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ON THE MACKEY FORMULAS FOR CYCLOTOMIC HECKE ALGEBRAS AND CATEGORIES $\mathcal{O}$ OF RATIONAL CHEREDNIK ALGEBRAS

Dedicated to Toshiaki Shoji on the occasion of his 70th birthday.

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Abstract

In this paper, we shall establish the Mackey formulas in the following two setups:

(i) on the tensor induction and restriction functors on the modules over cyclotomic Hecke algebras (Ariki-Koike algebras) and their standard subalgebras of parabolic subgroups;

(ii) on the Bezrukavnikov-Etingof induction and restriction functors [3] among categories $\mathcal{O}$ [11] of rational Cherednik algebras for the complex reflection group of type $G(r, 1, n)$ and their parabolic subgroups.

0. Introduction

The Mackey formula [17], [7, p.273] plays a very important role in representation theory :

$$\text{(0.0.1)} \qquad \mathrm{Res}_L \circ \mathrm{Ind}^G(M) \cong \bigoplus_{w \in L\backslash G/H} \mathrm{Ind}^L \circ \mathrm{Res}_{L\cap wHw^{-1}}(w \otimes M)$$

for a finite group G, its subgroups H, L and H-module M. In modular representation theory of finite groups, Green's vertex theory is based on this formula [loc.cit].

In finite reductive groups, Dipper and Fleischmann [9, (1.14) Theorem] established the Mackey formula on the Harish-Chandra induction and restriction for Levi subgroups, and used it as an important base for their modular Harish-Chandra theory. And, also in finite reductive groups, the Mackey formula on the Deligne-Lusztig induction and restriction has a very important implication for the Lusztig conjecture on the characters on these groups which is developed by C. Bonnafé (see [4]). This is an extension of Mackey formula on the Harish-Chandra induction and restriction, although it is at the level of characters. So, the Mackey formula is subject to a subgroup lattice Λ and a family of two kinds of functors Ind_A^B and Res_A^B labeled by the pairs (A, B) with $A \subset B$ in this lattice Λ.

In this paper we shall report yet another Mackey formula for the case where Λ is a set of parabolic subgroups of a complex reflection group. More precisely, we shall tackle proving the following conjecture:

Conjecture 0.1 (The Mackey formula for $\mathcal{O}$). *For any finite complex reflection group W, and its parabolic subgroups W_a and W_b, the Mackey formula with respect to the*

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Bezrukavnikov-Etingof induction and restriction holds. More precisely, at the level of representation categories, we have the following isomorphism of functors :

$$ {}^{\mathcal{O}}\mathrm{Res}^{W}_{W_a} \circ {}^{\mathcal{O}}\mathrm{Ind}^{W}_{W_b} \cong \bigoplus_{u \in {}^aW^b} {}^{\mathcal{O}}\mathrm{Ind}^{W_a}_{W_a \cap uW_bu^{-1}} \circ u(-) \circ {}^{\mathcal{O}}\mathrm{Res}^{W_b}_{u^{-1}W_au \cap W_b}, $$

where ${}^aW^b$ is a complete set of double coset representatives of $W_a \backslash W / W_b$.

Here, ${}^{\mathcal{O}}\mathrm{Res}^{W}_{W'}$ (resp. ${}^{\mathcal{O}}\mathrm{Ind}^{W}_{W'}$) is the Bezrukavnikov-Etingof restriction (resp. induction) functor [3] and $u(-)$ is the functor naturally induced by a conjugation (automorphism) by $u \in W$.

We write $W_{n,r}$ for the complex reflection group of type $G(r, 1, n)$ in Shephard-Todd notation. In this paper, we shall study the Mackey formula for the cyclotomic Hecke algebra $\mathscr{H}_{n,r} = \langle T_0, T_1, \dots, T_{n-1} \rangle$ of type $G(r, 1, n)$, also known as the Ariki-Koike algebra (see 3.1 for the precise definition) * and the categories $\mathcal{O}$ of cyclotomic rational Cherednik algebras associated with $W_{n,r}$ (in so-called $t = 1$ case) and establish the Mackey formulas in the following two set ups:

(i) Λ is the set of standard parabolic subgroups of $W_{n,r}$ and Ind^B_A is the tensor induction functor and Res^B_A is the restriction functor between Hecke algebras associated with A and B in Λ.

(ii) Λ is the set of parabolic subgroups of $W_{n,r}$. The induction and restriction are the Bezrukavnikov-Etingof induction ${}^{\mathcal{O}}\mathrm{Ind}$ and restriction ${}^{\mathcal{O}}\mathrm{Res}$ [3] respectively among categories $\mathcal{O}$ of cyclotomic rational Cherednik algebras for the complex reflection group $W_{n,r}$ and their parabolic subgroups.

The precise statement of (i) is Theorem 3.12. The precise statement of (ii) is Theorem 4.10, which supports Conjecture 0.1. The part (i) is given in a characteristic-free manner, even holds over $\mathbb{Z}[q, q^{-1}, Q_1, \dots, Q_r]$, where $q, Q_1, \dots, Q_r$ are indeterminate over $\mathbb{Z}$. On the contrary, the part (ii) heavily depends on the coefficient field $\mathbb{C}$, due to the use of KZ-functor, Riemann-Hilbert correspondence. In particular, the Mackey formulas for $\mathcal{O}$ do not imply the Mackey formulas for the Ariki-Koike algebras over the field with a positive characteristic. Rather, we use (i) to prove (ii). So, the statement of (i) is stronger than the statement of (ii) in this sense. Also, nowadays, the representation theory of Ariki-Koike algebras is an independent research area (e.g. [1]). So, if one is only interested in Ariki-Koike algebras, one may only read the proof for the statement of (i), which has not been known since the birth of Ariki-Koike algebras. It is well known that Ariki-Koike algebras is very strongly related to affine Hecke algebras of type GL as cyclotomic quotients. Indeed, any finite dimensional indecomposable module over a fixed affine Hecke algebra of type GL is a module for some Ariki-Koike algebra. So, via affine Hecke algebras, our result also goes to the representation theory of p-adic groups. So, we have an application to classical subjects. Next, we make remarks on the subgroup lattice Λ: Let W be a complex reflection group, and let $\mathfrak{h}$ be the reflection $\mathbb{C}$-representation of W. By a parabolic subgroup of W, we mean a stabilizer, in W, of some point in $\mathfrak{h}$. We mean by a standard parabolic subgroup of W a special parabolic subgroup $\langle I \rangle$ of W for some subset I of the set of simple reflections.

Very briefly we remark some known results related to the above (i) and (ii): In [15, 2.29], the Mackey formula on the 1-parameter Iwahori-Hecke algebras can be found. In

*The Ariki-Koike algebra is in an imprimitive class of the cyclotomic Hecke algebras.

[23], the Mackey formula on the cyclotomic Hecke algebras for the maximal co-rank 1 cases are treated, namely, it is with respect to two identical subgroups $W_a = W_{n-1,r}$ and $W_b = W_{n-1,r}$ of $W_{n,r}$. Since in our set up we can take any two standard parabolic subgroups, the part (i) is a strong generalization of her result. In [21, Lemma 2.5], at the level of the Grothendieck group the part (ii) is considered. However, this is a consequence of Mackey's original formula (0.0.1). Indeed, the KZ functor commutes with inductions and restriction in $\mathcal{O}$ and cyclotomic Hecke algebras. So, at the level of Grothendieck groups, the branching rule for (co)standard modules in $\mathcal{O}$ in terms of (co)standard modules is identical with the rule for Specht modules over (tensor products of) Ariki-Koike algebras in terms of Specht modules. And, moreover, at the level of Grothendieck groups, the rule on Ariki-Koike algebras depends only on the choices of parabolic subgroups. Therefore, one may take group algebras of complex reflection groups to detect the rule in question. † In [16, Theorem 2.7.2], they established Mackey formula for the categories $\mathcal{O}$ of rational Cherednik algebras of Coxeter groups.

In the case where W is a finite Coxeter group, to obtain the Mackey formula for corresponding Hecke algebras, we discuss by using reduced expressions of group elements, the distinguished minimal coset representatives and their properties. However, in the case where W is a complex reflection group which is not a Coxeter group, we do not have enough properties for reduced expressions of group elements, and we do not know a good choice of coset representatives. These lacks of theory for complex reflection groups cause difficulty to obtain the Mackey formula for cyclotomic Hecke algebras. In this paper, we give a solution of this problem for complex reflection groups of type $G(r, 1, n)$.

Regarding applications, as in first paragraph, the role of the Mackey formula in rational Cherednik algebras similar to the one in [9, 10] is expected. And, as the Bezrukavnikov-Etingof induction functor sends projective resolutions in $\mathcal{O}$ to projective resolutions, an obvious application is for a study of cohomology groups $\mathrm{Ext}^i_{\mathcal{O}(W)}({}^{\mathcal{O}}\mathrm{Ind}^W_{W_a}(M), {}^{\mathcal{O}}\mathrm{Ind}^W_{W_b}(N))$ via Eckmann-Shapiro lemma

$$(0.1.1)\quad \mathrm{Ext}^i_{\mathcal{O}(W)}({}^{\mathcal{O}}\mathrm{Ind}^W_{W_a}(M), {}^{\mathcal{O}}\mathrm{Ind}^W_{W_b}(N)) \cong \bigoplus_{u\in {}^aW^b} \mathrm{Ext}^i_{\mathcal{O}(W_a)}(M, {}^{\mathcal{O}}\mathrm{Ind}^{W_a}_{W_a\cap uW_bu^{-1}} \circ u(-) \circ {}^{\mathcal{O}}\mathrm{Res}^{W_b}_{u^{-1}W_au\cap W_b} N).$$

Here, $\mathcal{O}(W')$ is the category $\mathcal{O}$ for a complex reflection group W' defined in [11]. Especially, it is useful to study the endomorphism ring of an induced module. When $i = 0$, $W_a = W_b \neq W$ at (0.1.1), finding a basis of the right hand side of (0.1.1) is easier than that of the left hand side of (0.1.1). For a parabolic subgroup W_b of W with X being finite dimensional simple in $\mathcal{O}(W_b)$, the endomorphism ring $\mathrm{End}_{\mathcal{O}(W)}({}^{\mathcal{O}}\mathrm{Ind}^W_{W_b}(X))$ is studied in [16]. They call it a generalized Hecke algebra (see [16, Theorem 3.2.4, Definition 3.2.5]). Their strategy is very traditional like [12, 13], Harish-Chandra philosophy, inducing cuspidals and decompose them by the endomorphism rings, but tactics is new, such as geometrical properties of the categories $\mathcal{O}$. In the case where W is a Coxeter group, they obtained an explicit description of the generalized Hecke algebra. In their argument, Mackey formula has an important role to detect the explicit rank of endomorphism ring of an induced cuspidal module: For

†Another way to detect the rule is to deform the Cherednik algebra in question to another algebra so that its $\mathcal{O}$ is semisimple.

$i = 0, a = b, X = M = N$, on the right hand side of (0.1.1) one may only need to take the sum over $u \in {}^bW^b \cap N_W(W_b)$ since the restriction of X to $\mathcal{O}$ on any proper parabolic subgroup of W_b is zero. We may follow their arguments to calculate the rank of a generalized Hecke algebra for a complex reflection group $W_{n,r}$.

This paper is organized as follows.

In §1, we review some known facts on symmetric groups, and also give a technical result (Lemma 1.3) which shall be used in §2.

In §2, we shall determine a complete set of representatives of double cosets $W_1\backslash W_{n,r}/W_2$ over two standard parabolic subgroups W_1, W_2 of $W_{n,r}$. Throughout this paper, we use a expression of elements of $W_{n,r}$ being along the semidirect product $W_{n,r} = \mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$. This expression will be used in §3 to construct a basis of the cyclotomic Hecke algebra associated with $W_{n,r}$, so called Ariki-Koike basis. This basis is not standard. (By a standard basis, we mean a basis which are labeled by a group W and does not depend on the choice of reduced expressions in terms of a specific set of generator of W.) Our coset representatives are compatible with this expression, and they have a good behavior in the arguments for Hecke algebras. One of important properties of our coset representatives appears in Proposition 2.13. In this proposition, for each our representative u of $W_1\backslash W_{n,r}/W_2$, we prove that the subgroup $W_1 \cap uW_2u^{-1}$ is a standard parabolic subgroup of $W_{n,r}$. An advantage for taking a not only parabolic but also standard one is that we can find the associated subalgebra in the cyclotomic Hecke algebra. As another important property of our coset representatives, we may construct a slightly new basis $\{\widetilde{T}_w\}$ of the Ariki-Koike algebra by multiplying the Ariki-Koike basis $\{T_w\}$. This basis, *a priori*, depends on two standard parabolic subgroups W_1 and W_2. We remark that our coset representatives are not the distinguished minimal coset representatives in the case where $r = 2$ (i.e. $W_{n,r}$ is Weyl group of type B_n). Thus, our representatives are not a generalization of the distinguished minimal coset representatives for finite Coxeter groups.

In §3, we shall establish the Mackey formula for Ariki-Koike algebras (cyclotomic Hecke algebras of type $G(r, 1, n)$) in Theorem 3.12.

In §4, we discuss the Mackey formula for the categories $\mathcal{O}$ of rational Cherednik algebras associated with parabolic subgroups of $W_{n,r}$. By using lifting argument which has been employed in [16, Theorem 2.7.2] for Coxeter group case, one can lift the Mackey formula for the Hecke algebras to the Mackey formula for the categories $\mathcal{O}$. The Mackey formula for the categories $\mathcal{O}$ is given in Proposition 4.8 and Theorem 4.10. We remark that although we lack standard basis in the Ariki-Koike algebra, we may make the desired lifting thanks to a good property of our coset representatives.

In Appendix B, we compare known results on the coset representatives in [20] with the ones in §2 for some special cases (i.e. the case where $\mu = (1^{n-l})$ and $\nu = (1^{n-m})$) as a sort of independent interest. We remark that the coset representatives in [20] follows from notion of root systems of type $G(r, 1, n)$. However, in [20], they give the coset representatives only for special cases where $\mu = (1^{n-l})$ and $\nu = (1^{n-m})$, and we do not know whether we can obtain the coset representatives by using root systems in general. We also remark that our coset representatives are not generalization of ones in [20] (see Remark B.9). For the reader being only interested in the Mackey formula, he or she can skip this appendix.

1. The symmetric groups

In this section, we review some known results on symmetric groups which follow from the general theory of Coxeter groups (see e.g. [14], [8, Chapter 4]) except (1.2.1) and Lemma 1.3.

1.1. Let $\mathfrak{S}_n$ be the symmetric group on n letters. We consider the natural left action of $\mathfrak{S}_n$ on $\{1, 2, \dots, n\}$. So, when $x \in \mathfrak{S}_n$ sends i to j, we denote it by $x(i) = j$. For $i = 1, 2, \dots, n-1$, let $s_i = (i, i+1)$ be the adjacent transposition. Then $S = \{s_1, s_2, \dots, s_{n-1}\}$ is a set of simple reflections of $\mathfrak{S}_n$. For $x \in \mathfrak{S}_n$, we denote the length of x by $\ell(x)$. We denote the Bruhat order on $\mathfrak{S}_n$ by $\geq$.

For integers $k_1 \leq k_2 \in \mathbb{Z}$, we denote the interval $\{k_1, k_1+1, \dots, k_2\}$ in $\mathbb{Z}$ by $[k_1, k_2]$. For $1 \leq k_1 \leq k_2 \leq n$, we denote by $\mathfrak{S}_{[k_1,k_2]}$ the subgroup of $\mathfrak{S}_n$ generated by $\{s_{k_1}, s_{k_1+1}, \dots, s_{k_2-1}\}$, namely $\mathfrak{S}_{[k_1,k_2]}$ is the subgroup permuting the set $\{k_1, k_1+1, \dots, k_2\}$.

A composition of n is a sequence of non-negative integers $\mu = (\mu_1, \mu_2, \dots)$ such that $\sum_i \mu_i = n$, and we denote it by $\mu \vDash n$. We also denote $|\mu| = \sum_i \mu_i$.

For $\mu = (\mu_1, \mu_2, \dots, \mu_l) \vDash n$, let $\mathfrak{S}_\mu$ be the standard parabolic subgroup of $\mathfrak{S}_n$ associated with μ, namely $\mathfrak{S}_\mu$ is the subgroup of $\mathfrak{S}_n$ generated by the reflections

$$S_\mu := S \setminus \{s_j \mid j = \sum_{i=1}^{k} \mu_i \text{ for some } k \geq 1\}.$$

We have $\mathfrak{S}_\mu \cong \mathfrak{S}_{\mu_1} \times \dots \times \mathfrak{S}_{\mu_l}$. For $\mu \vDash n$, put

$$\mathfrak{S}^\mu = \{x \in \mathfrak{S}_n \mid \ell(xs) > \ell(x) \text{ for all } s \in S_\mu\},$$
$${}^\mu\mathfrak{S} = \{x \in \mathfrak{S}_n \mid \ell(sx) > \ell(x) \text{ for all } s \in S_\mu\},$$

then $\mathfrak{S}^\mu$ (resp. ${}^\mu\mathfrak{S}$) is the set of distinguished coset representatives of the coset $\mathfrak{S}_n/\mathfrak{S}_\mu$ (resp. $\mathfrak{S}_\mu \backslash \mathfrak{S}_n$). In particular, we have

(1.1.1)
$$\ell(xy) = \ell(x) + \ell(y) \text{ for } x \in \mathfrak{S}^\mu, y \in \mathfrak{S}_\mu,$$
$$\ell(xy) = \ell(x) + \ell(y) \text{ for } y \in {}^\mu\mathfrak{S}, x \in \mathfrak{S}_\mu.$$

For $\mu, \nu \vDash n$, put ${}^\mu\mathfrak{S}^\nu = {}^\mu\mathfrak{S} \cap \mathfrak{S}^\nu$, then ${}^\mu\mathfrak{S}^\nu$ is a complete set of representatives of the double cosets $\mathfrak{S}_\mu \backslash \mathfrak{S}_n / \mathfrak{S}_\nu$.

For $x \in {}^\mu\mathfrak{S}^\nu$, let $\tau(x) \vDash n$ be the composition determined by the equation $S_{\tau(x)} = S_\mu \cap xS_\nu x^{-1}$. Then it is known that $\mathfrak{S}_\mu \cap x\mathfrak{S}_\nu x^{-1}$ is generated by $S_{\tau(x)}$. In particular, we have $\mathfrak{S}_\mu \cap x\mathfrak{S}_\nu x^{-1} = \mathfrak{S}_{\tau(x)}$. By the general theory of Coxeter groups, we see that $w \in \mathfrak{S}_n$ is uniquely written as $w = yxz$ ($x \in {}^\mu\mathfrak{S}^\nu$, $y \in (\mathfrak{S}_\mu)^{\tau(x)}$, $z \in \mathfrak{S}_\nu$), and we have

(1.1.2)
$$\ell(yxz) = \ell(y) + \ell(x) + \ell(z) \quad (x \in {}^\mu\mathfrak{S}^\nu,\ y \in (\mathfrak{S}_\mu)^{\tau(x)},\ z \in \mathfrak{S}_\nu).$$

1.2. The distinguished coset representatives $\mathfrak{S}^\mu$ (resp. ${}^\mu\mathfrak{S}$) is described by a standard combinatorics as follows. For $\mu \vDash n$, the diagram of μ is the set $[\mu] = \{(i, j) \in \mathbb{Z}^2_{\geq 0} \mid i \geq 1, 1 \leq j \leq \mu_i\}$. Here, we take the English fashion for treating the element of $[\mu]$, for example, we say that there are μ_i boxes in the i-th row of $[\mu]$, we also say that $(i, 1)$ is the left most box of the i-th row if $(i, 1) \in [\mu]$, etc. For $\mu \vDash n$, a μ-tableau is a bijection $\mathfrak{t} : [\mu] \to \{1, 2, \dots, n\}$. The symmetric group $\mathfrak{S}_n$ acts on the set of μ-tableaux from left by permuting the entries inside a given tableau, namely, for $x \in \mathfrak{S}_n$ and μ-tableau $\mathfrak{t}$,

$$(x \cdot \mathfrak{t})(i, j) = x(\mathfrak{t}(i, j)) \quad ((i, j) \in [\mu]).$$

We say that a μ-tableau $\mathfrak{t}$ is row-standard if $\mathfrak{t}(i, j) < \mathfrak{t}(i, j+1)$ for all $(i, j) \in [\mu]$ such that $(i, j+1) \in [\mu]$, namely if the entries in $\mathfrak{t}$ increase from left to right in each row.

For $\mu \vDash n$, let $\mathfrak{t}^\mu$ be the μ-tableau such that the bijection $[\mu] \to \{1, \dots, n\}$ is given by

$$\mathfrak{t}^\mu(i, j) = \sum_{k=1}^{i-1} \mu_k + j \quad ((i, j) \in [\mu]).$$

Then, we have

$$\begin{aligned} \mathfrak{S}^\mu &= \{x \in \mathfrak{S}_n \mid x \cdot \mathfrak{t}^\mu \text{ is row-standard}\}, \\ {}^\mu\mathfrak{S} &= \{x \in \mathfrak{S}_n \mid x^{-1} \cdot \mathfrak{t}^\mu \text{ is row-standard}\} \end{aligned} \tag{1.2.1}$$

(see [18, Proposition 3.3]).

For a convenience in later arguments, for $0 \le l \le n$ and $\mu \vDash n - l$, we put $(l, \mu) = (l, \mu_1, \mu_2, \dots) \vDash n$. Then, we have the following lemma:

Lemma 1.3. *For $x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)}$ for some $0 \le l, m \le n$, $\mu \vDash n - l$, $\nu \vDash n - m$, put*

$$c = \min\{i \ge 0 \mid x(i+1) \ne i+1 \text{ or } i = n\} \text{ and } k = \min\{c, l, m\}.$$

Then we have $x \in \mathfrak{S}_{[k+1,n]}$ and $[1, l] \cap \{x(1), x(2), \dots, x(m)\} = [1, k]$.

Proof. If $c \ge \min\{l, m\}$, it is clear. Suppose $c < \min\{l, m\}$ (note that $k = c$ in this case), we have

$$x(c) = c < c + 1 < x(c+1) < x(c+2) < \cdots < x(m) \tag{1.3.1}$$

and there exists $b > m$ such that $x(b) = c + 1$ since $x \cdot \mathfrak{t}^{(m,\nu)}$ is row-standard by (1.2.1).

If $x(c+1) > l$, then $x \in \mathfrak{S}_{[c+1,n]}$ and $[1, l] \cap \{x(1), \dots, x(m)\} = [1, c]$ by (1.3.1).

If $x(c+1) \le l$, we have $c + 1 < x(c+1) \le l$. Then we deduce that both $c + 1$ and $x(c+1)$ appear in the first row of $\mathfrak{t}^{(l,\mu)}$. On the other hand, we have

$$x^{-1}(c+1) = b > m \ge c + 1 = x^{-1}(x(c+1)).$$

This contradicts that $x^{-1} \cdot \mathfrak{t}^{(l,\mu)}$ is row-standard. Thus this case does not occur. □

2. The complex reflection group of type $G(r, 1, n)$

In this section, we study the complex reflection group $W_{n,r}$ of type $G(r, 1, n)$. For standard parabolic subgroups $W_{(l,\mu)}$ and $W_{(m,\nu)}$ of $W_{n,r}$, we shall find a complete set of representatives of the cosets $W_{n,r}/W_{(l,\mu)}$ and the double cosets $W_{(l,\mu)}\backslash W_{n,r}/W_{(m,\nu)}$. These representatives will be used in the next section to obtain the Mackey formula for cyclotomic Hecke algebras.

2.1. The complex reflection group of type $G(r, 1, n)$ is the semidirect product $W_{n,r} = \mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, where $\mathfrak{S}_n$ acts on $(\mathbb{Z}/r\mathbb{Z})^n$ via the permutation of factors. The group $W_{n,r}$ has a presentation such that $W_{n,r}$ is generated by $s_0, s_1, \dots, s_{n-1}$ subject to the defining relations

$$s_0^r = 1,\ s_i^2 = 1\ (1 \le i \le n-1),$$
$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} (1 \le i \le n-2),\ s_i s_j = s_j s_i\ (|i - j| > 1).$$

The relations in the second row are called the braid relations. Put

$$t_i = s_{i-1}s_{i-2}\dots s_1 s_0 s_1 \dots s_{i-2}s_{i-1}$$

for $i = 1, 2, \dots, n$. Then $S = \{s_1, s_2, \dots, s_{n-1}\}$ generates $\mathfrak{S}_n$, and t_i generates $\mathbb{Z}/r\mathbb{Z}$, the i-th factor of $(\mathbb{Z}/r\mathbb{Z})^n$. Then we have

$$W_{n,r} = \{xt_1^{a_1}t_2^{a_2}\dots t_n^{a_n} \mid x \in \mathfrak{S}_n,\ a_1, a_2, \dots, a_n \in [0, r-1]\}.$$

From the definitions, we have the following relations

$$\begin{aligned} &t_i t_j = t_j t_i \quad (1 \le i, j \le n), \\ &x t_i x^{-1} = t_{x(i)} \quad (x \in \mathfrak{S}_n,\ 1 \le i \le n). \end{aligned} \tag{2.1.1}$$

2.2. For $0 \le l \le n$ and $\mu \vDash n-l$, let $W_{(l,\mu)}$ be the subgroup of $W_{n,r}$ generated by

$$X_{(l,\mu)} = \{s_0, s_1, \dots, s_{n-1}\} \setminus \{s_j \mid j = l + \sum_{i=0}^{k} \mu_i \text{ for some } k \ge 0\},$$

where we put $\mu_0 = 0$. It is well-known that any parabolic subgroup of $W_{n,r}$ is conjugate to $W_{(l,\mu)}$ for some $0 \le l \le n$ and $\mu \vDash n-l$.

Set

$$S_{(l,\mu)} = X_{(l,\mu)} \cap S, \quad S_{(l)} = \{s_1, \dots, s_{l-1}\}, \quad S^{[l]}_\mu = S_{(l,\mu)} \setminus S_{(l)},$$

where we put $S_{(l)} = \emptyset$ if $l \le 1$. We easily see that

- the subgroup generated by $\{s_0, s_1, \dots, s_{l-1}\}$ is $W_{l,r}$, where we put $W_{l,r} = 1$ if $l = 0$,
- the subgroup generated by $S_{(l,\mu)}$ (resp. $S_{(l)}$) is the parabolic subgroup $\mathfrak{S}_{(l,\mu)}$ (resp. $\mathfrak{S}_{(l)}$) of $\mathfrak{S}_n \subset W_{n,r}$ associated with (l,μ) (resp. (l)),
- the subgroup generated by $S^{[l]}_\mu$ is the parabolic subgroup $\mathfrak{S}^{[l]}_\mu$ of $\mathfrak{S}_{[l+1,n]}$ associated with μ.

Note that $\mathfrak{S}^{[l]}_\mu$ is contained in the centralizer of $W_{l,r}$, we have

$$W_{(l,\mu)} = W_{l,r} \times \mathfrak{S}^{[l]}_\mu \cong (\mathfrak{S}_l \ltimes (\mathbb{Z}/r\mathbb{Z})^l) \times \mathfrak{S}_\mu,$$

and

$$W_{(l,\mu)} = \{xt_1^{a_1}t_2^{a_2}\dots t_l^{a_l} \mid x \in \mathfrak{S}_{(l,\mu)},\ a_1, a_2, \dots, a_l \in [0, r-1]\}.$$

Set

$$\begin{aligned} W^{(l,\mu)} &= \{xt_{l+1}^{a_{l+1}}t_{l+2}^{a_{l+2}}\dots t_n^{a_n} \mid x \in \mathfrak{S}^{(l,\mu)},\ a_{l+1}, a_{l+2}, \dots, a_n \in [0, r-1]\}, \\ {}^{(l,\mu)}W &= \{t_n^{a_n}\dots t_{l+2}^{a_{l+2}}t_{l+1}^{a_{l+1}}x \mid x \in {}^{(l,\mu)}\mathfrak{S},\ a_{l+1}, a_{l+2}, \dots, a_n \in [0, r-1]\}. \end{aligned}$$

Lemma 2.3. *The set $W^{(l,\mu)}$ (resp. ${}^{(l,\mu)}W$) is a complete set of representatives of the coset $W_{n,r}/W_{(l,\mu)}$ (resp. $W_{(l,\mu)}\backslash W_{n,r}$).*

Proof. We prove only the claim for $W^{(l,\mu)}$ since the claim for ${}^{(l,\mu)}W$ is proven in a similar way.

For $w = xt_1^{a_1}t_2^{a_2}\dots t_n^{a_n} \in W_{n,r}$ ($x \in \mathfrak{S}_n, a_1, \dots, a_n \in [0, r-1]$), we can write

$$x = x_1x_2 \quad (x_1 \in \mathfrak{S}^{(l,\mu)},\ x_2 \in \mathfrak{S}_{(l,\mu)}) \text{ and } x_2 = y_1y_2 \quad (y_1 \in \mathfrak{S}_{(l)},\ y_2 \in \mathfrak{S}^{[l]}_\mu).$$

Note that $y_1 \in \mathfrak{S}_{[1,l]}$ and $y_2 \in \mathfrak{S}_{[l+1,n]}$, the relations (2.1.1) imply that

$$\begin{aligned} w &= xt_1^{a_1}t_2^{a_2}\dots t_n^{a_n} \\ &= x_1y_1y_2t_1^{a_1}t_2^{a_2}\dots t_n^{a_n} \\ &= x_1(t_{y_1(1)}^{a_1}\dots t_{y_1(l)}^{a_l})(t_{y_2(l+1)}^{a_{l+1}}\dots t_{y_2(n)}^{a_n})y_1y_2 \\ &= x_1(t_{y_2(l+1)}^{a_{l+1}}\dots t_{y_2(n)}^{a_n})(t_{y_1(1)}^{a_1}\dots t_{y_1(l)}^{a_l})y_1y_2 \\ &= x_1(t_{y_2(l+1)}^{a_{l+1}}\dots t_{y_2(n)}^{a_n})y_1y_2(t_1^{a_1}\dots t_l^{a_l}) \\ &= x_1(t_{y_2(l+1)}^{a_{l+1}}\dots t_{y_2(n)}^{a_n})x_2(t_1^{a_1}\dots t_l^{a_l}), \end{aligned}$$

and we see that $x_1(t_{y_2(l+1)}^{a_{l+1}}\dots t_{y_2(n)}^{a_n}) \in W^{(l,\mu)}$ and $x_2(t_1^{a_1}\dots t_l^{a_l}) \in W_{(l,\mu)}$. Thus, we have

$$W_{n,r} = \bigcup_{u\in W^{(l,\mu)}} uW_{(l,\mu)}. \tag{2.3.1}$$

On the other hand, note that $|W_{n,r}| = |\mathfrak{S}_n|r^n$, $|W_{(l,\mu)}| = |\mathfrak{S}_{(l,\mu)}|r^l$ and $|W^{(l,\mu)}| = |\mathfrak{S}^{(l,\mu)}|r^{n-l}$, and thus

$$[W_{n,r} : W_{(l,\mu)}] = |W_{n,r}|/|W_{(l,\mu)}| = (|\mathfrak{S}_n|/|\mathfrak{S}_{(l,\mu)}|)r^{n-l} = |\mathfrak{S}^{(l,\mu)}|r^{n-l} = |W^{(l,\mu)}|. \tag{2.3.2}$$

Then, (2.3.1) and (2.3.2) imply the desired result for $W^{(l,\mu)}$. □

Remark 2.4. In the case where $r = 2$, the group $W_{n,2}$ is the Weyl group of type B_n. In this case, $W^{(l,\mu)}$ (resp. ${}^{(l,\mu)}W$) is not the set of distinguished coset representatives in general. For example, take $l = 0$ and $\mu \vDash n$ such that $\mu_1 > 2$. Then $W_{(0,\mu)}$ is generated by S_μ. In this case, $s_1 \in S_\mu$, and $t_2 \in W^{(l,\mu)}$. However, we have $\ell(t_2) = \ell(s_1s_0s_1) = 3$ and $\ell(t_2s_1) = \ell(s_1s_0) = 2$. Thus, t_2 is not a distinguished coset representative.

2.5. For $x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)}$ $(0 \le l, m \le n, \mu \vDash n-l, \nu \vDash n-m)$, put

$$I(x) = [m+1, n] \cap \{x^{-1}(l+1), x^{-1}(l+2), \dots, x^{-1}(n)\}.$$

For $xt_{m+1}^{a_{m+1}}\dots t_n^{a_n} \in W^{(m,\nu)}$ $(x \in \mathfrak{S}^{(m,\nu)})$, we have $xt_{m+1}^{a_{m+1}}\dots t_n^{a_n} = t_{x(m+1)}^{a_{m+1}}\dots t_{x(n)}^{a_n}x$ by (2.1.1). Thus we deduce that $xt_{m+1}^{a_{m+1}}\dots t_n^{a_n} \in {}^{(l,\mu)}W \cap W^{(m,\nu)}$ if and only if $x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)}$ and $x(k) \in [l+1, n]$ for $k \in [m+1, n]$ whenever $a_k \neq 0$. This implies that

$${}^{(l,\mu)}W \cap W^{(m,\nu)} = \{x\prod_{i\in I(x)} t_i^{a_i} \mid x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)},\ a_i \in [0, r-1]\}.$$

For $x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)}$, recall that $\tau(x)$ is the composition such that

$$S_{\tau(x)} = S_{(l,\mu)} \cap xS_{(m,\nu)}x^{-1}, \tag{2.5.1}$$

and we have $\mathfrak{S}_{\tau(x)} = \mathfrak{S}_{(l,\mu)} \cap x\mathfrak{S}_{(m,\nu)}x^{-1}$.

For $z = xyx^{-1} \in \mathfrak{S}_{\tau(x)}$ $(z \in \mathfrak{S}_{(l,\mu)}, y \in \mathfrak{S}_{(m,\nu)})$, we see that $y(i) \in [m+1, n]$ if $i \in [m+1, n]$ since $y \in \mathfrak{S}_{(m,\nu)}$. We also obtain

$$y(i) \in \{x^{-1}(l+1), \dots, x^{-1}(n)\} \text{ if } i \in \{x^{-1}(l+1), \dots, x^{-1}(n)\}$$

since $yx^{-1}(l+j) = x^{-1}z(l+j)$ and $z \in \mathfrak{S}_{(l,\mu)}$. These imply that

(2.5.2) $$y(i) \in I(x) \text{ if } i \in I(x)$$

for $z = xyx^{-1} \in \mathfrak{S}_{\tau(x)}$. For $x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)}$, put

$$^{(l,\mu)}W \cap W^{(m,\nu)}(x) = \{x \prod_{i \in I(x)} t_i^{a_i} \mid a_i \in [0, r-1]\}.$$

Then, we have

$$^{(l,\mu)}W \cap W^{(m,\mu)} = \bigcup_{x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)}} {}^{(l,\mu)}W \cap W^{(m,\mu)}(x).$$

Thanks to (2.5.2), we can define an action of $\mathfrak{S}_{\tau(x)}$ on ${}^{(l,\mu)}W \cap W^{(m,\nu)}(x)$ by

(2.5.3) $$z \odot (x \prod_{i \in I(x)} t_i^{a_i}) = x \prod_{i \in I(x)} t_{y(i)}^{a_i}$$

for $z = xyx^{-1} \in \mathfrak{S}_{\tau(x)}$. We remark that, for $z = xyx^{-1} \in \mathfrak{S}_{\tau(x)}$, we have

(2.5.4) $$z \odot (x \prod_{i \in I(x)} t_i^{a_i}) = x \prod_{i \in I(x)} t_{y(i)}^{a_i} = z(x \prod_{i \in I(x)} t_i^{a_i}) y^{-1},$$

where $z \in \mathfrak{S}_{(l,\mu)}$ and $y \in \mathfrak{S}_{(m,\nu)}$. Thus, for $z \in \mathfrak{S}_{\tau(x)}$ and $u \in {}^{(l,\mu)}W \cap W^{(m,\nu)}(x)$, the elements u and $z \odot u$ belong to the same $(W_{(l,\mu)}, W_{(m,\nu)})$-double coset.

For $u \in {}^{(l,\mu)}W \cap W^{(m,\mu)}(x)$, let $O(u) = \{z \odot u \mid z \in \mathfrak{S}_{\tau(x)}\}$ be the $\mathfrak{S}_{\tau(x)}$-orbit under the action (2.5.3).

For $u = x \prod_{i \in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W \cap W^{(m,\mu)}$, set $\mathbf{a}(u) = (a_1, a_2, \dots, a_n) \in [0, r-1]^n$, where we put $a_i = 0$ if $i \notin I(x)$. Let $\geq$ be the lexicographic order on $\mathbb{Z}^n$. We define a partial order $\geq$ on ${}^{(l,\mu)}W \cap W^{(m,\nu)}$ by

(2.5.5) $$u \geq u' \Leftrightarrow x = x' \text{ and } \mathbf{a}(u) \geq \mathbf{a}(u')$$

for $u = x \prod_{i \in I(x)} t_i^{a_i}, u' = x' \prod_{i \in I(x')} t_i^{a'_i} \in {}^{(l,\mu)}W \cap W^{(m,\nu)}$.

Now we introduce a set ${}^{(l,\mu)}W^{(m,\nu)}$ turning out to be a complete set of double coset representatives in Proposition 2.8. It plays a key role to establish the Mackey formula for the Hecke algebra associated with $W_{n,r}$, and it is a main new ingredient in this paper.

Definition 2.6. We define

$$^{(l,\mu)}W^{(m,\nu)} = \{u \in {}^{(l,\mu)}W \cap W^{(m,\nu)} \mid u \text{ is minimal in } O(u)\}.$$

From the definition, any element of ${}^{(l,\mu)}W \cap W^{(m,\nu)}$ is obtained from ${}^{(l,\mu)}W^{(m.\nu)}$ by the action of $\mathfrak{S}_{\tau(x)}$ for $x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)}$.

Lemma 2.7. *If $\nu = (1^{n-m})$, we have ${}^{(l,\mu)}W^{(m,\nu)} = {}^{(l,\mu)}W \cap W^{(m,\nu)}$.*

Proof. For any $z = xyx^{-1} \in \mathfrak{S}_{\tau(x)}$ ($x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)}$), we have

$$z \odot (x \prod_{i \in I(x)} t_i^{a_i}) = x \prod_{i \in I(x)} t_{y(i)}^{a_i} = x \prod_{i \in I(x)} t_i^{a_i}$$

since $y \in \mathfrak{S}_{(m,\nu)}$ and $I(x) \subset [m+1, n]$ together with $\nu = (1^{n-m})$. This implies that $O(u) = \{u\}$ for any $u \in {}^{(l,\mu)}W \cap W^{(m,\nu)}$, and we have the lemma. $\square$

Proposition 2.8. *The set* ${}^{(l,\mu)}W^{(m,\nu)}$ *is a complete set of representatives of the double cosets* $W_{(l,\mu)}\backslash W_{n,r}/W_{(m,\nu)}$.

Proof. For $w = xt_1^{a_1}t_2^{a_2}\dots t_n^{a_n} \in W_{n,r}$ where $x \in \mathfrak{S}_n$ and $a_1,\dots,a_n \in [0, r-1]$, we can write

$$x = x_1x_2x_3 \quad (x_1 \in \mathfrak{S}_{(l,\mu)},\ x_2 \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)},\ x_3 \in \mathfrak{S}_{(m,\nu)}).$$

The relations (2.1.1) imply that

$$w = x_1x_2x_3t_1^{a_1}t_2^{a_2}\dots t_n^{a_n} = x_1x_2(t_{x_3(m+1)}^{a_{m+1}}t_{x_3(m+2)}^{a_{m+2}}\dots t_{x_3(n)}^{a_n})x_3t_1^{a_1}t_2^{a_2}\dots t_m^{a_m},$$

where we have $\{x_3(m+1), x_3(m+2),\dots,x_3(n)\} = [m+1, n]$ since $x_3 \in \mathfrak{S}_{(m,\nu)}$. Put $I(x_2)^c = [m+1, n] \setminus I(x_2)$. Then, we obtain

$$\begin{aligned}(2.8.1)\qquad w &= x_1x_2(t_{x_3(m+1)}^{a_{m+1}}t_{x_3(m+2)}^{a_{m+2}}\dots t_{x_3(n)}^{a_n})x_3t_1^{a_1}t_2^{a_2}\dots t_m^{a_m}\\ &= x_1x_2(\prod_{x_3(i)\in I(x_2)^c} t_{x_3(i)}^{a_i})(\prod_{x_3(i)\in I(x_2)} t_{x_3(i)}^{a_i})x_3t_1^{a_1}t_2^{a_2}\dots t_m^{a_m}\\ &= (x_1\prod_{x_3(i)\in I(x_2)^c} t_{x_2x_3(i)}^{a_i})(x_2\prod_{x_3(i)\in I(x_2)} t_{x_3(i)}^{a_i})(x_3t_1^{a_1}t_2^{a_2}\dots t_m^{a_m}),\end{aligned}$$

and

$$(2.8.2)\qquad \{x_2x_3(i) \mid x_3(i) \in I(x_2)^c,\ m+1 \le i \le n\} \subset [1, l]$$

from the definition of $I(x_2)^c$.

Take $z = x_2yx_2^{-1} \in \mathfrak{S}_{\tau(x_2)}$ such that $z\odot(x_2\prod t_{x_3(i)}^{a_i})$ is minimal in $O(x_2\prod t_{x_3(i)}^{a_i})$, then (2.8.1) and (2.5.4) imply

$$\begin{aligned}(2.8.3)\qquad w &= (x_1\prod_{x_3(i)\in I(x_2)^c} t_{x_2x_3(i)}^{a_i})(x_2\prod_{x_3(i)\in I(x_2)} t_{x_3(i)}^{a_i})(x_3t_1^{a_1}t_2^{a_2}\dots t_m^{a_m})\\ &= (x_1\prod_{x_3(i)\in I(x_2)^c} t_{x_2x_3(i)}^{a_i})z^{-1}(z\odot(x_2\prod_{x_3(i)\in I(x_2)} t_{x_3(i)}^{a_i}))y(x_3t_1^{a_1}t_2^{a_2}\dots t_m^{a_m})\\ &= (x_1z^{-1}\prod_{x_3(i)\in I(x_2)^c} t_{zx_2x_3(i)}^{a_i})(z\odot(x_2\prod_{x_3(i)\in I(x_2)} t_{x_3(i)}^{a_i}))(yx_3t_1^{a_1}t_2^{a_2}\dots t_m^{a_m}),\end{aligned}$$

where we have $\{zx_2x_3(i) \mid x_3(i) \in I(x_2)^c\} \subset [1, l]$ by (2.8.2) and $z \in \mathfrak{S}_{(l,\mu)}$. From the above argument, we conclude that

$$\begin{aligned}(2.8.4)\qquad & z\odot(x_2\prod_{x_3(i)\in I(x_2)} t_{x_3(i)}^{a_i}) \in {}^{(l,\mu)}W^{(m,\nu)},\\ & (x_1z^{-1}\prod_{x_3(i)\in I(x_2)^c} t_{zx_2x_3(i)}^{a_i}) \in W_{(l,\mu)} \text{ and } (yx_3t_1^{a_1}t_2^{a_2}\dots t_m^{a_m}) \in W_{(m,\nu)}.\end{aligned}$$

The equations (2.8.3) and (2.8.4) imply that

$$(2.8.5)\qquad W_{n,r} = \bigcup_{u\in {}^{(l,\mu)}W^{(m,\nu)}} W_{(l,\mu)}uW_{(m,\nu)}.$$

Finally, we prove that distinct elements of ${}^{(l,\mu)}W^{(m,\nu)}$ belong to distinct $(W_{(l,\mu)}, W_{(m,\nu)})$-double cosets.

For $u = x\prod_{i\in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$ and $u' = x'\prod_{i\in I(x')} t_i^{a'_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, suppose that u and u' belong to the same $(W_{(l,\mu)}, W_{(m,\nu)})$-double coset, namely $u' = w_1uw_2$ for some $w_1 =$

$z\prod_{i=1}^{l} t_i^{b_i} \in W_{(l,\mu)}$ ($z \in \mathfrak{S}_{(l,\mu)}$) and $w_2 = y\prod_{i=1}^{m} t_i^{c_i} \in W_{(m,\mu)}$ ($y \in \mathfrak{S}_{(m,\nu)}$). Then we see that

$$x' \prod_{i\in I(x')} t_i^{a'_i} = (z\prod_{i=1}^{l} t_i^{b_i})(x\prod_{i\in I(x)} t_i^{a_i})(y\prod_{i=1}^{m} t_i^{c_i}) = zxy\prod_{i=1}^{l} t_{y^{-1}x^{-1}(i)}^{b_i} \prod_{i\in I(x)} t_{y^{-1}(i)}^{a_i} \prod_{i=1}^{m} t_i^{c_i}.$$

This implies that

$$x' = zxy \text{ and } \prod_{i\in I(x')} t_i^{a'_i} = \prod_{i=1}^{l} t_{y^{-1}x^{-1}(i)}^{b_i} \prod_{i\in I(x)} t_{y^{-1}(i)}^{a_i} \prod_{i=1}^{m} t_i^{c_i}. \tag{2.8.6}$$

Note that $z \in \mathfrak{S}_{(l,\mu)}$ and $y \in \mathfrak{S}_{(m,\nu)}$, and thus x and x' belong to the same $(\mathfrak{S}_{(l,\mu)}, \mathfrak{S}_{(m,\nu)})$-double coset. Then we have $x = x'$ since $x, x' \in {}^{(l,\mu)}\mathfrak{S}^{(m,\mu)}$. We also have $[1,m] \cap I(x) = \emptyset$ and $y^{-1}x^{-1}(i) = x^{-1}z(i) \in [m+1,n] \setminus I(x)$ for $i \in [1,l]$ by $z \in \mathfrak{S}_{(l,\mu)}$ and the definition of $I(x)$. Thus (2.8.6) implies that

$$x' = x = zxy, \quad \prod_{i\in I(x)} t_i^{a'_i} = \prod_{i\in I(x)} t_{y^{-1}(i)}^{a_i} \text{ and } \prod_{i=1}^{l} t_{y^{-1}x^{-1}(i)}^{b_i} \prod_{i=1}^{m} t_i^{c_i} = 1,$$

and we deduce

$$u' = x\prod_{i\in I(x)} t_i^{a'_i} = x\prod_{i\in I(x)} t_{y^{-1}(i)}^{a_i} = z \odot (x\prod_{i\in I(x)} t_i^{a_i}) = z \odot u$$

since $z = xy^{-1}x^{-1} \in \mathfrak{S}_{\tau(x)} = \mathfrak{S}_{(l,\mu)} \cap x\mathfrak{S}_{(m,\nu)}x^{-1}$. Thus we obtain $u' \in O(u)$. On the other hand, both of u and u' are minimal in $O(u)$ since $u, u' \in {}^{(l,\mu)}W^{(m,\nu)}$, and we conclude that $u = u'$ since a minimal element in $O(u)$ is unique by the definition. □

Lemma 2.9. *For $u = x\prod_{i=1}^{n} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$ ($a_i = 0$ if $i \notin I(x)$) and $y \in \mathfrak{S}_{(m,\nu)}$, we have the following.*

(i) $uyu^{-1} = xyx^{-1}\prod_{i=1}^{n} t_{x(i)}^{a_{y(i)}-a_i}$.

(ii) $ut_ju^{-1} = t_{x(j)}$ *for* $j = 1, 2, \ldots, n$.

(iii) $a_{y(i)} = a_i = 0$ *if* $i \in [1,m]$.

(iv) $a_i = 0$ *if* $x(i) \le l$.

(v) $a_{y(i)} = 0$ *if* $x(i) \le l$ *and* $xyx^{-1} \in \mathfrak{S}_{(l,\mu)}$.

Proof. (i). Note that $u^{-1} = \prod_{i=1}^{n} t_i^{-a_i}x^{-1} = x^{-1}\prod_{i=1}^{n} t_{x(i)}^{-a_i}$, and we obtain

$$uyu^{-1} = (x\prod_{i=1}^{n} t_i^{a_i})y(x^{-1}\prod_{i=1}^{n} t_{x(i)}^{-a_i}) = xyx^{-1}(\prod_{i=1}^{n} t_{xy^{-1}(i)}^{a_i})(\prod_{i=1}^{n} t_{x(i)}^{-a_i}) = xyx^{-1}\prod_{i=1}^{n} t_{x(i)}^{a_{y(i)}-a_i}.$$

(ii). For $j = 1, 2, \ldots, n$, we have

$$ut_ju^{-1} = (x\prod_{i=1}^{n} t_i^{a_i})t_j(\prod_{i=1}^{n} t_i^{-a_i}x^{-1}) = xt_jx^{-1} = t_{x(j)}.$$

(iii). Note that $y \in \mathfrak{S}_{(m,\nu)}$ and $[1,m] \cap I(x) = \emptyset$, and thus $a_{y(i)} = a_i = 0$ if $i \in [1,m]$.

(iv). If $a_i \ne 0$, we have $i \in I(x)$. Thus, we can write $i = x^{-1}(l+j)$ for some $j \ge 1$. This implies that $x(i) > l$ if $a_i \ne 0$.

(v). If $a_{y(i)} \ne 0$, we can write $y(i) = x^{-1}(l+j)$ for some $j \ge 1$. This implies that $x(i) = xy^{-1}x^{-1}(l+j)$. Since $xy^{-1}x^{-1} = (xyx^{-1})^{-1} \in \mathfrak{S}_{(l,\mu)}$, we conclude that $x(i) > l$ if

$a_{y(i)} \neq 0$. □

2.10. For $u = x\prod_{i \in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, set

$$c(u) = \min\{c \geq 0 \mid x(c+1) \neq c+1 \text{ or } c = n\} \text{ and } k(u) = \min\{c(u), l, m\},$$

Define the set of elements

$$\Gamma(u) = (S_{(l,\mu)} \cap \{xs_jx^{-1} \in xS_{(m,\nu)}x^{-1} \mid a_j = a_{j+1}\}) \cup \{t_1, t_2, \dots, t_{k(u)}\}, \tag{2.10.1}$$

where we put $a_i = 0$ if $i \notin I(x)$.

By the definition of $I(x)$, we have $1, 2, \dots, k(u) \notin I(x)$ since $k(u) \leq m$, and thus

$$a_1 = a_2 = \cdots = a_{k(u)} = 0. \tag{2.10.2}$$

We also see that

$$s_j = xs_jx^{-1} \in S_{(l,\mu)} \cap xS_{(m,\nu)}x^{-1} \text{ for } j = 1, 2, \dots, k(u) - 1 \tag{2.10.3}$$

since $k(u) \leq l, m$ and $x \in \mathfrak{S}_{[k(u)+1,n]}$ by Lemma 1.3.

On the other hand, we have

$$xs_jx^{-1}(k(u)) = \begin{cases} x(k(u)+1) & \text{if } j = k(u), \\ k(u) - 1 & \text{if } j = k(u) - 1, \\ k(u) \text{ otherwise} & \end{cases}$$

for $j = 1, 2, \dots, n$ since $x \in \mathfrak{S}_{[k(u)+1,n]}$ by Lemma 1.3. This implies that

$$j = k(u) \text{ and } x(k(u)+1) = k(u)+1 \text{ if } s_{k(u)} = xs_jx^{-1}. \tag{2.10.4}$$

By (2.10.2), (2.10.3) and (2.10.4), we obtain that

$$\{s_1, s_2, \dots, s_{k(u)-1}\} \subset \Gamma(u) \text{ and } s_{k(u)} \notin \Gamma(u), \tag{2.10.5}$$

where we note that $s_m \notin S_{(m,\nu)}$, $s_l \notin S_{(l,\mu)}$ and $x(c(u)+1) \neq c(u)+1$.

We define a composition $\pi(u)$ of $n - k(u)$ by

$$S_{(k(u),\pi(u))} = \Gamma(u) \cap S. \tag{2.10.6}$$

We remark that

$$\mathfrak{S}_{(k(u),\pi(u))} \subset \mathfrak{S}_{\tau(x)} = \mathfrak{S}_{(l,\mu)} \cap x\mathfrak{S}_{(m,\nu)}x^{-1} \tag{2.10.7}$$

since $\mathfrak{S}_{\tau(x)}$ is generated by $S_{(l,\mu)} \cap xS_{(m,\nu)}x^{-1}$ (see (2.5.1)).

Thanks to (2.10.5), the subgroup of $W_{n,r}$ generated by $\Gamma(u)$ coincides with the standard parabolic subgroup $W_{(k(u),\pi(u))} \cong (\mathfrak{S}_{k(u)} \ltimes (\mathbb{Z}/r\mathbb{Z})^{k(u)}) \times \mathfrak{S}_{\pi(u)}$. We remark that $W_{(k(u),\pi(u))}$ is also a parabolic subgroup of $W_{(l,\mu)}$.

2.11. For $u = x\prod_{i=1}^n t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, it is clear that $u^{-1}W_{(k(u),\pi(u))}u$ is generated by $u^{-1}\Gamma(u)u$ as a subgroup of $W_{n,r}$. For $s_{j'} = xs_jx^{-1} \in \Gamma(u) \cap S$, we have

$$u^{-1}s_{j'}u = x^{-1}s_{j'}x\prod_{i=1}^n t_{x^{-1}(i)}^{a_{x^{-1}(i)} - a_{x^{-1}s_{j'}(i)}} = s_j\prod_{i=1}^n t_{x^{-1}(i)}^{a_{x^{-1}(i)} - a_{s_jx^{-1}(i)}} = s_j = x^{-1}s_{j'}x, \tag{2.11.1}$$

where we note that $a_i = a_{s_j(i)}$ for all $i = 1, 2, \dots, n$ by $s_{j'} = x s_j x^{-1} \in \Gamma(u)$. On the other hand, we see that

(2.11.2) $$u^{-1} t_i u = t_i \text{ for } i \in [1, k(u)]$$

by Lemma 2.9 (ii) and the definition of $k(u)$. As a consequence, we have

(2.11.3) $$u^{-1}\Gamma(u)u = (x^{-1} S_{(l,\mu)} x \cap \{s_j \in S_{(m,\nu)} \mid a_j = a_{j+1}\}) \cup \{t_1, t_2, \dots, t_{k(u)}\}.$$

Moreover, we deduce that

(2.11.4) $$j' = k(u) \text{ and } x^{-1}(k(u)+1) = k(u)+1 \text{ if } s_{k(u)} = x^{-1} s_{j'} x.$$

in a similar way to (2.10.4). By (2.10.2), (2.10.3) and (2.11.4), we obtain that

(2.11.5) $$\{s_1, s_2, \dots, s_{k(u)-1}\} \subset u^{-1}\Gamma(u)u \text{ and } s_{k(u)} \notin u^{-1}\Gamma(u)u,$$

where we note that $s_m \notin S_{(m,\nu)}$, $s_l \notin S_{(l,\mu)}$ and $x^{-1}(c(u)+1) \neq c(u)+1$.

We define a composition $\pi^\sharp(u)$ of $n - k(u)$ by $S_{(k(u),\pi^\sharp(u))} = u^{-1}\Gamma(u)u \cap S$. Then we conclude that

(2.11.6) $$S_{(k(u),\pi^\sharp(u))} = x^{-1} S_{(k(u),\pi(u))} x$$

and

(2.11.7) $$u^{-1} W_{(k(u),\pi(u))} u = W_{(k(u),\pi^\sharp(u))} \cong (\mathfrak{S}_{k(u)} \ltimes (\mathbb{Z}/r\mathbb{Z})^{k(u)}) \times \mathfrak{S}_{\pi^\sharp(u)}$$

by (2.11.3) and (2.11.5). In particular, $u^{-1} W_{(k(u),\pi(u))} u$ is a standard parabolic subgroup of $W_{(m,\nu)}$. Recall the definition of $X_{(l,\mu)}$, the set of generators of $W_{(l,\mu)}$, from 2.2.

Proposition 2.12. *For $u = x \prod_{i \in I(X)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, we have $W_{(k(u),\pi(u))} = u W_{(k(u),\pi^\sharp(u))} u^{-1}$ and $X_{(k(u),\pi(u))} = u X_{(k(u),\pi^\sharp(u))} u^{-1}$. In particular, for $s_j \in X_{(k(u),\pi(u))}$, there exists $s_{\psi(j)} \in X_{(k(u),\pi^\sharp(u))}$ such that $s_j = u s_{\psi(j)} u^{-1}$.*

Moreover, the identity

$$s_j (s_{i_1} s_{i_2} \dots s_{i_l} \prod_{i \in I(X)} t_i^{a_i}) = (s_{i_1} s_{i_2} \dots s_{i_l} \prod_{i \in I(X)} t_i^{a_i}) s_{\psi(j)} \textit{ for } s_j \in X_{(k(u),\pi(u))}$$

follows only from the braid relations associated with $W_{n,r}$, where $x = s_{i_1} s_{i_2} \dots s_{i_l}$ is a reduced expression of $x \in \mathfrak{S}_n$.

Proof. For $u = x \prod_{i \in I(X)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, we have already seen that $W_{(k(u),\pi(u))} = u W_{(k(u),\pi^\sharp(u))} u^{-1}$ and $X_{(k(u),\pi(u))} = u X_{(k(u),\pi^\sharp(u))} u^{-1}$. Thus, for $s_j \in X_{(k(u),\pi(u))}$, there exists $s_{\psi(j)} \in X_{(k(u),\pi^\sharp(u))}$ such that $s_j = u s_{\psi(j)} u^{-1}$. Let $x = s_{i_1} s_{i_2} \dots s_{i_l}$ be a reduced expression of $x \in \mathfrak{S}_n$.

It is easy to check that the relations

(2.12.1) $$\begin{aligned} &s_i s_j = s_j s_i \text{ if } |i-j| > 1, \\ &t_i t_j = t_j t_i \ (1 \le i, j \le n), \\ &s_i t_j = t_j s_i \text{ if } j \neq i, i+1 \\ &s_i t_i t_{i+1} = t_i t_{i+1} s_i \end{aligned}$$

follow only from the braid relations associated with $W_{n,r}$ by direct calculation.

Note that $x \in \mathfrak{S}_{[k(u)+1,n]}$ by Lemma 1.3 and $s_0 = t_1$. So, we have

$$s_0(s_{i_1}s_{i_2}\dots s_{i_l}\prod_{i\in I(x)} t_i^{a_i}) = (s_{i_1}s_{i_2}\dots s_{i_l}\prod_{i\in I(x)} t_i^{a_i})s_0$$

if $k(u) \neq 0$ and this identity follows only from the braid relations.

For $s_j \in X_{(k(u),\pi(u))} \setminus \{s_0\}$, we have $s_j = xs_{\psi(j)}x^{-1}$ and $a_{\psi(j)} = a_{\psi(j)+1}$ by (2.11.1). Moreover, we have $\ell(s_jx) = \ell(x)+1 = \ell(xs_{\psi(j)})$ since $s_j \in \mathfrak{S}_{(l,\mu)}$, $s_{\psi(j)} \in \mathfrak{S}_{(m,\nu)}$ (see (2.10.1) and (2.11.3)), and $x \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)} = {}^{(l,\mu)}\mathfrak{S} \cap \mathfrak{S}^{(m,\nu)}$. Thus, the identity $s_j(s_{i_1}s_{i_2}\dots s_{i_l}) = (s_{i_1}s_{i_2}\dots s_{i_l})s_{\psi(j)}$ follows only from the braid relations associated with $\mathfrak{S}_n$ by the general theory of Coxeter groups. So, we conclude that the identity $s_j(s_{i_1}s_{i_2}\dots s_{i_l}\prod_{i\in I(X)} t_i^{a_i}) = (s_{i_1}s_{i_2}\dots s_{i_l}\prod_{i\in I(X)} t_i^{a_i})s_{\psi(j)}$ follows only from the braid relations (by noting that $a_{\psi(j)} = a_{\psi(j)+1}$). □

Proposition 2.13. *For $u = x\prod_{i\in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, the subgroup $W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1}$ of $W_{n,r}$ is generated by $\Gamma(u)$. In particular, we have*

$$W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1} = W_{(k(u),\pi(u))} \cong (\mathfrak{S}_{k(u)} \ltimes (\mathbb{Z}/r\mathbb{Z})^{k(u)}) \times \mathfrak{S}_{\tau(u)}.$$

Proof. Put $a_j = 0$ for $j \notin I(x)$. For $w = y\prod_{i=1}^m t_i^{b_i} \in W_{(m,\nu)}$ $(y \in \mathfrak{S}_{(m,\nu)})$, we have

$$uwu^{-1} = uyu^{-1}\prod_{i=1}^m (ut_iu^{-1})^{b_i} = xyx^{-1}(\prod_{i=1}^n t_{x(i)}^{a_{y(i)}-a_i})(\prod_{i=1}^m t_{x(i)}^{b_i})$$

by Lemma 2.9 (i) and (ii). Note that

$$a_{y(i)} = a_i = 0 \text{ for } i \in [1,m] \tag{2.13.1}$$

by Lemma 2.9 (iii), and thus we have

$$uwu^{-1} = xyx^{-1}(\prod_{i=1}^m t_{x(i)}^{b_i})(\prod_{i=m+1}^n t_{x(i)}^{a_{y(i)}-a_i}). \tag{2.13.2}$$

Suppose that $uwu^{-1} \in W_{(l,\mu)} = (\mathfrak{S}_l \ltimes (\mathbb{Z}/r\mathbb{Z})^l) \times \mathfrak{S}_\mu^{[l]}$. Then we deduce

$$xyx^{-1} \in \mathfrak{S}_{(l,\mu)}, \quad a_{y(i)} = a_i \text{ if } x(i) > l, \quad b_i = 0 \text{ if } x(i) > l \tag{2.13.3}$$

by (2.13.2). Since $xyx^{-1} \in \mathfrak{S}_{(l,\mu)}$, we have

$$a_{y(i)} = a_i = 0 \text{ if } x(i) \le l \tag{2.13.4}$$

by Lemma 2.9 (iv) and (v). Moreover, we see that

$$[1,l] \cap \{x(1),\dots,x(m)\} = [1,k(u)] \tag{2.13.5}$$

by Lemma 1.3, where $x(i) = i$ for $i \in [1,k(u)]$. As a consequence of (2.13.1), (2.13.2), (2.13.3), (2.13.4) and (2.13.5), we conclude that

$$xyx^{-1} \in \mathfrak{S}_{(l,\mu)},\ a_{y(i)} = a_i\,(1 \le i \le n) \text{ and } b_j = 0\,(j > k(u)) \tag{2.13.6}$$

if $uwu^{-1} \in W_{(l,\mu)}$. On the other hand, it is clear that $uwu^{-1} \in W_{(l,\mu)}$ if (2.13.6) holds for $w = y\prod_{i=1}^m t_i^{b_i} \in W_{(m,\nu)}$ $(y \in \mathfrak{S}_{(m,\nu)})$. Thus, we may deduce that $W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1}$ is generated by

$$\begin{aligned}\hat{\varGamma}(u) :=& (\mathfrak{S}_{(l,\mu)} \cap \{xyx^{-1} \mid y \in \mathfrak{S}_{(m,\nu)} \text{ such that } a_{y(i)} = a_i \ (1 \le i \le n)\}) \\ & \cup \{t_1, t_2, \dots, t_{k(u)}\}.\end{aligned}$$

For $z = xyx^{-1} \in \mathfrak{S}_{\tau(x)} = \mathfrak{S}_{(l,\mu)} \cap x\mathfrak{S}_{(m,\nu)}x^{-1}$, let $y = s_{i_1}s_{i_2}\dots s_{i_p}$ be a reduced expression. Then, $xs_{i_j}x^{-1} \in \mathfrak{S}_{\tau(x)}$ $(j = 1, 2, \dots, p)$ since $\mathfrak{S}_{\tau(x)}$ is generated by $S_{(l,\mu)} \cap xS_{(m,\nu)}x^{-1}$. We claim that

$$a_{i_j} = a_{i_{j+1}} \ (1 \le j \le p) \text{ if } a_{y(i)} = a_i \ (1 \le i \le n). \tag{2.13.7}$$

Then (2.13.7) implies that $\hat{\varGamma}(u) \supset \varGamma(u)$, and we easily see that $W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1}$ is generated by $\varGamma(u)$.

We shall prove the claim (2.13.7). We have

$$\{i_1, i_2, \dots, i_k\} = \bigcup_{i<y(i)} \{i, i+1, \dots, y(i)-1\} \cup \bigcup_{i>y(i)} \{y(i), y(i)+1, \dots, i-1\}. \tag{2.13.8}$$

Suppose that $i < y(i)$ and $a_{y(i)} = a_i$. By (2.13.8), we see that $xs_jx^{-1} \in \mathfrak{S}_{\tau(x)}$ $(i \le j < y(i))$, and we obtain

$$(xs_jx^{-1}) \odot u = x(t_1^{a_1} \dots t_{j-1}^{a_{j-1}})(t_j^{a_{j+1}} t_{j+1}^{a_j})(t_{j+2}^{a_{j+2}} \dots t_n^{a_n}) \in O(u).$$

If there exists j $(i \le j < y(i))$ such that $a_i \le a_{i+1} \le \dots \le a_j$ and $a_j > a_{j+1}$, we have $\mathbf{a}(u) \succ \mathbf{a}((xs_jx^{-1}) \odot u)$. This is a contradiction since u is minimal in $O(u)$. Thus, we conclude that $a_i \le a_{i+1} \le \dots \le a_{y(i)}$, and $a_i = a_{i+1} = \dots = a_{y(i)}$ by $a_{y(i)} = a_i$. Similarly, we have $a_{y(i)} = a_{y(i)+1} = \dots = a_i$ if $i > y(i)$ and $a_{y(i)} = a_i$. Then we obtain the desired result (2.13.7). □

2.14. For $u \in {}^{(l,\mu)}W^{(m,\nu)}$, the group $W_{(k(u),\pi(u))} = W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1}$ is a standard parabolic subgroup of $W_{(l,\mu)}$ by Proposition 2.13. Put

$$(W_{(l,\mu)})^{(k(u),\pi(u))} = \{xt_{k(u)+1}^{a_{k(u)+1}} t_{k(u)+2}^{a_{k(u)+2}} \dots t_l^{a_l} \mid x \in (\mathfrak{S}_{(l,\mu)})^{(k(u),\pi(u))}, \ a_{k(u)+1}, \dots, a_l \in [0, r-1]\},$$

where $(\mathfrak{S}_{(l,\mu)})^{(k(u),\pi(u))}$ is the set of distinguished coset representatives of the cosets $\mathfrak{S}_{(l,\mu)}/\mathfrak{S}_{(k(u),\pi(u))}$. Then $(W_{(l,\mu)})^{(k(u),\pi(u))}$ is a complete set of representatives of $W_{(l,\mu)}/W_{(k(u),\pi(u))}$ which is proven in a similar way to the proof of Lemma 2.3. We have the following corollary.

Corollary 2.15. *For each $u \in {}^{(l,\mu)}W^{(m,\nu)}$, the multiplication map (in W)*

$$(W_{(l,\mu)})^{(k(u),\pi(u))} \times \{u\} \times W_{(m,\nu)} \to W_{(l,\mu)}uW_{(m,\nu)}, \quad (w_1, u, w_2) \mapsto w_1uw_2$$

is a bijection.

Proof. By definitions, it is clear that the map is surjective. On the other hand, if $w_1uw_2 = w_1'uw_2'$ for $w_1, w_1' \in (W_{(l,\mu)})^{(k(u),\pi(u))}$ and $w_2, w_2' \in W_{(m,\nu)}$, we have

$$w_1^{-1}w_1' = uw_2w_2'^{-1}u^{-1} \in W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1} = W_{(k(u),\pi(u))}.$$

This implies that $w_1 = w_1'$ since $w_1, w_1' \in (W_{(l,\mu)})^{(k(u),\pi(u))}$. Thus we also have $w_2 = w_2'$, and therefore the map is injective. □

3. The Mackey formula for cyclotomic Hecke algebras

In this section, we construct various R-free basis of the cyclotomic Hecke algebra $\mathscr{H}_{n,r}$ associated with $W_{n,r}$ which are compatible with the decomposition of $W_{n,r}$ to the cosets $W_{n,r}/W_{(l,\mu)}$ and the double cosets $W_{(l,\mu)}\backslash W_{n,r}/W_{(m,\nu)}$. Then we establish the Mackey formula for cyclotomic Hecke algebras.

3.1. Let R be a unital commutative ring, and take parameters $q, Q_1, Q_2, \dots, Q_r \in R$ such that q is invertible in R. The cyclotomic Hecke algebra (Ariki-Koike algebra) $\mathscr{H}_{n,r} = \mathscr{H}(W_{n,r})$ associated with $W_{n,r}$ is the associative algebra with 1 over R generated by $T_0, T_1, \dots, T_{n-1}$ with the following defining relations:

$$
\begin{aligned}
\text{(3.1.1)}\qquad & (T_0 - Q_1)(T_0 - Q_2)\dots(T_0 - Q_r) = 0, \quad (T_i + 1)(T_i - q) = 0 \quad (1 \le i \le n-1),\\
& T_0T_1T_0T_1 = T_1T_0T_1T_0, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad (1 \le i \le n-2),\\
& T_iT_j = T_jT_i \quad (|i-j| > 1).
\end{aligned}
$$

The subalgebra of $\mathscr{H}_{n,r}$ generated by $T_1, T_2, \dots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra $\mathscr{H}(\mathfrak{S}_n)$ associated with $\mathfrak{S}_n$. For $x \in \mathfrak{S}_n$, put $T_x = T_{i_1}T_{i_2}\dots T_{i_l}$ for a reduced expression $x = s_{i_1}s_{i_2}\dots s_{i_l}$, and $\{T_x \mid x \in \mathfrak{S}_n\}$ is an R-free basis of $\mathscr{H}(\mathfrak{S}_n)$.

Set $L_i = q^{1-i}T_{i-1}\dots T_1T_0T_1\dots T_{i-1}$ for $i = 1, 2, \dots, n$. For $w = xt_1^{a_1}\dots t_n^{a_n} \in W_{n,r}$ where $x \in \mathfrak{S}_n$ and $a_1, \dots, a_n \in [0, r-1]$, put $T_w = T_xL_1^{a_1}L_2^{a_2}\dots L_n^{a_n}$. Then we have that $\{T_w \mid w \in W_{n,r}\}$ is an R-free basis of $\mathscr{H}_{n,r}$ by [2, Theorem 3.10].

For a parabolic subgroup $W_{(l,\mu)}$ of $W_{n,r}$, we define the subalgebra $\mathscr{H}_{(l,\mu)}$ of $\mathscr{H}_{n,r}$ generated by T_0 (in the case where $l \ge 1$) and T_x for $x \in \mathfrak{S}_{(l,\mu)}$. It is isomorphic to the cyclotomic Hecke algebra $\mathscr{H}(W_{(l,\mu)})$ associated with $W_{(l,\mu)}$. It is easy to see that $\{T_w \mid w \in W_{(l,\mu)}\}$ is an R-free basis of $\mathscr{H}_{(l,\mu)}$.

The following properties are well known, and one can check them by direct calculation using the defining relations.

Lemma 3.2. *We have the following.*

(i) *L_i and L_j commute with each other for any $1 \le i, j \le n$.*
(ii) *T_i and L_j commute with each other if $j \ne i, i+1$.*
(iii) *T_i commutes with both L_iL_{i+1} and $L_i + L_{i+1}$.*
(iv) *$L_{i+1}^bT_i = T_iL_i^b + (q-1)\sum_{c=0}^{b-1} L_i^cL_{i+1}^{b-c}$.*
(v) *$L_i^bT_i = T_iL_{i+1}^b - (q-1)\sum_{c=0}^{b-1} L_i^cL_{i+1}^{b-c}$.*

Lemma 3.2 implies the following lemma:

Lemma 3.3. *For $k \ge 0$, $x \in \mathfrak{S}_{(k,n-k)}$ and $a_{k+1}, \dots, a_n \in [0, r-1]$, we have*

$$
\begin{aligned}
& T_x(L_{k+1}^{a_{k+1}}L_{k+2}^{a_{k+2}}\dots L_n^{a_n})\\
&= (L_{x(k+1)}^{a_{k+1}}L_{x(k+2)}^{a_{k+2}}\dots L_{x(n)}^{a_n})T_x + \sum_{y<x}\ \sum_{(b_{k+1},\dots,b_n)\in[0,r-1]^{n-k}} r_y^{(b_{k+1},\dots,b_n)}T_y(L_{k+1}^{b_{k+1}}L_{k+2}^{b_{k+2}}\dots L_n^{b_n})
\end{aligned}
$$

for some $r_y^{(b_1,\dots,b_n)} \in R$.

Proof. We shall prove the lemma by the induction on $\ell(x)$. If $\ell(x) = 0$, it is clear. Suppose that $\ell(x) > 0$. Let $x = s_{i_1}s_{i_2}\dots s_{i_l}$ be a reduced expression, and put $x' = xs_{i_l}$. Note that $T_x = T_{x'}T_{i_l}$, and we have

$$T_x(L_{k+1}^{a_{k+1}}L_{k+2}^{a_{k+2}}\dots L_n^{a_n})$$
$$=\begin{cases} T_{x'}(L_{s_{i_l}(k+1)}^{a_{k+1}}L_{s_{i_l}(k+2)}^{a_{k+2}}\dots L_{s_{i_l}(n)}^{a_n})T_{i_l} & \text{if } a_{i_l}=a_{i_l+1},\\ T_{x'}(L_{s_{i_l}(k+1)}^{a_{k+1}}L_{s_{i_l}(k+2)}^{a_{k+2}}\dots L_{s_{i_l}(n)}^{a_n})T_{i_l} & \\ \quad +(q-1)\sum_{c=a_{i_l}}^{a_{i_l+1}-1}T_{x'}(L_{k+1}^{a_{k+1}}\dots L_{i_l-1}^{a_{i_l-1}})(L_{i_l}^{c}L_{i_l+1}^{a_{i_l}+a_{i_l+1}-c})(L_{i_l+2}^{a_{i_l+2}}\dots L_n^{a_n}) & \\ & \text{if } a_{i_l}<a_{i_l+1},\\ T_{x'}(L_{s_{i_l}(k+1)}^{a_{k+1}}L_{s_{i_l}(k+2)}^{a_{k+2}}\dots L_{s_{i_l}(n)}^{a_n})T_{i_l} & \\ \quad -(q-1)\sum_{c=a_{i_l+1}}^{a_{i_l}-1}T_{x'}(L_{k+1}^{a_{k+1}}\dots L_{i_l-1}^{a_{i_l-1}})(L_{i_l}^{c}L_{i_l+1}^{a_{i_l}+a_{i_l+1}-c})(L_{i_l+2}^{a_{i_l+2}}\dots L_n^{a_n}) & \\ & \text{if } a_{i_l}>a_{i_l+1} \end{cases}$$

by direct calculation using Lemma 3.2. Applying the assumption of the induction to $T_{x'}(L_{s_{i_l}(k+1)}^{a_{k+1}}L_{s_{i_l}(k+2)}^{a_{k+2}}\dots L_{s_{i_l}(n)}^{a_n})$, we have the lemma. □

Proposition 3.4. *For $W_{(l,\mu)}$, a parabolic subgroup of $W_{n,r}$, the elements*

$$\{T_{w_1}T_{w_2}\mid w_1\in W^{(l,\mu)},\ w_2\in W_{(l,\mu)}\}$$

is an R-free basis of $\mathscr{H}_{n,r}$. Moreover $\mathscr{H}_{n,r}$ is a free right $\mathscr{H}_{(l,\mu)}$-module with an $\mathscr{H}_{(l,\mu)}$-free basis $\{T_w\mid w\in W^{(l,\mu)}\}$.

Proof. For $w=xt_1^{a_1}\dots t_n^{a_n}\in W_{n,r}$ with $x\in\mathfrak{S}_n$, we can write $x=x_1x_2$ where $x_1\in\mathfrak{S}^{(l,\mu)}$, $x_2\in\mathfrak{S}_{(l,\mu)}$, and $x_2=y_1y_2$ where $y_1\in\mathfrak{S}_l$, $y_2\in\mathfrak{S}_\mu^{[l]}$. Note that $\ell(x)=\ell(x_1)+\ell(x_2)$ and $\ell(x_2)=\ell(y_1)+\ell(y_2)$. Then, we have

$$\begin{aligned} T_w &= T_xL_1^{a_1}\dots L_n^{a_n}\\ &= T_{x_1}T_{x_2}L_1^{a_1}\dots L_n^{a_n}\\ &= T_{x_1}T_{y_1}T_{y_2}L_1^{a_1}\dots L_n^{a_n}\\ &= T_{x_1}T_{y_2}(L_{l+1}^{a_{l+1}}L_{l+2}^{a_{l+2}}\dots L_n^{a_n})T_{y_1}(L_1^{a_1}L_2^{a_2}\dots L_l^{a_l}), \end{aligned}$$

where we use Lemma 3.2 (i) and (ii) in the last equation. Note that $y_2\in\mathfrak{S}_\mu^{[l]}$, and we obtain

$$\begin{aligned} &T_{y_2}(L_{l+1}^{a_{l+1}}L_{l+2}^{a_{l+2}}\dots L_n^{a_n})\\ &= L_{y_2(l+1)}^{a_{l+1}}L_{y_2(l+2)}^{a_{l+2}}\dots L_{y_2(n)}^{a_n}T_{y_2}+\sum_{z<y_2}\sum_{(b_{l+1},\dots,b_n)\in[0,r-1]^{n-l}} r_z^{(b_{l+1},\dots,b_n)}L_{l+1}^{b_{l+1}}\dots L_n^{b_n}T_z \end{aligned}$$

by using Lemma 3.3 repeatedly. Thus, we have

$$\begin{aligned} T_w &= T_{x_1}L_{y_2(l+1)}^{a_{l+1}}L_{y_2(l+2)}^{a_{l+2}}\dots L_{y_2(n)}^{a_n}T_{y_2}T_{y_1}(L_1^{a_1}L_2^{a_2}\dots L_l^{a_l})\\ &\quad+\sum_{z<y_2}\sum_{(b_{l+1},\dots,b_n)\in[0,r-1]^{n-l}} r_z^{(b_{l+1},\dots,b_n)}T_{x_1}L_{l+1}^{b_{l+1}}\dots L_n^{b_n}T_zT_{y_1}(L_1^{a_1}L_2^{a_2}\dots L_l^{a_l})\\ &= (T_{x_1}L_{y_2(l+1)}^{a_{l+1}}L_{y_2(l+2)}^{a_{l+2}}\dots L_{y_2(n)}^{a_n})(T_{x_2}L_1^{a_1}L_2^{a_2}\dots L_l^{a_l})\\ &\quad+\sum_{z<y_2}\sum_{(b_{l+1},\dots,b_n)\in[0,r-1]^{n-l}} r_z^{(b_{l+1},\dots,b_n)}(T_{x_1}L_{l+1}^{b_{l+1}}\dots L_n^{b_n})(T_{y_1z}L_1^{a_1}L_2^{a_2}\dots L_l^{a_l}), \end{aligned} \tag{3.4.1}$$

where we note that $y_1z<x_2=y_1y_2$.

We define a preorder $\geq$ on $W_{(l,\mu)}$ by $w=xt_1^{a_1}\dots t_l^{a_l}\geq w'=x't_1^{a'_1}\dots t_l^{a'_l}$ if $x>x'$. Then we have

$$T_w = T_{w_1}T_{w_2} + \sum_{\substack{w'_1 \in W^{(l,\mu)}, w'_2 \in W_{(l,\mu)} \\ w'_2 < w_2}} r_{w'_1,w'_2} T_{w'_1} T_{w'_2}$$

by the equations (3.4.1), where $w_1 = x_1 t_{y_2(l+1)}^{a_l+1} t_{y_2(l+2)}^{a_l+2} \dots t_{y_2(n)}^{a_n}$ and $w_2 = x_2 t_1^{a_1} \dots t_l^{a_l}$.

This implies that $\{T_{w_1}T_{w_2} \mid w_1 \in W^{(l,\mu)},\ w_2 \in W_{(l,\mu)}\}$ is an R-free basis of $\mathscr{H}_{n,r}$, and hence $\mathscr{H}_{n,r} = \bigoplus_{w \in W^{(l,\mu)}} T_w \mathscr{H}_{(l,\mu)}$ as right $\mathscr{H}_{(l,\mu)}$-modules. □

3.5. Recall that $W_{(k(u),\pi(u))} = W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1}$ for $u = x\prod_{i \in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, and we have

$$W_{(k(u),\pi(u))} = \{z t_1^{a_1} t_2^{a_2} \dots t_{k(u)}^{a_{k(u)}} \mid z \in \mathfrak{S}_{(k(u),\pi(u))},\ a_1, \dots, a_{k(u)} \in [0, r-1]\}.$$

Then, the subalgebra $\mathscr{H}_{(k(u),\pi(u))}$ has an R-free basis

$$\{T_z L_1^{a_1} L_2^{a_2} \dots L_{k(u)}^{a_{k(u)}} \mid z \in \mathfrak{S}_{(k(u),\pi(u))},\ a_1, \dots, a_{k(u)} \in [0, r-1]\}. \tag{3.5.1}$$

Proposition 3.6. *For each $u = x\prod_{i \in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, we have the following:*

(i) $L_i T_u = T_u L_i$ *for* $i = 1, 2, \dots, k(u)$.

(ii) $T_z T_u = T_u T_{x^{-1}zx}$ *for* $z \in \mathfrak{S}_{(k(u),\pi(u))}$.

In particular, $T_u\mathscr{H}_{(m,\nu)}$ has an $(\mathscr{H}_{(k(u),\pi(u))}, \mathscr{H}_{(m,\nu)})$-bimodule structure by multiplications in $\mathscr{H}_{n,r}$. More precisely, for $T_uY \in T_u\mathscr{H}_{(m,\nu)}$, we have

$$L_i(T_uY) = T_u(L_iY) \quad (1 \le i \le k(u)), \qquad T_z(T_uY) = T_u(T_{x^{-1}zx}Y) \quad (z \in \mathfrak{S}_{(k(u),\pi(u))}).$$

Proof. Recall the definition of the element $T_w \in \mathscr{H}_{n,r}$ for $w \in W_{n,r}$. Then, this proposition follows from Proposition 2.12 together with (2.11.1) and (2.11.2). □

3.7. For $u = x\prod_{i \in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, recall that $W_{(k(u),\pi(u))} = W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1}$ and $W_{(k(u),\pi^\sharp(u))} = u^{-1}W_{(k(u),\pi(u))}u$ (see (2.11.7)) are parabolic subgroups of $W_{n,r}$. Then, the subalgebra $\mathscr{H}_{(k(u),\pi(u))}$ (resp. $\mathscr{H}_{(k(u),\pi^\sharp(u))}$) has an R-free basis

$$\{T_z L_1^{a_1} \dots L_{k(u)}^{a_{k(u)}} \mid z \in \mathfrak{S}_{(k(u),\pi(u))},\ a_1, \dots a_{k(u)} \in [0, z-1]\}$$
$$(\text{resp. } \{T_y L_1^{a_1} \dots L_{k(u)}^{a_{k(u)}} \mid y \in \mathfrak{S}_{(k(u),\pi^\sharp(u))},\ a_1, \dots a_{k(u)} \in [0, z-1]\}),$$

where we note that $\mathfrak{S}_{(k(u),\pi^\sharp(u))} = x^{-1}\mathfrak{S}_{(k(u),\pi(u))}x$ by (2.11.6).

Corollary 3.8. *For $u = x\prod_{i \in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, we have the following:*

(i) *$T_u\mathscr{H}_{(k(u),\pi^\sharp(u))}$ has an $(\mathscr{H}_{(k(u),\pi(u))}, \mathscr{H}_{(k(u),\pi^\sharp(u))})$-bimodule structure by multiplications in $\mathscr{H}_{n,r}$.*

(ii) *We have the isomorphism of $(\mathscr{H}_{(k(u),\pi(u))}, \mathscr{H}_{(m,\nu)})$-bimodules*

$$T_u\mathscr{H}_{(m,\nu)} \cong T_u\mathscr{H}_{(k(u),\pi^\sharp(u))} \otimes_{\mathscr{H}_{(k(u),\pi^\sharp(u))}} \mathscr{H}_{(m,\nu)}.$$

Proof. (i) follows from Proposition 3.6 (note that $\mathfrak{S}_{(k(u),\pi^\sharp(u))} = x^{-1}\mathfrak{S}_{(k(u),\pi(u))}x$). Note that $u \in {}^{(l,\mu)}W^{(m,\nu)} \subset W^{(m,\nu)}$, and $\mathscr{H}_{(k(u),\pi^\sharp(u))}$ is a subalgebra of $\mathscr{H}_{(m,\nu)}$. Then, by Proposition 3.4, we see that $T_u\mathscr{H}_{(k(u),\pi^\sharp(u))} \cong \mathscr{H}_{(k(u),\pi^\sharp(u))}$ as right $\mathscr{H}_{(k(u),\pi^\sharp(u))}$-modules, and we obtain (ii). □

3.9. By Corollary 2.15, any element $w \in W_{n,r}$ is uniquely written as

$$w = w_1uw_2 \quad (u \in {}^{(l,\mu)}W^{(m,\nu)},\ w_1 \in (W_{(l,\mu)})^{(k(u),\pi(u))},\ w_2 \in W_{(m.\nu)}). \tag{3.9.1}$$

By using this decomposition, we define $\widetilde{T}_w \in \mathscr{H}_{n,r}$ by $\widetilde{T}_w = T_{w_1}T_uT_{w_2}$.

Proposition 3.10. *We have the following.*

(i) $\mathscr{H}_{n,r} = \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)}$.

(ii) $\{\widetilde{T}_w \mid w \in W_{n,r}\}$ *is an R-free basis of* $\mathscr{H}_{n,r}$.

Proof. We prove (i). First we prove that

$$T_v \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)} \text{ for any } v = x\prod_{i\in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W \cap W^{(m,\nu)} \tag{3.10.1}$$

by induction on the order $\geq$ on ${}^{(l,\mu)}W \cap W^{(m,\nu)}$.

If v is minimal in ${}^{(l,\mu)}W \cap W^{(m,\nu)}$, it is also minimal in $O(v)$. Then we have $v \in {}^{(l,\mu)}W^{(m,\nu)}$, and (3.10.1) is clear.

Suppose that v is not minimal in ${}^{(l,\mu)}W \cap W^{(m,\nu)}$. If v is minimal in $O(v)$, we have $v \in {}^{(l,\mu)}W^{(m,\nu)}$, and (3.10.1) is clear. We also suppose that v is not minimal in $O(v)$. Then, there exists $s_{j'} = xs_jx^{-1} \in S_{\tau(x)} = S_{(l,\mu)} \cap xS_{(m,\nu)}x^{-1}$ such that $a_j > a_{j+1}$ and $j, j+1 \in I(x)$ by definitions (see (2.5.2), (2.5.3) and (2.5.5)). Since $x \in {}^{(l,\mu)}\mathfrak{S}_n^{(m,\nu)}$, we have $\ell(s_{j'}x) = \ell(s_{j'}) + \ell(x)$ and $\ell(xs_j) = \ell(x) + \ell(s_j)$, and thus $T_{j'}T_x = T_{s_{j'}x} = T_{xs_j} = T_xT_j$. Put $a_i = 0$ if $i \notin I(x)$. Then we obtain

$$\begin{aligned} T_{j'}T_v &= T_{j'}(T_xL_1^{a_1}L_2^{a_2}\dots L_n^{a_n}) \\ &= T_xT_j(L_1^{a_1}L_2^{a_2}\dots L_n^{a_n}) \\ &= T_x(L_1^{a_1}\dots L_{j-1}^{a_{j-1}})(L_j^{a_{j+1}}L_{j+1}^{a_j})(L_{j+2}^{a_{j+2}}\dots L_n^{a_n})T_j \\ &\quad - (q-1)\sum_{c=a_{j+1}}^{a_j-1} T_x(L_1^{a_1}\dots L_{j-1}^{a_{j-1}})(L_j^cL_{j+1}^{a_j+a_{j+1}-c})(L_{j+2}^{a_{j+2}}\dots L_n^{a_n}), \end{aligned} \tag{3.10.2}$$

where we use Lemma 3.2 in the last equation (note $a_j > a_{j+1}$). Since

$$\begin{aligned} &x(t_1^{a_1}\dots t_{j-1}^{a_{j-1}})(t_j^{a_{j+1}}t_{j+1}^{a_j})(t_{j+2}^{a_{j+2}}\dots t_n^{a_n}) \prec v, \\ &x(t_1^{a_1}\dots t_{j-1}^{a_{j-1}})(t_j^c t_{j+1}^{a_j+a_{j+1}-c})(t_{j+2}^{a_{j+2}}\dots t_n^{a_n}) \prec v \quad (a_{j+1} \leq c \leq a_j - 1), \end{aligned}$$

the assumption of the induction implies

$$\begin{aligned} T_x(L_1^{a_1}\dots L_{j-1}^{a_{j-1}})(L_j^{a_{j+1}}L_{j+1}^{a_j})(L_{j+2}^{a_{j+2}}\dots L_n^{a_n}) &\in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)}, \\ T_x(L_1^{a_1}\dots L_{j-1}^{a_{j-1}})(L_j^cL_{j+1}^{a_j+a_{j+1}-c})(L_{j+2}^{a_{j+2}}\dots L_n^{a_n}) &\in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)}. \end{aligned}$$

Combining them with (3.10.2), we conclude that

$$T_v \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)},$$

where we note that $T_{j'}^{-1} \in \mathscr{H}_{(l,\mu)}$ and $T_j \in \mathscr{H}_{(m,\nu)}$. Thus we proved (3.10.1).

In order to prove (i), it is enough to show that

(3.10.3) $$T_w \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)} \text{ for any } w = xt_1^{a_1}t_2^{a_2}\dots t_n^{a_n} \in W_{n,r}.$$

We prove (3.10.3) by the induction on $\ell(x)$.

If $\ell(x) = 0$, we have

(3.10.4) $$T_w = L_1^{a_1}L_2^{a_2}\dots L_n^{a_n} = \begin{cases} (L_1^{a_1}L_2^{a_2}\dots L_l^{a_l})(L_{l+1}^{a_{l+1}}L_{l+2}^{a_{l+2}}\dots L_n^{a_n}) & \text{if } l \geq m, \\ (L_{m+1}^{a_{m+1}}L_{m+2}^{a_{m+2}}\dots L_n^{a_n})(L_1^{a_1}L_2^{a_2}\dots L_m^{a_m}) & \text{if } l < m. \end{cases}$$

We see that

$$(t_{l+1}^{a_{l+1}}t_{l+2}^{a_{l+2}}\dots t_n^{a_n}) \in {}^{(l,\mu)}W \cap W^{(m,\nu)} \text{ if } l \geq m,$$
$$(t_{m+1}^{a_{m+1}}t_{m+2}^{a_{m+2}}\dots t_n^{a_n}) \in {}^{(l,\mu)}W \cap W^{(m,\nu)} \text{ if } l < m,$$

where we note that $I(e) = [m+1, n] \cap \{l+1, l+2, \dots, n\}$ for the identity element $e \in \mathfrak{S}_n$. Thus we have

$$(L_1^{a_1}L_2^{a_2}\dots L_l^{a_l}) \in \mathscr{H}_{(l,\mu)},$$
$$(L_1^{a_1}L_2^{a_2}\dots L_m^{a_m}) \in \mathscr{H}_{(m,\nu)},$$
$$(L_{l+1}^{a_{l+1}}L_{l+2}^{a_{l+2}}\dots L_n^{a_n}) \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)} \text{ if } l \geq m,$$
$$(L_{m+1}^{a_{m+1}}L_{m+2}^{a_{m+2}}\dots L_n^{a_n}) \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)} \text{ if } l < m$$

by (3.10.1). Then, using the above facts together with (3.10.4), we obtain $T_w \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)}$.

Suppose that $\ell(x) > 0$. We can uniquely write $x = x_1x_2x_3$ for some $x_2 \in {}^{(l,\mu)}\mathfrak{S}^{(m,\nu)}$, $x_1 \in (\mathfrak{S}_{(l,\mu)})^{\tau(x_2)}$ and $x_3 \in \mathfrak{S}_{(m,\nu)}$. It implies $\ell(x) = \ell(x_1) + \ell(x_2) + \ell(x_3)$ by the general theory of Coxeter group. Thus we deduce

$$T_w = T_{x_1}T_{x_2}T_{x_3}L_1^{a_1}L_2^{a_2}\dots L_n^{a_n} = T_{x_1}T_{x_2}T_{x_3}(L_{m+1}^{a_{m+1}}L_{m+2}^{a_{m+2}}\dots L_n^{a_n})(L_1^{a_1}L_2^{a_2}\dots L_m^{a_m}).$$

Applying Lemma 3.3, we have

(3.10.5)
$$\begin{aligned} T_w &= T_{x_1}T_{x_2}(L_{x_3(m+1)}^{a_{m+1}}L_{x_3(m+2)}^{a_{m+2}}\dots L_{x_3(n)}^{a_n})T_{x_3}(L_1^{a_1}L_2^{a_2}\dots L_m^{a_m}) \\ &\quad + \sum_{y_3<x_3}\sum_{(b_{m+1},\dots b_n)\in[0,r-1]^{n-m}} r_{y_3}^{(b_{m+1},\dots,b_n)}T_{x_1}T_{x_2}T_{y_3}(L_{m+1}^{b_{m+1}}L_{m+2}^{b_{m+2}}\dots L_n^{b_n})(L_1^{a_1}L_2^{a_2}\dots L_m^{a_m}). \end{aligned}$$

Since $T_{x_1}T_{x_2}T_{x_3} = T_x$, we obtain

(3.10.6) $$\begin{aligned} &\sum_{y_3<x_3}\sum_{(b_{m+1},\dots b_n)\in[0,r-1]^{n-m}} r_{y_3}^{(b_{m+1},\dots,b_n)}T_{x_1}T_{x_2}T_{y_3}(L_{m+1}^{b_{m+1}}L_{m+2}^{b_{m+2}}\dots L_n^{b_n})(L_1^{a_1}L_2^{a_2}\dots L_m^{a_m}) \\ &\qquad \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)}T_u\mathscr{H}_{(m,\nu)} \end{aligned}$$

by the assumption of the induction. Note that $\{x_3(m+1), x_3(m+2), \dots, x_3(n)\} = [m+1, n]$ by $x_3 \in \mathfrak{S}_{(m,\nu)}$. By Lemma 3.3, we have

$$T_{x_1}T_{x_2}(L_{x_3(m+1)}^{a_{m+1}}L_{x_3(m+2)}^{a_{m+2}}\dots L_{x_3(n)}^{a_n}) = T_{x_1}T_{x_2}(L_{m+1}^{a'_{m+1}}L_{m+2}^{a'_{m+2}}\dots L_n^{a'_n})$$

$$
\begin{aligned}
&= T_{x_1}(L_{x_2(m+1)}^{a'_{m+1}} L_{x_2(m+2)}^{a'_{m+2}} \dots L_{x_2(n)}^{a'_n}) T_{x_2} \\
&\quad + \sum_{y_2 < x_2} \sum_{(b_1,\dots,b_n)\in[0,r-1]^n} r_{y_2}^{(b_1,\dots,b_n)} T_{x_1} T_{y_2}(L_1^{b_1} L_2^{b_2} \dots L_n^{b_n}),
\end{aligned}
$$

where we put $a'_i = a_{x_3^{-1}(i)}$ for $i = m+1,\dots,n$. By the assumption of the induction, we deduce

$$
\sum_{y_2 < x_2} \sum_{(b_1,\dots,b_n)\in[0,r-1]^n} r_{y_2}^{(b_1,\dots,b_n)} T_{x_1} T_{y_2}(L_1^{b_1} L_2^{b_2} \dots L_n^{b_n}) \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)} T_u \mathscr{H}_{(m,\nu)}.
$$

By Lemma 3.3, we also have

$$
\begin{aligned}
T_{x_1}(L_{x_2(m+1)}^{a'_{m+1}} L_{x_2(m+2)}^{a'_{m+2}} \dots L_{x_2(n)}^{a'_n}) T_{x_2} &= T_{x_1}(\prod_{\substack{m+1\le i\le n \\ x_2(i)\le l}} L_{x_2(i)}^{a'_i})(\prod_{\substack{m+1\le i\le n \\ x_2(i)> l}} L_{x_2(i)}^{a'_i}) T_{x_2} \\
&= T_{x_1}(\prod_{\substack{m+1\le i\le n \\ x_2(i)\le l}} L_{x_2(i)}^{a'_i}) T_{x_2}(\prod_{i\in I(x_2)} L_i^{a'_i}) \\
&\quad - T_{x_1}(\prod_{\substack{m+1\le i\le n \\ x_2(i)\le l}} L_{x_2(i)}^{a'_i})(\sum_{y_2<x_2} \sum_{(b_1,\dots,b_n)\in[0,r-1]^n} r_{y_2}^{(b_1,\dots,b_n)} T_{y_2}(L_1^{b_1} L_2^{b_2} \dots L_n^{b_n})) \\
&\in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)} T_u \mathscr{H}_{(m,\nu)}
\end{aligned}
$$

since $T_{x_1}(\prod_{\substack{m+1\le i\le n \\ x_2(i)\le l}} L_{x_2(i)}^{a'_i}) \in \mathscr{H}_{(l,\mu)}$, $T_{x_2}(\prod_{i\in I(x_2)} L_i^{a'_i}) \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)} T_u \mathscr{H}_{(m,\nu)}$ by (3.10.1), and $T_{y_2}(L_1^{b_1} L_2^{b_2} \dots L_n^{b_n}) \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)} T_u \mathscr{H}_{(m,\nu)}$ for $y_2 < x_2$ again by the assumption of the induction. As a consequence, we obtain

$$
T_{x_1} T_{x_2}(L_{x_3(m+1)}^{a_{m+1}} L_{x_3(m+2)}^{a_{m+2}} \dots L_{x_3(n)}^{a_n}) \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)} T_u \mathscr{H}_{(m,\nu)},
$$

and this implies that

$$
T_{x_1} T_{x_2}(L_{x_3(m+1)}^{a_{m+1}} L_{x_3(m+2)}^{a_{m+2}} \dots L_{x_3(n)}^{a_n}) T_{x_3}(L_1^{a_1} L_2^{a_2} \dots L_m^{a_m}) \in \sum_{u\in {}^{(l,\mu)}W^{(m,\nu)}} \mathscr{H}_{(l,\mu)} T_u \mathscr{H}_{(m,\nu)}, \tag{3.10.7}
$$

since $T_{x_3}(L_1^{a_1} L_2^{a_2} \dots L_m^{a_m}) \in \mathscr{H}_{(m,\nu)}$. Thanks to (3.10.5), (3.10.6) and (3.10.7), we obtain (3.10.3), and hence we proved (i).

We prove (ii). For each $u \in {}^{(l,\mu)}W^{(m,\nu)}$, the set of elements

$$
\{T_{w_1} T_v \mid w_1 \in (W_{(l,\mu)})^{(k(u),\pi(u))},\ v \in W_{(k(u),\pi(u))}\}
$$

is an R-free basis of $\mathscr{H}_{(l,\mu)}$ by Proposition 3.4. Note that $T_u \mathscr{H}_{(m,\nu)}$ is a left $\mathscr{H}_{(k(u),\pi(u))}$-module by Proposition 3.6, then (i) implies that $\mathscr{H}_{n,r}$ is spanned by

$$
\{T_{w_1} T_u T_{w_2} \mid u \in {}^{(l,\mu)}W^{(m,\nu)},\ w_1 \in (W_{(l,\mu)})^{(k(u),\pi(u))},\ w_2 \in W_{(m,\nu)}\} = \{\widetilde{T}_w \mid w \in W_{n,r}\}
$$

as an R-module. Then we can define the surjective homomorphism of R-modules $\phi : \mathscr{H}_{n,r} \to \mathscr{H}_{n,r}$ such that $\phi(T_w) = \widetilde{T}_w$ $(w \in W_{n,r})$, and ϕ is an isomorphism by [19, Theorem 2.4]. Therefore, $\{\widetilde{T}_w \mid w \in W_{n,r}\}$ is an R-free basis of $\mathscr{H}_{n,r}$. □

3.11. For a parabolic subgroup $W_{(l,\mu)}$ of $W_{n,r}$ (resp. $W_{(k(u),\pi^\sharp(u))}$ of $W_{(m,\nu)}$), we define the restriction functor

$$^{\mathscr{H}}\mathrm{Res}^{W_{n,r}}_{W_{(l,\mu)}} : \mathscr{H}_{n,r}\text{-mod} \to \mathscr{H}_{(l,\mu)}\text{-mod}$$

$$(\text{resp. } ^{\mathscr{H}}\mathrm{Res}^{W_{(m,\nu)}}_{W_{(k(u),\pi^\sharp(u))}} : \mathscr{H}_{(m,\nu)}\text{-mod} \to \mathscr{H}_{(k(u),\pi^\sharp(u))}\text{-mod})$$

by the restriction of the action. We also define the induction functor

$$^{\mathscr{H}}\mathrm{Ind}^{W_{n,r}}_{W_{(l,\mu)}} = \mathscr{H}_{n,r} \otimes_{\mathscr{H}_{(l,\mu)}} - : \mathscr{H}_{(l,\mu)}\text{-mod} \to \mathscr{H}_{n,r}\text{-mod},$$

where we regard $\mathscr{H}_{n,r}$ as an $(\mathscr{H}_{n,r}, \mathscr{H}_{(l,\mu)})$-bimodule by multiplications.

For $u \in {}^{(l,\mu)}W^{(m,\nu)}$, we consider the induction functor

$$^{\mathscr{H}}\mathrm{Ind}^{W_{(l,\mu)}}_{W_{(k(u),\pi(u))}} = \mathscr{H}_{(l,\mu)} \otimes_{\mathscr{H}_{(k(u),\pi(u))}} - : \mathscr{H}_{(k(u),\pi(u))}\text{-mod} \to \mathscr{H}_{(l,\mu)}\text{-mod},$$

where we note that $\mathscr{H}_{(k(u),\pi(u))}$ is a subalgebra of $\mathscr{H}_{(l,\mu)}$. We also introduce the functor $T_u(-) : \mathscr{H}_{(k(u),\pi^\sharp(u))}\text{-mod} \to \mathscr{H}_{(k(u),\pi(u))}\text{-mod}$ by

$$T_u(-) = T_u\mathscr{H}_{(k(u),\pi^\sharp(u))} \otimes_{\mathscr{H}_{(k(u),\pi^\sharp(u))}} - : \mathscr{H}_{(k(u),\pi^\sharp(u))}\text{-mod} \to \mathscr{H}_{(k(u),\pi(u))}\text{-mod}\,.$$

We see that any functor defined in the above is exact by Proposition 3.4. Then now we obtain the first main theorem of this paper.

Theorem 3.12 (The Mackey formula for cyclotomic Hecke algebras)**.** *For $0 \le l, m \le n$, $\mu \vDash n-l$ and $\nu \vDash n-m$, we have the following:*

(i) *There exists an isomorphism of $(\mathscr{H}_{(l,\mu)}, \mathscr{H}_{(m,\nu)})$-bimodules*

$$\mathscr{H}_{n,r} \to \bigoplus_{u\in {}^{(l,\mu)}W^{(m,\nu)}} (\mathscr{H}_{(l,\mu)} \otimes_{\mathscr{H}_{(k(u),\pi(u))}} T_u\mathscr{H}_{(m,\nu)})$$

given by $\widetilde{T}_w = T_{w_1}T_uT_{w_2} \mapsto T_{w_1} \otimes T_uT_{w_2}$ where $u \in {}^{(l,\mu)}W^{(m,\nu)}$, $w_1 \in (W_{(l,\mu)})^{(k(u),\pi(u))}$ and $w_2 \in W_{(m,\nu)}$.

(ii) *For a left $\mathscr{H}_{(m,\nu)}$-module M, we have a natural isomorphism of left $\mathscr{H}_{(l,\mu)}$-modules*

$$\mathscr{H}_{n,r} \otimes_{\mathscr{H}_{(m,\nu)}} M \cong \bigoplus_{u\in {}^{(l,\mu)}W^{(m,\nu)}} (\mathscr{H}_{(l,\mu)} \otimes_{\mathscr{H}_{(k(u),\pi(u))}} T_u\mathscr{H}_{(m,\nu)}) \otimes_{\mathscr{H}_{(m,\nu)}} M.$$

(iii) *We have an isomorphism of functors*

$$^{\mathscr{H}}\mathrm{Res}^{W_{n,r}}_{W_{(l,\mu)}} \circ {}^{\mathscr{H}}\mathrm{Ind}^{W_{n,r}}_{W_{(m,\nu)}} \cong \bigoplus_{u\in {}^{(l,\mu)}W^{(m,\nu)}} {}^{\mathscr{H}}\mathrm{Ind}^{W_{(l,\mu)}}_{W_{(k(u),\pi(u))}} \circ T_u(-) \circ {}^{\mathscr{H}}\mathrm{Res}^{W_{(m,\nu)}}_{W_{(k(u),\pi^\sharp(u))}}.$$

Proof. We prove (i). Since $\{\widetilde{T}_w \mid w \in W_{n,r}\}$ is an R-free basis of $\mathscr{H}_{n,r}$ by Proposition 3.10 (ii), we can define a homomorphism of R-modules

$$\Phi : \mathscr{H}_{n,r} \to \bigoplus_{u\in {}^{(l,\mu)}W^{(m,\nu)}} (\mathscr{H}_{(l,\mu)} \otimes_{\mathscr{H}_{(k(u),\pi(u))}} T_u\mathscr{H}_{(m,\nu)}),$$

by $\widetilde{T}_w = T_{w_1}T_uT_{w_2} \mapsto T_{w_1} \otimes T_uT_{w_2}$ ($u \in {}^{(l,\mu)}W^{(m,\nu)}$, $w_1 \in (W_{(l,\mu)})^{(k(u),\pi(u))}$, $w_2 \in W_{(m,\nu)}$). In order to define the inverse map of Φ, for $u \in {}^{(l,\mu)}W^{(m,\nu)}$, let

$$\Psi'_u : \mathscr{H}_{(l,\mu)} \times T_u\mathscr{H}_{(m,\nu)} \to \mathscr{H}_{n,r},$$

be the multiplication map in $\mathscr{H}_{n,r}$. Since $T_u\mathscr{H}_{(m,\nu)}$ (resp. $\mathscr{H}_{(l,\mu)}$) is a left (resp. right) $\mathscr{H}_{(k(u),\pi(u))}$-module by multiplications in $\mathscr{H}_{n,r}$ (see Proposition 3.6), it is clear that Ψ'_u is a $\mathscr{H}_{(k(u),\pi(u))}$-balanced map. Thus we have the homomorphism of R-modules

$$\Psi_u : \mathscr{H}_{(l,\mu)} \otimes_{\mathscr{H}_{(k(u),\pi(u))}} T_u \mathscr{H}_{(m,\nu)} \to \mathscr{H}_{n,r}, \quad X \otimes Y \mapsto XY.$$

Then it is clear that $\Psi = \bigoplus_{u \in {}^{(l,\mu)}W^{(m,\nu)}} \Psi_u$ is the inverse map of Φ, and we see that Φ is isomorphism. Obviously, Ψ is an isomorphism of $(\mathscr{H}_{(l,\mu)}, \mathscr{H}_{(m,\nu)})$-bimodules since actions in both sides are given by multiplications. Thus Φ is also an isomorphism of $(\mathscr{H}_{(l,\mu)}, \mathscr{H}_{(m,\nu)})$-bimodules. (ii) follows from (i), and (iii) follows from (i) together with Corollary 3.8. □

4. The Mackey formula for the categories $\mathcal{O}$ of rational Cherednik algebras of type $G(r, 1, n)$

In this section, we discuss the Mackey formula for the categories $\mathcal{O}$ of the rational Cherednik algebras associated with the complex reflection group $W_{n,r}$.

4.1. Let W be a finite complex reflection group and let $\mathfrak{h}$ be the $\mathbb{C}$-vector space on which W acts by reflections. Let $\mathcal{A}_W$ be the set of reflection hyperplanes, and let $\mathfrak{h}_W^{reg} = \mathfrak{h} \setminus \bigcup_{H \in \mathcal{A}_W} H$ be its complement. We denote by $\mathcal{S}_W$ the set of reflections in W. For $s \in \mathcal{S}_W$, write λ_s for the non-trivial eigenvalue of s in $\mathfrak{h}^*$. For $s \in \mathcal{S}_W$, let $\alpha_s \in \mathfrak{h}^*$ be a generator of $\mathrm{Im}(s|_{\mathfrak{h}^*} - 1)$ and let $\alpha_s^\vee$ be the generator of $\mathrm{Im}(s|_{\mathfrak{h}} - 1)$ such that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$ where $\langle\ ,\ \rangle$ is the standard pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. Let $\mathcal{D}(\mathfrak{h}_W^{reg})$ be the $\mathbb{C}$-algebra of algebraic differential operators on the smooth affine manifold $\mathfrak{h}_W^{reg}$. The action of the group W on $\mathfrak{h}$ induces an action of W on the $\mathbb{C}$-algebra $\mathcal{D}(\mathfrak{h}_W^{reg})$. We denote the smash product of the algebra $\mathcal{D}(\mathfrak{h}_W^{reg})$ and the group W by $\mathcal{D}(\mathfrak{h}_W^{reg}) \rtimes W$. The rational Cherednik algebra $H(W) = H(W, \mathfrak{h})$ associated with W is a subalgebra of $\mathcal{D}(\mathfrak{h}_W^{reg}) \rtimes W$ which is generated by elements of $\mathbb{C}[\mathfrak{h}]$, elements of W and the Dunkl operators D_ξ for $\xi \in \mathfrak{h}$:

$$D_\xi = \partial_\xi + \sum_{s \in \mathcal{S}_W} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(\xi)}{\alpha_s}(s - 1) \in \mathcal{D}(\mathfrak{h}_W^{reg}) \rtimes W$$

where $\{c_s\}_{s \in \mathcal{S}_W}$ is the parameter of the algebra $H(W)$.

4.2. The category $\mathcal{O}(W)$ is a full subcategory of the category of finitely generated $H(W)$-modules on which object the Dunkl operators acts locally nilpotently. For a module $M \in \mathcal{O}(W)$, we consider the localization $M^{an} = \mathcal{O}^{an}_{\mathfrak{h}_W^{reg}} \otimes_{\mathbb{C}[\mathfrak{h}]} M$ where $\mathcal{O}^{an}_{\mathfrak{h}_W^{reg}}$ is the sheaf of holomorphic functions on $\mathfrak{h}_W^{reg}$. Since we have $\mathbb{C}[\mathfrak{h}_W^{reg}] \otimes_{\mathbb{C}[\mathfrak{h}]} H(W) = \mathcal{D}(\mathfrak{h}_W^{reg}) \rtimes W$, the algebra $\mathcal{D}(\mathfrak{h}_W^{reg}) \rtimes W$ acts on M^{an}. Considering the W-equivariant local system of horizontal sections together with the monodromy action of the fundamental group $\pi_1(\mathfrak{h}_W^{reg}/W, \bar{p}_0)$ for a certain fixed point $\bar{p}_0 \in \mathfrak{h}_W^{reg}/W$, we obtain the finite-dimensional vector space $\mathrm{KZ}_W(M)$. By [11], the monodromy action factors through a Hecke algebra $\mathscr{H}(W)$ associated with W with a parameter q determined by the formula in [11, Section 5.2], and we have the functor

$$\mathrm{KZ}_W : \mathcal{O}(W) \to \mathscr{H}(W)\text{-mod}, \quad M \mapsto \mathrm{KZ}_W(M).$$

4.3. For a parabolic subgroup W' of W, Bezrukavnikov and Etingof introduced the functors of parabolic restriction ${}^{\mathcal{O}}\mathrm{Res}^W_{W'}$ and induction ${}^{\mathcal{O}}\mathrm{Ind}^W_{W'}$ for modules of the category $\mathcal{O}$ in [3]. They are exact functors ${}^{\mathcal{O}}\mathrm{Res}^W_{W'} : \mathcal{O}(W) \longrightarrow \mathcal{O}(W')$, ${}^{\mathcal{O}}\mathrm{Ind}^W_{W'} : \mathcal{O}(W') \longrightarrow \mathcal{O}(W)$ between the categories $\mathcal{O}$ for the rational Cherednik algebras $H(W)$ and $H(W')$.

4.4. For a parabolic subgroup W' of W and an element $x \in W$, we have a $\mathbb{C}$-algebra isomorphism $\theta^{(x)}_{W'} : H(xW'x^{-1}) \longrightarrow H(W')$ given by $f \mapsto x^{-1}fx$ for $f \in H(xW'x^{-1})$. We define a functor

$$\Theta^{(x)}_{W'} : \mathcal{O}(W') \to \mathcal{O}(xW'x^{-1}), \quad M \mapsto M^{\theta^{(x)}_{W'}},$$

where $M^{\theta^{(x)}_{W'}} = M$ as vector spaces and the action is twisted by $\theta^{(x)}_{W'}$. We sometimes denote the functor $\Theta^{(u)}_{W'}$ by $u(-)$ and also denote $M^{\theta^{(u)}_{W'}}$ by uM when we need not notify the subgroup W'.

4.5. Consider the parabolic subgroups $W_{(l,\mu)}$ and $W_{(m,\nu)}$ of $W_{n,r}$. For a double coset representative $u = x \prod_{i \in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, we have $W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1} = W_{(k(u),\pi(u))}$ by Proposition 2.13, and $u^{-1}W_{(k(u),\pi(u))}u = W_{(k(u),\pi^\sharp(u))}$ by (2.11.7). Recall that we denote by $X_{(k(u),\pi(u))} \subset \{s_0, s_1, \dots, s_{n-1}\}$ (resp. $X_{(k(u),\pi^\sharp(u))}$) the set of standard generators of the parabolic subgroup $W_{(k(u),\pi(u))}$ (resp. $W_{(k(u),\pi^\sharp(u))}$).

4.6. Let $\mathfrak{h} = \mathbb{C}^n$ be the reflection representation of the complex reflection group $W_{n,r}$. The group $W_{n,r}$ is naturally identified with a finite quotient group of the fundamental group $B_{n,r} = \pi_1(\mathfrak{h}^{reg}_{W_{n,r}}/W_{n,r}, \bar{p}_0)$ of $\mathfrak{h}^{reg}_{W_{n,r}}/W_{n,r}$ with a fixed base point $\bar{p}_0 \in \mathfrak{h}^{reg}_{W_{n,r}}/W_{n,r}$. Similarly, the Hecke algebra $\mathscr{H}_{n,r}$ is the finite-dimensional quotient algebra of the group algebra $\mathbb{C}B_{n,r}$. For $u \in {}^{(l,\mu)}W^{(m,\nu)}$ and $s_j \in X_{(k(u),\pi(u))}$, we see that the identity $s_j u = u s_{\psi(j)}$ in Proposition 2.12 also holds in $B_{n,r}$ since the identity follows only from the braid relations, and so does in $\mathscr{H}_{n,r}$. The following lemma follows from the definition of KZ functors and the identity in $B_{n,r}$. Since this lemma is key to prove the Mackey formula for the category $\mathcal{O}$ by the lifting argument and it may not so clear for non-experts, we will give the proof of the lemma later in Appendix A.

Lemma 4.7. *For a double coset representative $u \in {}^{(l,\mu)}W^{(m,\nu)}$ and a module $M \in \mathcal{O}(W_{(k(u),\pi^\sharp(u))})$, we have the following isomorphism of functors* :

$$\mathrm{KZ}_{W_{(k(u),\pi(u))}} \circ u(-) \cong T_u(-) \circ \mathrm{KZ}_{W_{(k(u),\pi^\sharp(u))}} .$$

By the above lemma, we obtain the Mackey formula for the categories $\mathcal{O}$ as a corollary of Theorem 3.12.

Proposition 4.8. *We have the following isomorphism of functors* :

$$ {}^{\mathcal{O}}\mathrm{Res}^{W_{n,r}}_{W_{(l,\mu)}} \circ {}^{\mathcal{O}}\mathrm{Ind}^{W_{n,r}}_{W_{(m,\nu)}} \cong \bigoplus_{u \in {}^{(l,\mu)}W^{(m,\nu)}} {}^{\mathcal{O}}\mathrm{Ind}^{W_{(l,\mu)}}_{W_{(k(u),\pi(u))}} \circ u(-) \circ {}^{\mathcal{O}}\mathrm{Res}^{W_{(m,\nu)}}_{W_{(k(u),\pi^\sharp(u))}} .$$

Proof. The proof is the same with the proof of [16, Theorem 2.7.2]. By [22, Theorem 2.1] and Lemma 4.7, the KZ functors commute with the parabolic restriction functors and the twisting functors. Thus, we have the isomorphism of functors

$$\mathrm{KZ}_{W_{(l,\mu)}} \circ {}^{\mathcal{O}}\mathrm{Res}^{W_{n,r}}_{W_{(l,\mu)}} \circ {}^{\mathcal{O}}\mathrm{Ind}^{W_{n,r}}_{W_{(m,\nu)}} \cong \bigoplus_{u \in {}^{(l,\mu)}W^{(m,\nu)}} \mathrm{KZ}_{W_{(l,\mu)}} \circ {}^{\mathcal{O}}\mathrm{Ind}^{W_{(l,\mu)}}_{W_{(k(u),\pi(u))}} \circ u(-) \circ {}^{\mathcal{O}}\mathrm{Res}^{W_{(m,\nu)}}_{W_{(k(u),\pi^\sharp(u))}}$$

by Theorem 3.12. By [22, Lemma 2.4], this isomorphism implies the isomorphism of the proposition. □

4.9. In order to obtain the Mackey formula for Hecke algebras, we should take standard parabolic subgroups since we use explicit calculations using Ariki-Koike basis of $\mathscr{H}_{n,r}$. We also should take standard parabolic subgroups for its lifting to categories $\mathcal{O}$ in Proposition 4.8 since we need suitable corresponding identity in $B_{n,r}$ to one in Proposition 2.12. However, once we obtain the formula for standard parabolic subgroups in categories $\mathcal{O}$, we can extend it to the formula for any parabolic subgroups as follows. Note that any parabolic subgroup of $W_{n,r}$ coincides with $xW_{(l,\mu)}x^{-1}$ for some $l \geq 0$, $\mu \vDash n-l$ and $x \in W_{n,r}$. By applying the twisting functors $\Theta^{(x)}_{W_{(l,\mu)}}$, $\Theta^{(y^{-1})}_{yW_{(m,\nu)}y^{-1}}$ to the isomorphism of Proposition 4.8 and using the equivariance of the parabolic induction and restriction, we finally obtain the second main theorem of this paper, which supports Conjecture 0.1 :

Theorem 4.10 (The Mackey formula for $\mathcal{O}$ over cyclotomic rational Cherednik algebras)**.** *Let W_a, W_b be parabolic subgroups of $W_{n,r}$, and ${}^aW^b$ be a complete set of double coset representatives of $W_a\backslash W_{n,r}/W_b$. Then we have the following isomorphism of functors* :

$$^{\mathcal{O}}\mathrm{Res}^W_{W_a} \circ {}^{\mathcal{O}}\mathrm{Ind}^W_{W_b} \cong \bigoplus_{u\in {}^aW^b} {}^{\mathcal{O}}\mathrm{Ind}^{W_a}_{W_a\cap uW_bu^{-1}} \circ u(-) \circ {}^{\mathcal{O}}\mathrm{Res}^{W_b}_{u^{-1}W_au\cap W_b}.$$

Appendix A Proof of Lemma 4.7

In this appendix, we discuss the proof of Lemma 4.7. Though the argument is straightforward from definitions, we give a proof here for readers who are not so familiar with the KZ functor. In order to give the proof, we need to review the definition of the monodromy action of the Hecke algebra on the KZ functor, so most part of this appendix is devoted to review of the results of [6] and [11].

A.1. First, we review the definition of the action of the Hecke algebra on the KZ functor. For a module $M \in \mathcal{O}(W)$, we consider the localization $M^{an} = \mathcal{O}^{an}_{\mathfrak{h}^{reg}_W} \otimes_{\mathbb{C}[\mathfrak{h}]} M$ where $\mathcal{O}^{an}_{\mathfrak{h}^{reg}_W}$ is the sheaf of holomorphic functions on $\mathfrak{h}^{reg}_W$. Since we have $\mathbb{C}[\mathfrak{h}^{reg}_W] \otimes_{\mathbb{C}[\mathfrak{h}]} H(W) = \mathcal{D}(\mathfrak{h}^{reg}_W) \rtimes W$, the algebra $\mathcal{D}(\mathfrak{h}^{reg}_W) \rtimes W$ acts on M^{an}. Namely, M^{an} is a vector bundle on $\mathfrak{h}^{reg}_W$ with a W-equivariant flat connection. Let $(M^{an})^{\nabla}$ be the W-equivariant local system of horizontal sections of M^{an}. For any point $p \in \mathfrak{h}^{reg}_W$, the stalk $(M^{an})^{\nabla}_p$ at the point p is a finite-dimensional vector space over $\mathbb{C}$. Fix a point $p_0 \in \mathfrak{h}^{reg}_W$, and then we set $\mathrm{KZ}_W(M) = (M^{an})^{\nabla}_{p_0}$ as a vector space. Let $\bar{p}_0 \in \mathfrak{h}^{reg}_W/W$ be the image of p_0 under the projection $\mathfrak{h}^{reg}_W \longrightarrow \mathfrak{h}^{reg}_W/W$, and let $\pi_1(\mathfrak{h}^{reg}_W/W, \bar{p}_0)$ be the fundamental group of the space $\mathfrak{h}^{reg}_W/W$ with the base point $\bar{p}_0$. The vector space $\mathrm{KZ}_W(M)$ is naturally equipped with the action of the fundamental group $\pi_1(\mathfrak{h}^{reg}_W/W, \bar{p}_0)$ via monodromy as follows.

Let $[0,1] \subset \mathbb{R}$ be the closed interval between 0 and 1 (not an interval in $\mathbb{Z}$). For a path $\gamma : [0,1] \longrightarrow \mathfrak{h}^{reg}_W$ and a germ $v \in (M^{an})^{\nabla}_{\gamma(0)}$ at $\gamma(0)$, we have its analytic continuation $v' \in (M^{an})^{\nabla}_{\gamma(1)}$ at $\gamma(1)$ through the path γ. Then, we define an operator of analytic continuation

$$S_M(\gamma) : (M^{an})^{\nabla}_{\gamma(0)} \longrightarrow (M^{an})^{\nabla}_{\gamma(1)}, \qquad v \mapsto v'.$$

Following [6, §2.B.], recall how we obtain a homomorphism of $\pi_1(\mathfrak{h}^{reg}_W/W, \bar{p}_0)$ to W: Note that, for a loop $\sigma \in \pi_1(\mathfrak{h}^{reg}_W/W, \bar{p}_0)$ and a point $p \in Wp_0$, we have a unique path ${}^p\widetilde{\sigma}$: $[0,1] \longrightarrow \mathfrak{h}^{reg}_W$ such that ${}^p\widetilde{\sigma}(0) = p$ and its image in $\mathfrak{h}^{reg}_W/W$ coincides with σ. The path ${}^p\widetilde{\sigma}$ in $\mathfrak{h}^{reg}_W$ is called a lift of the element $\sigma \in \pi_1(\mathfrak{h}^{reg}_W/W, \bar{p}_0)$. As [6], we describe elements of $\pi_1(\mathfrak{h}^{reg}_W/W, \bar{p}_0)$ by their lifts (see [6, Appendix A]). For the above loop σ, we set $\overline{\sigma} = w \in W$

where w is an element with ${}^{p}\widetilde{\sigma}(1) = w(p)$.

For loops $\sigma, \sigma' \in \pi_1(\mathfrak{h}_W^{reg}/W, \bar{p}_0)$, we denote the composition of loops σ and σ' by $\sigma' \cdot \sigma$; i.e.

$$(\sigma' \cdot \sigma)(t) = \begin{cases} \sigma(2t) & (0 \leq t \leq 1/2) \\ \sigma'(2t-1) & (1/2 \leq t \leq 1) \end{cases}$$

Similarly, for paths γ, γ' in $\mathfrak{h}_W^{reg}$ with $\gamma(0) = p$, $\gamma(1) = \gamma'(0) = p'$ and $\gamma'(1) = p''$, define the composite path $\gamma' \cdot \gamma$ by

$$(\gamma' \cdot \gamma)(t) = \begin{cases} \gamma(2t) & (0 \leq t \leq 1/2) \\ \gamma'(2t-1) & (1/2 \leq t \leq 1) \end{cases}$$

It is a path from p to p''. For a path γ in $\mathfrak{h}_W^{reg}$ and an element $g \in W$, let $g(\gamma)$ be a path given by $(g(\gamma))(t) = g(\gamma(t))$.

For loops $\sigma, \sigma' \in \pi_1(\mathfrak{h}_W^{reg}/W, \bar{p}_0)$, the path $\overline{\sigma}({}^{p_0}\widetilde{\sigma}')$ is a lift of σ' with initial point $\overline{\sigma}(p_0)$, and thus the composite path $\overline{\sigma}({}^{p_0}\widetilde{\sigma}') \cdot {}^{p_0}\widetilde{\sigma}$ is a lift of the composite loop $\sigma' \cdot \sigma$. Then, we have

$$(\overline{\sigma}({}^{p_0}\widetilde{\sigma}') \cdot {}^{p_0}\widetilde{\sigma})(1) = \overline{\sigma}(\overline{\sigma}'(p_0)),$$

and hence $\overline{\sigma' \cdot \sigma} = \overline{\sigma}\,\overline{\sigma}' \in W$. That is, we have a homomorphism of groups ([6, (2.10)])

$$\overline{(-)} : \pi_1(\mathfrak{h}_W^{reg}/W, \bar{p}_0)^{\mathrm{opp}} \longrightarrow W.$$

Now we consider an action of the fundamental group given by monodromy

$$\widetilde{T}_M : \pi_1(\mathfrak{h}_W^{reg}/W, \bar{p}_0)^{\mathrm{opp}} \longrightarrow GL((M^{an})^{\nabla}_{p_0}), \qquad \sigma \mapsto S_M(({}^{p_0}\widetilde{\sigma})^{-1})\overline{\sigma}.$$

By [6, Theorem 4.12] and [11, Theorem 5.13], the linearly extended homomorphism $\widetilde{T}_M$ factors through an algebra homomorphism $\widetilde{T}_M : \mathscr{H}(W) \longrightarrow \mathrm{End}_{\mathbb{C}}((M^{an})^{\nabla}_{p_0})$. Then we obtain the functor

$$\mathrm{KZ}_W : \mathcal{O}(W) \to \mathscr{H}(W)\text{-mod}, \quad M \mapsto \mathrm{KZ}_W(M).$$

A.2. Let $W' \subset W$ be a parabolic subgroup and let $x \in W$ be an element. Recall the functor $\Theta^{(x)}_{W'} : H(xW'x^{-1}) \longrightarrow H(W')$ introduced in 4.4. For a module $M \in \mathcal{O}(W')$ and a point $p \in \mathfrak{h}_{W'}^{reg}$, the functor $\Theta^{(x)}_{W'}$ induces an isomorphism $\widehat{\Theta}^{(x)}_{W'} : (M^{an})^{\nabla}_p \longrightarrow (\Theta^{(x)}_{W'} M^{an})^{\nabla}_p \simeq (M^{an})^{\nabla}_{x(p)}$ of vector spaces, and we have the following commutative diagram:

$$\begin{array}{ccc} (M^{an})^{\nabla}_{\gamma(0)} & \xrightarrow{S_M(\gamma)} & (M^{an})^{\nabla}_{\gamma(1)} \\ \Big\downarrow{\scriptstyle \widehat{\Theta}^{(x)}_{W'}} & & \Big\downarrow{\scriptstyle \widehat{\Theta}^{(x)}_{W'}} \\ (M^{an})^{\nabla}_{x(\gamma)(0)} & \xrightarrow[S_{\Theta^{(x)}_{W'}(M)}(x(\gamma))]{} & (M^{an})^{\nabla}_{x(\gamma)(1)} \end{array}$$

for a path γ in $\mathfrak{h}_{W'}^{reg}$. Here $x(\gamma)$ is the path given by $x(\gamma)(t) = x(\gamma(t))$.

A.3. Now we consider the case of the complex reflection group $W_{n,r}$. Consider the parabolic subgroups $W_{(l,\mu)}$ and $W_{(m,\nu)}$ of $W_{n,r}$. For a double coset representative $u = x \prod_{i \in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, we have $W_{(l,\mu)} \cap uW_{(m,\nu)}u^{-1} = W_{(k(u),\pi(u))}$ by Proposition 2.13, and $u^{-1}W_{(k(u),\pi(u))}u =$

$W_{(k(u),\pi^\sharp(u))}$ by (2.11.7). Recall that we denote by $X_{(k(u),\pi(u))} \subset \{s_0, s_1, \ldots, s_{n-1}\}$ (resp. $X_{(k(u),\pi^\sharp(u))}$) the set of standard generators of the parabolic subgroup $W_{(k(u),\pi(u))}$ (resp. $W_{(k(u),\pi^\sharp(u))}$). We sometimes denote the functor $\Theta^{(u)}_{W_{(k(u),\pi^\sharp(u))}}$ by $u(-)$ and also denote $M^{\theta^{(u)}_{W_{(k(u),\pi^\sharp(u))}}}$ by uM when we need not notify the subgroup $W_{(k(u),\pi^\sharp(u))}$.

A.4. Let $B_{n,r} = \pi_1(\mathfrak{h}^{reg}_{W_{n,r}}/W_{n,r}, \bar{p}_0)$ be the fundamental group of the space $\mathfrak{h}^{reg}_{W_{n,r}}/W_{n,r}$. It is the braid group associated with $W_{n,r}$. For $j = 0, 1, \ldots, n-1$, we fix a generator $\sigma_j \in B_{n,r}$ of the braid group given in [6, §2B] such that $\overline{\sigma}_j = s_j$. Then the image of $\sigma_0, \ldots, \sigma_{n-1}$ in $\mathscr{H}_{n,r} = \mathscr{H}(W_{n,r})$ are $T_0, \ldots, T_{n-1} \in \mathscr{H}_{n,r}$, the generators of the Hecke algebra $\mathscr{H}_{n,r}$ which we introduced in 3.1. For $i = 1, \ldots, n$, we set $\gamma_i = \sigma_{i-1}\sigma_{i-2}\ldots\sigma_1\sigma_0\sigma_1\ldots\sigma_{i-1}$, an element in $B_{n,r}$. Then, its image in $\mathscr{H}_{n,r}$ is $q^{i-1}L_i$ and we have $\overline{\gamma}_i = t_i$. Note that these elements $\gamma_1, \ldots, \gamma_n$ mutually commute since the commutativity of $t_1, \ldots, t_n \in W_{n,r}$ is obtained only by using the braid relations. For the double coset representative $u = x\prod_{i\in I(x)} t_i^{a_i} \in {}^{(l,\mu)}W^{(m,\nu)}$, we consider an element $\omega = (\prod_{i\in I(x)} \gamma_i^{a_i})\sigma_{i_l}\ldots\sigma_{i_1} \in B_{n,r}$ where $x = s_{i_1}\ldots s_{i_l}$ is a reduced expression of $x \in \mathfrak{S}_n$. Then, we have $\overline{\omega} = u$. By Proposition 2.12, for $s_j \in X_{(k(u),\pi(u))}$, there exists $s_{\psi(j)} \in X_{(k(u),\pi^\sharp(u))}$ such that $s_j(s_{i_1}s_{i_2}\ldots s_{i_l}\prod_{i\in I(x)} t_i^{a_i}) = (s_{i_1}s_{i_2}\ldots s_{i_l}\prod_{i\in I(x)} t_i^{a_i})s_{\psi(j)}$, and this identity can be lifted to the identity

$$\omega\sigma_j = \sigma_{\psi(j)}\omega \tag{A.4.1}$$

in the braid group $B_{n,r}$.

By the embedding of [6, §2D], we identify the braid group $B_{(k(u),\pi(u))}$ (resp. $B_{(k(u),\pi^\sharp(u))}$) associated with the parabolic subgroup $W_{(k(u),\pi(u))}$ (resp. $W_{(k(u),\pi^\sharp(u))}$) with the subgroup of $B_{n,r}$ generated by the standard generators $\{\sigma_j \mid s_j \in X_{(k(u),\pi(u))}\}$ (resp. $\{\sigma_j \mid s_j \in X_{(k(u),\pi^\sharp(u))}\}$). See also [22, Section 2.2] for the embedding of parabolic subgroups.

Now we prove Lemma 4.7.

Lemma A.5 (Lemma 4.7). *For a double coset representative $u \in {}^{(l,\mu)}W^{(m,\nu)}$ and a module $M \in \mathcal{O}(W_{(k(u),\pi^\sharp(u))})$, we have the following isomorphism of functors* :

$$\mathrm{KZ}_{W_{(k(u),\pi(u))}} \circ u(-) \cong T_u(-) \circ \mathrm{KZ}_{W_{(k(u),\pi^\sharp(u))}}.$$

Proof. Note that an $\mathscr{H}_{(k(u),\pi(u))}$-module T_uN for an $\mathscr{H}_{(k(u),\pi^\sharp(u))}$-module N is isomorphic to N as a vector space by the map $N \longrightarrow T_uN$, $v \mapsto T_uv$ and the action of $T_z \in \mathscr{H}_{(k(u),\pi(u))}$ corresponding to $z \in W_{(k(u),\pi(u))}$ on T_uN is given by $T_zT_uv = T_u(T_{u^{-1}zu}v)$ for $v \in N$. For a module $M \in \mathcal{O}(W_{(k(u),\pi^\sharp(u))})$, we define a map

$$\begin{aligned}\kappa^{(u)} : (T_u(-) \circ \mathrm{KZ}_{W_{(k(u),\pi^\sharp(u))}})(M) &\longrightarrow (\mathrm{KZ}_{W_{(k(u),\pi(u))}} \circ \Theta^{(u)}_{W_{(k(u),\pi^\sharp(u))}})(M),\\ T_u\, v &\mapsto \widehat{\Theta}^{(u)}_{W_{(k(u),\pi^\sharp(u))}} \circ S_M(u^{-1}(({}^{p_0}\widetilde{\omega})^{-1}))(v)\end{aligned}$$

for $v \in \mathrm{KZ}_{W_{(k(u),\pi^\sharp(u))}}(M) = (M^{an})^{\nabla}_{p_0}$. Here we remark that we have $\overline{\omega} = u$ and $u^{-1}(({}^{p_0}\widetilde{\omega})^{-1})$ is a path from p_0 to $u^{-1}(p_0)$. Obviously $\kappa^{(u)}$ is an isomorphism of $\mathbb{C}$-vector spaces. We see that the map $\kappa^{(u)}$ commutes with the action of the Hecke algebra $\mathscr{H}_{(k(u),\pi(u))}$ by direct computation as follows: For $v \in \mathrm{KZ}_{W_{(k(u),\pi^\sharp(u))}}(M) = (M^{an})^{\nabla}_{p_0}$ and $T_i \in \mathscr{H}_{(k(u),\pi(u))}$ corresponding to the generator $s_i \in X_{(k(u),\pi(u))}$, we have

$$\kappa^{(u)}(T_i(T_u v)) = \kappa^{(u)}(T_u(T_{\psi(i)}v))$$
$$= \widehat{\Theta}^{(u)}_{W_{(k(u),\pi^\sharp(u))}} \circ S_M(u^{-1}(({}^{p_0}\widetilde{\omega})^{-1}))(S_M(({}^{p_0}\widetilde{\sigma}_{\psi(i)})^{-1})s_{\psi(i)}v)$$
$$= S_{uM}(({}^{p_0}\widetilde{\omega})^{-1} \cdot u(({}^{p_0}\widetilde{\sigma}_{\psi(i)})^{-1}))s_i \circ \widehat{\Theta}^{(u)}_{W_{(k(u),\pi^\sharp(u))}}(v).$$

Here we can deduce that the path $({}^{p_0}\widetilde{\omega})^{-1} \cdot u(({}^{p_0}\widetilde{\sigma}_{\psi(i)})^{-1}) = (u({}^{p_0}\widetilde{\sigma}_{\psi(i)}) \cdot {}^{p_0}\widetilde{\omega})^{-1} = ({}^{p_0}\widetilde{\sigma_{\psi(i)} \cdot \omega})^{-1}$ is lifted from the element $(\sigma_{\psi(i)} \cdot \omega)^{-1} \in B_{n,r}$, being equal to $(\omega \cdot \sigma_i)^{-1} \in B_{n,r}$ by (A.4.1). Thus, by the uniqueness of lifting from the fixed initial base point $(us_{\psi(i)})(p_0) = (s_i u)(p_0)$, we have

$$S_{uM}(({}^{p_0}\widetilde{\sigma_{\psi(i)} \cdot \omega})^{-1}) = S_{uM}(({}^{p_0}\widetilde{\omega \cdot \sigma_i})^{-1}) = S_{uM}(({}^{p_0}\widetilde{\sigma}_i)^{-1} \cdot s_i(({}^{p_0}\widetilde{\omega})^{-1})).$$

Therefore, we have

$$\kappa^{(u)}(T_i(T_u v)) = S_{uM}(({}^{p_0}\widetilde{\omega})^{-1} \cdot u(({}^{p_0}\widetilde{\sigma}_{\psi(i)})^{-1}))s_i \circ \widehat{\Theta}^{(u)}_{W_{(k(u),\pi^\sharp(u))}}(v)$$
$$= S_{uM}(({}^{p_0}\widetilde{\sigma}_i)^{-1} \cdot s_i(({}^{p_0}\widetilde{\omega})^{-1}))s_i \circ \widehat{\Theta}^{(u)}_{W_{(k(u),\pi^\sharp(u))}}(v)$$
$$= (S_{uM}(({}^{p_0}\widetilde{\sigma}_i)^{-1})s_i)(S_{uM}(({}^{p_0}\widetilde{\omega})^{-1}) \circ \widehat{\Theta}^{(u)}_{W_{(k(u),\pi^\sharp(u))}}(v)) = T_i \cdot \kappa^{(u)}(T_u v).$$

That is, the map $\kappa^{(u)}$ is a homomorphism of $\mathscr{H}_{(k(u),\pi(u))}$-modules. It is clear from the definition that $\kappa^{(u)}$ is functorial, and hence we have the desired isomorphism of functors. □

Appendix B A root system for $G(r, 1, n)$

In this appendix, we explain some connection with a root system for the complex reflection group of type $G(r, 1, n)$ introduced in [5]. We use notation and results given in [20].

B.1. Let V be a complex vector space with a basis $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$. Let $\zeta = \exp(2\pi\sqrt{-1}/r)$ be the primitive r-th root of unity. Then $W_{n,r}$ acts on V by

$$(xt_1^{a_1} \dots t_n^{a_n}) \cdot \epsilon_i = \zeta^{a_i}\epsilon_{x(i)} \quad (x \in \mathfrak{S}_n,\ 0 \le a_1, \dots, a_n \le r-1,\ 1 \le i \le n).$$

For $i = 1, \dots, n-1$, a vector $\epsilon_{i+1} - \epsilon_i$ is orthogonal to the reflection hyperplane corresponding the reflection s_i, and a vector ϵ_1 is orthogonal to the reflection hyperplane corresponding the reflection s_0. Put

$$\overline{\Delta} = \{\epsilon_{i+1} - \epsilon_i \mid 1 \le i \le n-1\} \cup \{\epsilon_1\}$$

and put $\overline{\Phi} = W_{n,r} \cdot \overline{\Delta}$. Then we have

$$\overline{\Phi} = \{\zeta^a\epsilon_i - \zeta^b\epsilon_j \mid 1 \le i \ne j \le n,\ 0 \le a, b \le r-1\} \cup \{\zeta^a\epsilon_i \mid 1 \le i \le n,\ 0 \le a \le r-1\}.$$

B.2. In this appendix, we identify elements $a \in \mathbb{Z}/r\mathbb{Z}$ with integers $0 \le a \le r-1$. We consider a set $X = \{e_i^{(a)} \mid 1 \le i \le n,\ a \in \mathbb{Z}/r\mathbb{Z}\}$, where $e_i^{(a)}$ is just a symbol indexed by i and a. One can define an action of $W_{n,r}$ on X by

$$(xt_1^{a_1} \dots t_n^{a_n}) \cdot e_i^{(a)} = e_{x(i)}^{(a+a_i)} \quad (x \in \mathfrak{S}_n,\ 0 \le a_1, \dots, a_n \le r-1,\ 1 \le i \le n,\ a \in \mathbb{Z}/r\mathbb{Z}).$$

We also define another action of $W_{n,r}$ on X by

$$(xt_1^{a_1} \dots t_n^{a_n}) * e_i^{(a)} = e_{x(i)}^{(a-a_i)} \quad (x \in \mathfrak{S}_n,\ 0 \le a_1, \dots, a_n \le r-1,\ 1 \le i \le n,\ a \in \mathbb{Z}/r\mathbb{Z}).$$

We express an element $(e_i^{(a)}, e_j^{(b)}) \in X \times X$ as $e_i^{(a)} - e_j^{(b)}$ in the case where $i \neq j$. Then we define a root system Φ for $W_{n,r}$ by

$$\Phi = \{e_i^{(a)} - e_j^{(b)} \mid 1 \leq i \neq j \leq n,\ a, b \in \mathbb{Z}/r\mathbb{Z}\} \cup \{e_i^{(a)} \mid 1 \leq i \leq n,\ a \in \mathbb{Z}/r\mathbb{Z}\}.$$

We define subsets Φ_0, Ω and Δ of Φ by

$$\begin{aligned}
\Phi_0 &= \{e_i^{(a)} - e_j^{(b)} \in \Phi \mid i > j,\ a = 0\} \cup \{e_i^{(a)} - e_j^{(b)} \mid i < j,\ b \neq 0\} \cup \{e_i^{(0)} \mid 1 \leq i \leq n\},\\
\Omega &= \{e_i^{(0)} - e_j^{(b)} \mid 1 \leq j < i \leq n,\ b \in \mathbb{Z}/r\mathbb{Z}\} \cup \{e_i^{(0)} \mid 1 \leq i \leq n\},\\
\Delta &= \{e_{i+1}^{(0)} - e_i^{(0)} \mid 1 \leq i \leq n-1\} \cup \{e_1^{(0)}\}
\end{aligned}$$

Let $\varphi : \Phi \to V$ be a map such that $\varphi(e_i^{(a)} - e_j^{(b)}) = \zeta^a \epsilon_i - \zeta^b \epsilon_j$ and $\varphi(e_i^{(a)}) = \zeta^a \epsilon_i$, then we see that $\varphi(\Phi) = \overline{\Phi}$ and $\varphi(\Delta) = \overline{\Delta}$.

REMARK B.3. (i). In the case where $r = 2$ (in this case, $W_{n,2}$ coincides with the Weyl group of type B_n), we see that $\overline{\Phi}$ (resp. $\overline{\Delta}$) coincides with a root system (resp. a set of simple roots) for the Weyl group of type B_n. Moreover, $\varphi(\Omega)$ coincides with the set of positive roots with respect to $\overline{\Delta}$ of the Weyl group of type B_n.

(ii). In general case, Ω is not a positive root in the sense of [5], but Ω plays the role of positive roots. Moreover, in this appendix, we follow notion in [20], and the definitions of Φ and Ω are different from them in [5]. See [20, Remark 1.4] for these differences.

B.4. For $0 \leq l \leq n$ and $\mu \vDash n - l$, we obtain the root system $\Phi_{(l,\mu)}$ and subsets $\Omega_{(l,\mu)}, \Delta_{(l,\mu)} \subset \Phi_{(l,\mu)}$ for the parabolic subgroup $W_{(l,\mu)}$ of $W_{n,r}$ by

$$\begin{aligned}
\Phi_{(l,\mu)} &= \{e_i^{(a)} - e_j^{(b)} \mid 1 \leq i \neq j \leq l,\ a, b \in \mathbb{Z}/r\mathbb{Z}\} \cup \{e_i^{(a)} \mid 1 \leq i \leq l,\ a \in \mathbb{Z}/r\mathbb{Z}\}\\
&\quad \cup \bigcup_{p=1}^{\ell(\mu)} \{e_i^{(0)} - e_j^{(0)} \mid l + |\mu|_{p-1} + 1 \leq i \neq j \leq l + |\mu|_p\},\\
\Omega_{(l,\mu)} &= \{e_i^{(0)} - e_j^{(b)} \mid 1 \leq j < i \leq l,\ b \in \mathbb{Z}/r\mathbb{Z}\} \cup \{e_i^{(0)} \mid 1 \leq i \leq l\}\\
&\quad \cup \bigcup_{p=1}^{\ell(\mu)} \{e_i^{(0)} - e_j^{(0)} \mid l + |\mu|_{p-1} + 1 \leq j < i \leq l + |\mu|_p\},\\
\Delta_{(l,\mu)} &= \{e_{i+1}^{(0)} - e_i^{(0)} \mid 1 \leq i \leq l-1\} \cup \{e_1^{(0)} \mid l \neq 0\}\\
&\quad \cup \bigcup_{p=1}^{\ell(\mu)} \{e_{i+1}^{(0)} - e_i^{(0)} \mid l + |\mu|_{p-1} + 1 \leq i \leq l + |\mu|_p - 1\},
\end{aligned}$$

where we put $|\mu|_p = \sum_{k=1}^{p} \mu_k$ with $|\mu|_0 = 0$, and $\{e_1^{(0)} \mid l \neq 0\} = \begin{cases} \{e_i^{(0)}\} & \text{if } l \neq 0, \\ \emptyset & \text{if } l = 0. \end{cases}$ We also define

$$\begin{aligned}
\widetilde{\Omega}_{(l,\mu)} &= \{e_i^{(0)} - e_j^{(b)} \mid 1 \leq j < i \leq l,\ b \in \mathbb{Z}/r\mathbb{Z}\} \cup \{e_i^{(0)} \mid 1 \leq i \leq l\}\\
&\quad \cup \bigcup_{p=1}^{\ell(\mu)} \{e_i^{(0)} - e_j^{(b)} \mid l + |\mu|_{p-1} + 1 \leq j < i \leq l + |\mu|_p,\ b \in \mathbb{Z}/r\mathbb{Z}\}.
\end{aligned}$$

Then we have $\Omega_{(l,\mu)} \subset \widetilde{\Omega}_{(l,\mu)}$, and we also have $\Omega_{(l,\mu)} = \widetilde{\Omega}_{(l,\mu)}$ if and only if $\mu = (1^{n-l})$.

B.5. For $w \in W_{n,r}$, let $\ell(w)$ be the smallest number k such that w is expressed as a product $w = s_{i_1} \dots s_{i_k}$ ($s_{i_j} \in \{s_0, s_1, \dots, s_{n-1}\}$).

For $0 \le l \le n$ and $\mu \vDash n-l$, we define a subsets $\mathcal{R}_{(l,\mu)}$, $\mathcal{R}^*_{(l,\mu)}$, $\mathcal{R}^0_{(l,\mu)}$ and $\mathcal{R}^{*0}_{(l,\mu)}$ of $W_{n,r}$ by

$$\mathcal{R}_{(l,\mu)} = \{w \in W \mid w(\Omega_{(l,\mu)}) \subset \Phi_0\}, \qquad \mathcal{R}^*_{(l,\mu)} = \{w \in W \mid w^{-1} * \Omega_{(l,\mu)} \subset \Phi_0\},$$
$$\mathcal{R}^0_{(l,\mu)} = \{w \in W \mid w(\widetilde{\Omega}_{(l,\mu)}) \subset \Phi_0\}, \qquad \mathcal{R}^{*0}_{(l,\mu)} = \{w \in W \mid w^{-1} * \widetilde{\Omega}_{(l,\mu)} \subset \Phi_0\}.$$

Since $\Omega_{(l,\mu)} \subset \widetilde{\Omega}_{(l,\mu)}$, we have $\mathcal{R}^0_{(l,\mu)} \subset \mathcal{R}_{(l,\mu)}$ (resp. $\mathcal{R}^{*0}_{(l,\mu)} \subset \mathcal{R}^*_{(l,\mu)}$). The following proposition is proven in [20, Lemma 1.27, Proposition 1.28, Corollary 1.29].

Proposition B.6. *For $0 \le l \le n$ and $\mu \vDash n-l$, we have the following:*

(i) (a) *For $w \in W_{n,r}$, we have $w(\Omega_{(l,\mu)}) \subset \Phi_0$ if and only if $w(\Delta_{(l,\mu)}) \subset \Phi_0$.*
(b) *For $w \in W_{n,r}$, we have $w^{-1} * \Omega_{(l,\mu)} \subset \Phi_0$ if and only if $w^{-1} * \Delta_{(l,\mu)} \subset \Phi_0$.*

(ii) (a) *For $w \in \mathcal{R}^0_{(l,\mu)}$ and $w' \in W_{(l,\mu)}$, we have $\ell(ww') = \ell(w) + \ell(w')$.*
(b) *For $w \in \mathcal{R}^*_{(l,\mu)}$ and $w' \in W_{(l,\mu)}$, we have $\ell(w'w) = \ell(w') + \ell(w)$.*

(iii) (a) *For $w \in W_{n,r}$, if $\ell(w)$ is minimal among all elements in $wW_{(l,\mu)}$, we have $w \in \mathcal{R}_{(l,\mu)}$.*
(b) *For $w \in W_{n,r}$, if $\ell(w)$ is minimal among all elements in $W_{(l,\mu)}w$, we have $w \in \mathcal{R}^*_{(l,\mu)}$.*

(iv) (a) *In the case where $\mathcal{R}_{(l,\mu)} = \mathcal{R}^0_{(l,\mu)}$, the set $\mathcal{R}_{(l,\mu)}$ is a complete set of coset representatives for $W_{n,r}/W_{(l,\mu)}$.*
(b) *In the case where $\mathcal{R}^*_{(l,\mu)} = \mathcal{R}^{*0}_{(l,\mu)}$, the set $\mathcal{R}^*_{(l,\mu)}$ is a complete set of coset representatives for $W_{(l,\mu)}\backslash W_{n,r}$.*

B.7. Assume that $l \neq 0$ and $\mu = (1^{n-l})$. In this case, we have $W_{(l,\mu)} = W_{l,r}$, $\Omega_{(l,\mu)} = \widetilde{\Omega}_{(l,\mu)}$, $\mathcal{R}_{(l,\mu)} = \mathcal{R}^0_{(l,\mu)}$ and $\Delta_{(l,\mu)} = \{e^{(0)}_{i+1} - e^{(0)}_i \mid 1 \le i \le l-1\} \cup \{e^{(0)}_1\}$. For $xt^{a_{l+1}}_{l+1} t^{a_{l+2}}_{l+2} \dots t^{a_n}_n \in W^{(l,\mu)}$ and $e^{(0)}_{i+1} - e^{(0)}_i$ ($1 \le i \le l-1$), we have

$$(xt^{a_{l+1}}_{l+1} t^{a_{l+2}}_{l+2} \dots t^{a_n}_n) \cdot (e^{(0)}_{i+1} - e^{(0)}_i) = e^{(0)}_{x(i+1)} - e^{(0)}_{x(i)},$$

and $x(i+1) > x(i)$ since $x \in \mathfrak{S}^{(l,\mu)}$ and $s_i \in S_{(l,\mu)}$. We also have

$$(xt^{a_{l+1}}_{l+1} t^{a_{l+2}}_{l+2} \dots t^{a_n}_n) \cdot e^{(0)}_1 = e^{(0)}_{x(1)}.$$

Thus we see that $(xt^{a_{l+1}}_{l+1} t^{a_{l+2}}_{l+2} \dots t^{a_n}_n)(\Delta_{(l,\mu)}) \subset \Phi_0$ for any $xt^{a_{l+1}}_{l+1} t^{a_{l+2}}_{l+2} \dots t^{a_n}_n \in W^{(l,\mu)}$. Then, by Proposition B.6 (i), we have that $W^{(l,\mu)} \subset \mathcal{R}_{(l,\mu)}$. On the other hand, $W^{(l,\mu)}$ (resp. $\mathcal{R}_{(l,\mu)}$) is a complete set of representatives for $W/W_{(l,\mu)}$ by Lemma 2.3 (resp. Proposition B.6 (iv)). Thus we have $W^{(l,\mu)} = \mathcal{R}_{(l,\mu)}$ if $\mu = (1^{n-l})$. Similarly, we have ${}^{(l,\mu)}W = \mathcal{R}^*_{(l,\mu)}$ if $\mu = (1^{n-l})$. Moreover we have ${}^{(l,\mu)}W^{(m,\nu)} = {}^{(l,\mu)} W \cap W^{(m,\nu)}$ if $\nu = (1^{n-m})$ by Lemma 2.7. As a consequence, we have the following corollary.

Corollary B.8. *Assume that $\mu = (1^{n-l})$ and $\nu = (1^{n-m})$. Then we have ${}^{(l,\mu)}W = \mathcal{R}^*_{(l,\mu)}$ and $W^{(m,\nu)} = \mathcal{R}_{(m,\nu)}$. Moreover, $\mathcal{R}^*_{(l,\mu)} \cap \mathcal{R}_{(m,\nu)} = {}^{(l,\mu)}W \cap W^{(m,\nu)}$ is a complete set of representatives for $W_{(l,\mu)}\backslash W_{n,r}/W_{(m,\nu)}$.*

Remark B.9. (i). In the case where $\mu \neq (1^{n-l})$, there exists $i > l$ such that $e_{i+1}^{(0)} - e_i^{(0)} \in \Delta_{(l,\mu)}$. For $e_{i+1}^{(0)} - e_i^{(0)} \in \Delta_{(l,\mu)}$ such that $i > l$ and $xt_{l+1}^{a_{l+1}} \dots t_n^{a_n} \in W^{(l,\mu)}$, we have

$$(xt_{l+1}^{a_{l+1}} \dots t_n^{a_n}) \cdot (e_{i+1}^{(0)} - e_i^{(0)}) = e_{x(i+1)}^{(a_{i+1})} - e_{x(i)}^{(a_i)},$$

and $x(i+1) > x(i)$ since $x \in \mathfrak{S}^{(l,\mu)}$ and $s_i \in S_{(l,\mu)}$. Moreover, $e_{x(i+1)}^{(a_{i+1})} - e_{x(i)}^{(a_i)} \notin \Phi_0$ if $a_{i+1} \neq 0$. Thus, we see that $W^{(l,\mu)} \not\subset \mathcal{R}_{(l,\mu)}$ if $\mu \neq (1^{n-l})$. Similarly, we have ${}^{(l,\mu)}W \not\subset \mathcal{R}^*_{(l,\mu)}$ if $\mu \neq (1^{n-l})$.

(ii). In general case, we do not know if we can characterize the set $W^{(l,\mu)}$ (or another complete set of representatives for $W/W_{(l,\mu)}$) by using the root system Φ.

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