{B_{p,q}^{\gamma+\frac{d}{p}-\delta+\kappa}}\\ &\lesssim \frac{C}{t}\,|u|_{B_{p,q}^{\gamma+\frac{d}{p}-\delta+\kappa}}.\end{aligned}$$

Now we are following the same argument of the proof of Corollary 5.1 to complete the argument. Fix $\gamma < (n-1)\delta-\kappa-d$ and let $p \ge 1$ such that $\frac{d}{p} < (n-1)\delta-\kappa-\gamma$. Fix $1 \le q < \infty$ arbitrary. Then, we know, firstly (see [38, p. 14])

$$C^\gamma(\mathbb{R}^d) = B^\gamma_{\infty,\infty}(\mathbb{R}^d), \quad \gamma \notin \mathbb{N}.$$

Secondly, we apply the embedding $B_{p,q}^{\gamma+\frac{d}{p}}(\mathbb{R}^d) \hookrightarrow B^\gamma_{\infty,\infty}(\mathbb{R}^d)$ (see [38, Chapter 2.2.3], [1, Section 6.4]), and, finally, we apply Theorem 5.1 n times to get,

$$\begin{aligned}|\mathcal{P}_t\, m(D)\,u|_{C^\gamma_b} &\le |\mathcal{P}_t\, m(D)\,u|_{B^\gamma_{\infty,\infty}(\mathbb{R}^d)} \le |\mathcal{P}_t\, m(D)\,u|_{B_{p,q}^{\gamma+\frac{d}{p}}}\\ &= \left|(\mathcal{P}_{\frac{t}{n}})^n\, m(D)\,u\right|_{B_{p,q}^{\gamma+\frac{d}{p}}} \le \frac{(nC)^n}{t^n}\,|u|_{B_{p,q}^{\gamma+\frac{d}{p}-n\delta+\kappa}}.\end{aligned} \tag{5.4}$$

By means of [1, Excercise 6.25, Corollary 6.14],

$$B^\theta_{p',1}(\mathbb{R}^d) \hookrightarrow B^{\frac{d}{p'}}_{p',1}(\mathbb{R}^d) \hookrightarrow C^0(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d),$$

where $\theta = n\delta - \frac{d}{p} - \gamma + \kappa$ and $p' = \frac{p}{p-1}$. Now, applying [1, Lemma 6.5] and the duality property of the Besov spaces give that

$$B^\theta_{p',1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \hookrightarrow B^{-\theta}_{p,\infty}(\mathbb{R}^d) = B_{p,\infty}^{\gamma+\frac{d}{p}-n\delta+\kappa}(\mathbb{R}^d) \hookrightarrow B_{p,1}^{\gamma+\frac{d}{p}-(n-1)\delta+\kappa}(\mathbb{R}^d).$$

Finally by fixing $q = 1$ we get

$$|\mathcal{P}_t\, m(D)\,u|_{C^\gamma_b} \le \frac{(nC)^n}{t^n}\,|u|_{B_{p,1}^{\gamma+\frac{d}{p}-n\delta+\kappa}} \le \frac{(nC)^n}{t^n}\,|u|_{B_{p,1}^{\gamma+\frac{d}{p}-(n-1)\delta+\kappa}} \le \frac{(nC)^n}{t^n}\,|u|_{L^\infty}.$$

This completes the proof. □

If L is an α stable process, the problem appears that only the moments up to $p < \alpha$ are bounded. Therefore, the symbol is not necessarily uniformly differentiable up to order $d+1$ in any neighbourhood of $\xi = 0$. However, if $\alpha > 1$, then this problem can be solved.

Corollary 5.4. *Let L be a Lévy process L with Blumenthal–Getoor index* $1 < \delta < 2$ *of order* $2d+4$. *Let* $\sigma \in C_b^{d+3}(\mathbb{R}^d)$ *be bounded away from zero and* $b \in C_b^{d+3}(\mathbb{R}^d)$. *Then, the Markovian semigroup* $(\mathcal{P}_t)_{t\ge 0}$ *of the process defined by* (5.1) *is strong Feller.*

Proof of Corollary 5.4. In order to deal with the large jumps we decompose the Lévy process into a Lévy process recollecting the jumps smaller than one and a second Lévy process, recollecting the jumps larger than one. In doing so, we split the Lévy measure. Let ν_0 be the Lévy measure defined by

$$\nu_0 : \mathcal{B}(\mathbb{R}^d) \ni U \mapsto \nu(U \cap Z_0),$$

and ν_1 be the Lévy measure defined by

$$\nu_1 : \mathcal{B}(\mathbb{R}^d) \ni U \mapsto \nu(U \cap Z_1),$$

where $Z_0 = \{z \in \mathbb{R}^d : |z| \leq 1\}$ and $Z_1 = \{z \in \mathbb{R}^d : |z| > 1\}$. Since the proof of this theorem mainly rely on the analysis of the decomposition of the small and large jumps it is important to decompose also the probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. Let $\tilde{\eta}_0$ be a compensated Poisson random measure on $(Z_0\times\mathbb{R}_+, \mathcal{B}(Z_0)\otimes\mathcal{B}(\mathbb{R}_+))$ over $\mathfrak{A}^0 = (\Omega^0, \mathcal{F}^0, \{\mathcal{F}_t^0\}_{t\in[0,T]}, \mathbb{P}^0)$ with intensity measure ν_0 where

$$\mathcal{F}^0 = \sigma\{\eta(B, [0, s]) : B \in \mathcal{B}(Z_0), s \in [0, T]\}$$

and for $0 \leq t \leq T$

$$\mathcal{F}_t^0 = \sigma\{\eta(B, [0, s]) : B \in \mathcal{B}(Z_0), s \in [0, t]\}.$$

Furthermore, let $\tilde{\eta}_1$ be a compensated Poisson random measure on $(Z_1 \times\mathbb{R}_+, \mathcal{B}(Z_1)\otimes\mathcal{B}(\mathbb{R}_+))$ over $\mathfrak{A}^1 = (\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}_{t\in[0,T]}, \mathbb{P}^1)$ with finite intensity measure ν_1 where

$$\mathcal{F}^1 = \sigma\{\eta(B, [0, s]) : B \in \mathcal{B}(Z_1), s \in [0, T]\}$$

and for $0 \leq t \leq T$

$$\mathcal{F}_t^1 = \sigma\{\eta(B, [0, s]) : B \in \mathcal{B}(Z_1), s \in [0, t]\}.$$

Let $\Omega := (\Omega^0 \times \Omega^1)$, $\mathcal{F} := \mathcal{F}^0 \otimes \mathcal{F}^1$, $\mathcal{F}_t := \mathcal{F}_t^0 \otimes \mathcal{F}_t^1$, $P = P^0 \otimes P^1$ and $\mathbb{E} = \mathbb{E}^0 \otimes \mathbb{E}^1$. We denote the to ν_0 and ν_1 associated Lévy processes by L_0 and L_1. It is clear by the independent scattered property of a Poisson random measure, that L_0 and L_1 are independent. Since ν_1 is a finite measure, L_1 can be represented as a sum over its jumps. In particular, let $\rho = \nu_1(\mathbb{R}^d)$, $\{\tau_n : n \in \mathbb{N}\}$ be a family of independent exponential distributed random variables with parameter ρ,

$$T_n = \sum_{j=1}^{n} \tau_j, \quad n \in \mathbb{N}, \tag{5.5}$$

and $\{N(t) : t \geq 0\}$ be the counting process defined by

$$N(t) := \sum_{j=1}^{\infty} 1_{[T_j,\infty)}(t), \quad t \geq 0.$$

Observe, for any $t > 0$, the random variable $N(t)$ is a Poisson distributed random variable with parameter ρt. Let $\{Y_n : n \in \mathbb{N}\}$ be a family of independent, ν_0/ρ distributed random variables. Then the Lévy process L_1 given by (see [37, Chapter 3])

$$L_1(t) = \int_0^t \int_{Z_1} z\,\tilde{\eta}_1(dz, ds), \quad t \geq 0,$$

can be represented as

$$L_1(t) = \begin{cases} -z_0 t & \text{for } N(t) = 0, \\ \sum_{j=1}^{N(t)} Y_j - z_0 t & \text{for } N(t) > 0, \end{cases}$$

where $z_0 = \int_{\mathbb{R}^d} z\,\nu_1(dz)$. Let $(\mathcal{P}^0_t)$ the Markovian semigroup of the solution process X^x_0 given by

$$\left\{ \begin{array}{rcl} dX^x_0(t) & = & b(X_0(t-))\,dt + \sigma(X_0(t-))dL_0(t), \\ X^x_0(0) & = & x, \quad x \in \mathbb{R}^d. \end{array} \right. \tag{5.6}$$

Now, we have for $u(t) = \mathbb{E}\phi(X(t))$, where X is the solution to the original equation (5.1), the following identity

$$u(t) = \mathbb{E}(\mathcal{P}^0_t \phi)(x) + \mathbb{E}\sum_{i=1}^{N(t)} \mathcal{P}^0_{t-T_i} B_{Y_i} u(T_i^-) - \int_0^t \mathcal{P}^0_{t-s} Du(s)[z_0]\,ds, \tag{5.7}$$

where $(B_y\phi)(x) = \phi(x+y) - \phi(x)$. To verify formula (5.7), observe that in the time interval $[0, T_1)$ the solution of u is given by

$$u(t) \;=\; (\mathcal{P}^0_t \phi)(x) + \int_0^t \mathcal{P}^0_{t-s} Du(s)[z_0]\,ds, \qquad t \in [0, T_1].$$

In particular, u solves on the time interval $[0, T_1)$ the equation

$$\left\{ \begin{array}{rcl} \dot u(t) & = & a(x, D)u(t) + Du(t)z_0\,, \quad t \in [0, T_1), \\ u(0) & = & \phi. \end{array} \right. \tag{5.8}$$

Let us denote the solution of (5.8) on the first time interval $[0, T_1]$ by u_1. At time T_1 the first large jump occurred. Hence, on the time interval $[T_1, T_2)$, u solves

$$\left\{ \begin{array}{rcl} \dot u(t) & = & a(x, D)u(t) + Du(t)z_0\,, \quad t \in (T_1, T_2), \\ u(T_1) & = & \mathbb{E}u_1(T_1, \cdot + Y_1). \end{array} \right. \tag{5.9}$$

Let us denote the solution of (5.9) by u_2. The variation of constant formula gives for $t \in (T_1, T_2)$

$$u_2(t) = \mathcal{P}^0_{t-T_1} u_2(T_1) + \int_{T_1}^t \mathcal{P}^0_{t-s} Du_2(s)\, z_0\, ds.$$

Let us put

$$u(t) := \begin{cases} u_1(t), & \text{if } t \in [0, T_1), \\ u_2(t), & \text{if } t \in [T_1, T_2). \end{cases}$$

Since

$$u_1(T_1) = \mathcal{P}^0_{T_1}\phi + \int_0^{T_1} \mathcal{P}^0_{T_1 - s} Du(s)\, z_0\, ds,$$

and $u_2(T_1, x) = u_1(T_1, x + Y_1)$, $x \in \mathbb{R}^d$, it follows

$$\begin{aligned} u(t) \;&=\; \mathcal{P}^0_{t-T_1}\mathcal{P}^0_{T_1}\phi + \mathcal{P}^0_{t-T_1}\int_0^{T_1} \mathcal{P}^0_{T_1-s} Du(s)\, z_0\, ds \\ &\quad + \int_{T_1}^t \mathcal{P}^0_{t-s} Du(s)\, z_0\, ds + \mathbb{E}\mathcal{P}^0_{t-T_1} u(T_1^-, \cdot + Y_1) - \mathbb{E}\mathcal{P}^0_{t-T_1} u(T_1^-, \cdot). \\ &=\; \mathcal{P}^0_t \phi + \int_0^t \mathcal{P}^0_{t-s} Du(s)\, z_0\, ds + \mathbb{E}\mathcal{P}^0_{t-T_1}\left[u(T_1^-, \cdot + Y_1) - u(T_1^-, \cdot)\right]. \end{aligned}$$

Repeating these calculations successively for all time intervals gives formula (5.7).

Since, given $N(t) = k$, the random variables $\{Y_1, Y_2, \ldots, Y_k\}$ and $\{T_1, T_2, \ldots, T_k\}$ are mutually independent and T_i, $i = 1, \ldots, k$, are uniform distributed on $[0, t]$, it follows that

$$\begin{aligned}
\mathbb{E} \sum_{i=1}^{N(t)} \mathcal{P}^0_{t-s} B_{Y_i} u(T_i) &= \sum_{k=1}^{\infty} \mathbb{P}\left(N(t) = k\right) \mathbb{E}\left[\sum_{l=1}^{N(t)} \mathbb{E}^1 \left[\mathcal{P}^0_{t-T_l} B_{Y_l} u(T_l) \mid N(t) = k\right]\right] \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left(N(t) = k\right) \mathbb{E}^1 \left[\sum_{l=1}^{k} \mathcal{P}^0_{t-T_l} B_{Y_l} u(T_l)\right] \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left(N(t) = k\right) \mathbb{E}^1 \left[\int_0^t \mathcal{P}^0_{t-s} B_Y u(s)\right],
\end{aligned}$$

where Y is distributed as ν_0/ρ. Thus, we get for $\gamma \leq \delta - 1$

$$|u(t)|_{B^\gamma_{\infty,\infty}} = \left|\mathcal{P}^0_t \phi + C \int_0^t \mathcal{P}^0_{t-s} \mathcal{B}_y u(s)\, ds + \int_0^t \mathcal{P}^0_{t-s} D u(s)[z_0]\, ds\right|_{B^\gamma_{\infty,\infty}},$$

where $(\mathcal{B}_y \phi)(x) = \int_{\mathbb{R}^d} [\phi(x + y) - \phi(x)]\, \nu_0(dy)$ and C is a constant depending on ρ and ν_0. By applying Minkowski inequality and Theorem 5.1, resp. Corollary 5.3 with $m(D) = D$ and $m(D)u = \mathcal{B}_y u$, gives for some $p > 1$ with $n > \frac{\gamma+1+d}{\delta} + 1$

$$\begin{aligned}
|u(t)|_{B^\gamma_{\infty,\infty}} &\leq \frac{1}{t} |\phi|_{B^0_{\infty,\infty}} + \int_0^t \left|\mathcal{P}^0_{t-s} \mathcal{B}_y u(s)\right|_{B^\gamma_{\infty,\infty}} ds + |z_0| \int_0^t \left|\mathcal{P}^0_{t-s} D u(s)\right|_{B^\gamma_{\infty,\infty}} ds \\
&\leq \frac{1}{t} |\phi|_{B^0_{\infty,\infty}} + K_1 \int_0^t (t-s)^{-n}\, |u(s)|_{B^\gamma_{\infty,\infty}}\, ds + K_2 |z_0| \int_0^t (t-s)^{-n}\, |u(s)|_{B^\gamma_{\infty,\infty}}\, ds \\
&\leq \frac{1}{t} |\phi|_{B^0_{\infty,\infty}} + K(1 + |z_0|) \left[\int_0^t (t-s)^{-p}\, ds\right]^{\frac{1}{p}} \left[\int_0^t |u(s)|^{\frac{p}{p-1}}_{B^\gamma_{\infty,\infty}}\, ds\right]^{\frac{p-1}{p}} \\
&\leq \frac{1}{t} |\phi|_{B^0_{\infty,\infty}} + \frac{C_1}{t^{\frac{np-1}{p}}} \left[\int_0^t |u(s)|^{\frac{p}{p-1}}_{B^\gamma_{\infty,\infty}}\, ds\right]^{\frac{p-1}{p}}.
\end{aligned}$$

Rearranging gives

$$|u(t)|^{\frac{p}{p-1}}_{B^\gamma_{\infty,\infty}} \leq \frac{1}{t^{\frac{p}{p-1}}} |\phi|^{\frac{p}{p-1}}_{B^0_{\infty,\infty}} + \frac{C_2}{t^{\frac{np-1}{p-1}}} \int_0^t |u(s)|^{\frac{p}{p-1}}_{B^\gamma_{\infty,\infty}}\, ds.$$

A simple application of Gronwall's Lemma gives

$$|u(t)|_{B^\gamma_{\infty,\infty}} \leq C(t, p, n) |\phi|_{B^0_{\infty,\infty}}.$$

By the definition of $B^0_{\infty,\infty}(\mathbb{R}^d)$, it follows that the process is strong Feller. □

6. The second application: Error Estimates for Monte-Carlo Simulation

Given the intensity (or Lévy) measure of a Lévy process, in most of the cases, one does not know the distribution of $L(t)$ for a fixed time point $t \geq 0$. However, simulating stochastic differential equations driven by a Lévy process using the explicit or implicit Euler-Marayuama scheme, one has to simulate the increments $\Delta^n_\tau L := L(n\tau) - L((n-1)\tau)$ for $\tau > 0$

small. Here, one can apply several strategies to simulate the random variables $\Delta_\tau^n L$, $n \in \mathbb{N}$, to generate a so-called Lévy walk. In general, the distribution of $\Delta_\tau^k L$ is not known, such that $\Delta_\tau^k L$ cannot be simulated directly. One way is to cut off the jumps being smaller than a given ε and to simulate the corresponding compound Poisson process directly. Now, one has two possibilities, to neglect the small jumps or to replace the small jumps by a Gaussian random variable. Doing so, one gets a new Lévy process, denoted by $\hat{L}_\varepsilon$. This method was introduced by Tsuchiya [50]. Asmussen and Rosinki [3] investigated the process generated only by the small jumps. They investigate under which conditions this process converges in distribution to a Wiener process. Hence, one can improve the weak error by not neglecting the small jumps and simulating instead of the small jumps a Wiener process. The second advantage is that replacing the small jumps by a Wiener process leads to the fact that the Markovian semigroup of the approximation is analytic. Therefore, the error for t small improves. This we will present it in this section theoretically and verify practically.

To be more precise, let us cut off all jumps being in the unit ball with radius ε, denoted in the following by B_ε, i.e, let $B_\varepsilon = \{z \in \mathbb{R}^d : |z| \leq \varepsilon\}$, $|\cdot|$ denotes a norm on $\mathbb{R}^d$. Then, $\Delta_\tau^n L$ is the sum over N random variables $\{Y_1, \dots, Y_N\}$, where N is Poisson distributed with parameter $\nu(\mathbb{R}^d \setminus B_\varepsilon)$ and the random variables $\{Y_1, \dots, Y_N\}$ are identical and mutually independent distributed with

$$Y_i \overset{d}{=} \frac{\nu(\cdot \cap B_\varepsilon^c)}{\nu(B_\varepsilon^c)}, \quad i = 1, \dots N.$$

Now, we can replace the neglected small jumps by increments of a Wiener process. Here, the rate of convergence for the strong error will not be improved. However, calculating the weak error the quality of the approximation will be improved. One of the reason is that the Markovian semigroup of the approximation where the small jumps are approximated by a Wiener process is analytic. To explain the implication of this, let us consider the function $\phi : \mathbb{R} \ni x \mapsto \mathbb{1}_{[a,\infty)}(x)$. Then, for $t > 0$ we know $\mathbb{P}\left(X_t^{x_0} \geq a\right) = \mathbb{E}\mathbb{1}_{[a,\infty)}(X_t^{x_0})$, where X solves the stochastic differential equation ($b : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous, $\sigma > 0$)

$$\text{(6.1)} \qquad \begin{cases} dX^{x_0}(t) &= b(X^{x_0}(t-))\, dt + \sigma(X^{x_0}(t-))\, dL(t), \\ X^{x_0}(0) &= x_0, \quad x_0 \in \mathbb{R}. \end{cases}$$

Let us denote the approximation of X, where we replaced the small jumps by a Wiener process, by $\hat{X}$. Then, the function $\phi : \mathbb{R} \ni x_0 \mapsto \mathbb{E}\mathbb{1}_{[a,\infty)}(X_t^{x_0})$ is infinitely often differentiable and we can use the Taylor approximation to get a nice error estimate. In this way, the analyticity of the Markovian semigroup has a strong impact on the quality of the approximation.

Fix a truncation parameter $0 < \epsilon < 1$. Let us define the approximate Lévy measure

$$\nu^\varepsilon : \mathcal{B}(\mathbb{R}) \ni C \mapsto \nu\Big(C \cap B_\varepsilon^c\Big).$$

Let $\hat{L}_\varepsilon$ be the Lévy process induced by truncating the small jumps. In particular, $\hat{L}_\varepsilon$ is a Lévy process having intensity ν^ε. Not to neglect the small jumps, we generate at each time-step $k \in \mathbb{N}$ a Gaussian random variable $\Delta_\tau^k W_\epsilon$, where

$$\text{(6.2)} \qquad \Delta_\tau^k W_\epsilon \sim \mathcal{N}\Big(0, \Sigma^2(\varepsilon)\tau I_d\Big), \qquad \text{with} \qquad \Sigma^2(\varepsilon) = \int_{B_\varepsilon} \langle y, y\rangle\, \nu(dy)\,,$$

where I_d denotes the d-by-d identity matrix. Then, the increments of the Lévy process are approximated by

$$\left(\hat{\Delta}^0_\tau L_{\epsilon,1} + \Delta^0_\tau W_\epsilon, \hat{\Delta}^1_\tau L_{\epsilon,1} + \Delta^1_\tau W_\epsilon, \hat{\Delta}^2_\tau L_{\epsilon,1} + \Delta^2_\tau W_\epsilon, \ldots, \hat{\Delta}^k_\tau L_{\epsilon,1} + \Delta^k_\tau W_\epsilon, \ldots\right).$$

In the following we give an error estimate of the two processes X^{x_0} and $\hat{X}^{x_0}$, where X^{x_0} solves (6.1) and $\hat{X}^{x_0}$ solves

$$\left\{\begin{array}{rcl} d\hat{X}^{x_0}_\varepsilon(t) &=& b(\hat{X}^{x_0}_\varepsilon(t-))\,dt + \sigma(\hat{X}^{x_0}_\varepsilon(t-))d\hat{L}_\varepsilon(t) + \sigma(\hat{X}^{x_0}_\varepsilon(t-))dW_\varepsilon(t), \\ X^{x_0}(0) &=& x_0, \quad x_0 \in \mathbb{R}^d. \end{array}\right. \tag{6.3}$$

Here W_ε is a Wiener process with covariance $\Sigma(\varepsilon)$. We suppose that the Lévy process is a self–decomposable Lévy process. In particular, we assume that $L(1)$ is self–decomposable.

Definition 6.1. A probability measure on $\mathbb{R}^d$ is self–decomposable, iff for any $b > 1$, there exists a probability measure ρ_b on $\mathbb{R}^d$ such that

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\rho}_b(z), \quad z \in \mathbb{R}^d.$$

By Theorem 15.10 of [39, p. 95], we know that there exists

- a finite measure λ on the sphere $\mathbb{S} = \{x \in \mathbb{R}^d : |x| = 1\}$
- and a measurable function $k : S \times \mathbb{R}^+ \to \mathbb{R}^+_0$, decreasing in the second variable,

such that the Lévy measure ν of L has the following representation

$$\nu(B) = \int_{\mathbb{S}} \int_0^\infty 1_B(rx)k(r,x)\frac{dr}{r}\,\lambda(dx), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Again let us define the corresponding Markovian semigroups. Let $(\mathcal{P}_t)_{t\ge 0}$ be the Markovian semigroup of the process (6.1), i.e.

$$\mathcal{P}_t\phi(x_0) \;:=\; \mathbb{E}\phi(X^{x_0}(t)), \quad t \ge 0,\ x_0 \in \mathbb{R}^d, \tag{6.4}$$

and let $(\hat{\mathcal{P}}^\varepsilon_t)_{t\ge 0}$ be the Markovian semigroup of the process (6.3), i.e.

$$\hat{\mathcal{P}}^\varepsilon_t\phi(x_0) \;:=\; \mathbb{E}\phi(\hat{X}^{x_0}_\varepsilon(t)), \quad t \ge 0,\ x_0 \in \mathbb{R}^d. \tag{6.5}$$

The first proposition shows that the semigroup $(\hat{\mathcal{P}}^\varepsilon_t)_{t\ge 0}$ of the approximation $\hat{X}$ is analytic on $B^m_{p,q}(\mathbb{R}^d)$.

Proposition 6.1. *Let L be a Lévy process, such that $L(1)$ has a self–decomposable distribution. Let $\alpha \in (1,2)$ be the Blumenthal-Getoor index of L. In particular, we assume that there exists a finite measure λ on $\mathbb{S}$ and a measurable function $k : \mathbb{S} \times \mathbb{R}^+ \to \mathbb{R}^+_0$, slowly varying at zero and monotone decreasing for $x \to \infty$ in the second variable such that*

$$\nu(B) = \int_{\mathbb{S}} \int_0^\infty 1_B(rx)k(r,x)\frac{dr}{r^{1+\alpha}}\,\lambda(dx), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Let us assume that there exists some $c_0 > 0$ such that $\lambda(\{|\langle x, e_j\rangle| > \cos(\pi/4)\}) \ge c_0$ for $j = 1, \ldots, d$, where $e_j = (e^1_j, e^2_j, \ldots, e^d_j)$ with $e^k_j = 0$ iff $j \ne k$ and $e^j_j = 1$. Let us assume that $\sigma \in C^{d+3}_b(\mathbb{R}^d)$ is bounded away from zero, i.e. $\sigma(x) \ge \delta I$ for all $x \in \mathbb{R}^D$, and $b \in C^{d+3}_b(\mathbb{R}^d)$.

- *Then, for all $1 \le p, q < \infty$, the Markovian semigroup $(\hat{\mathcal{P}}^\varepsilon_t)_{t\ge 0}$ is an analytic semigroup in $B^m_{p,q}(\mathbb{R}^d)$ for all $m \in \mathbb{R}$.*

- *Let $\vartheta(D)$ be a pseudo–differential operator with symbol $\vartheta(\xi)$, where $\vartheta \in S^{\kappa}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\kappa \in \mathbb{R}$. Then, we have for $u \in B^m_{p,q}(\mathbb{R}^d)$*

$$\left|\hat{\mathcal{P}}^\varepsilon_t \vartheta(D) u\right|_{B^m_{p,q}} \le \frac{C}{t} |u|_{B^{m-\alpha+\kappa}_{p,q}}. \tag{6.6}$$

Remark 6.1. By Theorem 14.3 [39], the Lévy process L from Proposition 6.1 is α–stable and the Blumenthal-Getoor index and α coincides.

As mentioned before, due to the fact that the Markovian semigroup is analytic, the weak error can be improved.

Theorem 6.1. *Let us assume that $\sigma \in C^{d+3}_b(\mathbb{R}^d)$ and $b \in C^{d+3}_b(\mathbb{R}^d)$. Let σ be bounded away from zero and let $\alpha \in (1, 2)$ be the Blumenthal-Getoor index of L. Then, for $\delta = 2(\alpha - 1)$, $r_1, r_2 \in (0, 1)$ such that $r_1 + r_2 > 1$ and $2r_1 > r_2$ with $\delta_1 = \frac{\delta r_2}{2}$ and $\delta_2 = \delta(r_1 - \frac{r_2}{2})$,*

$$\left\|\mathcal{P}_t - \hat{\mathcal{P}}^\varepsilon_t\right\|_{L(B^{-\delta_2}_{\infty,\infty}, B^{\delta_1}_{\infty,\infty})} \le C t^{2(\delta-1)(r_1+r_2)-1} \varepsilon^{(2-\delta)}.$$

To illustrate Theorem 6.1 we postponed the proofs of Theorem 6.1, Proposition 6.1, and present the following simulations. Here, we took as underlying process

$$\begin{cases} dX^{x_0}(t) &= -aX^{x_0}(t-)dt + dL(t), \\ X^{x_0}(0) &= x_0, \quad x_0 \in \mathbb{R}, \end{cases} \tag{6.7}$$

where $a = 3$ and $(L_t)_{t\ge 0}$ is a strictly α–stable process, where we specify the value of α later and let run α from one to two (see Figure 5). Note, α coincide with the Blumenthal-Getoor index of L, compare Remark 6.1.

In addition, we compared the error for different α, i.e. we let run α from one to two (see Figure 5). Summarizing, there is a significant improvement by adding a Wiener process. In particular, we approximate this process once by cutting off the small jumps and replacing the small jumps by an independent Wiener process described by

$$\begin{cases} d\hat{X}^{x_0}_\varepsilon(t) &= -a\hat{X}^{x_0}_\varepsilon(t-)dt + d\hat{L}_\varepsilon(t) + dW_\varepsilon(t), \\ \hat{X}^{x_0}_\varepsilon(0) &= x_0, \quad x_0 \in \mathbb{R}, \end{cases} \tag{6.8}$$

and secondly, by only cutting off the small jumps, i.e. by

$$\begin{cases} d\bar{X}^{x_0}_\varepsilon(t) &= -a\bar{X}^{x_0}_\varepsilon(t-)dt + d\hat{L}_\varepsilon(t), \\ \bar{X}^{x_0}_\varepsilon(0) &= x_0, \quad x_0 \in \mathbb{R}. \end{cases} \tag{6.9}$$

Let $(\mathcal{P}_t)_{t\ge 0}$ be the Markovian semigroup of the process (6.7), i.e.

$$\mathcal{P}_t \phi(x_0) \;:=\; \mathbb{E}\phi(X^{x_0}(t)), \quad t \ge 0, \tag{6.10}$$

let $(\hat{\mathcal{P}}^\varepsilon_t)_{t\ge 0}$ be the Markovian semigroup of the process (6.8), i.e.

$$\hat{\mathcal{P}}^\varepsilon_t \phi(x_0) \;:=\; \mathbb{E}\phi(\hat{X}^{x_0}_\varepsilon(t)), \quad t \ge 0, \tag{6.11}$$

and, finally,

$$\bar{\mathcal{P}}^\varepsilon_t \phi(x_0) \;:=\; \mathbb{E}\phi(\bar{X}^{x_0}_\varepsilon(t)), \quad t \ge 0, \tag{6.12}$$

be the Markovian semigroup of the process (6.9). We simulate the Markovian semigroup for two different functions

$$\phi_1(x) := \mathbb{1}_{x\geq 0.5} \quad \text{and} \quad \phi_2(x) = \frac{x^2}{1+x^2},\ x\in\mathbb{R}.$$

In Figure 1 and Figure 2, we simulate $\mathcal{P}_t\phi_i(x_0)$, $\hat{\mathcal{P}}^\varepsilon_t\phi_i(x_0)$, $\bar{\mathcal{P}}^\varepsilon_t\phi_i(x_0)$ and the absolute error of approximations $\bar{\mathcal{P}}^\varepsilon_t\phi_i(x_0)$ and $\hat{\mathcal{P}}^\varepsilon_t\phi_i(x_0)$ for $\alpha = 1.05$ and a sample size 6×10^7 with varying ε. Here, one can observe that, if ε decreases, then the error also decreases. In Figure 3 and Figure 4, we simulate $\mathcal{P}_t\phi_i(x_0)$, $\hat{\mathcal{P}}^\varepsilon_t\phi_i(x_0)$, $\bar{\mathcal{P}}^\varepsilon_t\phi_i(x_0)$ and the absolute error of approximations $\bar{\mathcal{P}}^\varepsilon_t\phi_i(x_0)$ and $\hat{\mathcal{P}}^\varepsilon_t\phi_i(x_0)$, i.e. $|\mathcal{P}_t\phi_i(x_0)-\hat{\mathcal{P}}^\varepsilon_t\phi_i(x_0)|$ and $|\mathcal{P}_t\phi_i(x_0)-\bar{\mathcal{P}}^\varepsilon_t\phi_i(x_0)|$ for $\varepsilon = 0.1$; sample size 1.5×10^6; $x_0 = 0$ and $x_0 = 0.45$; $i = 1,2$; and $\alpha = 1.05, 1.95$. It is observed that replacing the small jumps by a Wiener process improves the quality of the approximation. Especially, if α is closed to two. This we could also verify by Figure 5, where we simulate the logarithm of absolute error of $\hat{\mathcal{P}}^\varepsilon_1\phi_i(x_0)$ and $\bar{\mathcal{P}}^\varepsilon_1\phi_i(x_0)$ for $i = 1,2$, $x_0 = 0$ and $x_0 = 0.45$.

Before presenting the proof, we want to give some remarks. Note, that $(\mathcal{P}_t)_{t\geq 0}$ has generator given by the symbol $a(x,\xi) = \psi(\sigma(x)^T\xi)$ and $(\hat{\mathcal{P}}_t)_{t\geq 0}$ has generator given by the symbol $\hat{a}_\varepsilon(x,\xi) = \psi_\varepsilon(\sigma(x)^T\xi) - \frac{1}{2}\Sigma^2(\varepsilon)\langle\sigma(x)^T\xi,\sigma(x)^T\xi\rangle$, where

$$\psi_\varepsilon(\xi) = \int_{\mathbb{R}^d\setminus B_\varepsilon} (e^{i\langle y,\xi\rangle} - 1 - i\langle y,\xi\rangle)\nu(dy), \quad \xi\in\mathbb{R}^d,$$

and

$$\Sigma^2(\varepsilon) = \int_{B_\varepsilon} \langle y,y\rangle\,\nu(dy).$$

Proof of Proposition 6.1. Let us assume that the support of ν belongs to $\{x\in\mathbb{R}^d \mid |x|\leq 1\}$. Since the large jumps are simulated precisely, this is no restriction. The symbol for the approximation is given by

$$\hat{a}_\varepsilon(x,\xi) = \psi_\varepsilon(\sigma(x)^T\xi) - \frac{1}{2}\Sigma^2(\varepsilon)\langle\sigma(x)^T\xi,\sigma(x)^T\xi\rangle,$$

where

$$\psi_\varepsilon(\xi) := \int_{\mathbb{S}}\int_\varepsilon^1 (e^{ir\langle y,\xi\rangle} - 1 - ir\langle y,\xi\rangle)\,k(r,y)\frac{dr}{r^{1+\alpha}}\,\lambda(dy)$$

and

$$\Sigma^2(\varepsilon) = \int_{\mathbb{S}}\int_0^\varepsilon \langle ry,ry\rangle k(r,y)\frac{dr}{r^{1+\alpha}}\,\lambda(dy).$$

The aim in the following calculations is to show that $\hat{a}_\varepsilon(x,\xi)$ belongs to $\mathrm{Hyp}^\alpha_{2d+4,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$. Throughout this proof, we denote by C a varying positive constant. First, we will show that for any $\xi\in\mathcal{U}_1$ there exist some constants $R>0$ and $C>0$ such that

$$|\hat{a}_\varepsilon(x,\gamma\xi)| = |\psi_\varepsilon(\sigma(x)^T\gamma\xi) - \frac{\gamma^2}{2}\Sigma^2(\varepsilon)\langle\sigma(x)^T\xi,\sigma(x)^T\xi\rangle| \geq C|\gamma|^\alpha, \quad \xi\in\mathcal{U}_1, \quad \gamma\geq R.$$

Let $\xi\in\mathcal{U}_1$ and $\xi' = \gamma\xi$. By the Euler identity, we obtain

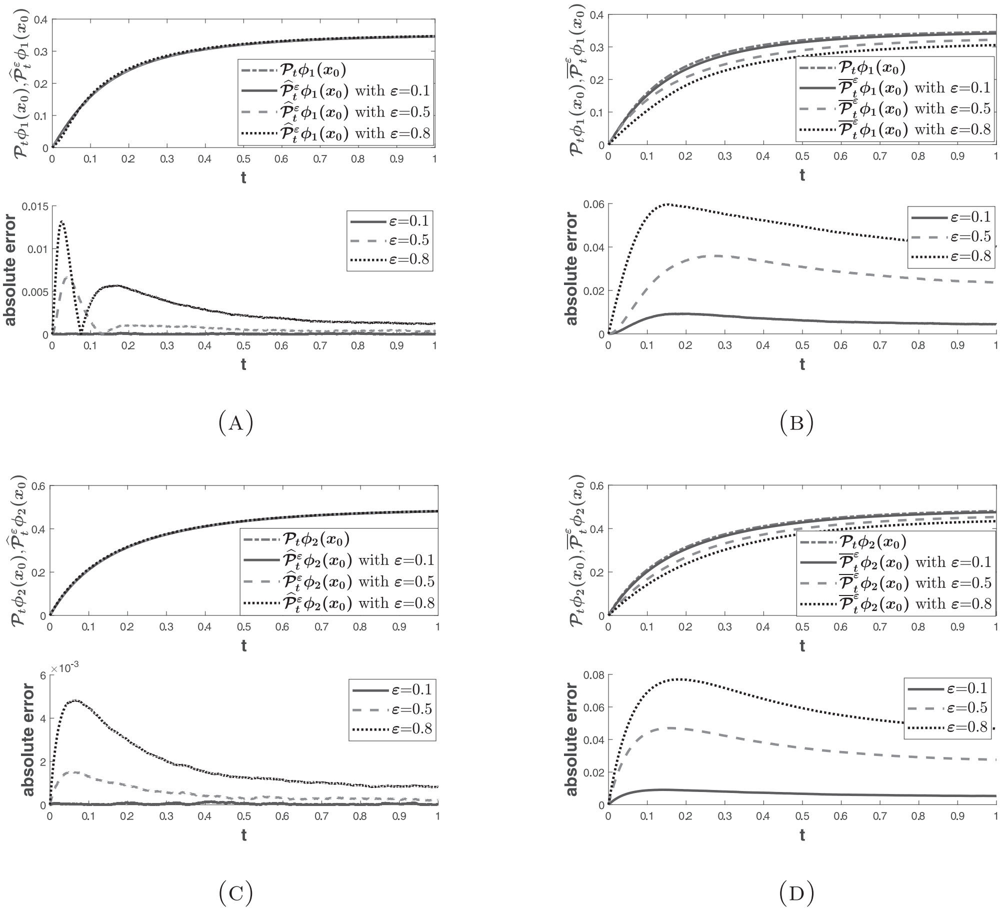

Fig. 1. (A) The Markovian semigroups $\mathcal{P}_t\phi_1(x_0)$, $\hat{\mathcal{P}}_t^\varepsilon\phi_1(x_0)$ and the absolute error of the approximations $\hat{\mathcal{P}}_t^\varepsilon\phi_1(x_0)$ for $x_0 = 0$ and ε is varying; (B) the Markovian semigroups $\mathcal{P}_t\phi_1(x_0)$, $\bar{\mathcal{P}}_t^\varepsilon\phi_1(x_0)$ and the absolute error of the approximations $\bar{\mathcal{P}}_t^\varepsilon\phi_1(x_0)$ for $x_0 = 0$ and ε varying; (C) the Markovian semigroups $\mathcal{P}_t\phi_2(x_0)$ and $\hat{\mathcal{P}}_t^\varepsilon\phi_2(x_0)$ and the absolute error of the approximations $\hat{\mathcal{P}}_t^\varepsilon\phi_2(x_0)$ for $x_0 = 0$ and ε varying; (D) the Markovian semigroups $\mathcal{P}_t\phi_2(x_0)$ and $\bar{\mathcal{P}}_t^\varepsilon\phi_2(x_0)$ and the absolute error of approximations $\bar{\mathcal{P}}_t^\varepsilon\phi_2(x_0)$ for $x_0 = 0$ and ε varying.

$$\begin{aligned}
|\hat{a}_\varepsilon(x,\xi')| \quad = \quad & \bigg| \int_{\mathbb{S}}\int_\varepsilon^1 (1-\cos(r\langle y,\sigma(x)^T\xi'\rangle))\, k(r,y)\frac{dr}{r^{1+\alpha}}\,\lambda(dy) \\
& + \int_{\mathbb{S}}\int_\varepsilon^1 i(-\sin(r\langle y,\sigma(x)^T\xi'\rangle) + r\langle y,\sigma(x)^T\xi'\rangle)\, k(r,y)\frac{dr}{r^{1+\alpha}}\,\lambda(dy) \\
& + \frac{1}{2}\Sigma^2(\varepsilon)\langle\sigma(x)^T\xi',\sigma(x)^T\xi'\rangle \bigg|.
\end{aligned}$$

Using the fact that $|a+ib| > |a|$ and $\cos(a) \le 1$ for all $a, b \in \mathbb{R}$ we obtain

$$|\hat{a}_\varepsilon(x,\xi')| \quad \ge \quad \left| \int_{\mathbb{S}}\int_\varepsilon^1 (1-\cos(r\langle y,\sigma(x)^T\xi'\rangle))\, k(r,y)\frac{dr}{r^{1+\alpha}}\,\lambda(dy) \right|$$

$$+\frac{1}{2}\Big|\Sigma^2(\varepsilon)\langle\sigma(x)^T\xi',\sigma(x)^T\xi'\rangle\Big|$$
$$:= \quad |\psi^1_\varepsilon(\sigma(x)^T\xi')| + \frac{1}{2}\Big|\Sigma^2(\varepsilon)\langle\sigma(x)^T\xi',\sigma(x)^T\xi'\rangle\Big|.$$

Note, that there exists some constant $c_0 > 0$ such that for all $1 \le j \le d$ we have $\lambda(\{x \in \mathbb{S} : |\langle x, e_j\rangle| > \cos(\pi/4)\}) \ge c_0$. Let us write $\xi' = \xi\gamma$, where $\xi \in \mathcal{U}_1$. Then, due to the shape of $\mathcal{U}_1$, there exists some $j_0 \in \{1,\ldots,d\}$ and some $\tilde{c} > 0$ with $\langle \xi, x'\rangle \ge \tilde{c}$ for all $x' \in \{x \in \mathbb{S} : |\langle x, e_j\rangle| > \cos(\pi/4)\}$. Next, we take into account that $\sigma(x) \ge \delta I$ for all $x \in \mathbb{R}^d$. For $\gamma \in (2\pi, \varepsilon^{-1})$ we get

$$|\psi^1_\varepsilon(\sigma(x)^T\xi')| \quad \ge \quad C\delta\gamma^\alpha \int_{\mathbb{S}} |\langle y,\xi\rangle|^\alpha \int_{\varepsilon\gamma|\langle y,\xi\rangle|}^{\gamma|\langle y,\xi\rangle|} (1-\cos(r))\frac{dr}{r^{1+\alpha}}\,\lambda(dy)$$

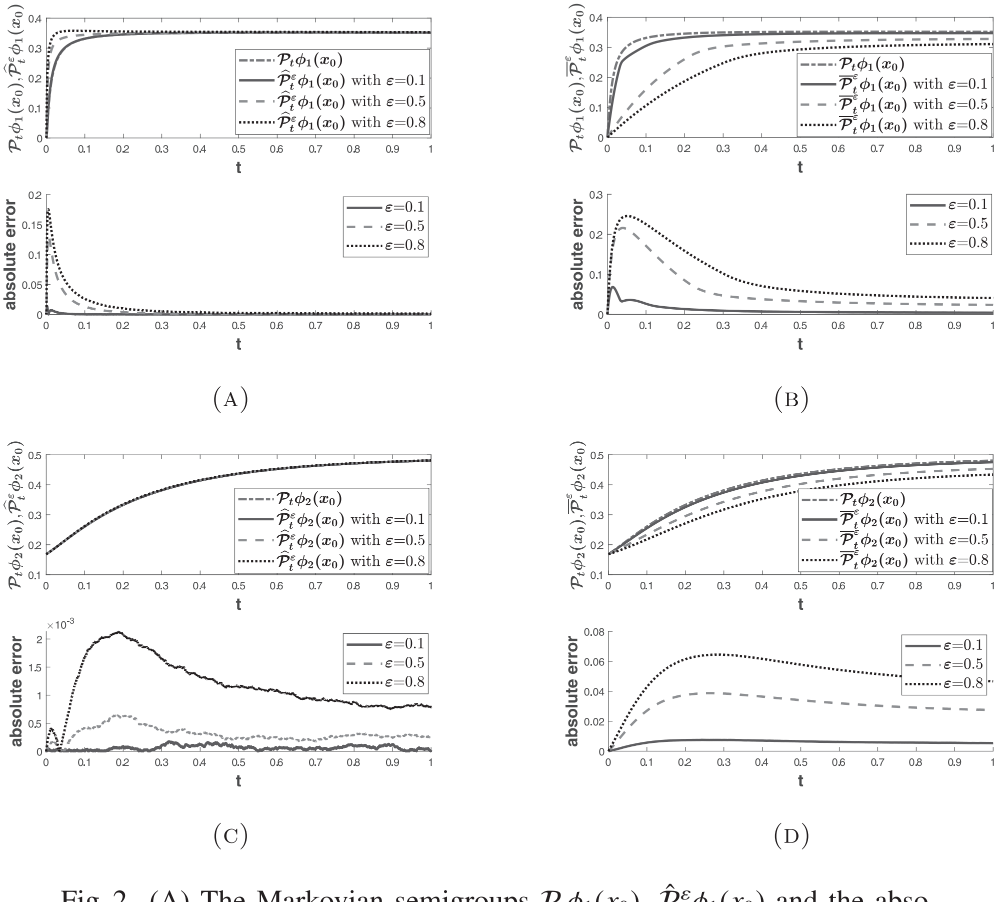

Fig. 2. (A) The Markovian semigroups $\mathcal{P}_t\phi_1(x_0)$, $\hat{\mathcal{P}}^\varepsilon_t\phi_1(x_0)$ and the absolute error of the approximations $\hat{\mathcal{P}}^\varepsilon_t\phi_1(x_0)$ for $x_0 = 0.45$ and ε varying; (B) the Markovian semigroups $\mathcal{P}_t\phi_1(x_0)$, $\bar{\mathcal{P}}^\varepsilon_t\phi_1(x_0)$ and the absolute error of the approximations $\bar{\mathcal{P}}^\varepsilon_t\phi_1(x_0)$ for $x_0 = 0.45$ and ε varying; (C) the Markovian semigroups $\mathcal{P}_t\phi_2(x_0)$ and $\hat{\mathcal{P}}^\varepsilon_t\phi_2(x_0)$ and the absolute error of the approximations $\hat{\mathcal{P}}^\varepsilon_t\phi_2(x_0)$ for $x_0 = 0.45$ and ε varying; (D) the Markovian semigroups $\mathcal{P}_t\phi_2(x_0)$ and $\bar{\mathcal{P}}^\varepsilon_t\phi_2(x_0)$ and the absolute error of approximations $\bar{\mathcal{P}}^\varepsilon_t\phi_2(x_0)$ for $x_0 = 0.45$ and ε varying.

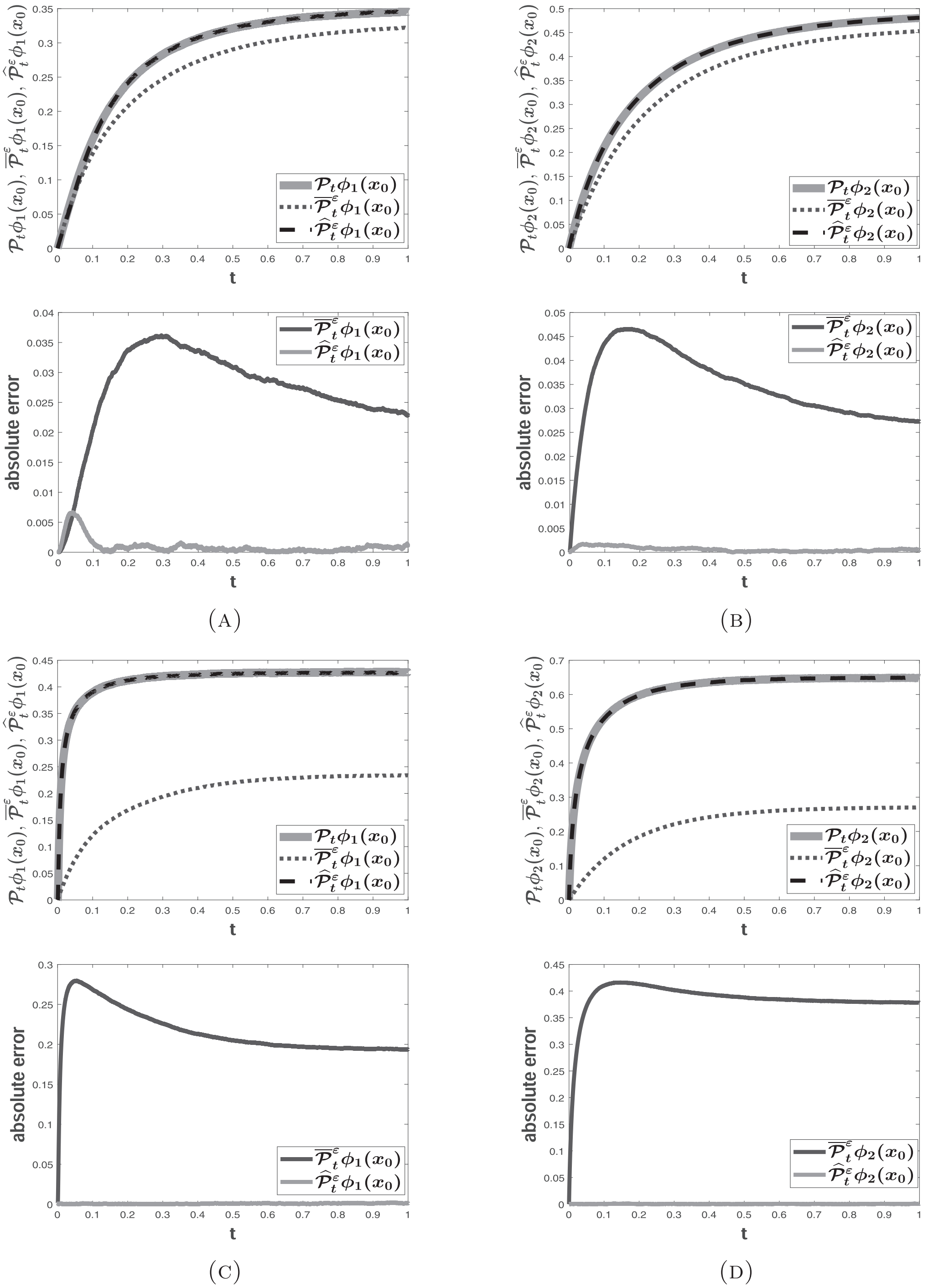

Fig. 3. The Markovian semigroups $\mathcal{P}_t\phi(x_0)$, $\hat{\mathcal{P}}_t^{\varepsilon}\phi(x_0)$, and $\bar{\mathcal{P}}_t^{\varepsilon}\phi(x_0)$ and the absolute error of the approximations $\bar{\mathcal{P}}_t^{\varepsilon}\phi(x_0)$ and $\hat{\mathcal{P}}_t^{\varepsilon}\phi(x_0)$ for $\varepsilon = 0.1$, $x_0 = 0$, and (A) $\alpha = 1.05$, $\phi = \phi_1$; (B) $\alpha = 1.05$, $\phi = \phi_2$; (C) $\alpha = 1.95$, $\phi = \phi_1$; (D) $\alpha = 1.95$, $\phi = \phi_2$

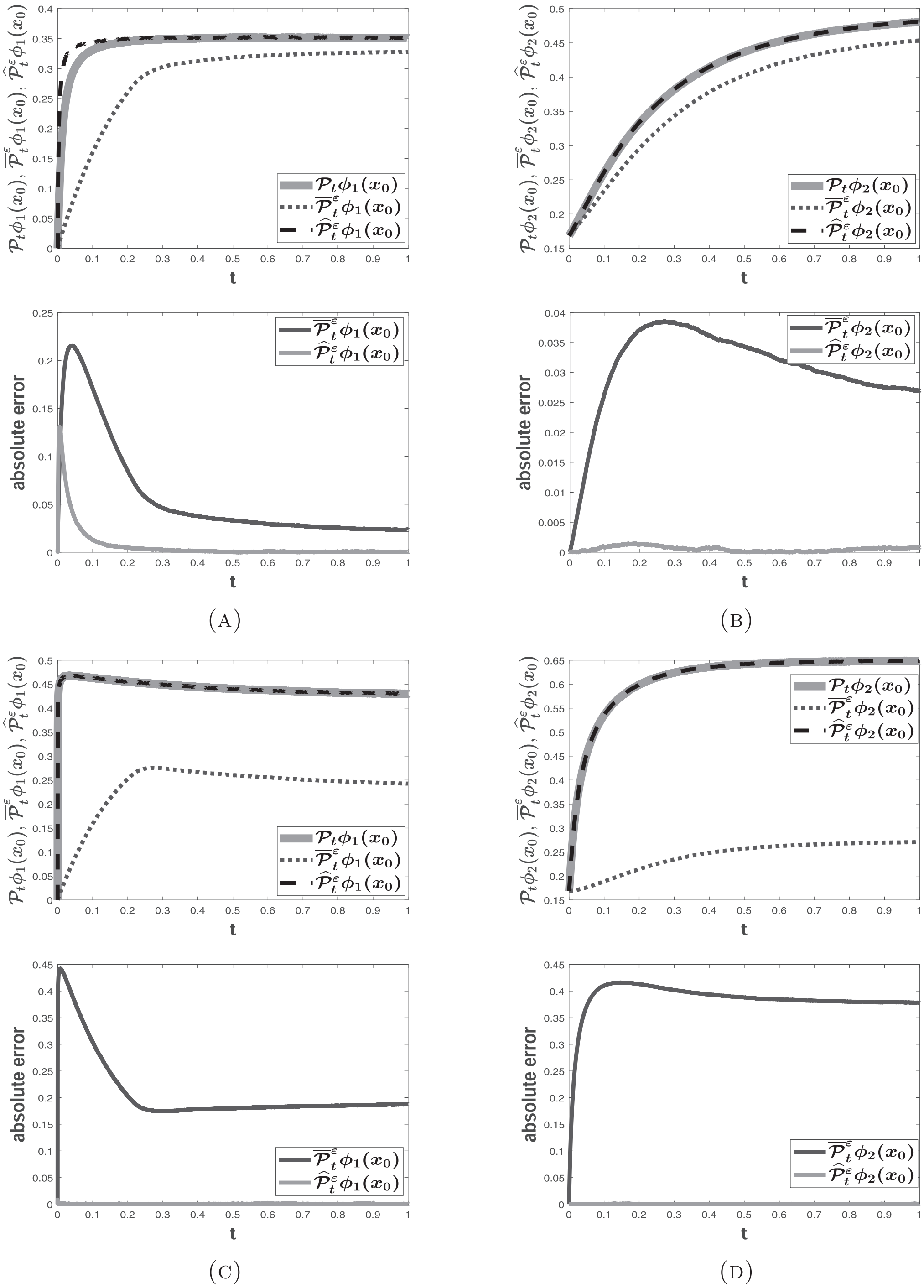

Fig. 4. The Markovian semigroups $\mathcal{P}_t\phi(x_0)$, $\hat{\mathcal{P}}_t^{\varepsilon}\phi(x_0)$, and $\bar{\mathcal{P}}_t^{\varepsilon}\phi(x_0)$ and the absolute error of the approximations $\bar{\mathcal{P}}_t^{\varepsilon}\phi(x_0)$ and $\hat{\mathcal{P}}_t^{\varepsilon}\phi(x_0)$ for $\varepsilon = 0.1$, $x_0 = 0.45$ and (A) $\alpha = 1.05$, $\phi = \phi_1$; (B) $\alpha = 1.05$, $\phi = \phi_2$; (C) $\alpha = 1.95$, $\phi = \phi_1$; (D) $\alpha = 1.95$, $\phi = \phi_2$.

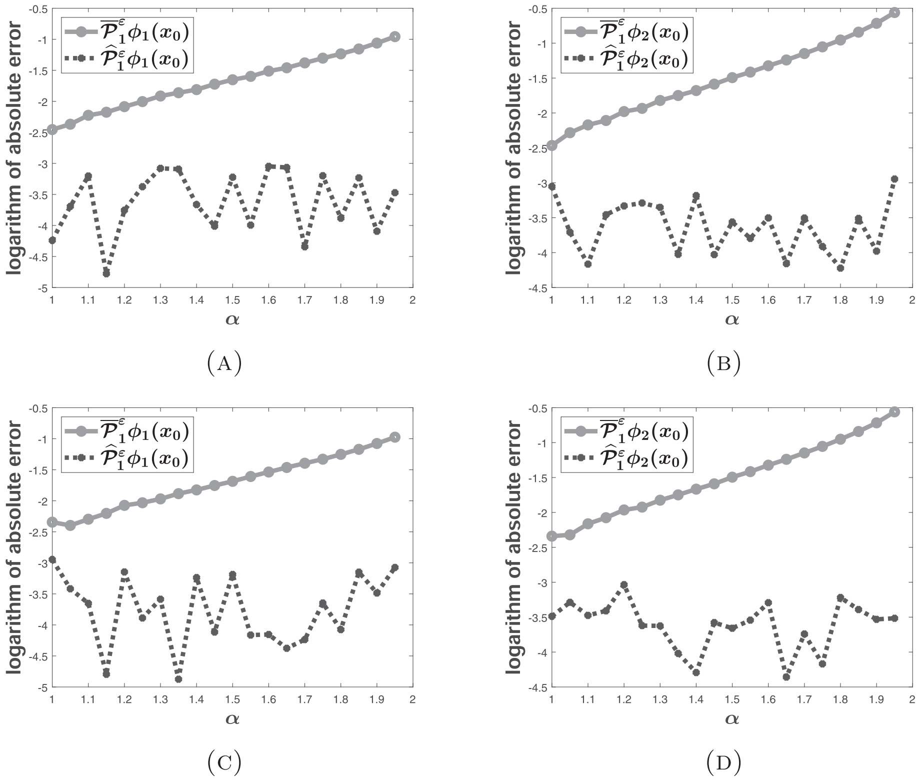

Fig. 5. The logarithm of the absolute error of the approximations $\hat{\mathcal{P}}_1^{\varepsilon}\phi(x_0)$ and $\bar{\mathcal{P}}_1^{\varepsilon}\phi(x_0)$ for $\varepsilon = 0.1$ with sample size 1.5×10^6 and (A) $\phi = \phi_1$, $x_0 = 0$; (B) $\phi = \phi_2$, $x_0 = 0$; (C) $\phi = \phi_1$, $x_0 = 0.45$; (D) $\phi = \phi_2$, $x_0 = 0.45$.

$$
\begin{aligned}
&\geq \; C\delta\gamma^{\alpha} \int_{\mathbb{S}} |\langle y, \xi\rangle|^{\alpha} \int_{|\langle y,\xi\rangle|}^{\gamma|\langle y,\xi\rangle|} (1-\cos(r))\frac{dr}{r^{1+\alpha}}\,\lambda(dy) \\
&\geq \; Cc_0\delta\tilde{c}^{\alpha}\gamma^{\alpha} \int_1^{2\pi} (1-\cos(r))\frac{dr}{r^{1+\alpha}} \geq Cc_0\delta\tilde{c}^{\alpha}\gamma^{\alpha}\frac{2\pi - 1 + \sin(1)}{(2\pi)^{1+\alpha}}.
\end{aligned}
$$

For $\gamma > \varepsilon^{-1}$ we have

$$
\left|\Sigma^2(\varepsilon)\langle \sigma(x)^T\xi', \sigma(x)^T\xi'\rangle\right| \geq C\frac{\varepsilon^{2-\alpha}}{2-\alpha}|\xi'|^2 \geq C\frac{\gamma^{\alpha}}{2-\alpha}.
$$

Hence, there exists a constant $C > 0$ such that we have for all $\xi' = \gamma\xi$ and $\xi \in \mathcal{U}_1$

$$
|\hat{a}_{\varepsilon}(x, \xi')| \geq C\langle\gamma\rangle^{\alpha}. \tag{6.13}
$$

Now we will show that

$$
\left|\partial_{\xi}^{\beta}\left[\hat{a}_{\varepsilon}(x,\xi)\right]\right| \leq C\langle|\xi|\rangle^{\alpha-|\beta|}
$$

with $1 \leq |\beta| \leq 2d+4$ and $\xi \in R\mathcal{U}_1$ where $R \geq \varepsilon^{-1}$. Now, let us consider $|\beta| = 1$. We have for $1 \leq j \leq d$

$$
\left|\frac{\partial}{\partial\xi_j}\left[\hat{a}_{\varepsilon}(x,\xi)\right]\right| = \left|\frac{d}{d\xi_j}\left[\psi_{\varepsilon}(\sigma(x)^T\xi) - \frac{1}{2}\Sigma^2(\varepsilon)\langle\sigma(x)^T\xi, \sigma(x)^T\xi\rangle\right]\right|
$$

$$
\begin{aligned}
&= \left|\int_{\mathbb{R}^d\setminus B_\varepsilon} \frac{d}{d\xi_j}\left[e^{i\langle y,\sigma(x)^T\xi\rangle}-1-i\langle y,\sigma(x)^T\xi\rangle\right]\nu(dy)\right| \\
&\quad + \left|\int_{B_\varepsilon} \frac{d}{d\xi_j}\left[\frac{1}{2}\langle \sigma^T(x)\xi,\sigma(x)^T\xi\rangle\langle y,y\rangle\right]\nu(dy)\right| \\
&= \left|\int_{\mathbb{R}^d\setminus B_\varepsilon} \sum_{k=1}^d i y_k\sigma_{jk}(x)\left[e^{i\langle y,\sigma(x)^T\xi\rangle}-1\right]\nu(dy)\right| \\
&\quad + \left|\int_{B_\varepsilon} \sum_{k=1}^d \sigma_{jk}(x)\left[\sum_{l=1}^d \sigma_{lk}(x)\xi_l\right]\langle y,y\rangle\nu(dy)\right| \\
&\le C\left[\int_{\mathbb{R}^d\setminus B_\varepsilon}|y|\,\nu(dy)+|\xi|\int_{B_\varepsilon}|y|^2\,\nu(dy)\right]\le C\left[\int_\varepsilon^1 r\,\frac{dr}{r^{1+\alpha}}+|\xi|\int_0^\varepsilon |r|^2\,\frac{dr}{r^{1+\alpha}}\right] \\
&\le C\left[(1-\varepsilon^{1-\alpha})+|\xi|\varepsilon^{2-\alpha}\right]\le C\left[(1+|\xi|^2)^{\frac{\alpha-1}{2}}+|\xi|\,|\xi|^{\alpha-2}\right] \\
&\le C\langle|\xi|\rangle^{\alpha-1}.
\end{aligned}
$$

We now investigate for $|\beta| = 2$ and then for $2 < |\beta| \le 2d + 4$. Here, we get the following sequence of calculations for $1 \le l, j \le d$

(6.14)

$$
\begin{aligned}
&\left|\frac{\partial^{(j,l)}}{\partial_{\xi_j}\partial_{\xi_l}}\left[\hat{a}_\varepsilon(x,\xi)\right]\right| \\
&= \left|\frac{\partial^{(j,l)}}{\partial_{\xi_j}\partial_{\xi_l}}\left[\psi_\varepsilon(\sigma(x)^T\xi)-\frac{1}{2}\Sigma^2(\varepsilon)\langle\sigma^T(x)\xi,\sigma^T(x)\xi\rangle\right]\right|. \\
&= \left|\int_{\mathbb{R}^d\setminus B_\varepsilon} \frac{\partial^{(j,l)}}{\partial_{\xi_j}\partial_{\xi_l}}\left[e^{i\langle y,\sigma(x)^T\xi\rangle}-1-i\langle y,\sigma(x)^T\xi\rangle\right]\nu(dy)\right| \\
&\quad + \left|\int_{B_\varepsilon} \frac{\partial^{(j,l)}}{\partial_{\xi_j}\partial_{\xi_l}}\left[\frac{1}{2}\langle\sigma(x)^T\xi,\sigma(x)^T\xi\rangle\langle y,y\rangle\right]\nu(dy)\right| \\
&\le \left|\int_{\mathbb{R}^d\setminus B_\varepsilon}\Big(\sum_{k=1}^d (y_k\sigma_{lk}(x))\Big(\sum_{k=1}^d (y_k\sigma_{jk}(x))\Big)e^{i\langle y,\sigma(x)^T\xi\rangle}\nu(dy)\right| + |C(\sigma,x)|\int_{B_\varepsilon}|(y,y)|\,\nu(dy) \\
&\le C\int_{\mathbb{R}^d\setminus B_\varepsilon}|y|^2\,\nu(dy)+|C(\sigma,x)|\int_{B_\varepsilon}|(y,y)|\,\nu(dy)\le C\left[\int_\varepsilon^1 r^2\,\frac{dr}{r^{1+\alpha}}+|\xi|\int_0^\varepsilon |r|^2\,\frac{dr}{r^{1+\alpha}}\right] \\
&\le C\left[(1-\varepsilon^{2-\alpha})+\varepsilon^{2-\alpha}\right]\le C\left[(1+|\xi|^2)^{\frac{\alpha-2}{2}}+|\xi|^{\alpha-2}\right] \\
&\le C\langle|\xi|\rangle^{\alpha-2}.
\end{aligned}
$$

Now, let $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ with $m = |\beta| \in (3, 2d + 4)$. Then we get

$$
\begin{aligned}
&\left|\frac{\partial^\beta}{\partial_{\xi_{\beta_1}}\partial_{\xi_{\beta_2}}\ldots\partial_{\xi_{\beta_m}}}\left[\hat{a}_\varepsilon(x,\xi)\right]\right| \\
&\quad = \left|\frac{\partial^\gamma}{\partial_{\xi_{\beta_1}}\partial_{\xi_{\beta_2}}\ldots\partial_{\xi_{\beta_m}}}\left[\psi_\varepsilon(\sigma(x)^T\xi)-\frac{1}{2}\Sigma^2(\varepsilon)\langle\sigma^T(x)\xi,\sigma^T(x)\xi\rangle\right]\right|.
\end{aligned}
$$

Continuing, we get

$$\text{(6.15)} \qquad \ldots = \left| \int_{\mathbb{R}^d \setminus B_\varepsilon} \frac{\partial^\gamma}{\partial_{\xi_{\beta_1}} \partial_{\xi_{\beta_2}} \ldots \partial_{\xi_{\beta_m}}} \left[e^{i\langle y, \sigma(x)^T \xi \rangle} \right] \nu(dy) \right|$$

$$\leq C \int_{\mathbb{R}^d \setminus B_\varepsilon} |y|^m \nu(dy) \leq C \left[\int_\varepsilon^1 r^m \frac{dr}{r^{1+\alpha}} \right]$$

$$\leq C \left[(1 - \varepsilon^{m-\alpha}) \right] \leq C \left[(1 + \xi^2)^{\frac{\alpha-m}{2}} \right]$$

$$\leq C \langle |\xi| \rangle^{\alpha - m}.$$

Observe, due to $\sigma \in C_b^{d+3}(\mathbb{R}^d)$ we can write

$$\text{(6.16)} \qquad \left| \partial_\xi^\beta \partial_x^\varrho \left[\hat{a}_\varepsilon(x, \xi) \right] \right| \leq C \langle |\xi| \rangle^{\alpha - |\beta|}$$

with $1 \leq |\beta| \leq 2d + 4$ and $1 \leq |\varrho| \leq d + 3$. Now we will show that

$$\left| \partial_\xi^\beta \left[\frac{1}{\hat{a}_\varepsilon(x, \xi)} \right] \right| \leq C \langle |\xi| \rangle^{-\alpha - |\beta|}$$

with $1 \leq |\beta| \leq 2d + 4$. Let us consider $|\beta| = 1$, i.e. $\beta = j$ for some $1 \leq j \leq d$. We obtain

$$\left| \partial_\xi^\beta \left[\frac{1}{\hat{a}_\varepsilon(x, \xi)} \right] \right| \leq C |\hat{a}_\varepsilon(x, \xi)|^{-2} \left| \frac{\partial}{\partial \xi_j} \left[\hat{a}_\varepsilon(x, \xi) \right] \right| \leq \langle |\xi| \rangle^{-2\alpha} \langle |\xi| \rangle^{\alpha - |\beta|} \leq \langle |\xi| \rangle^{-\alpha - |\beta|}.$$

Next, let $|\beta| = 2$, i.e. $\beta = (\xi_l, \xi_k)$ for $k, l \in \{1, \ldots, d\}$. Then,

$$\partial_\xi^\beta \left[\frac{1}{\hat{a}_\varepsilon(x, \xi)} \right] = \partial_{\xi_l} \partial_{\xi_k} \left[\frac{1}{\hat{a}_\varepsilon(x, \xi)} \right] = \frac{2 \partial_{\xi_l} \hat{a}_\varepsilon(x, \xi) \partial_{\xi_k} \hat{a}_\varepsilon(x, \xi)}{\hat{a}_\varepsilon(x, \xi)^3} - \frac{\partial^2_{\xi_l \xi_k} \hat{a}_\varepsilon(x, \xi)}{\hat{a}_\varepsilon(x, \xi)^2}.$$

Hence, we have

$$\left| \partial_\xi^\beta \left[\frac{1}{\hat{a}_\varepsilon(x, \xi)} \right] \right| \leq C \left\{ |\hat{a}_\varepsilon(x, \xi)|^{-3} \langle |\xi| \rangle^{2\alpha - 2} + |\hat{a}_\varepsilon(x, \xi)|^{-2} \langle |\xi| \rangle^{\alpha - 2} \right\}$$

$$\leq C \left\{ \langle |\xi| \rangle^{-3\alpha} \langle |\xi| \rangle^{2\alpha - 2} + \langle |\xi| \rangle^{-2\alpha} \langle |\xi| \rangle^{\alpha - 2} \right\} \leq C \langle |\xi| \rangle^{-\alpha - 2}.$$

Again, let $\beta = (\xi_l, \xi_k, \xi_j)$ for $j, k, l \in \{1, \ldots, d\}$. Then,

$$\partial_\xi^\beta \left[\frac{1}{\hat{a}_\varepsilon(x, \xi)} \right] = -\frac{6 \partial_{\xi_l} \hat{a}_\varepsilon(x, \xi) \partial_{\xi_k} \hat{a}_\varepsilon(x, \xi) \partial_{\xi_j} \hat{a}_\varepsilon(x, \xi)}{\hat{a}_\varepsilon(x, \xi)^4} - \frac{\partial^3_{\xi_l \xi_k \xi_j} \hat{a}_\varepsilon(x, \xi)}{\hat{a}_\varepsilon(x, \xi)^2}$$

$$+2 \frac{\partial_{\xi_j} \hat{a}_\varepsilon(x, \xi) \partial^2_{\xi_l \xi_k} \hat{a}_\varepsilon(x, \xi) + \partial_{\xi_k} \hat{a}_\varepsilon(x, \xi) \partial^2_{\xi_l \xi_j} \hat{a}_\varepsilon(x, \xi) + \partial_{\xi_l} \hat{a}_\varepsilon(x, \xi) \partial^2_{\xi_k \xi_j} \hat{a}_\varepsilon(x, \xi)}{\hat{a}_\varepsilon(x, \xi)^3}.$$

Thus we obtain

$$\left| \partial_\xi^\beta \left[\frac{1}{\hat{a}_\varepsilon(x, \xi)} \right] \right| \leq C \left\{ |\hat{a}_\varepsilon(x, \xi)|^{-4} \langle |\xi| \rangle^{3\alpha - 3} + |\hat{a}_\varepsilon(x, \xi)|^{-3} \langle |\xi| \rangle^{2\alpha - 3} + |\hat{a}_\varepsilon(x, \xi)|^{-2} \langle |\xi| \rangle^{\alpha - 3} \right\}$$

$$\leq C \left\{ \langle |\xi| \rangle^{-4\alpha} \langle |\xi| \rangle^{3\alpha - 3} + \langle |\xi| \rangle^{-3\alpha} \langle |\xi| \rangle^{2\alpha - 3} + \langle |\xi| \rangle^{-2\alpha} \langle |\xi| \rangle^{\alpha - 3} \right\}$$

$$\leq C \langle |\xi| \rangle^{-\alpha - 3}.$$

Now, let us consider $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ where β is a multi-index with $m = |\beta| \in (4, 2d + 4)$. By observing the pattern of the above derivative we can identify the general derivative $\partial_\xi^\beta [\frac{1}{a(x,\xi)}]$ and get the following estimate

$$
\begin{aligned}
\left|\partial_\xi^\beta\left[\frac{1}{\hat{a}_\varepsilon(x,\xi)}\right]\right| &\lesssim |\hat{a}_\varepsilon(x,\xi)|^{-|\beta|-1}\langle|\xi|\rangle^{\alpha|\beta|-|\beta|} + |\hat{a}_\varepsilon(x,\xi)|^{-|\beta|}\langle|\xi|\rangle^{\alpha(|\beta|-1)-|\beta|} \\
&\quad + \cdots + |\hat{a}_\varepsilon(x,\xi)|^{-2}\langle|\xi|\rangle^{\alpha-|\beta|} \\
&\lesssim \langle|\xi|\rangle^{-\alpha-|\beta|}.
\end{aligned}
$$

Observe, due to $\sigma \in C_b^{d+3}(\mathbb{R}^d)$ we can write

$$
\left|\partial_\xi^\beta \partial_x^\varrho\left[\frac{1}{\hat{a}_\varepsilon(x,\xi)}\right]\right| \leq C\langle|\xi|\rangle^{-\alpha-|\beta|} \tag{6.17}
$$

with $1 \leq |\beta| \leq 2d+4$ and $1 \leq |\varrho| \leq d+3$. Therefore we can conclude that $\hat{a}_\varepsilon(x,\xi)$ belongs to $\mathrm{Hyp}^\alpha_{2d+4,d+3;1,0}(\mathbb{R}^d \times \mathbb{R}^d)$. Hence, for all $1 \leq p,q < \infty$, the Markovian semigroup $(\hat{\mathcal{P}}_t^\varepsilon)_{t\geq 0}$ is an analytic semigroup in $B^m_{p,q}(\mathbb{R}^d)$ for all $m \in \mathbb{R}$.

Now, we will prove (6.6). Let $f = \vartheta(D)u$. We will use the representation of the semigroup $(\hat{\mathcal{P}}_t^\varepsilon)_{t\geq 0}$ in terms of the contour integrals which is already successfully applied in [11], [19], or [13]. Let $\theta' \in (0,\theta)$, $\rho \in (0,\infty)$, and

$$
\Gamma_{\theta'}(\rho, M) = \Gamma^{(1)}_{\theta',\rho} + \Gamma^{(2)}_{\theta',\rho} + \Gamma^{(3)}_{\theta',\rho},
$$

where $\Gamma^{(1)}_{\theta',\rho}$ and $\Gamma^{(2)}_{\theta',\rho}$ are the rays $re^{i(\frac{\pi}{2}+\theta')}$ and $re^{-i(\frac{\pi}{2}+\theta')}$, $\rho \leq r \leq M < \infty$, and $\Gamma^{(3)}_{\theta',\rho} = Me^{i\eta}$, $\eta \in [-\frac{\pi}{2}-\theta', \frac{\pi}{2}+\theta']$. It follows from [35, Theorem 1.7.7] and Fubini's Theorem that for $t > 0$ and $v \in B^m_{p,q}(\mathbb{R}^d)$ we have

$$
\hat{\mathcal{P}}_t^\varepsilon v = \lim_{M\to\infty} \frac{1}{2\pi i}\int_{\Gamma_{\theta'}(\rho,M)} e^{\lambda t} R(\lambda : \hat{a}_\varepsilon(x,D)) v\, d\lambda,
$$

where $R(\lambda : \hat{a}_\varepsilon(x,D))$ be the resolvent $[\lambda + \hat{a}_\varepsilon(x,D)]^{-1}$ of an operator $\hat{a}_\varepsilon(x,D)$. From the previous result, we know that $(\hat{\mathcal{P}}_t^\varepsilon)_{t\geq 0}$ is an analytic semigroup in $B^m_{p,q}(\mathbb{R}^d)$. Therefore, for any element $v \in B^m_{p,q}(\mathbb{R}^d)$, the limit exists and is well defined. Let $\{v_n : n \in \mathbb{N}\}$ be a sequence such that $v_n \in B^m_{p,q}(\mathbb{R}^d)$ and $v_n \to f$ in $B^{m-\kappa}_{p,q}(\mathbb{R}^d)$. By a change of variables, we obtain

$$
\begin{aligned}
&\lim_{M\to\infty}\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta'}(\rho,M)} e^{\lambda t} R(\lambda : \hat{a}_\varepsilon(x,D)) v_n\, d\lambda\right|_{B^m_{p,q}} \\
&\leq \lim_{M\to\infty}\left|\frac{1}{2\pi i t}\int_\rho^M e^{re^{-i(\frac{\pi}{2}+\theta')}} R(\frac{r}{t}e^{-i(\frac{\pi}{2}+\theta')}, \hat{a}_\varepsilon(x,D))\, v_n e^{i(\frac{\pi}{2}+\theta')} dr\right|_{B^m_{p,q}} \\
&+ \lim_{M\to\infty}\left|\frac{1}{2\pi i t}\int_\rho^M e^{re^{i(\frac{\pi}{2}+\theta')}} R(\frac{r}{t}e^{i(\frac{\pi}{2}+\theta')}, \hat{a}_\varepsilon(x,D))\, v_n e^{-i(\frac{\pi}{2}+\theta')} dr\right|_{B^m_{p,q}} \\
&+ \left|\frac{1}{2\pi i t}\int_{-\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho e^{i\beta}} R(\frac{\rho}{t}e^{i\beta}, \hat{a}_\varepsilon(x,D))\, v_n \rho^{-1} e^{i\beta} d\beta\right|_{B^m_{p,q}}.
\end{aligned}
$$

The Minkowski inequality gives

$$
\begin{aligned}
\ldots &\leq \frac{1}{2t\pi}\int_\rho^\infty e^{-r\sin\theta'} \left|R(\frac{r}{t}e^{-i(\frac{\pi}{2}+\theta')}, \hat{a}_\varepsilon(x,D))v_n\right|_{B^m_{p,q}} dr \\
&+ \frac{1}{2t\pi}\int_\rho^\infty e^{-r\sin\theta'} \left|R(\frac{r}{t}e^{i(\frac{\pi}{2}+\theta')}, \hat{a}_\varepsilon(x,D))v_n\right|_{B^m_{p,q}} dr
\end{aligned} \tag{6.18}
$$

$$+\quad \frac{\rho^{-1}}{2t\pi}\int_{-\frac{\pi}{2}-\theta'}^{\frac{\pi}{2}+\theta'} e^{\rho\cos\beta}\left|R(\frac{\rho}{t}e^{i\beta},\hat{a}_\varepsilon(x,D))v_n\right|_{B^m_{p,q}}d\beta.$$

We analyse the right-hand side of the estimate above by analysing the operator $R(\frac{\rho}{t}e^{i\beta}, \hat{a}_\varepsilon(x,D))$ by applying Theorem 7.1. Using (6.13) we have

$$|\lambda+\hat{a}_\varepsilon(x,\xi)|\geq C\langle|\xi|\rangle^{\alpha},\quad \xi\in R\mathcal{U}_1.$$

In this way we obtain

$$\left|\frac{1}{\lambda+\hat{a}_\varepsilon(x,\xi)}\right| \leq C(\sigma,\delta)\,\langle|\xi|\rangle^{-\alpha}.$$

Let $k\in\{1,\dots,d\}$. From (6.16), we obtain

$$\left|\partial_{\xi_k}\left[\frac{1}{\lambda+\hat{a}_\varepsilon(x,\xi)}\right]\right| = \left|\frac{\partial_{\xi_k}\hat{a}_\varepsilon(x,\xi)}{(\lambda+\hat{a}_\varepsilon(x,\xi))^2}\right|\leq\left|\frac{\langle|\xi|\rangle^{\alpha-1}}{(\lambda+\langle|\xi|\rangle^{\alpha})^2}\right|\leq C(\sigma,\alpha)\,\langle|\xi|\rangle^{-\alpha-1}.$$

Next, let $k,l\in\{1,\dots,d\}$. Then,

$$\partial_{\xi_l}\partial_{\xi_k}\left[\frac{1}{\lambda+\hat{a}_\varepsilon(x,\xi)}\right]=-\frac{\partial^2_{\xi_l\xi_k}\hat{a}_\varepsilon(x,\xi)}{(\lambda+\hat{a}_\varepsilon(x,\xi))^2}+\frac{2\partial_{\xi_l}\hat{a}_\varepsilon(x,\xi)\partial_{\xi_k}\hat{a}_\varepsilon(x,\xi)}{(\lambda+\hat{a}_\varepsilon(x,\xi))^3}.$$

Hence, using (6.15) we have

$$\left|\partial_{\xi_l}\partial_{\xi_k}\left[\frac{1}{\lambda+\hat{a}_\varepsilon(x,\xi)}\right]\right| \leq C(\sigma,\alpha)\left\{\frac{\langle|\xi|\rangle^{\alpha-2}}{(\lambda+\langle|\xi|\rangle^{\alpha})^2}+\frac{\langle|\xi|\rangle^{\alpha-2}}{(\lambda+\langle|\xi|\rangle^{\alpha})^2}\right\}\leq C(\sigma,\alpha)\langle|\xi|\rangle^{-\alpha-2}.$$

Let $\beta=(\beta_1,\cdots,\beta_m)$ be a multi-index with $m=|\beta|\in(3,d+1)$. By observing the pattern of the above derivative we can identify the general derivative $\partial^\beta_\xi[\frac{1}{\lambda+\hat{a}_\varepsilon(x,\xi)}]$ and get the following estimate. There exist $C_1,C_2,\cdots C_{|\beta|}>0$ depending on σ and α such that

$$\begin{aligned}\left|\partial^\beta_\xi\left[\frac{1}{\lambda+\hat{a}_\varepsilon(x,\xi)}\right]\right|&\leq C_1|\lambda+\hat{a}_\varepsilon|^{-|\beta|-1}\langle\xi\rangle^{\alpha|\beta|-|\beta|}\\&+C_2|\lambda+\hat{a}_\varepsilon|^{-|\beta|}\langle\xi\rangle^{\alpha(|\beta|-1)-|\beta|}+C_3|\lambda+\hat{a}_\varepsilon|^{-|\beta|+1}\langle\xi\rangle^{\alpha(|\beta|-2)-|\beta|}+\cdots+C_{|\beta|}|\lambda+\hat{a}_\varepsilon|^{-2}\langle\xi\rangle^{\alpha-|\beta|}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\left|\partial^\beta_\xi\left[\frac{1}{\lambda+\hat{a}_\varepsilon(x,\xi)}\right]\right|\langle\xi\rangle^{-\alpha+|\beta|}&\leq C_1|\lambda+\hat{a}_\varepsilon|^{-|\beta|-1}\langle\xi\rangle^{\alpha|\beta|-\alpha}\\&+C_2|\lambda+\hat{a}_\varepsilon|^{-|\alpha|}\langle\xi\rangle^{\alpha|\beta|-2\alpha}+C_3|\lambda+\hat{a}_\varepsilon|^{-|\beta|+1}\langle\xi\rangle^{\alpha|\beta|-3\alpha}+\cdots+C_{|\beta|}|\lambda+\hat{a}_\varepsilon|^{-2},\end{aligned}$$

with $1\leq|\beta|\leq d+1$. From the last line we obtain that $\lambda+\hat{a}_\varepsilon(x,\xi)\in \mathrm{Hyp}^\alpha_{d+1,0;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$. It remains to estimate the norm of the symbol $\lambda+\hat{a}_\varepsilon(x,\xi)$ in $\tilde{\mathcal{A}}^{1,1}_{2d+4,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$. Due to the fact that one has to take at least once the derivative with respect to ξ, the constant λ has no influence on the norm in $\tilde{\mathcal{A}}$. From (6.16) we have $\hat{a}_\varepsilon\in\mathcal{A}^\alpha_{k_1,k_2;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$ and

$$\|\lambda+\hat{a}_\varepsilon\|_{\tilde{\mathcal{A}}^{\alpha,1}_{k_1,k_2;1,0}} = \sup_{1\leq|\beta|\leq k_1,|\varrho|\leq k_2}\ \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d}\left|\partial^\beta_\xi\partial^\varrho_x(\lambda+\hat{a}_\varepsilon(x,\xi))\right|\langle|\xi|\rangle^{|\beta|-\alpha}<\infty,$$

where $k_1=2d+4$ and $k_2=d+3$. Therefore we can conclude that $\lambda+\hat{a}_\varepsilon(x,\xi)\in\mathcal{A}^{1,1}_{2d+4,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$. Finally from Theorem 7.1 and (6.18) we can conclude that

$$\lim_{M\to\infty}\left|\frac{1}{2\pi i}\int_{\Gamma_{\theta'}(\rho,M)} e^{\lambda t}R(\lambda:\hat{a}_\varepsilon(x,D))v_n d\lambda\right|_{B^m_{p,q}} \leq \frac{C(\sigma,\alpha)}{2t\pi}|v_n|_{B^{m-\alpha}_{p,q}}.$$

Taking the limit $n\to\infty$ and using Theorem A.1 and the fact $\vartheta\in S^\kappa_{1,0}(\mathbb{R}^d\times\mathbb{R}^d)$, we get

$$\left|\hat{\mathcal{P}}^\varepsilon_t\vartheta(D)u\right|_{B^m_{p,q}} \leq \frac{C(\sigma,\alpha)}{2t\pi}|f|_{B^{m-\alpha}_{p,q}} \leq \frac{C(\sigma,\alpha)}{2t\pi}|u|_{B^{m-\alpha+\kappa}_{p,q}},$$

which is the assertion. □

Proof of Theorem 6.1. Firstly, we have to show that

$$|a(x,\xi)-\hat{a}_\varepsilon(x,\xi)|\leq|\xi|^2\,\varepsilon^{(2-\delta)}. \tag{6.19}$$

In particular, we have to show for any multi-index γ with $|\gamma|\leq d+1$, we have $\left|\frac{\partial^\gamma}{\partial_\xi}[a(x,\xi)-\hat{a}_\varepsilon(x,\xi)]\right|\leq C\,\varepsilon^{|\gamma|-\delta}$. By this we can then conclude that $a(x,\xi)-\hat{a}_\varepsilon(x,\xi)\in S^2_{1,0}(\mathbb{R}^d\times\mathbb{R}^d)$. Throughout this proof we denote by C a varying constant. Let us start with $\gamma=0$. Straightforward calculations give

$$\begin{aligned}
|a(x,\xi)-\hat{a}_\varepsilon(x,\xi)| &= \left|\psi(\sigma(x)^T\xi)-\psi_\varepsilon(\sigma(x)^T\xi)+\frac{1}{2}\Sigma^2(\varepsilon)\langle\sigma(x)^T\xi,\sigma(x)^T\xi\rangle\right| \\
&= \Bigg|\int_{\mathbb{R}^d\setminus\{0\}}\left[e^{i\langle y,\sigma(x)^T\xi\rangle}-1-i\langle y,\sigma(x)^T\xi\rangle\right]\nu(dy)-\int_{\mathbb{R}^d\setminus B_\varepsilon}\left[e^{i\langle y,\sigma(x)^T\xi\rangle}-1-i\langle y,\sigma(x)^T\xi\rangle\right]\nu(dy) \\
&\quad+\frac{1}{2}\int_{B_\varepsilon}\langle\sigma(x)^T\xi,\sigma(x)^T\xi\rangle\langle y,y\rangle\nu(dy)\Bigg| \\
&= \left|\int_{B_\varepsilon}\left[e^{i\langle y,\sigma(x)^T\xi\rangle}-1-i\langle y,\sigma(x)^T\xi\rangle+\frac{1}{2}\langle\sigma(x)^T\xi,\sigma(x)^T\xi\rangle\langle y,y\rangle\right]\nu(dy)\right|.
\end{aligned}$$

The triangle inequality gives

$$\begin{aligned}
&\leq \int_{B_\varepsilon}\left|e^{i\langle y,\sigma(x)^T\xi\rangle}-1-i\langle y,\sigma(x)^T\xi\rangle\right|\nu(dy)+\frac{1}{2}\left|\langle\sigma(x)^T\xi,\sigma(x)^T\xi\rangle\right|\int_{B_\varepsilon}|y|^2\,\nu(dy). \\
&\leq \int_{B_\varepsilon}\left|(y,\sigma(x)^T\xi)\right|^2\nu(dy)+C_1(d)\left|(\xi,\sigma(x)^T\xi)\right|^2\int_{B_\varepsilon}|y|^2\,\nu(dy) \\
&\leq C\left[|\xi|^2\left|\sigma(x)^T\right|^2+|\xi|^2\left|\sigma(x)^T\right|^2\right]\int_{B_\varepsilon}|y|^2\,\nu(dy) \\
&\leq C\,|\xi|^2\int_{B_\varepsilon}|y|^2\,\nu(dy)\leq C\,|\xi|^2\int_{\mathbb{S}}|\eta|^2\,\lambda(d\eta)\int_0^\varepsilon r^2\frac{dr}{r^{1+\delta}}\leq C\,|\xi|^2\,\varepsilon^{2-\delta},
\end{aligned}$$

where $|y|=\sqrt{y_1^2+\cdots+y_d^2}$. The second inequality from the top of the above estimate is due to the first estimate in the proof of the Lemma 15.1.7 in [27]. Since ν is a δ-stable Lévy measure, we can apply result 14.7 in p.79 [39] to get the last estimate.

Now, let us consider $\gamma=1$. We have for $1\leq j\leq d$

$$\begin{aligned}
\left|\frac{d}{d\xi_j}[a(x,\xi)-\hat{a}_\varepsilon(x,\xi)]\right| &= \left|\frac{d}{d\xi_j}\left[\psi(\sigma(x)^T\xi)-\psi_\varepsilon(\sigma(x)^T\xi)+\frac{1}{2}\Sigma^2(\varepsilon)\langle\sigma(x)^T\xi,\sigma(x)^T\xi\rangle\right]\right| \\
&= \left|\int_{B_\varepsilon}\frac{d}{d\xi_j}\left[e^{i\langle y,\sigma(x)^T\xi\rangle}-1-i\langle y,\sigma(x)^T\xi\rangle+\frac{1}{2}\langle\sigma^T(x)\xi,\sigma(x)^T\xi\rangle\langle y,y\rangle\right]\nu(dy)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left|\int_{B_\varepsilon} \sum_{k=1}^{d} \left[i y_k \sigma_{jk}(x) \left[e^{i\langle y, \sigma(x)^T \xi\rangle} - 1 \right] + \sigma_{jk}(x) \left[\sum_{l=1}^{d} \sigma_{lk}(x) \xi_l \right] \langle y, y\rangle \right] \nu(dy) \right| \\
&\le \int_{B_\varepsilon} \sum_{k=1}^{d} \Bigg[\left| y_k \sigma_{jk}(x) \right| \left| e^{i\langle y, \sigma(x)^T \xi\rangle} - 1 - i\langle y, \sigma(x)^T \xi\rangle \right| \\
&\quad + \left| y_k \sigma_{jk}(x) \right| \left| \langle y, \sigma(x)^T \xi\rangle \right| + \sigma_{jk}(x) \left[\sum_{l=1}^{d} \sigma_{lk}(x) \xi_l \right] \langle y, y\rangle \Bigg] \nu(dy) \\
&\le \int_{B_\varepsilon} \sum_{k=1}^{d} \Bigg[\left| y_k \sigma_{jk}(x) \right| \left| \langle y, \sigma(x)^T \xi\rangle \right|^2 + \left| y_k \sigma_{jk}(x) \right| \left| \langle y, \sigma(x)^T \xi\rangle \right| \\
&\quad + \sigma_{jk}(x) \left[\sum_{l=1}^{d} \sigma_{lk}(x) \xi_l \right] \langle y, y\rangle \Bigg] \nu(dy) \\
&\le \int_{B_\varepsilon} \sum_{k=1}^{d} \left| y_k \sigma_{jk}(x) \right| \left[\left| \langle y, \sigma(x)^T \xi\rangle \right| + \left| \langle y, \sigma(x)^T \xi\rangle \right|^2 \right] \nu(dy) + |\xi| \left| \sigma(x)^T \right|^2 \int_{B_\varepsilon} |y|^2 \, \nu(dy) \\
&\le C \sum_{n=2}^{3} \int_{B_\varepsilon} |\xi|^n \, |y|^n \, \nu(dy) \le C \sum_{n=2}^{3} |\xi|^n \int_{\mathbb{S}} |\eta|^n \, \lambda(d\eta) \int_0^\varepsilon r^n \frac{dr}{r^{1+\delta}} \le C \sum_{n=2}^{3} |\xi|^n \, \varepsilon^{n-\delta}.
\end{aligned}
$$

To get the last estimate we applied the similar steps as in previous estimate. Now we estimate

$$
\left| \frac{\partial^\gamma}{\partial^{\alpha_1} \xi_1 \ldots \partial^{\alpha_d} \xi_d} \left[a(x, \xi) - \hat{a}_\varepsilon(x, \xi) \right] \right|
$$

first for $|\gamma| = 2$ and then for $2 < |\gamma| \le d + 1$. Here, we get the following sequence of calculations for $1 \le l, j \le d$

$$
\begin{aligned}
&\left| \frac{\partial^{(j,l)}}{\partial_{\xi_j} \partial_{\xi_l}} \left[a(x, \xi) - \hat{a}_\varepsilon(x, \xi) \right] \right| \\
&\quad = \left| \frac{\partial^{(j,l)}}{\partial_{\xi_j} \partial_{\xi_l}} \left[\psi(\sigma(x)^T \xi) - \psi_\varepsilon(\sigma(x)^T \xi) + \frac{1}{2} \Sigma^2(\varepsilon) \langle \sigma^T(x) \xi, \sigma^T(x) \xi \rangle \right] \right| \\
&\quad = \left| \int_{B_\varepsilon} \frac{\partial^{(j,l)}}{\partial_{\xi_j} \partial_{\xi_l}} \left[e^{i\langle y, \sigma(x)^T \xi\rangle} - 1 - i\langle y, \sigma(x)^T \xi\rangle + \frac{1}{2} \langle \sigma(x)^T \xi, \sigma(x)^T \xi \rangle \langle y, y \rangle \right] \nu(dy) \right| \\
&\quad \le \left| \int_{B_\varepsilon} \Big(\sum_{k=1}^{d} (y_k \sigma_{lk}(x)) \Big(\sum_{k=1}^{d} (y_k \sigma_{jk}(x)) \Big) e^{i\langle y, \sigma(x)^T \xi\rangle} \nu(dy) \right| + |C(\sigma, x)| \int_{B_\varepsilon} |\langle y, y\rangle| \, \nu(dy) \\
&\quad \le C \int_{B_\varepsilon} |y|^2 \, \nu(dy) + |C(\sigma, x)| \int_{B_\varepsilon} |y|^2 \, \nu(dy) \le C \int_{\mathbb{S}} |\eta|^2 \, \lambda(d\eta) \int_0^\varepsilon r^2 \frac{dr}{r^{1+\delta}} \le C \varepsilon^{2-\delta}.
\end{aligned}
$$

Now, let $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ with $m = |\gamma| \in (3, d)$. Then we get

$$
\begin{aligned}
&\left| \frac{\partial^\gamma}{\partial_{\xi_{\alpha_1}} \partial_{\xi_{\alpha_2}} \ldots \partial_{\xi_{\alpha_d}}} \left[a(x, \xi) - \hat{a}_\varepsilon(x, \xi) \right] \right| \\
&\quad = \left| \frac{\partial^\gamma}{\partial_{\xi_{\alpha_1}} \partial_{\xi_{\alpha_2}} \ldots \partial_{\xi_{\alpha_m}}} \left[\psi(\sigma(x)^T \xi) - \psi_\varepsilon(\sigma(x)^T \xi) + \frac{1}{2} \Sigma^2(\varepsilon) \langle \sigma^T(x) \xi, \sigma^T(x) \xi \rangle \right] \right| \\
&\quad = \left| \int_{B_\varepsilon} \frac{\partial^\gamma}{\partial_{\xi_{\alpha_1}} \partial_{\xi_{\alpha_2}} \ldots \partial_{\xi_{\alpha_m}}} \left[e^{i\langle y, \sigma(x)^T \xi\rangle} \right] \nu(dy) \right|
\end{aligned}
$$

$$\leq \quad C\int_{B_\varepsilon}|y|^m\nu(dy)\leq C\int_{B_\varepsilon}|y|^m\,\nu(dy)\leq C\int_{\mathbb{S}}|\eta|^m\,\lambda(d\eta)\int_0^\varepsilon r^m\frac{dr}{r^{1+\delta}}\leq C\varepsilon^{m-\delta}.$$

Observe, due to [35, Proposition 3.1.2, in p.77] we can write

$$\left[\mathcal{P}_t-\hat{\mathcal{P}}_t^\varepsilon\right]u=\int_0^t\hat{\mathcal{P}}_{t-s}^\varepsilon\left[a(x,D)-\hat{a}_\varepsilon(x,D)\right]\mathcal{P}_s u\,ds.$$

Now let $r_1,r_2\in(0,1)$ such that $r_1+r_2>1$ and $2r_1>r_2$. Then we get for $u\in B_{\infty,\infty}^{-\delta(r_1-\frac{r_2}{2})}(\mathbb{R}^d)$

$$\begin{aligned}&\left|\left[\mathcal{P}_t-\hat{\mathcal{P}}_t^\varepsilon\right]u\right|_{B_{\infty,\infty}^{\frac{\delta r_2}{2}}}\\&\leq\quad\int_0^t\left|\hat{\mathcal{P}}_{t-s}^\varepsilon\left[a(x,D)-\hat{a}_\varepsilon(x,D)\right]\mathcal{P}_s u\right|_{B_{\infty,\infty}^{\frac{\delta r_2}{2}}}\,ds.\end{aligned}$$

Since $\hat{\mathcal{P}}_{t-s}^\varepsilon$ is bounded from $B_{\infty,\infty}^\rho(\mathbb{R}^d)$ into itself we can write

$$\begin{aligned}&\left|\left[\mathcal{P}_t-\hat{\mathcal{P}}_t^\varepsilon\right]u\right|_{B_{\infty,\infty}^{\frac{\delta r_2}{2}}}\\&\leq\quad\int_0^t\left\|\hat{\mathcal{P}}_{t-s}^\varepsilon\right\|_{L\left(B_{\infty,\infty}^{-\frac{\delta r_2}{2}},B_{\infty,\infty}^{\frac{\delta r_2}{2}}\right)}\|a(x,D)-\hat{a}_\varepsilon(x,D)\|_{L\left(B_{\infty,\infty}^{\frac{\delta r_2}{2}},B_{\infty,\infty}^{-\frac{\delta r_2}{2}}\right)}\\&\times\quad\|\mathcal{P}_s\|_{L\left(B_{\infty,\infty}^{-\delta(r_1-\frac{r_2}{2})},B_{\infty,\infty}^{\frac{\delta r_2}{2}}\right)}|u|_{B_{\infty,\infty}^{-\delta(r_1-\frac{r_2}{2})}}\,ds\\&\leq\quad\|a(x,D)-\hat{a}_\varepsilon(x,D)\|_{L\left(B_{\infty,\infty}^{\frac{\delta r_2}{2}},B_{\infty,\infty}^{-\frac{\delta r_2}{2}}\right)}|u|_{B_{\infty,\infty}^{-\delta(r_1-\frac{r_2}{2})}}\int_0^t(t-s)^{-\delta r_2}s^{-\delta r_1}\,ds.\end{aligned}$$

Integrating gives

$$\begin{aligned}&\left|\left[\mathcal{P}_t-\hat{\mathcal{P}}_t^\varepsilon\right]u\right|_{B_{\infty,\infty}^{\frac{\delta r_2}{2}}}\\&\leq\quad t^{\delta(r_1+r_2)-1}\|a(x,D)-\hat{a}_\varepsilon(x,D)\|_{L\left(B_{\infty,\infty}^{\frac{\delta r_2}{2}},B_{\infty,\infty}^{-\frac{\delta r_2}{2}}\right)}|u|_{B_{\infty,\infty}^{-\delta(r_1-\frac{r_2}{2})}}\int_0^1(1-s)^{-\delta r_2}s^{-\delta r_1}\,ds\\&\leq\quad t^{\delta(r_1+r_2)-1}\|a(x,D)-\hat{a}_\varepsilon(x,D)\|_{L\left(B_{\infty,\infty}^{\frac{\delta r_2}{2}},B_{\infty,\infty}^{-\frac{\delta r_2}{2}}\right)}|u|_{B_{\infty,\infty}^{-\delta(r_1-\frac{r_2}{2})}}B(1-\delta r_1,1-\delta r_2)\\&\leq\quad Ct^{\delta(r_1+r_2)-1}|u|_{B_{\infty,\infty}^{-\delta(r_1-\frac{r_2}{2})}}.\end{aligned}$$

For the last inequality we used the fact that we have already proven (6.19). Therefore, it follows from Theorem 6.19 in [1] that for any $m\in\mathbb{R}$

$$(a(x,D)-\hat{a}_\varepsilon(x,D)):B_{\infty,\infty}^{2+m}(\mathbb{R}^d)\to B_{\infty,\infty}^m(\mathbb{R}^d)$$

is a bounded operator. Therefore we have

$$\|a(x,D)-\hat{a}_\varepsilon(x,D)\|_{L\left(B_{\infty,\infty}^{\frac{\delta r_2}{2}},B_{\infty,\infty}^{-\frac{\delta r_2}{2}}\right)}\leq C\varepsilon^{(2-\delta)}.$$

Let $\delta_1=\frac{\delta r_2}{2}$ and $\delta_2=\delta(r_1-\frac{r_2}{2})$. Rewriting above gives

$$\left|\left[\mathcal{P}_t-\hat{\mathcal{P}}_t^\varepsilon\right]u\right|_{C_b^0}\leq\left|\left[\mathcal{P}_t-\hat{\mathcal{P}}_t^\varepsilon\right]u\right|_{B_{\infty,\infty}^{\delta_1}}\leq t^{\delta(r_1+r_2)-1}\varepsilon^{(2-\delta)}|u|_{B_{\infty,\infty}^{-\delta_2}}\leq t^{\delta(r_1+r_2)-1}\varepsilon^{(2-\delta)}|u|_{C_b^0}.$$

□

7. Invertibility of pseudo–differential operators

In this section, we study under which conditions the pseudo-differential operator is invertible. To investigate the inverse of a pseudo-differential operator one has to introduce the set of elliptic and hypoelliptic symbols. For the reader's convenience, we define the elliptic and hypoelliptic symbols in this section.

We are now interested, under which condition an operator $a(x, D)$ is invertible. To be more precise, we aim to answer the following questions. Given $f \in B^s_{p,r}(\mathbb{R}^d)$, does there exists an element $u \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$a(x, D)u(x) = f(x), \quad x \in \mathbb{R}^d, \tag{7.1}$$

and to which Besov space belongs u?

The invertibility is used for giving bounds of the resolvent of an operator $a(x, D)$. Here, one is interested not only in the invertibility of $a(x, D)$ but also in the invertibility of $\lambda + a(x, D)$, $\lambda \in \rho(a(x, D))$. In particular, we are interested in the norm of the operator $[\lambda + a(x, D)]^{-1}$ uniformly for all λ belonging to the set of resolvents. However, executing a careful analysis, we can see that certain constants depend only on the first or second derivative on the symbol of $\lambda + a(x, D)$, which has the effect that this norm is independent of λ. Hence, it is necessary to introduce the additional class $\tilde{\mathcal{A}}^{m,\kappa}_{k_1,k_2;\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$.

Definition 7.1. Let ρ, δ be two real numbers such that $0 \le \rho \le 1$ and $0 \le \delta \le 1$. Let $m \in \mathbb{R}$ and $\kappa \in \mathbb{N}_0$. Let $\tilde{\mathcal{A}}^{m,\kappa}_{k_1,k_2;\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ be the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where

- ◦ $a(x, \xi)$ is k_1–times differentiable in ξ and k_2 times differentiable in x;
- ◦ for any two multi-indices α and β, with $|\alpha| \ge \kappa$, there exists $C_{\alpha,\beta}$ such that

$$\left| \partial^\alpha_{\xi'} \partial^\beta_x a(x, \xi')\Big|_{\xi'=\gamma\xi} \right| \le C_{\alpha,\beta} \langle|\gamma\xi|\rangle^{m-\rho|\alpha|} \langle|x|\rangle^{\delta|\beta|}, \quad x \in \mathbb{R}^d, \xi \in \mathcal{U}_1, \gamma \ge 1.$$

For $k_1, k_2 \in \mathbb{N}_0$, we also introduce the following semi–norm for $a \in \tilde{\mathcal{A}}^{m,\kappa}_{k_1,k_2;\rho,\delta}(\mathbb{R}^d, \mathbb{R}^d)$ by

$$\|a\|_{\tilde{\mathcal{A}}^{m,\kappa}_{k_1,k_2;\rho,\delta}} = \sup_{\kappa \le |\alpha| \le k_1, |\beta| \le k_2} \; \sup_{(x,\xi,\gamma) \in \mathbb{R}^d \times \mathcal{U}_1 \times [1,\infty)} \left| \partial^\alpha_{\xi'} \partial^\beta_x a(x, \xi')\Big|_{\xi'=\gamma\xi} \right| \langle|\gamma\xi|\rangle^{\rho|\alpha|-m} \langle|x|\rangle^{-\delta|\beta|}.$$

Now we are ready to state the main result of this section.

Remark 7.1. The outline of the proof of following theorem, i.e. Theorem (7.1) is quite similar to the proof of Theorem 5.4 in [30], however there is an important difference. We have to introduced a symbol class $\tilde{\mathcal{A}}^{m,\kappa}_{k_1,k_2;\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$, since we needed to construct the parametrix of the resolvent of $[\lambda + a(x, D)]$ of an operator $a(x, D)$, where λ belongs to $\rho(a(x, D))$ and can be quite large.

Theorem 7.1. *Let $k \ge 0$, $m \in \mathbb{R}$, $1 \le p, r < \infty$. Let $a(x, \xi)$ be a symbol such that $a \in \tilde{\mathcal{A}}^{1,1}_{2d+4,d+3;1,0}(\mathbb{R}^d \times \mathbb{R}^d) \cap Hyp^\kappa_{d+1,0;1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ for $\kappa = [k]$. Let $R \in \mathbb{N}$ such that*

$$R \ge 10 \times d \times \|a\|_{\tilde{\mathcal{A}}^{1,1}_{2d+1,d+1;1,0}} \|a\|_{Hyp^\kappa_{2d+1,0;1,0}} \tag{7.2}$$

and

$$(7.3) \qquad \langle|\gamma\xi|\rangle^\kappa \quad \le \quad \frac{|a(x,\gamma\xi)|}{|a|_{\mathcal{A}^\kappa_{0,0;1,0}}} \quad \text{for all} \quad x \in \mathbb{R}^d \quad \text{and} \quad \xi \in \mathcal{U}_1 \quad \text{with} \quad \gamma \ge R.$$

Then, there exists a bounded pseudo–differential operator $B : B^m_{p,r}(\mathbb{R}^d) \to B^m_{p,r}(\mathbb{R}^d)$ with symbol $b(x,D)$, such that

- $\{(\xi,x) \in \mathbb{R}^d \times \mathbb{R}^d : \sup_{x\in\mathbb{R}^d} b(x,\xi) > 0\} \subset \{|\xi| \le 2R\}$,
- *B has norm R^k on $B^m_{p,r}(\mathbb{R}^d)$ into itself,*
- $a(x,D) = A + B$,

and, for any given $f \in B^m_{p,r}(\mathbb{R}^d)$, the problem

$$(7.4) \qquad Au(x) = f(x), \quad x \in \mathbb{R}^d$$

has a unique solution u belonging to $B^{m+\kappa}_{p,r}(\mathbb{R}^d)$. In addition, there exists a constant $C_1 > 0$ such that for all $f \in B^m_{p,r}(\mathbb{R}^d)$ and u solving (7.4) *we have*

$$|u|_{B^{m+\kappa}_{p,r}} \le C_1 \, \|a\|_{Hyp^\kappa_{d+1,0;1,0}} |f|_{B^m_{p,r}}, \quad f \in B^m_{p,r}(\mathbb{R}^d).$$

Remark 7.2. Since $a(x,\xi)$ is elliptic, we can find a number $R > 0$ satisfying (7.2) and (7.3).

Remark 7.3. In fact, analysing the resolvent $[\lambda + a(x,D)]^{-1}$ of an operator $a(x,D)$, it will be important that in the estimate for $R > 0$ the norm of $\tilde{\mathcal{A}}^{m,1}_{k_1,k_2,\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ and not the norm of $\mathcal{A}^m_{k_1,k_2,\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ appears. As mentioned in Remark 7.1, the reason is that calculating the norm in $\tilde{\mathcal{A}}^{m,1}_{k_1,k_2,\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ the first derivative has to be taken. Therefore, the norm in $\tilde{\mathcal{A}}^{m,1}_{k_1,k_2,\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ is independent of λ.

Proof. Note, that, for convenience for the reader, we summarized several definition and results necessary for the proof in the appendix A. For simplicity, let $E = B^m_{p,r}(\mathbb{R}^d)$. Let $\chi \in C^\infty_b(\mathbb{R}^+_0)$ such that

$$\chi(\xi) = \begin{cases} 0 & \text{if } |\xi| \le 1, \\ 1 & \text{if } |\xi| \ge 2, \\ \in (0,1) & \text{if } |\xi| \in (1,2). \end{cases}$$

Let us put $\chi_R(\xi) := \chi(\xi/R)$, $\xi \in \mathbb{R}^d$. In addition, let us set

$$p_R(x,\xi) := a(x,\xi)\chi_R(\xi), \quad b(x,\xi) := a(x,\xi)(1-\chi_R(\xi)), \quad \text{and} \quad q_R(x,\xi) := \frac{1}{a(x,\xi)}\chi_R(\xi).$$

Due to the condition on R, the function $q_R(x,\xi)$ is bounded and as a symbol, it is well defined.

Let us consider the following problem: Given $f \in B^m_{p,r}(\mathbb{R}^d)$, find an element $u \in \mathcal{S}'(\mathbb{R}^d)$ such that we have

$$(7.5) \qquad p_R(x,D)u(x) = f(x), \quad x \in \mathbb{R}^d.$$

Observe, that on one hand for a solution u of (7.5) we have

$$[q(x,D)p_R(x,D)]\, u = q(x,D)f,$$

and, on the other hand, by Remark A.2, the symbol for $q(x,D)\, p_R(x,D)$ is given by

$$(q \circ p_R)(x,\xi) = q(x,\xi)p_R(x,\xi) + C(x,\xi) + \sum_{|\gamma|=\max(d-k+2,k)} \text{Os--}\iint e^{-i\langle y,\eta\rangle}\eta^\gamma r_\gamma(x,\xi,y,\eta)\,dy\,d\eta,$$

where

$$C(x,\xi) = \sum_{1\le|\rho|\le\max(d-k+1,k-1)} \partial_\xi^\rho q(x,\xi)\partial_x^\rho p_R(x,\xi) \tag{7.6}$$

and

$$r_\gamma(x,\xi,y,\eta) = \int_0^1 \left[\partial_{\xi'}^\gamma q(x',\xi')\Big|_{\substack{\xi'=\xi-\theta\eta\\ x'=x}} \partial_{x'}^\gamma p_R(x',\xi')\Big|_{\substack{\xi'=\xi\\ x'=x-y}}\right] d\theta. \tag{7.7}$$

Observe, firstly, for $\xi \in r\mathcal{U}_1$, $r \ge 1$, we have

$$\begin{aligned} &|\partial_\xi^\rho q(x,\xi)\partial_x^\rho p_R(x,\xi)| \\ \le\ & 2|a|_{\mathrm{Hyp}^\kappa_{|\rho|,0;1,0}}\langle|\xi|\rangle^{-\kappa-|\rho|}\,|p|_{\tilde{\mathcal{A}}^{\kappa,1}_{0,\rho;0,0}}\langle|\xi|\rangle^\kappa \le 2|a|_{\mathrm{Hyp}^\kappa_{|\rho|,0;1,0}}\,|p|_{\tilde{\mathcal{A}}^{\kappa,1}_{0,\rho;0,0}}\langle|\xi|\rangle^{-1}. \end{aligned} \tag{7.8}$$

Observe, secondly, that by integration by part we have

$$\text{Os--}\iint e^{-i\langle y,\eta\rangle}\eta^\gamma r_\gamma(x,\xi,y,\eta)\,dy\,d\eta = -\,\text{Os--}\iint e^{-i\langle y,\eta\rangle}\partial_\zeta^\gamma r_\gamma(x,\xi,\zeta,\eta)\Big|_{\zeta=y}\,dy\,d\eta.$$

Putting

$$m_R(x,\xi) := \sum_{|\gamma|=1}\text{Os--}\iint e^{-i\langle y,\eta\rangle}\partial_y^\gamma r_\gamma(x,\xi,y,\eta)\,dy\,d\eta,$$

one can verify that $m_R(x,\xi) \in \tilde{\mathcal{A}}^{-1,1}_{d+1,0;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$. In fact, since $a \in \tilde{\mathcal{A}}^{\kappa,1}_{2d+4,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d) \cap \mathrm{Hyp}^\kappa_{d+1,0;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$, we know by Theorem A.2 that

$$\partial_y^\gamma r_\gamma(\cdot,\cdot,x,\xi) \in \mathcal{A}^{-1}_{d+1,d+3;1,0}(\mathbb{R}^d\times\mathbb{R}^d) \cap \mathcal{A}^0_{d+1,0;1,0}(\mathbb{R}^d\times\mathbb{R}^d).$$

This can be seen by straightforward calculations. First, by the definition of the hypoelliptic norm we have for any multi-index α and $\xi \in \delta\mathcal{U}_1$, $\delta > 1$

$$\partial_\xi^{(\gamma,\alpha)}\left[\frac{1}{p_R(x,\xi)}\right] \le \|a\|_{\mathrm{Hyp}^\kappa_{|\alpha|+|\gamma|,0;1,0}}\langle|\xi|\rangle^{-\kappa-|\alpha|-|\gamma|}.$$

Next, by the definition of the norm in $\tilde{\mathcal{A}}^{\kappa,1}_{1,1;1,0}(\mathbb{R}^d\times\mathbb{R}^d)$ we have for any multi-index α

$$\partial_x^{(\gamma,\alpha)}p_R(x,\xi) \le \|a\|_{\tilde{\mathcal{A}}^{\kappa,1}_{|\alpha|+|\gamma|,1;1,0}}\langle|\xi|\rangle^\kappa, \quad \xi\in\delta\mathcal{U}_1,\ \delta>1.$$

Going back to the operator $m_R(x,D)$. By Theorem 3.13 in [1, p. 50], we can interchange the derivatives with the oscillatory integral. That is

$$\partial_\xi^\alpha m_R(x,\xi) = \iint e^{-i\langle y,\eta\rangle}\int_0^1 \partial_\xi^\alpha\left[\partial_{\xi'}^\gamma q(x',\xi')\mid_{\substack{\xi'=\xi+\theta\eta\\ x'=x}} \partial_{x'}^\gamma p_R(x',\xi')\mid_{\substack{\xi'=\xi\\ x'=x+y}}\right] d\theta\,dy\,d\eta.$$

Secondly, by the Young inequality for a product, we know for $s > 0$

$$\langle\xi+\theta\eta\rangle^{-2s} \le \langle\xi\rangle^{-s}\langle\theta\eta\rangle^{-s},$$

and by the Peetre inequality (see [1, Lemma 3.7, p. 44]), we know for $s > 0$

$$\langle \xi + \theta\eta\rangle^s \leq \langle \xi\rangle^s \langle \theta\eta\rangle^s.$$

Next, straightforward calculations gives for $s > d$

$$\int_{\mathbb{R}^d} \langle \eta\rangle^{-s}\, d\eta \leq C.$$

Using $\partial_\eta^\rho e^{-i\langle y,\eta\rangle} = (-y)^\rho e^{-i\langle y,\eta\rangle}$, where ρ is a multi-index, and integration by parts, gives

$$\begin{aligned} &\partial_\xi^\alpha m_R(x,\xi) \\ &= \sum \iint (-y)^{-\rho} e^{-i\langle y,\eta\rangle} \int_0^1 \partial_\xi^\alpha \partial_\eta^\rho \left[\partial_{\xi'}^\gamma q(x',\xi') \mid_{\substack{\xi'=\xi+\theta\eta \\ x'=x}} \partial_{x'}^\gamma p_R(x',\xi') \mid_{\substack{\xi'=\xi \\ x'=x+y}} \right] d\theta\, dy\, d\eta \\ &= \sum \iint (-y)^{-\rho} e^{-i\langle y,\eta\rangle} \int_0^1 \partial_\xi^\alpha \theta^{d+1} \left[\partial_{\xi'}^{\gamma+\rho} q(x',\xi') \mid_{\substack{\xi'=\xi+\theta\eta \\ x'=x}} \partial_{x'}^\gamma p_R(x',\xi') \mid_{\substack{\xi'=\xi \\ x'=x+y}} \right] d\theta\, dy\, d\eta. \end{aligned}$$

Here the sum runs over all multi-index of the form $(d+1, 0, \ldots, 0)$, $(0, d+1, \ldots, 0)$, $\cdots$, $(0, \ldots, d+1)$. Analysing the proof of Theorem 3.9 [1] we see that we have to estimate

$$\begin{aligned} |\partial_\xi^\alpha m_R(x,\xi)| &\leq \iint \int_0^1 \theta^{d+1} |y|^{-(d+1)} \left[\langle \xi + \theta\eta\rangle^{-(|\gamma|+|\alpha|+\kappa+(d+1))} \langle \xi\rangle^{\kappa-|\alpha|} \right] d\theta\, dy\, d\eta \\ &\leq \int \int_0^1 \theta^{d+1} \left[\langle \xi\rangle^{-|\alpha|} \langle \xi\rangle^{-\frac12(|\gamma|+\kappa+(d+1))} \langle \theta\eta\rangle^{-\frac12(|\gamma|+\kappa+(d+1))} \langle \xi\rangle^{\kappa-|\alpha|} \right] d\theta\, d\eta \\ &\leq \langle \xi\rangle^{-2|\alpha|-(d+1)} \int \int_0^1 \theta^{d+1} \langle \theta\eta\rangle^{-\frac12(|\gamma|+\kappa+(d+1))}\, d\theta\, d\eta \\ &\leq \langle \xi\rangle^{-2|\alpha|-(d+1)} \int \int_0^1 \theta \langle \eta\rangle^{-\frac12(|\gamma|+\kappa+(d+1))}\, d\theta\, d\eta. \end{aligned}$$

The calculation above gives that for $|\gamma| > d - k + 1$, the integration with respect to η and θ is finite. Taking into account Theorem A.2, we can verify for $\xi \in \delta\mathcal{U}_1$, $\delta > 1$ that

$$\begin{aligned} &\sup_{|\alpha|\leq d+1} \left| \partial_\xi^\alpha m_R(x,\xi) \right| \langle |\xi| \rangle^{|\alpha|+1} \\ &\lesssim \sup_{\substack{1\leq|\alpha|\leq d+1 \\ 1\leq|\beta|\leq 2d+1}} \left| \partial_\xi^\alpha \partial_\xi^\beta \left[q_R(x,\xi) \right] \right| \sup_{|\delta|\leq d+1} \left| \partial_x^\delta p_R(x,\xi) \right|. \end{aligned}$$

Hence, by the generalized Leibniz rule (see [1, p. 200, (A.1)]) we have

$$\|m_R\|_{\mathcal{A}^{-1}_{d+1,0;1,0}} \leq \|q_R\|_{\mathrm{Hyp}^\kappa_{2d+1,0;1,0}} \|p_R\|_{\tilde{\mathcal{A}}^{\kappa,1}_{2d+1,2d+1;1,0}},$$

from what it follows that $m_R(x, D)$ is a bounded operator with from $B^m_{p,r}(\mathbb{R}^d)$ to $B^{m+1}_{p,r}(\mathbb{R}^d)$. In addition, by the same analysis, we get

$$\|m_R\|_{\mathcal{A}^{0}_{d+1,0;1,0}} \leq \|q\|_{\mathrm{Hyp}^\kappa_{2d+1,0;1,0}} \|p_R\|_{\tilde{\mathcal{A}}^{\kappa-1,1}_{2d+1,2d+1;1,0}}.$$

Observe, that

$$\|p_R\|_{\tilde{\mathcal{A}}^{\kappa-1,1}_{2d+4,d+3;1,0}} \leq \frac{1}{R} \|p_R\|_{\tilde{\mathcal{A}}^{\kappa,1}_{2d+1,2d+1;1,0}}.$$

Therefore, analysing the symbols, $m_R(x, D)$ is a bounded operator from $B^m_{p,r}(\mathbb{R}^d)$ to $B^m_{p,r}(\mathbb{R}^d)$ having norm

$$\|m_R\|_{\mathcal{A}^0_{d+1,0;1,0}} \le \frac{1}{R}\, \|m_R\|_{\mathcal{A}^{-1}_{d+1,0;1,0}}.$$

Now, let us go back to a slightly modified problem to verify for a given $f \in B^m_{p,r}(\mathbb{R}^d)$, the regularity of u, where u solves

$$p_R(x, D)u(x) = f(x), \quad x \in \mathbb{R}^d. \tag{7.9}$$

From before, we know that

$$(q \circ p_R)(x, \xi) = q(x, \xi)p_R(x, \xi) + C(x, \xi) + m_R(x, \xi) = I + C(x, \xi) + m_R(x, \xi).$$

A careful analysis (see (7.8)) shows that $C \in \mathcal{A}^{-1}_{d+1;0;1,0}$, and

$$\|C\|_{\mathcal{A}^1_{d+1;0,1,0}} \le \frac{1}{R}|C|_{\mathcal{A}^{-1}_{d+1;0;1,0}}.$$

Due to the assumption on R we know that R is such large that

$$\left(\|C\|_{\mathcal{A}^1_{d+1;0,1,0}} + \|m_R\|_{\mathcal{A}^0_{d+1,0;1,0}}\right) \le \frac{1}{6}.$$

First, we will show that if $f \in B^m_{p,r}(\mathbb{R}^d)$, then it follows that $u \in B^m_{p,r}(\mathbb{R}^d)$. This we will proof by contradiction. Suppose u is unbounded in $B^m_{p,r}(\mathbb{R}^d)$, in particular, suppose for any $M \in \mathbb{N}$ we have $|u|_{B^m_{p,r}} \ge M$. Since, from before we know that

$$(I + C(x, D) + m_R(x, D))u = q(x, D)f,$$

we get

$$|(I + C(x, D) + m_R(x, D))u|_{B^m_{p,r}} \ge \left| |u|_{B^m_{p,r}} - \frac{1}{R}|u|_{B^m_{p,r}} \right| \ge \frac{5}{6}|u|_{B^m_{p,r}}.$$

On the other side,

$$|(I + C(x, D) + m_R(x, D))u|_{B^m_{p,r}} = |q(x, D)f|_{B^m_{p,r}} \le \|q\|_{\mathcal{A}^0_{d+1,0;1,0}} |f|_{B^m_{p,r}} < \infty,$$

which leads to a contradiction, since we assumed that for any $M \in \mathbb{N}$ we have $|u|_{B^m_{p,r}} \ge M$. Hence, we know that $u \in B^m_{p,r}(\mathbb{R}^d)$. In the next step, we will show that we have even $u \in B^{m+\kappa}_{p,r}(\mathbb{R}^d)$ and calculate its norm in this space. Using similar arguments as above, we know by Theorem A.1 and Remark A.1 that

$$|q(x, D)f|_{B^{m+\kappa}_{p,r}} \le \|q\|_{\mathcal{A}^{-\kappa}_{d+1,0;1,0}} |f|_{B^m_{p,r}}$$

Similar as in the proof of Theorem 3.24 in [1, p. 59] we define

$$\tilde{q}(x, D) := \sum_{j=0}^{k} (-1)^j (C(x, D) + m_R(x, D))^j q(x, D),$$

where

$$(C(x, D) + m_R(x, D))^j = \underbrace{(C(x, D) + m_R(x, D)) \ldots (C(x, D) + m_R(x, D))}_{j \text{ times}}.$$

Since the right-hand side is an alternating sum, it follows by the identity

$$q(x, D)p_R(x, D) = I + C(x, D) + m_R(x, D)u(x)$$

that

$$\tilde{q}(x,D)p_R(x,D) = I + (-1)^{k+1}(C(x,D)+m_R(x,D))^{k+1}. \tag{7.10}$$

On the other side, since

$$u(x) = q(x,D)f(x) - (C(x,D)+m_R(x,D))u(x),$$

we have

$$q(x,D)f(x) \;=\; q(x,D)p_R(x,D)u(x) = (I + C(x,D) + m_R(x,D))\,u(x).$$

Since $|m_R(x,\xi)|_{\mathcal{A}^0_{d+1,0;1,0}} \le \frac{1}{6}$, the sequence $\{u_N : n\in\mathbb{N}\}$ defined by

$$u_N(x) = \left(I + \sum_{k=1}^{N}(-1)^k (C(x,D)+m_R(x,D))^k\right) q(x,D)f(x),$$

is bounded and a Cauchy sequence. Therefore, there exists a u with $u_N \to u$ strongly and we can write

$$\begin{aligned}
|u|_{B^{m+\kappa}_{p,r}} &\lesssim \|q\|_{\mathrm{Hyp}^\kappa_{d+1,0;1,0}}\left(1+\sum_{k=1}^{\infty}\|C(x,D)+m_R\|^k_{\tilde{\mathcal{A}}^{-1,1}_{d+1,0;1,0}}\right)|f|_{B^m_{p,r}}\\
&\lesssim \|q\|_{\mathrm{Hyp}^\kappa_{d+1,0;1,0}}\left(1+\sum_{k=1}^{\infty}\left(\frac{1}{6}\right)^k\right)|f|_{B^m_{p,r}}\\
&\lesssim \frac{6}{5}\,\|q\|_{\mathrm{Hyp}^\kappa_{d+1,0;1,0}}|f|_{B^m_{p,r}}.
\end{aligned}$$

This gives the assertion. □

Appendix A Some important facts about pseudo–differential operators

In this section, we introduce the definitions and theorems which are necessary for our purpose. However, we suppose that the reader is familiar with the definitions already introduced in section 3.

To start, let $a(x,\xi)$ be a symbol. Clearly, $a(x,D)$ is bounded from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$. In the following Corollary we will investigate its boundedness in Sobolev spaces.

Corollary A.1. *Let $u\in H^m_2(\mathbb{R}^d)$ for all $m\in\mathbb{R}$. Then*

$$a(x,D)\,u(x) := \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}a(x,\xi)\,\hat{u}(\xi)\,d\xi,$$

is well defined with $a(x,D)$ being a pseudo-differential operator.

Proof. Let $v,\phi\in\mathcal{S}(\mathbb{R}^d)$. Then consider,

$$\begin{aligned}
(a(x,D)v,\phi)_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}a(x,\xi)\,\hat{v}(\xi)\,d\xi\overline{\phi(x)}\,dx\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix^T\xi}a(x,\xi)\overline{\phi(x)}\,dx\,\hat{v}(\xi)\,d\xi.\\
&= \int_{\mathbb{R}^d}\hat{v}(\xi)\overline{\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}\overline{a(x,\xi)}\phi(x)\,dx}\;d\xi,
\end{aligned}$$

where we use the Fubini theorem and the fact that $\phi, \hat{v} \in \mathcal{S}(\mathbb{R}^d)$. In Lemma 3.31 in [1] showed that

$$w(\xi) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} \overline{a(x,\xi)} \phi(x)\, dx \in \mathcal{S}(\mathbb{R}^d),$$

where $a(x,\xi) \in S^m_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ with $m \in \mathbb{R}$. Therefore we have,

$$(a(x,D)v, \phi)_{L^2(\mathbb{R}^d)} = (v, a^*(x,D)\phi)_{L^2(\mathbb{R}^d)},$$

such that $a^*(x,D)\phi \in \mathcal{S}(\mathbb{R}^d)$. Now let $u \in H^m_2(\mathbb{R}^d)$. There exist $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ such that (see Corollary 3.42 in [1]),

$$\lim_{n\to\infty} \langle u_n - u, \phi\rangle = 0,$$

for any $\phi \in \mathcal{S}(\mathbb{R}^d)$. Therefore due to the above facts we have

$$\lim_{n\to\infty} \langle a(x,D)u_n, \phi\rangle = \lim_{n\to\infty} \langle u_n, a^*(x,D)\phi\rangle = \langle u, a^*(x,D)\phi\rangle = \langle a(x,D)u, \phi\rangle < \infty.$$

Hence we conclude that the Fourier integral representation of $a(x,D)u$ is well defined in $H^m_2(\mathbb{R}^d)$ with $m \in \mathbb{R}$. See Theorem 3.41 in [1] as well. □

One can easily see under which conditions $a(x,D)$ is also bounded from $L^p(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$. To see it, first, observe that the operator can also be represented by a kernel of the form

$$a(x,D)f(x) = \int_{\mathbb{R}^d} k(x, x-y) f(y)\, dy, \quad x \in \mathbb{R}^d,$$

where the kernel is given by the inverse Fourier transform

$$k(x,z) = \mathcal{F}_{\xi\to z}\left[a(x,\xi)\right](z)^4.$$

Differentiation gives the following estimate

$$|k(x,z)| \le C \left|\partial^\alpha_\xi p(x,\xi)\right| |z|^{-\alpha}.$$

By this estimate and the Young inequality for convolutions, one can calculate bounds of the operator between Lebesgue spaces, like

$$|a(x,D)f|_{L^q} \le ||a||_{\mathcal{A}^0_{\gamma,0;1,0}} |f|_{L^q},$$

for $\gamma \ge d+1$. In case, we have additional regularity of the functions, or the function is a distribution, it is not that obvious. The next Theorem gives characterize the action of a pseudo–differential operator on Besov spaces.

Theorem A.1 (compare [1, Theorem 6.19, p. 164]). *Let $\kappa, m \in \mathbb{R}$, $a(x,\xi) \in S^\kappa_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ and $1 \le p, r \le \infty$. Then, $a(x,D) : B^{\kappa+m}_{p,r}(\mathbb{R}^d) \to B^m_{p,r}(\mathbb{R}^d)$ is a linear and bounded operator.*

REMARK A.1. Tracing step by step of the proof of Theorem 6.19 in [1, p. 164], one can see that for all $\kappa, m \in \mathbb{R}$, $a(x,\xi) \in S^\kappa_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ and $1 \le p, r \le \infty$ and any $k \ge d+1$ the following inequality holds

4 $\mathcal{F}_{\xi\to z}[a(x,\xi)](z) = \int_{\mathbb{R}^d} e^{-2\pi i\langle \xi,z\rangle} a(x,\xi)\, d\xi$.

$$|a(x,D)f|_{B^m_{p,r}} \leq \|a\|_{\mathcal{A}^{\kappa}_{k,0;\delta,0}} \; |f|_{B^{\kappa+m}_{p,r}} .$$

To analyse the composition of two operators of given symbols, one has to evaluate a so-called oscillatory integral. In particular, for any $\chi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi(0,0) = 1$ and $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, we define the oscillatory integral by

$$\text{Os–}\iint e^{-iy\eta} a(y,\eta)\, dy\, d\eta := \lim_{\varepsilon \to 0} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \chi(\varepsilon y, \varepsilon \eta)\, e^{-i\langle y,\eta\rangle} a(y,\eta)\, dy\, d\eta.$$

To calculate the oscillatory integral, the following Theorem is essential.

Theorem A.2 (compare [1, Theorem 3.9, p. 46])**.** *Let $m \in \mathbb{R}$, $a \in \mathcal{A}^m_{(d+1+m)\wedge 0, d+1;1,0}(\mathbb{R}^d \times \mathbb{R}^d)$, and let $\chi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi(0,0) = 1$. Then the oscillatory integral*

$$\text{Os–}\iint e^{-i\langle y,\eta\rangle} a(y,\eta)\, dy\, d\eta$$

exists and

$$\left| \text{Os–}\iint e^{-i\langle y,\eta\rangle} a(y,\eta)\, dy\, d\eta \right| \leq C_{m,d}\, \|a\|_{\mathcal{A}^m_{(d+1+m)\wedge 0, d+1;1,0}} .$$

Corollary A.2 (compare [1, Corollary 3.10, p. 48])**.** *Let $a_j \in S^m_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ be a bounded sequence in $\mathcal{A}^m_{d+1+m,d+1;\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ such that there exists some $a \in \mathcal{A}^m_{d+1+m,d+1;\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$*

$$\lim_{j\to\infty} \partial^\alpha_\eta \partial^\beta_y a_j(y,\eta) = \partial^\alpha_\eta \partial^\beta_y a(y,\eta),$$

for any $|\alpha| \leq d + m + 1$, $|\beta| \leq d + 1$, $y \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^d$. Then

$$\lim_{j\to\infty} \text{Os–}\iint e^{-i\langle y,\eta\rangle} a_j(y,\eta)\, dy\, d\eta = \text{Os–}\iint e^{-i\langle y,\eta\rangle} a(y,\eta)\, dy\, d\eta.$$

With the help of the oscillatory integral, one can show that the composition of two pseudo-differential operators is again a pseudo-differential operator. Using formal calculations, an application of the Taylor formula leads to the following characterization.

Theorem A.3 (compare [1, Theorem 3.16, p. 55])**.** *Let $a_1(x,\xi) \in S^{m_1}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ and $a_2(x,\xi) \in S^{m_2}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$. Then the composition $a_1(x,D)\, a_2(x,D)$ is again a pseudo–differential operator, whose symbol we denote by $[a_1 \circ a_2](x,\xi)$, and we have*

$$[a_1 \circ a_2](x,\xi) \in S^{m_1+m_2}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d).$$

Moreover, it can be expanded asymptotically as follows

$$[a_1 \circ a_2](x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \left(\partial^\alpha_\xi a_1(x,\xi)\right) \left(\partial^\alpha_x a_2(x,\xi)\right). \tag{A.1}$$

To be more precise, equation (A.1) *means that*

$$[a_1 \circ a_2](x,\xi) - \sum_{|\alpha|\leq N} \frac{1}{\alpha!} \left(\partial^\alpha_\xi a_1(x,\xi)\right) \left(\partial^\alpha_x a_2(x,\xi)\right) \tag{A.2}$$

belongs to $S^{m_1+m_2-N}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ for every positive integer N.

Remark A.2. Following the proof of Theorem 3.16 [1, p. 53], one observes that

$$[a_1 \circ a_2](x,\xi) - \sum_{|\alpha|\le N} \frac{1}{\alpha!}\left(\partial_\xi^\alpha a_1(x,\xi)\right)(\partial_x^\alpha a_2(x,\xi)) \tag{A.3}$$

$$= (N+1)\sum_{|\alpha|=N+1}\frac{1}{\alpha!}\text{Os–}\iint e^{-i\langle y,\eta\rangle}\eta^\alpha r_\alpha(x,\xi,y,\eta)\,dy\,d\eta \tag{A.4}$$

$$= (N+1)\sum_{|\alpha|=N+1}\frac{1}{\alpha!}\text{Os–}\iint e^{-i\langle y,\eta\rangle}D_y^\alpha r_\alpha(x,\xi,y,\eta)\,dy\,d\eta$$

with

$$r_\alpha(x,\xi,y,\eta) = \int_0^1 \left[\partial_{\xi'}^\alpha p_1(x',\xi')\,|_{\substack{\xi'=\xi+\theta\eta\\ x'=x}}\ \partial_{x'}^\alpha p_2(x',\xi')\,|_{\substack{\xi'=\xi\\ x'=x+y}}\ (1-\theta)^N\right] d\theta.$$

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