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COHERENT STATES ASSOCIATED TO THE JACOBI GROUP - A VARIATION ON A THEME BY ERICH KÄHLER

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Abstract. Using the coherent states attached to the complex Jacobi group – the semi-direct product of the Heisenberg-Weyl group with the real symplectic group – we study some of the properties of coherent states based on the manifold which is the product of the n-dimensional complex plane with the Siegel upper half plane.

1. Introduction

In this paper we continue the investigation of the Jacobi group [7,8,16] – the semi-direct product of the Heisenberg-Weyl group and the symplectic group – started in [4,5], using Perelomov's coherent states (CS). The Jacobi group is an important object in connection with Quantum Mechanics, Geometric Quantization, Optics, etc., [2, 9, 14, 15, 18, 20].

Applying the methods developed in [3], in [4] we have constructed generalized CS attached to the Jacobi group $G^J_1 = H_1 \rtimes \mathrm{SU}(1,1)$, based on the homogeneous Kähler manifold $\mathcal{D}^J_1 = H_1/\mathbb{R} \times \mathrm{SU}(1,1)/\mathrm{U}(1) = \mathbb{C}^1 \times \mathcal{D}_1$. Here $\mathcal{D}_1$ denotes the unit disk $\mathcal{D}_1 = \{w \in \mathbb{C};\, |w| < 1\}$, and H_n is the $(2n+1)$-dimensional real Heisenberg-Weyl group with Lie algebra $\mathfrak{h}_n$. In [4] we have also emphasized that, when expressed in appropriate coordinates on the manifold $\mathcal{X}^J_1 = \mathbb{C} \times \mathcal{H}_1$, $\mathcal{H}_1 = \{v \in \mathbb{C};\, \mathrm{Im}(v) > 0\}$, the Kähler two-form ω_1 is identical with the one considered by Kähler-Berndt [6, 7, 10–12].

In [5] we have considered coherent states attached to the Jacobi group $G^J_n = H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$, based on the manifold $\mathcal{D}^J_n = \mathbb{C}^n \times \mathcal{D}_n$, where $\mathcal{D}_n$ is the Siegel ball $\mathcal{D}_n = \{W \in M(n,\mathbb{C});\, W = W^t, 1 - W\bar{W} > 0\}$. In this paper we calculate the Kähler two-form ω'_n on the manifold $\mathcal{X}^J_n = \mathbb{C}^n \times \mathcal{H}_n$, where $\mathcal{H}_n$ is the Siegel upper half plane obtained by the Cayley transform of the Siegel ball $\mathcal{D}_n$. This ω'_n is a "n"-dimensional generalization of Kähler-Berndt's two-form ω'_1 on $\mathcal{X}^J_1$ to the corresponding one on $\mathcal{X}^J_n$. The physical relevance of these results follows from the fact that 'the "gaussons" [18] can be considered as CSs indexed as the points of the manifold $\mathcal{X}_n^J$ (cf. §10.1 in the first reference [1]).

Let us recall that the denomination of "Jacobi group"– the group which realize the "squeezed states" [20] of Quantum Optics [1] – was firstly introduced by mathematicians [8]. The same group is known to physicists under other names, as the Schrödinger group [17], [5] or the "Weyl-symplectic" group [21].

The paper is laid out as follows. For self-contentedness, Section 2 recalls the basic facts established in [4] about the algebra $\mathfrak{g}_1^J$ and its holomorphic differential representation. Section 3 is devoted to a comparison of our approach in [4] with that of Kähler-Berndt. We have included in Remark 3 the differential action of the generators of the Jacobi algebra $\mathfrak{g}_1^J$ expressed in the Kähler-Berndt variables on $\mathcal{X}_1^J$. Section 4 starts recalling some facts established in [5] about holomorphic representation of the Jacobi algebra $\mathfrak{g}_n^J$. Then we present the Kähler two-form ω'_n on $\mathcal{X}_n^J$, a generalization of Kähler-Berndt's construction of ω'_1 on $\mathcal{X}_1^J$ [6,7,10–13]. These results have significance for the geometry of the manifold on which the "gaussian pure states" [18] are constructed.

2. A Holomorphic Representation of the Jacobi Algebra $\mathfrak{g}_1^J$

The Jacobi algebra is defined as the the semi-direct sum of the Lie algebra $\mathfrak{h}_1$ of the Heisenberg-Weyl Lie group and the Lie algebra of the group $\mathrm{SU}(1,1)$

$$\mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1). \tag{1}$$

The Heisenberg-Weyl algebra $\mathfrak{h}_1 =< s1 + xa^+ - \bar{x}a >_{s\in\mathbb{R},x\in\mathbb{C}}$ is an ideal in $\mathfrak{g}_1^J$, determined by the commutation relations:

$$[a, a^+] = 1 \tag{2a}$$
$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_0 \tag{2b}$$
$$[a, K_+] = a^+, \qquad \left[K_-, a^+\right] = a \tag{2c}$$
$$\left[K_+, a^+\right] = [K_-, a] = 0 \tag{2d}$$
$$\left[K_0,\ a^+\right] = \frac{1}{2}a^+, \qquad [K_0, a] = -\frac{1}{2}a \tag{2e}$$

where a (a^+) are the boson annihilation (creation) operators and $K_{0,+,-}$ are the generators of $\mathrm{SU}(1,1)$.

We impose to the cyclic vector e_0 to verify simultaneously the conditions

$$ae_0 = 0, \quad \boldsymbol{K}_-e_0 = 0, \quad \boldsymbol{K}_0e_0 = ke_0, \quad k > 0, \quad 2k = 2,3,.... \tag{3}$$

Perelomov's coherent state vectors associated to the group G^J_1 with Lie algebra the Jacobi algebra (1), based on the manifold M

$$M \;:=\; H_1/\mathbb{R}\times \mathrm{SU}(1,1)/\mathrm{U}(1) \tag{4a}$$
$$M \;=\; \mathcal{D}^J_1 := \mathbb{C}\times\mathcal{D}_1 \tag{4b}$$

are defined as

$$e_{z,w} := \mathrm{e}^{za^+ + w\boldsymbol{K}_+}e_0, \qquad z,w\in\mathbb{C},\ |w|<1. \tag{5}$$

We summarize some of the results established in [4]

Proposition 1. *The differential action of the generators* (2a)-(2e) *of the Jacobi algebra* (1) *is given by the formulas*

$$\boldsymbol{a} = \frac{\partial}{\partial z}, \qquad \boldsymbol{a}^+ = z + w\frac{\partial}{\partial z} \tag{6a}$$
$$\mathbb{K}_- = \frac{\partial}{\partial w}, \qquad \mathbb{K}_0 = k + \frac{1}{2}z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w} \tag{6b}$$
$$\mathbb{K}_+ = \frac{1}{2}z^2 + 2kw + zw\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w} \tag{6c}$$

where $z,w\in\mathbb{C}$, $|w|<1$.
The reproducing kernel $K(z,w;\bar{z}',\bar{w}') := (e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'})$ is

$$K(z,w;\bar{z}',\bar{w}') = (1-w\bar{w}')^{-2k}\exp\frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2 w}{2(1-w\bar{w}')}\cdot \tag{7}$$

The Kähler two-form ω_1 is given by the formula

$$-\mathrm{i}\omega_1 = \frac{2k}{(1-w\bar{w})^2}\mathrm{d}w\wedge\mathrm{d}\bar{w} + \frac{A\wedge\bar{A}}{1-w\bar{w}},\ A = \mathrm{d}z + \bar{\alpha}_0\mathrm{d}w,\ \alpha_0 = \frac{z+\bar{z}w}{1-w\bar{w}}\cdot \tag{8}$$

3. Kähler-Berndt's Construction for $\mathbb{C}\times\mathcal{H}_1$

Rolf Berndt has studied the real Jacobi group $G^J(\mathbb{R})$ [6, 7]. The Jacobi group appears in the context of the so called *Poincaré group* or *The New Poincaré group* – the double cover of the de Sitter group $\mathrm{SO}_0(4,1)$ – investigated by Erich Kähler as the ten-dimensional group G^K which invariates a hyperbolic metric [10–12]. Kähler and Berndt have investigated the Jacobi group $G^J_0(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})\ltimes\mathbb{R}^2$ acting on the manifold $\mathcal{X}^J_1$.

The main ingredient in the proof of Remark 1 below is the Iwasawa decomposition. Let us also mention that Iwasawa decomposition was largely used in applications in Optics, see e.g. [19].

Remark 1. *The action of* $G^J_0(\mathbb{R})$ *on* $\mathcal{X}^J_1$ *is given by* $(g,(v,z)) \rightarrow (v_1,z_1)$*,* $g = (M,l)$*,* $l=(l_1,l_2)\in\mathbb{R}^2$*, where*

$$v_1=\frac{av+b}{cv+d},\qquad z_1=\frac{z+l_1v+l_2}{cv+d},\qquad M=\left(\begin{array}{cc} a & b\\ c & d\end{array}\right)\in \mathrm{SL}_2(\mathbb{R}). \quad (9)$$

Let us now recall that

$$C^{-1}\mathrm{SL}_2(\mathbb{R})C=\mathrm{SU}(1,1),\quad \text{where } C=\left(\begin{array}{cc} \mathrm{i} & \mathrm{i}\\ -1 & 1\end{array}\right). \quad (10)$$

If M is given by the third equation (9), then, under the transformation (10)

$$M_*=C^{-1}MC=\left(\begin{array}{cc} \alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{array}\right),\qquad \alpha,\beta\in\mathbb{C}, |\alpha|^2-|\beta|^2=1 \quad (11)$$

where

$$2\alpha=a+d+\mathrm{i}(b-c),\qquad 2\beta=a-d-\mathrm{i}(b+c). \quad (12)$$

Now we pass to the complex group $G^J_{\mathbb{C}}=C^{-1}G^J(\mathbb{R})C$. We have

Remark 2. *The action* $C^{-1}G^J_0(\mathbb{R})C$*, descends on the basis to the biholomorphic map:* $\check{C}^{-1}:\mathcal{X}^J_1:=\mathcal{H}_1\times\mathbb{C}\rightarrow\mathcal{D}^J_1:=\mathcal{D}_1\times\mathbb{C}$

$$w=\frac{v-\mathrm{i}}{v+\mathrm{i}},\qquad z=\frac{2\mathrm{i}u}{v+\mathrm{i}},\qquad w\in\mathcal{D}_1,\ v\in\mathcal{H}_1,\ z\in\mathbb{C}. \quad (13)$$

When expressed in the coordinates $(v,u)\in\mathcal{X}^J_1$ *which are related to the coordinates* $(w,z)\in\mathcal{D}^J_1$ *by the map* (13) *given by* Remark 2*, the Kähler two-form* (8) *is identical with the one considered by Kähler-Berndt*

$$-\mathrm{i}\,\omega'_1=-\frac{2k}{(\bar{v}-v)^2}\mathrm{d}v\wedge\mathrm{d}\bar{v}+\frac{2}{\mathrm{i}(\bar{v}-v)}B\wedge\bar{B},\qquad B=\mathrm{d}u-\frac{u-\bar{u}}{v-\bar{v}}\mathrm{d}v. \quad (14)$$

Remark 3. *When expressed in the coordinates* $(v,u)\in\mathcal{X}^J_1=\mathcal{H}_1\times\mathbb{C}$*, related with the coordinates* $(w,z)\in\mathcal{D}^J_1=\mathcal{D}_1\times\mathbb{C}$ *by* (13)*, the differential action of the generators* (2a)-(2e) *of the Jacobi algebra* (1)*, given by* Lemma 1*, becomes*

$$\boldsymbol{a}=\frac{v+\mathrm{i}}{2\mathrm{i}}\frac{\partial}{\partial v},\qquad \boldsymbol{a}^+=\frac{2\mathrm{i}u}{v+\mathrm{i}}+\frac{v-\mathrm{i}}{2\mathrm{i}}\frac{\partial}{\partial u} \quad (15a)$$

$$\mathbb{K}_-=\frac{(v+\mathrm{i})^2}{2\mathrm{i}}\frac{\partial}{\partial v}+\frac{v+\mathrm{i}}{2\mathrm{i}}u\frac{\partial}{\partial u},\qquad \mathbb{K}_0=k+\frac{uv}{2\mathrm{i}}\frac{\partial}{\partial u}+\frac{v^2+1}{2\mathrm{i}}\frac{\partial}{\partial v} \quad (15b)$$

$$\mathbb{K}_+=-\frac{2u^2}{(v+\mathrm{i})^2}+\frac{2k(v-\mathrm{i})}{v+\mathrm{i}}+\frac{u(v-\mathrm{i})}{2\mathrm{i}}\frac{\partial}{\partial u}+\frac{(v-\mathrm{i})^2}{2\mathrm{i}}\frac{\partial}{\partial v}. \quad (15c)$$

4. The Jacobi Group G_n^J

The Jacobi algebra is the the semi-direct sum

$$\mathfrak{g}_n^J := \mathfrak{h}_n \rtimes \mathfrak{sp}(n,\mathbb{R}) \tag{16}$$

$$\mathfrak{h}_n = \langle \mathrm{i}s1 + \sum_{i=1}^{n}(x_i a_i^+ - \bar{x}_i a_i)\rangle_{s\in\mathbb{R},x_i\in\mathbb{C}} \tag{17}$$

$$[a_i, a_j^+] = \delta_{ij}, \qquad [a_i, a_j] = [a_i^+, a_j^+] = 0 \tag{18}$$

$$[a_k^+, K_{ij}^+] = [a_k, K_{ij}^-] = 0, \qquad 2[a_i, K_{kj}^+] = \delta_{ik}a_j^+ + \delta_{ij}a_k^+ \tag{19a}$$

$$2[K_{ij}^0, a_k^+] = \delta_{jk}a_i^+, \qquad 2[a_k, K_{ij}^0] = \delta_{ik}a_j \tag{19b}$$

$$2[K_{kj}^-, a_i^+] = \delta_{ik}a_j + \delta_{ij}a_k. \tag{19c}$$

Above $K_{ij}^{0,+,-}$ represent the generators of the group $\mathrm{Sp}(n,\mathbb{R})$.

Perelomov's coherent state vectors associated to the group G_n^J with Lie algebra the Jacobi algebra (16), based on the complex N-dimensional manifold ($N = \frac{n(n+3)}{2}$)

$$M := H_n/\mathbb{R} \times \mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n) \tag{20a}$$

$$M = \mathcal{D}_n^J := \mathbb{C}^n \times \mathcal{D}_n \tag{20b}$$

are defined as

$$e_{z,W} = \exp(\boldsymbol{X})e_0, \quad \boldsymbol{X} := \sum_i z_i a_i^+ + \sum_{ij} w_{ij}\boldsymbol{K}_{ij}^+, \quad z\in\mathbb{C}^n; W\in\mathcal{D}_n. \tag{21}$$

The vector e_0 verify

$$a_i e_o = 0,\ i = 1,\cdots,n, \quad \boldsymbol{K}_{ij}^+ e_0 \neq 0, \quad \boldsymbol{K}_{ij}^- e_0 = 0, \quad \boldsymbol{K}_{ij}^0 e_0 = \frac{k}{4}\delta_{ij}e_0. \tag{22}$$

The scalar product of functions in the symmetric Fock space is [5]

$$(\phi,\psi) = \Lambda_n \int_{z\in\mathbb{C}^n;1-W\bar{W}>0} \bar{f}_\phi(z,W)f_\psi(z,W)QK^{-1}\mathrm{d}z\mathrm{d}W \tag{23}$$

where

$$Q = \det(1 - W\bar{W})^{-(n+2)}$$

$$K = \det(M)^{\frac{k}{2}}\exp\frac{1}{2}[2\langle z, Mz\rangle + \langle W\bar{z}, Mz\rangle + \langle z, MW\bar{z}\rangle] \tag{24}$$

$$M = (1 - W\bar{W})^{-1}$$

$$\mathrm{d}z = \prod_{i=1}^{n} \mathrm{dRe}z_i \mathrm{dIm}z_i, \qquad \mathrm{d}W = \prod_{1\le i\le j\le n} \mathrm{dRe}w_{ij}\mathrm{dIm}w_{ij} \tag{25}$$

$$\Lambda_n = \frac{k-3}{2\pi^{\frac{n(n+3)}{2}}} \prod_{i=1}^{n-1} \frac{(\frac{k-3}{2} - n + i)\Gamma(k+i-2)}{\Gamma[k+2(i-n-1)]}.$$

Comparatively with the case of the symplectic group, a shift of p to $p-1/2$ in the normalization constant $\Lambda_n = \pi^{-n}J^{-1}(p)$ is obtained [5].

On the manifold $\mathcal{D}^J_n$, we have the Kähler two-form [5]

$$-\mathrm{i}\,\omega_n = \frac{k}{2}\mathrm{tr}(C \wedge \bar{C}) + \mathrm{tr}(A^t \bar{M} \wedge \bar{A}) \tag{26}$$

$$\begin{aligned} &A = \mathrm{d}z + \mathrm{d}W\bar{x}, \qquad C = \, M\mathrm{d}W, \qquad M = (1 - W\bar{W})^{-1} \\ &x = (1 - W\bar{W})^{-1}(z + W\bar{z}), \qquad W \in \mathcal{D}_n, z \in \mathbb{C}^n. \end{aligned}$$

Now we consider the real Jacobi group $G^J_n(\mathbb{R}) = \mathrm{Sp}(n,\mathbb{R}) \ltimes H_n(\mathbb{R})$, where $H_n(\mathbb{R})$ is the real Heisenberg-Weyl group of real dimension $(2n+1)$. Let $g = (M, X, k), g' = (M', X', k') \in G^J_n(\mathbb{R})$, where $X = (\lambda, \mu) \in \mathbb{R}^{2n}$ and $(X, k) \in H_n(\mathbb{R})$. Then the composition law in $G^J_n(\mathbb{R})$ is

$$gg' = (MM', XM' + X', k + k' + XM'JX'^t). \tag{27}$$

We shall also consider the restricted real Jacobi group $G^J_n(\mathbb{R})_0$, consisting only of elements of the form above, but $g = (M, X)$.

We consider also the manifold $\mathcal{X}^J_n := \mathcal{H}_n \times \mathbb{R}^{2n}$, where $\mathcal{H}_n$ is Siegel upper half-plane

$$\mathcal{H}_n := \{Z \in M(n, \mathbb{C});\, Z = U + \mathrm{i}V, U, V \in M(n, \mathbb{R}), U^t = U, V^t = V, V > 0\}$$

Let us consider an element $g = (M, l)$ in $G^J_n(\mathbb{R})_0$, i.e.,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}) \; l = (l_1, l_2) \in \mathbb{R}^{2n} \tag{28}$$

and $v \in \mathcal{H}_n,\; z \in \mathbb{C}^n \equiv \mathbb{R}^{2n}$. Then the action of the group $G^J_n(\mathbb{R})_0$ on the base manifold $\mathcal{X}^J_n$ is given by the formula $(M, l) \times (v, z) \rightarrow (v_1, z_1) \in \mathcal{X}^J_n$, where

$$v_1 = (Av + B)(Cv + D)^{-1}, \qquad z_1 = (z + vl^t_1 + l^t_2)(Cv + D)^{-1}. \tag{29}$$

Now we consider the transformation

$$w = (v - \mathrm{i})(v + \mathrm{i})^{-1}, \qquad z = 2\mathrm{i}(v + \mathrm{i})^{-1}u \tag{30}$$

of $\mathcal{X}_n^J \to \mathcal{D}_n^J$. The first equation (30) is nothing else than the linear fractional transformation (29) - the fameous "abcd" law for laser beams [14, 18] - corresponding to the matrix $M = C^{-1}$ where C is the Cayley transform of the Siegel half-plane $\mathcal{H}_n$ into the Siegel unit ball $\mathcal{D}_n$

$$A \in M(2n,\mathbb{R}) \to A_{\mathbb{C}} \in M(2n,\mathbb{R})_{\mathbb{C}}, \qquad A_{\mathbb{C}} = C^{-1}AC,\; C = \begin{pmatrix} \mathrm{i}1 & \mathrm{i}1 \\ -1 & 1 \end{pmatrix}.$$

Under the same transformation, $C^{-1}\mathrm{Sp}(n,\mathbb{R})C \to \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$, and the linear fractional transformation (29) on $\mathcal{H}_n$ determined by a matrix (28) becomes linear fractional transformation on $\mathcal{D}_n$ with the matrix $C^{-1}MC$.

Under the transformation (30), the two-form (8) on $\mathcal{D}_n^J$ becomes on $\mathcal{X}_n^J$

$$-\mathrm{i}\,\omega_n' = \frac{k}{2}\mathrm{tr}(p^t \wedge \bar{p}) + \frac{2}{\mathrm{i}}\mathrm{tr}(B^t D \wedge \bar{B}) \tag{31}$$

$$D = (\bar{v} - v)^{-1}, \qquad p = D\mathrm{d}v, \qquad B = \mathrm{d}u - \mathrm{d}vD(\bar{u} - u).$$

The form (31) *is a "n"-dimensional generalization of Berndt-Kähler two-form* (14).

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