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GROUP THEORY IN THE PROBLEMS OF MODELING AND CONTROL OF MULTI-BODY SYSTEMS

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Abstract. This work is a review of our research activity during the last ten years concerning the problems of modeling and control of multi-body mechanical systems. Because the treatment of the above topics is quite sensitive with respect to the different parameterizations of the rotation group in three dimensional space SO(3) and because the properties of the parameterization more or less influence the efficiency of the dynamic model, here the so called vector-parameter is used for parallel considerations. The consideration of the mechanical system in the configurational space of pure vector-parameters with a group structure opens the possibilities for the Lie group theory to be applied in the problems of the dynamics and control. The sections in this paper present independent parts of an unified scientific approach.

Contents

1. Introduction in Multi–Body Mechanical System

Here we consider an open loop mechanical system with n degrees of freedom (for example a manipulator system). We denote by $q := [q(1) \dots q(n)]^T$ the $n \times 1$ matrix of the generalized coordinates (joint displacements) of the manipulator system (MS) in the usual sense, where $q \in Q \subset \mathbb{R}^n$ and Q is the configurational manifold

$$Q := \{q \,;\, q(i)^{\min} \;\le\; q(i) \;\le\; q(i)^{\max}, \qquad i = 1, 2, ..., n\}. \tag{1}$$

The connection between the position vector $x \in \mathbb{R}^m$ and q, i.e., the direct kinematic problem (DKP) is specified by

$$x = F(q) \tag{2}$$

where $F \;:\; Q \to X$ is a smooth projection map over the target space

$$X = \{x \,;\; x = F(q), q \in Q \subset \mathbb{R}^n\} \tag{3}$$

which is the working space of the MS under consideration. Differentiating (2) with respect to t, one obtains

$$\dot{x} = \left[\frac{\partial F}{\partial q}\right]\dot{q} = J(q)\dot{q} \tag{4}$$

where $J(q) \in R^{m,n}(m < n)$ is the Jacobian matrix of the map F. All configurations for which the rank of $J(q) < m$ are called **singular**. In the configurational space Q the dynamic equations for rigid body manipulator look like

$$H(q)\ddot{q} + h(q, \dot{q}) = P \tag{5}$$

where $H := H(q)$ is the $n \times n$ inertia matrix, the $n \times 1$ matrix $h := h(q, \dot{q})$ takes into account Coriolis, centrifugal and gravitational forces, and P is the $n \times 1$ matrix of the generalized forces and moments. According to the above notation, the end-effector (EE) location and its velocity have the following analytical forms

$$\begin{aligned} x &= [p \vdots o]^T = F(q) \in \mathbb{R}^3 \times \mathrm{SO}(3) \\ V &= [\dot{p} \vdots \omega]^T = J(q)\dot{q} \in \mathbb{R}^6. \end{aligned} \tag{6}$$

Alternatively

$$\begin{aligned} x &= [p \vdots \beta]^T = F(q) \in \mathbb{R}^3 \times \mathrm{SO}(3) \\ V &= [\dot{p} \vdots \dot{\beta}]^T. \end{aligned} \tag{7}$$

$\beta \in A \subset \mathbb{R}^3$ is a minimal representation of EE attitude. In general $\beta = f(o)$ which is many to one or undefined if the domain of $f(.)$ in SO(3) is not properly restricted. The reason is that the group SO(3) can not be covered by a single coordinate chart. Accordingly, β is not acceptable for all possible EE orientations and there will be a singularity of attitude representations unless we restrict EE attitude for some subregion of SO(3). Then β has to be defined in the image of admissible attitudes, namely in some $A \subset \mathbb{R}^3$. When $A \equiv \mathbb{R}^3$ as in the case of Euler Rodrigues parameters, singularities of attitude representation correspond to $\mid \beta \mid = \infty$. Nowadays, there is no doubt that a very important item in modeling and control of a mechanical system is its kinematical description [112]. It is well known that the rigid-body motion in $\mathbb{R}^3$ is described by the Euclidean group E(3), and that the SO(3) group cannot be avoided in the representation of orientations [110]. The appropriate parameterization of SO(3) is one of the most important practical problem in mechanics because it has a great influence over the overall efficiency of all methods. Angular velocity or momentum information is required by the most control strategies. It could be obtained using the derivatives of various orientation parameters β like Euler and Bryant angles, Euler or Cayley-Klein parameters, quaternions [12], [6], [57], [7], or the so called vector-parameter $\beta = c$, which as an element of a Lie group, having very nice and clear properties and simplifying the treatment of many problems [59], [61], [62]. After Fedorov [24] who introduces the vector-parameter in connection with representation theory of the Lorentz group a long standing program of studies of different group parameterizations of the rotational motion and their after–effects is started [59]. Using the vector-parameter language, the interrelations between vector and matrix transformations, screw geometry and dual algebra in description of Euclidean motions were outlined [59]. Because of the fact that every Euclidean motion may be represented as a screw motion, an useful interplay of screw geometry, dual algebra and vector and matrix transformations is proposed in [69], [72]. The special structure of a manipulator as a series of coupled bodies allows the specialization of the general line coordinate transformation matrix in the so called dual orthogonal matrix form as well the definition of the notion of dual vector-parameter to be introduced. The kinematical and dynamical equations of a manipulator system play an extremely important role as for motion simulation so in control [7]. The mathematical models which present kinematics, dynamics, trajectory planning and control of rigid body manipulators in vector-parameters are treated in details elsewhere [60–63], [65], [66], [78] and etc. Because of the fact that these parameters make a Lie group with a very simple and clear composition law, they are ideally suited for real time simulations and control modeling since through them the number of the elementary operations is reduced by more than $30\,\%$ (see Tables 1, 2 and 3).

The comparison between its computational cost and the other methods for simulation and control used till now shows its efficiency and feasibility for real time application. It is also proved that the computational effectiveness of the vector-parameter approach increases with the increasing number of the revolute joints [61]. This fact is heavily used in [89] and [73] where the problems of modeling and control of elastic joints manipulators are treated.

The dynamical modeling and control of manipulator systems in a "pure" vector-parameter configurational manifold as presented in [69] consists of describing the geometry, kinematics and dynamics of an open loop kinematic chain in a new extended configurational space with a group structure – the space of vectors describing the joints displacements (vector-parameters c and some translational vectors tr). In this way the transition operations from vector-parameters to generalized coordinates are saved and the kinematic and dynamic equations in "pure" vector – parameters are purely algebraic and the differential equations of motion "feel" the Lie group structure of the underlying configurational manifold.

The present work considers in detail the Euclidean motions of a rigid body in vector-parameters and presents the kinematics and dynamics of a multi-body mechanical system on a generalized manifold equipped with a group structure. An approach for diagonalization of real 3×3 symmetric matrix is considered and some mechanical applications are given. Attention is paid on modeling and control of nonholonomic systems, modeling of mobile robots, and control problems of the system of mobile platform and manipulator are also outlined.

Most scientific investigations can be roughly classified as investigations in to pure technical aspects, theoretical investigations (with possibilities for practical implementation) and studies from theory to practice as well from practice to theory. Our previous investigations demonstrate how, using a group-theoretical approach to the rotation motion presentation, one may reach computational effectiveness in kinematic and dynamic modeling of manipulator systems. Such studies can be considered as occupying some area between the second and third item in the above classification. It is worth to be mentioned that there is an analogy between the rigid body description through vector-parameter and this one realized in [31] on the base of screw operators. The intrinsic mathematical formalism in physical rigid body motions description is presented with the use of affine geometry together with Lie group theory and it is used for description of the mechanism kinematic pairs.

Table 1. Comparison of the matrix and vector-parameter method at the forming of an end-effector vector in the base frame.
m – multiplications, a – additions, tr. f. – transcendental functions

Number of operations		Matrix method			Vector-parameter method			
		B	Bp	$\prod_{i=1}^{n} B_i\, p$	c	$O(c)p$	c_1, c_2	Total
Eulerian angles	m	12	8	$20n$	3	14	10	$13n+4$
	a	4	6	$10n$	2	15	12	$14n+3$
	tr. f.	6	0	$6n$	5	0	0	$5n$
Brayant angles	m	14	8	$22n$	5	14	10	$15n+4$
	a	4	6	$10n$	0	15	12	$12n+3$
	tr. f.	6	0	$6n$	3	0	0	$3n$
Eulerian parameters	m	11	8	$19n$	3	14	10	$13n+4$
	a	10	6	$16n$	4	15	12	$16n+3$
	tr. f.	0	0	0	0	0	0	0
Denavit Hartenberg Parameters	m	6	8	$14n$	1	14	10	$11n+4$
	a	0	8	$8n$	0	15	12	$12n+3$
	tr. f.	4	0	$4n$	2	0	0	$2n$

Table 2. Comparison in computational aspect at Jacoby matrix forming using matrix and vector-parameter method
M – method, A – method of Vukobratović [132], [131], B – method of Waldron [133], C – method of Ribble [104], D – method of Renaud [103], E – method of Paul [96], [97], [98], F – method of Orin / Shrader [92], ${}^{j}J_i$ – index i denotes the coordinate frame according to which the Jacobian J is formed, index j – the frame according to which J is referred

M	J	Matrix method			Vector-parameter method		
		mult.	add.	tr. f.	mult.	add.	tr. f.
A	${}^{E}J_E$	$10n^2 + \ldots$	$n^2 + \ldots$	$2n$			
B	${}^{0}J_0$	$30n-55$	$15n-38$	$2n-2$	$24n-42$	$22n-41$	n
C	${}^{0}J_E$	$30n-11$	$18n-20$	$2n$	$24n-6$	$22n-3$	n
D	${}^{k}J_k$	$30n-87$	$15n-66$	$2n-2$	$24n-52$	$22n-60$	n
E	${}^{E}J_E$	$30n-25$	$15n-22$	$2n$	$24n-26$	$22n-26$	n
F	${}^{E}J_E$	$30n-18$	$14n-15$	$2n$	$24n-10$	$22n-12$	n

Table 3. Comparison of the number of the operations at dynamic modelling LG – Lagrange formalism, NE – Newton-Euler formalism, M – method, A – method of Uicker / Kahn [127], [40], B – method of Waters [139], C – method of Hollerbach (through homogeneous matrices 4×4 and rotation matrices 3×3) [32], [33], D – method of Luh / Armstrong [51], [4], E – method of Orin, McGhee and Vukobratović [91], F – method of Nakamura [84], G – vector-parameter method of Mladenova

M		multiplications	$n = 6$	additions	$n = 6$
LG	A	$32\frac{1}{2}n^4 + ...$	37823	$25n^4 + ...$	39462
	B	$106\frac{1}{2}n^2 + ...$	3256	$82n^2 + ...$	5652
	C				
	4×4	$830n - 592$	4388	$675n - 464$	3586
	3×3	$412n - 277$	2195	$320n - 201$	1719
NE	D,	$137n - 22$	852	$131n - 48$	738
	E	$150n - 48$	800	$101n - 11$	595
	F	$133n - 17$	781	$106n - 19$	617
	G	$138n - 53$	775	$143n - 66$	792

Using the knowledge of theoretical mechanics, matrix algebra, group theory, geometry and the basic methods in robotics, the paper presents our research activity in creating an unified approach for modeling and control of multi-body systems and it may serve as a nice tool for students, scientists and engineers from academia and industry experienced in this attractive area.

2. Rigid Body Kinematics

It is the well known that the rotations play an extremely decisive role in describing of the Euclidian motions [1], [5], [26], [53], [18]. The way of their representation defines the set of the problems which may be solved and the computational efficiency of the procedures. The rotations are met everywhere – in physics and engineering problems, computer simulations and visualizations, computer graphics, and therefore in the whole computerized world. Nowadays because of the modern and fast processors, the representation of the rotation group is still very important so that the hardware and the software to work efficiently together. The purpose of this work is to present the different parameterizations of the SO(3) group and to show their role in the efficient modelling in both the manipulator kinematics and dynamics and computer vision. The theoretical base is the refine knowledge of Lie groups and differential geometry in the problems of rigid body mechanics [42], [110].

We consider the special orthogonal group SO(3) and its different representations for description of the motion of a rigid body with a fixed point. Any displacement D of a rigid body may be considered as a screw motion, i.e., it takes a point from the body to the position A' through: translation along the axis AA' and a rotation R around point A. The point is called pole and $D = TR = RT$. Further, we will consider rotations mainly. According to a theorem due to Euler a displacement in a general sense is equivalent to a rotation around an axis through the point A [12], [129], [140]. There exist a few analytical representations of the rotations, i.e., any rotation is expressed by defining its action, equivalently, on a

i) vector x	$x^{'} = Ax,$	$A \in \mathrm{SO}(3)$
ii) quaternion x	$x^{'} = qxq^{-1},$	$q \in \mathrm{Sp}(1)$
iii) spinor ψ	$\psi' = U\psi,$	$U \in \mathrm{SU}(2)$
iv) matrix X	$X^{'} = U\, X\, U^{+}$	

where SO(3) denotes the group of real, orthogonal 3×3 matrices, Sp(1) the symplectic group of unit quaternions, SU(2) the unimodular unitary group of 2×2 complex matrices, ψ a one-index spinor and X a complex, Hermitian, traceless 2×2 matrix.

2.1. Vector Kinematics

Due to the theorem of Euler, the rigid body rotation in $\mathbb{R}^3$ is characterized through a fixed point O, a unit vector e and a right handed vector φ. The rotation of vector x in $x^{'}$ is defined through the Rodrigues formula (see [131], [137], [138])

$$x^{'} = \cos\,\varphi\,.\,x + (1 - \cos\varphi)\,(e, x)\,e + \sin\varphi\,[e, x]. \tag{8}$$

The symbols $(\ ,\ )$ and $[\ ,\ ]$ mean scalar and vector product respectively. Since $1 - \cos\varphi = 2\sin^2\varphi/2$, the terms of the vector $\beta = \sin\varphi/2\, e$ and the scalar $\beta_0 = \cos\varphi/2$, the equation 8 is transformed in the following way

$$x^{'} = (2\beta_0^2 - 1)\,x + 2(\beta, x) + 2\beta_0\,[\beta, x]. \tag{9}$$

We denote by $\beta = \mathrm{col}(\beta_1, \beta_2, \beta_3)$ a column-matrix. In a fixed coordinate system with a unit basis (e_1, e_2, e_3), the equation (9) may be written like

$$x^{'} = B\,x \tag{10}$$

where the transformation matrix $B = (\beta_0, \beta)$ is the real, proper, orthogonal matrix

$$B = \left[\begin{array}{ccc} \beta_0^2 + \beta_1^2 - \beta_2^2 - \beta_3^2 & 2(\beta_1\beta_2 - \beta_0\beta_3) & 2(\beta_1\beta_3 + \beta_0\beta_2) \\ 2(\beta_1\beta_2 + \beta_0\beta_3) & \beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2 & 2(\beta_2\beta_3 - \beta_0\beta_1) \\ 2(\beta_1\beta_3 - \beta_0\beta_2) & 2(\beta_2\beta_3 + \beta_0\beta_1) & \beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2 \end{array} \right]. \quad (11)$$

The four parameters, by definition, satisfy the condition

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1. \quad (12)$$

The rigid body position is defined by the matrix B, which is an element of the Lie group SO(3). The body motion corresponds to a curve $B(t)$ on the configuration space SO(3). The rotation velocity vector $\dot{B} = \mathrm{d}B/\mathrm{d}t$, is a tangent to the curve ant belongs to the tangent space TSO(3)$_B$ at the point B. Any curve satisfies $B(t)\,B^T(t) = I$ and differentiating it with respect to the time t, we get

$$\dot{B}(t)B^T(t) + B(t)\dot{B}^T(t) = \dot{B}(t)B^T(t) + (\dot{B}(t)B^T(t))^T = 0. \quad (13)$$

The matrix $\dot{B}(t)B^T(t)$ is skew-symmetric and is called angular velocity matrix relative to space. We denote it by Ω, so that

$$\Omega = \dot{B}(t)B^T(t) = \dot{B}(t)B^{-1}(t). \quad (14)$$

Having in mind the isomorphism between the space of skew-symmetric matrices and vector-space $\mathbb{R}^3$, we shall identify Ω with the instantaneous angular velocity vector in the following way

$$\Omega = \left[\begin{array}{ccc} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{array} \right] \longrightarrow \omega = \left[\begin{array}{c} \omega_1 \\ \omega_2 \\ \omega_3 \end{array} \right] \quad (15)$$

and it is valid

$$\Omega x = [\omega,\, x]. \quad (16)$$

In general the basic formulas describing the vector kinematics are

$$x^{'} = B\,x \quad (17)$$

$$\dot{x} = \Omega\,x, \qquad \left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{array} \right] = \left[\begin{array}{ccc} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] \quad (18)$$

$$\Omega = \dot{B}B^{-1}, \qquad \Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{19}$$

After differentiating $B^T(t)\,B(t) = I$ with respect to t, we obtain the following result for the vector of the angular velocity

$$\widetilde{\Omega} = B^{-1}\,\dot{B}, \qquad \widetilde{\Omega} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \tag{20}$$

where $\widetilde{\Omega} \simeq \widetilde{\omega} = \mathrm{col}(p, q, r)$.

2.2. Quaternions and Quaternion Kinematics

The real algebra of quaternions H is in one-to-one linear correspondence with the linear space $\mathbb{R}^4$, having a standard basis $(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$, and a quaternion product defined by the basis rules $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$. Hence, a quaternion q and its conjugate $\overline{q}$ are written as linear combinations: $q = q_0 + \mathrm{i}q_1 + \mathrm{j}q_2 + \mathrm{k}q_3$ and $\overline{q} = q_0 - \mathrm{i}q_1 - \mathrm{j}q_2 - \mathrm{k}q_3$. A symplectic inner product is defined in H by $\langle q, p\rangle = q\overline{p}$ so that the norm of a quaternion q is $\langle q, q\rangle = q_0^2 + q_1^2 + q_2^2 + q_3^2$ and every non-zero quaternion has an inverse defined by $q^{-1} = \overline{q}/\langle q, q\rangle$. The regular left representation of the group H^* of non-zero quaternions is the representation h of H^* on the real vector space H given by $h(q)q' = qq'$ (left multiplication by q), where h has matrix form

$$h(q) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}. \tag{21}$$

Any quaternion may be written as the sum $q = s + \underline{v}$, where $s \in \mathbb{R}$, $\underline{v} \in \mathbb{R}^3$. The product of two quaternions $q_1 = s_1 + \underline{v_1}$ and $q_2 = s_2 + \underline{v_2}$ is given by the rule

$$q_1 q_2 = s_1 s_2 - (v_1, v_2) + s_1\underline{v_1} + s_2\underline{v_2} + [v_1, v_2]. \tag{22}$$

The subgroup of quaternions H having unit norm, i.e., $q\overline{q} = 1$, form a compact non-Abelian Lie group $\mathrm{Sp}(1)$, called the symplectic group. This manifold is the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ $(q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1)$. It easy to be proved, that the automorphism of H: $x \to q \times q^{-1}$, with $q \in \mathrm{Sp}(1)$ and $x = (0, x) \in \mathbb{R}^3$ is a rotation of $\mathbb{R}^3$. The components of q are the Eulerian parameters of displacement.

The equation (9) after some circumstances may be brought to an equation from the type

$$x' = \beta_0(\beta, x) - (\beta, \beta_0 x + [\beta, x]) + \beta_0\beta_0 x + [\beta, x] + (\beta, x)\beta + [\beta, \beta_0 x + [\beta, x]]$$
or (23)
$$x' = s_1 s_2 - (v_1, v_2) + s_1 v_2 + s_2 v_1 + [v_1, v_2]$$

where we have substituted

$$s_1 = \beta_0, \qquad v_1 = \beta, \qquad s_2 = (\beta, x), \qquad v_2 = \beta_0 x + [\beta, x]. \tag{24}$$

The right side of (23) coincides with the formula for product $q_1 q_2$ of two quaternions, and in a result we have

$$x' = (\beta_0 + \beta)(0 + x)(\beta_0 - \beta). \tag{25}$$

We denote by

a) $b = \mathrm{col}(\beta_0, \beta_1, \beta_2, \beta_3) = \mathrm{col}(\beta_0, \beta)$ $(b \in \mathrm{Sp}(1))$ – the matrix-column corresponding to the quaternion $\beta_0 + \beta_1\mathrm{i} + \beta_2\mathrm{j} + \beta_3\mathrm{k} = \beta_0 + \beta$, which characterize the rotation, and $b^{-1} = \mathrm{col}(\beta_0, -\beta)$ the matrix-column of the conjugate quaternion.

b) $x = \mathrm{col}(0, x)$ and $\omega = \mathrm{col}(0, \omega)$ – the position and angular velocity quaternions resp.

c) $\widehat{b}, \widehat{x}, \widehat{\omega}$ – 4×4 quaternion matrices from the type $h(b), h(x), h(\omega)$ according to (21).

Hence, using regular left representations we may express the active rotations (25) as

$$x^{'} = \widehat{b}\widehat{x}b^{-1}. \tag{26}$$

The main formulas of kinematics of the rotation in quaternion form are

$$
\begin{aligned}
x' &= \widehat{b}\,\widehat{x}\,b^{-1}\begin{bmatrix} 0 \\ x_1' \\ x_2' \\ x_4' \end{bmatrix} \\
&= \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}\begin{bmatrix} 0 & -x_1 & -x_2 & -x_3 \\ x_1 & 0 & -x_3 & x_2 \\ x_2 & x_3 & 0 & -x_1 \\ x_3 & -x_2 & x_1 & 0 \end{bmatrix}\begin{bmatrix} \beta_0 \\ -\beta_1 \\ -\beta_2 \\ -\beta_3 \end{bmatrix} \\
\dot{b} &= \frac{1}{2}\widehat{\omega} b\begin{bmatrix} \dot\beta_0 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & -\omega_3 & \omega_2 \\ \omega_2 & \omega_3 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \\
\omega &= 2\,\widehat{\dot{b}}\,b^{-1}\begin{bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = 2\begin{bmatrix} \dot\beta_0 & -\dot\beta_1 & -\dot\beta_2 & -\dot\beta_3 & \beta_0 \\ \dot\beta_1 & \dot\beta_0 & -\dot\beta_3 & \dot\beta_2 & -\beta_1 \\ \dot\beta_2 & \dot\beta_3 & \dot\beta_0 & -\dot\beta_1 & -\beta_2 \\ \dot\beta_3 & -\dot\beta_2 & \dot\beta_1 & \dot\beta_0 & -\beta_3 \end{bmatrix}\begin{bmatrix} \beta_0 \\ -\beta_1 \\ -\beta_2 \\ -\beta_3 \end{bmatrix} \\
\tilde{\omega} &= 2\,\widehat{b}^{-1}\dot{b}\begin{bmatrix} 0 \\ p \\ q \\ r \end{bmatrix} = 2\begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}\begin{bmatrix} \dot\beta_0 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \end{bmatrix}.
\end{aligned}
$$

From a group-theoretical view, (26) realizes surjective homomorphism of Lie groups: Sp(1) $\simeq \mathbb{S}^3 \to$ SO(3) with kernel $(1, -1)$, in other word, the sphere $\mathbb{S}^3$ is a two-fold covering of the configurational space of the rigid body – SO(3).

2.3. Spinors and Spinor Kinematics

We consider the orthogonal matrix of rotation (11) in Eulerian parameters β_0, β_1, β_2, β_3, satisfying the condition (12). We denote the first vector-column with b, the second with d, and the third one with a. The vectors b, d, a are images of three unit vectors e_h $(h = 1, 2, 3)$ at the rotation action. These vectors form an unit orthonormal right oriented triadic.

Let us denote $a\,(a_1,\ a_2,\ a_3)$, $b\,(b_1,\ b_2,\ b_3)$, $d\,(d_1,\ d_2,\ d_3)$. We consider the vector a. The element $-\beta_0\,\beta_1$ may be presented in the way: $-\beta_0\,\beta_1 = \mathrm{i}^2\,\beta_0\,\beta_1 =$

$\mathrm{i}\,(\mathrm{i}\,\beta_0)\beta_1$, and precisely we have

$$a_1 = (\beta_3 + \mathrm{i}\,\beta_0)\,(\beta_1 - \mathrm{i}\,\beta_2) + (\beta_1 + \mathrm{i}\,\beta_2)\,(\beta_3 - \mathrm{i}\,\beta_0) \tag{27}$$

$$a_2 = \mathrm{i}\,[(\beta_3 + \mathrm{i}\,\beta_0)\,(\beta_1 - \mathrm{i}\,\beta_2) - (\beta_1 + \mathrm{i}\,\beta_2)\,(\beta_3 - \mathrm{i}\,\beta_0)] \tag{28}$$

$$\begin{aligned} a_3 &= \beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2 = \beta_0^2 + \beta_3^2 - (\beta_1^2 + \beta_2^2) \\ &= (\beta_3 + \mathrm{i}\,\beta_0)\,(\beta_3 - \mathrm{i}\,\beta_0) - (\beta_1 + \mathrm{i}\,\beta_2)\,(\beta_1 - \mathrm{i}\,\beta_2) \end{aligned} \tag{29}$$

We set

$$\psi_1 = \beta_3 + \mathrm{i}\,\beta_0, \qquad \psi_2 = \beta_1 + \mathrm{i}\,\beta_2 \tag{30}$$

and then

$$a_1 = \psi_1\overline{\psi}_2 + \overline{\psi}_1\psi_2 \tag{31}$$

$$a_2 = \mathrm{i}\,(\psi_1\overline{\psi}_2 - \overline{\psi}_1\psi_2) \tag{32}$$

$$a_3 = \psi_1\overline{\psi}_2 - \overline{\psi}_1\psi_2. \tag{33}$$

Analogically the components of the vectors b and d are

$$b_1 = \frac{1}{2}\,(\psi_2^2 + \overline{\psi}_2^2 - \psi_1^2 - \overline{\psi}_1^2) \tag{34}$$

$$b_2 = \frac{\mathrm{i}}{2}\,(\overline{\psi}_1^2 + \overline{\psi}_2^2 - \overline{\psi}_1^2 - \overline{\psi}_2^2) \tag{35}$$

$$b_3 = \psi_1\psi_2 + \overline{\psi}_1\psi_2) \tag{36}$$

$$d_1 = \frac{\mathrm{i}}{2}(\psi_1^2 - \overline{\psi}_1^2 - \psi_2^2 + \overline{\psi}_2^2) \tag{37}$$

$$d_2 = -\frac{1}{2}\,(\psi_1^2 + \overline{\psi}_1^2 + \psi_2^2 - \overline{\psi}_2^2) \tag{38}$$

$$d_3 = \mathrm{i}\,(\overline{\psi}_1\overline{\psi}_2 - \psi_1\psi_2). \tag{39}$$

The couple of the complex numbers ψ_1 and ψ_2 make an unit spinor, namely:

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \begin{pmatrix}\beta_3 + \mathrm{i}\,\beta_0 \\ \beta_1 + \mathrm{i}\,\beta_2\end{pmatrix}. \tag{40}$$

The expressions for $a_1,\, a_2,\, a_3$ may be presented through the so called Pauli matrices $\sigma_1,\, \sigma_2$ and σ_3 in the following way

$$a_1 = \begin{pmatrix}\overline{\psi}_2 & \overline{\psi}_1\end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \begin{pmatrix}\overline{\psi}_1 & \overline{\psi}_2\end{pmatrix}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \psi^+\sigma_1\psi$$

$$a_2 = \mathrm{i}\,\begin{pmatrix}\psi_2 & -\psi_1\end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \begin{pmatrix}\psi_1 & \psi_2\end{pmatrix}\begin{bmatrix}0 & -\mathrm{i} \\ \mathrm{i} & 0\end{bmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \psi^+\sigma_2\psi$$

$$a_3 = \begin{pmatrix}\overline{\psi}_1 & -\overline{\psi}_2\end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \begin{pmatrix}\overline{\psi}_1 & \overline{\psi}_2\end{pmatrix}\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \psi^+\sigma_3\psi.$$

We make a projection of the sphere $x^2 + y^2 + z^2 = 1$ from the point $(0,0,1)$ on the plane $z = 0$. The points S' and P from the sphere have the coordinates $(0,0,-1)$ and $P(x,y,z)$. The point $P^{'}(X,Y,0)$ is a projection of the point P on the plane $z = 0$. The equation of the line through the points P and S is $S + t\,(P - S)$ or in coordinates it looks like $(0,0,-1) + t\,(x, y, z+1)$, from where it follows that

$$t = \frac{1}{z+1}, \qquad x\,t = X = \frac{x}{z+1}, \qquad y\,t = Y = \frac{y}{z+1}\cdot \tag{41}$$

The complex number $\overline{\xi} = X + \mathrm{i}\,Y$ is called stereographic projection, $\overline{\xi} \in \mathbb{C}$

$$\overline{\xi} = \frac{x + \mathrm{i}\,y}{z+1}. \tag{42}$$

If $P(a_1, a_2, a_3)$, then

$$X = \frac{a_1}{a_3+1}, \qquad Y = \frac{a_2}{a_3+1}$$

and

$$\overline{\xi} = \frac{a_1 + \mathrm{i}\,a_2}{1+a_3}, \qquad \xi = \frac{a_1 - \mathrm{i}\,a_2}{1+a_3} \tag{43}$$

or

$$1 + a_3 = \frac{a_1 + \mathrm{i}\,a_2}{\xi} = \frac{a_1 - \mathrm{i}\,a_2}{\overline{\xi}}\cdot \tag{44}$$

Since $a_1^2 + a_2^2 + a_3^2 = 1$, it follows from (44) that $(1+a_3)^2 = (1+a_3)\,(1-a_3)/\xi\,\overline{\xi}$, or

$$a_3 = \frac{1 - \xi\,\overline{\xi}}{1 + \xi\,\overline{\xi}}\cdot \tag{45}$$

From (44) we have $a_1 + \mathrm{i}\,a_2 = \xi\,(1 + a_3)$, or $a_1 - \mathrm{i}\,a_2 = \overline{\xi}\,(1 + a_3)$, so that

$$a_1 = \frac{1}{2}\,(\xi + \overline{\xi})\,(1 + a_3), \qquad a_2 = -\frac{\mathrm{i}}{2}\,(\xi - \overline{\xi})\,(1 + a_3). \tag{46}$$

After the substitution

$$\xi = \frac{\psi_2}{\psi_1} \tag{47}$$

and with the help of the equations (45) and (46) we get that $a_1 = \psi_1 \overline{\psi}_2 + \overline{\psi}_1 \psi_2, \quad a_2 = \mathrm{i}\,(\psi_1 \overline{\psi}_2 - \overline{\psi}_1 \psi_2)$, which are the components of a. Any rotation is a linear-fractional map, which is defined through the complex unimodular second order matrix

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \qquad \alpha\,\delta - \gamma\,\beta = 1$$

and

$$\xi' = \frac{\alpha\,\xi + \beta}{\gamma\xi + \delta} = \frac{(\beta_0 + \mathrm{i}\,\beta_3)\,\xi + (\beta_2 - \mathrm{i}\,\beta_1)}{-(\beta_2 + \mathrm{i}\,\beta_1)\xi + (\beta_0 + \mathrm{i}\,\beta_3)} \tag{48}$$

where $\beta_0,\, \beta_1,\, \beta_2,\, \beta_3$ are the Euler parameters, and $\alpha,\ \beta,\ \gamma,\ \delta$ – Cayley-Klein parameters.

If we substitute ξ from (47), the expression (48) is transformed in

$$\psi' = \begin{bmatrix} \beta_0 - \mathrm{i}\beta_3 & -(\beta_2 + \mathrm{i}\,\beta_1) \\ \beta_2 - \mathrm{i}\,\beta_1 & \beta_0 + \mathrm{i}\,\beta_3 \end{bmatrix} \psi. \tag{49}$$

This is the well known SU(2) transformation of the spinors generating the rotation in $\mathbb{R}^3$. If we associate to an arbitrary quaternion q a spinor Q by means of the correspondence $f:\ H \to S, \quad q \to Q$ defined by

$$f(q_0 + \mathrm{i}\,q_1 + \mathrm{j}\,q_2 + \mathrm{k}\,q_3) = \begin{pmatrix} q_3 + \mathrm{i}\,q_0 \\ q_1 + \mathrm{i}\,q_2 \end{pmatrix} \equiv Q \tag{50}$$

we obtain a spinor formulation of the kinematics of rotations, which extends the quaternion formulation given above. The correspondence (50) is linear and injective since

$$f(q+p) = f(q) + f(p), \qquad f(\lambda q) = \lambda f(q), \quad \lambda \in \mathbb{C}, \qquad \ker f = 0. \tag{51}$$

The norm $\overline{q}\,q$ of the quaternion q is equal to the norm of the associated spinor $Q^+Q = \overline{Q}_1\,Q_1 + \overline{Q}_2\,Q_2$. For a conjugate unit quaternion $\overline{q}$ we set

$$f(q^{-1}) = f(q_0 - q_1\,\mathrm{i} - q_2\,\mathrm{j} - q_3\,\mathrm{k}) = \begin{pmatrix} -q_3 + \mathrm{i}\,q_0 \\ -q_1 - \mathrm{i}\,q_2 \end{pmatrix} \equiv Q^{-1}. \tag{52}$$

We may now associate to the product of two quaternions $q\,p$ – in addition to the quaternion-matrix product $\widehat{q}\,p$, – a spinor-matrix product, namely

$$q\,p \to \widehat{q}\,p \to \ -\mathrm{i}\,\widehat{Q}\,q \tag{53}$$

where q is the spinor associated to the quaternion p via (50) and $\widehat{Q}$ is the complex, unitary, square matrix (with negative determinant $q\,\overline{q}$), defined by

$$\widehat{Q} = \begin{bmatrix} q_3 + \mathrm{i}\,q_0 & q_1 - \mathrm{i}\,q_2 \\ q_1 + \mathrm{i}\,q_2 & -q_3 + \mathrm{i}\,q_0 \end{bmatrix} = q_0\,\mathrm{i}\,\widehat{\sigma}_0 + q_h\,\widehat{\sigma}_h \tag{54}$$

where $\widehat{\sigma}_\alpha$, $\alpha = 0, 1, 2, 3$ are the Pauli matrices. Let us denote by σ_α the first column of a Pauli matrix $\widehat{\sigma}_\alpha$. According to (50), the four basic quaternions $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$ are represented, respectively, by $\mathrm{i}\,\sigma_0, \sigma_1, \sigma_2, \sigma_3$. The connection between a spinor Q and a square matrix (54) is given by

$$Q = \widehat{Q}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{55}$$

The correspondence (50), $Q = \tau q$, is realized by means of the rectangular block-matrix $\tau = [\mathrm{i}\,\sigma_0 \mid \sigma_1 \mid \sigma_2 \mid \sigma_3]$. A spinor transformation

$$Q' = s\,Q, \qquad s = \begin{bmatrix} z & W \\ -\overline{W} & \overline{z} \end{bmatrix} \tag{56}$$

corresponds to a quaternion transformation $q' = \widehat{a}\,q$, i.e., to a rotation $q' = a\,q$ in $\mathbb{R}^4$, the matrix $\widehat{a} = h(a)$ is skew-symmetric, orthogonal and unimodular, and represents the unit quaternion $a = (\mathrm{Re}z, -\mathrm{Im}W, -\mathrm{Re}W, -\mathrm{Im}z)$. So, to the quaternion representing the rotation of a rigid body (50), corresponds the spinor

$$b \to \psi = \begin{pmatrix} \beta_3 + \mathrm{i}\,\beta_0 \\ \beta_1 + \mathrm{i}\,\beta_2 \end{pmatrix} \longrightarrow \widehat{\psi} = \begin{bmatrix} \beta_3 + \mathrm{i}\,\beta_0 & \beta_1 - \mathrm{i}\,\beta_2 \\ \beta_1 + \mathrm{i}\,\beta_2 & \beta_3 + \mathrm{i}\,\beta_0 \end{bmatrix} \tag{57}$$

$$\omega \to \Omega = \begin{pmatrix} \omega_3 \\ \omega_1 + \mathrm{i}\,\omega_2 \end{pmatrix} \longrightarrow \widehat{\Omega} = \begin{bmatrix} \omega_3 & \omega_1 - \mathrm{i}\,\omega_2 \\ \omega_1 + \mathrm{i}\,\omega_2 & -\omega_3 \end{bmatrix}. \tag{58}$$

From the equations (53) and (57) the quaternion expression (26) for rotations becomes

$$X' = (-\mathrm{i})\,\widehat{\psi}\,(-\mathrm{i})\,\widehat{X}\,\psi^{-1} = -\widehat{\psi}\,\widehat{X}\,\psi^{-1}. \tag{59}$$

Analogically the relations for the spinor kinematics are obtained

$$\begin{aligned} X' &= -\widehat{B}\,\widehat{X}\,B^{-1} \\ \begin{pmatrix} x_3 \\ x_1' + \mathrm{i}\,x_2' \end{pmatrix} &= \begin{bmatrix} \beta_3 + \mathrm{i}\,\beta_0 & \beta_1 - \mathrm{i}\,\beta_2 \\ \beta_1 + \mathrm{i}\,\beta_2 & -\beta_3 + \mathrm{i}\,\beta_0 \end{bmatrix} \begin{bmatrix} x_3 & x_1 - \mathrm{i}\,x_2 \\ x_1 + \mathrm{i}\,x_2 & -x_3 \end{bmatrix} \\ &\quad \times \begin{pmatrix} -\beta_3 + \mathrm{i}\,\beta_0 \\ -\beta_1 - \mathrm{i}\,\beta_2 \end{pmatrix} \end{aligned} \tag{60}$$

$$\begin{aligned} \dot{\psi} &= -\frac{1}{2}\,\mathrm{i}\,\widehat{\Omega}\,\psi \\ \begin{pmatrix} \dot{\beta}_3 + \mathrm{i}\,\dot{\beta}_0 \\ \dot{\beta}_1 + \mathrm{i}\,\dot{\beta}_2 \end{pmatrix} &= -\frac{1}{2}\,\mathrm{i} \begin{bmatrix} \omega_3 & \omega_1 - \mathrm{i}\,\omega_2 \\ \omega_1 + \mathrm{i}\,\omega_2 & -\omega_3 \end{bmatrix} \begin{pmatrix} \beta_3 + \mathrm{i}\,\beta_0 \\ \beta_1 + \mathrm{i}\,\beta_2 \end{pmatrix} \end{aligned} \tag{61}$$

$$\begin{aligned} \Omega &= -2\mathrm{i}\,\widehat{\dot{\psi}}\,\psi_{-1} \\ \begin{pmatrix} \omega_3 \\ \omega_1 + \mathrm{i}\,\omega_2 \end{pmatrix} &= -2\mathrm{i} \begin{bmatrix} \dot{\beta}_3 + \mathrm{i}\,\dot{\beta}_0 & \dot{\beta}_0 - \mathrm{i}\,\dot{\beta}_2 \\ \dot{\beta}_1 + \mathrm{i}\,\dot{\beta}_2 & -\dot{\beta}_3 + \mathrm{i}\,\dot{\beta}_0 \end{bmatrix} \begin{pmatrix} -\beta_3 + \mathrm{i}\,\beta_0 \\ -\beta_1 + \mathrm{i}\,\beta_2 \end{pmatrix} \end{aligned} \tag{62}$$

$$\begin{aligned} \widetilde{\Omega} &= -2\mathrm{i}\,\widehat{\psi}^{-1}\,\dot{\psi} \\ \begin{pmatrix} r \\ p + \mathrm{i}\,q \end{pmatrix} &= -2\mathrm{i} \begin{bmatrix} -\beta_3 + \mathrm{i}\,\beta_0 & -\beta_1 + \mathrm{i}\,\beta_2 \\ -\beta_1 - \mathrm{i}\,\beta_2 & \beta_3 + \mathrm{i}\,\beta_0 \end{bmatrix} \begin{pmatrix} \dot{\beta}_3 + \mathrm{i}\,\dot{\beta}_0 \\ \dot{\beta}_1 + \mathrm{i}\,\dot{\beta}_2 \end{pmatrix}. \end{aligned} \tag{63}$$

2.4. Vector-Parameter Representations of Rotations

Let us consider the special orthogonal group SO(3)

$$\mathrm{SO}(3) = \{O \in \mathrm{Mat}(3, R)\ ;\ \det O = 1, \quad OO^T = I\} \tag{64}$$

where $\mathrm{Mat}(3, R)$ is the group of 3×3 real matrices together with its Lie algebra (infinitesimal generators) consisting of the real skew-symmetrical 3×3 matrices. If A belongs to the Lie algebra of SO(3), the matrix $I - A$ is invertible, and the Cayley transformation making the connection between the algebra and the group explicit is given by the formulas [49]

$$O = (I + A)(I - A)^{-1} = (2I - (I - A))(I - A)^{-1} = 2(I - A)^{-1} - I.$$

As an exception in the three-dimensional space, there exists a map between vectors and skew-symmetric matrices, i.e., if $c \in \mathbb{R}^3$, then $c \to c^\times$, where $c^\times$ is the corresponding skew–symmetric matrix. Then we may write the SO(3) matrix in the form

$$O = O(c) = (I + c^{\times})(I - c^{\times})^{-1} = \frac{(1 - c^2)I + 2c.c + 2\,c^{\times}}{1 + c^2} \tag{65}$$

and consider it as a mapping from $\mathbb{R}^3$ to SO(3) (see [107] for higher-dimensional generalizations) for which the smooth inverse is

$$c^{\times} = \frac{[O - O^T]}{1 + \mathrm{tr}(O)}. \tag{66}$$

Here I is the 3×3 identity matrix, $c.c$ means diadic, $\mathrm{tr}(O)$ is the trace of the matrix O and "T" is the symbol for transposition of a matrix. The formula above provides us with an explicit parameterization of SO(3). The vector c is called the **vector-parameter** [24]. It is parallel to the axis of rotation and its module $\|c\|$ is equal to $\tan(\alpha/2)$. The so defined vector-parameters form a Lie group with the following composition law

$$c' = \langle c_1, c_2 \rangle = \frac{c_1 + c_2 + c_1 \times c_2}{1 - c_1\,c_2}. \tag{67}$$

The symbol "$\times$" means cross product of vectors. Every component of c can take all values from $-\infty$ to $+\infty$ without any restrictions, which is a great advantage compared with the evident asymmetry in the Eulerian parameterization. Obviously, the vector $c \equiv 0$ corresponds to the identity matrix $O(0) \equiv I$ and $-c$ produces the inverse rotation $O(-c) \equiv O^{-1}(c)$. Conjugating with elements from the SO(3) group leads to linear transformations in the vector-parameter space

$$O(c)\,O(c')\,O^{-1}(c) = O(c'')$$

where $c'' = O(c)c' = O_c c'$. Such a parameterization in the Lie group theory is called natural. It is worth mentioning that no other parameterization possesses either this property or a manageable superposition law (see also [28]).

The exceptional case in (67), i.e., when $c_1\,c_2 = 1$, may be treated by replacing the vector c with a vector d from another chart of the SO(3) group manifold using the relation

$$d = \frac{c}{1 + c^2} \tag{68}$$

and

$$O(c) = O'(d) = I + 2(\sqrt{1 - d^2} + d^{\times})d^{\times}. \tag{69}$$

The direction of d coincides with that of c and $\|d\| = \sin(\alpha/2)$. The composition law in this case is

$$d' = \langle d_1\,,\, d_2 \rangle = d_1\sqrt{1 - d_2^2} + d_2\sqrt{1 - d_1^2} + d_1 \times d_2. \tag{70}$$

2.5. Vector-Parameters for Different SO(3) Parameterizations

Here we shall show how the vector-parameter looks in the case when one uses other parameterizations of the SO(3) group. In the case of a parameterization of the rotation group through Eulerian angles $(\psi,\ \vartheta,\ \varphi)$ one finds that

$$c=(-\tan\frac{\vartheta}{2}\cos\frac{\psi-\varphi}{2}/\cos\frac{\psi+\varphi}{2},\tan\frac{\vartheta}{2}\sin\frac{\psi-\varphi}{2}/\cos\frac{\psi+\varphi}{2},\tan\frac{\varphi+\psi}{2}). \tag{71}$$

If c' is a vector-parameter describing another rotation expressed in Eulerian angles $(\psi',\ \vartheta',\ \varphi')$, through the relation $c''=\langle c',c\rangle$, then the complex and practically unusable composition law in Eulerian angles can be obtained

$$\begin{aligned}
\psi'' &= -\arctan\frac{c_2''}{c_1''}-\arctan c_3'' \\
\varphi'' &= \arctan\frac{c_2''}{c_1''}-\arctan c_3'' \\
\vartheta'' &= 2\arctan\pm\cos\left(\arctan c_3''\right)\sqrt{c_1''^2+c_2''^2}
\end{aligned}$$

where $c''(c_1''\ c_2''\ c_3'')$ and

$$\begin{aligned}
c_1'' &= \frac{-\tan\frac{\vartheta}{2}\cos\frac{\psi-\varphi-\psi'-\varphi'}{2}-\tan\frac{\vartheta'}{2}\cos\frac{\psi+\varphi+\psi'-\varphi'}{2}}{\cos\frac{\psi+\varphi+\psi'+\varphi'}{2}-\tan\frac{\vartheta}{2}\tan\frac{\vartheta'}{2}\cos\frac{\psi-\varphi-\psi'+\varphi'}{2}} \\
&= -\frac{1}{\cos\frac{\psi''+\varphi''}{2}}\tan\frac{\vartheta''}{2}\cos\frac{\psi''-\varphi''}{2} \\
c_2'' &= \frac{\tan\frac{\vartheta}{2}\sin\frac{\psi-\varphi-\psi'-\varphi'}{2}+\tan\frac{\vartheta'}{2}\sin\frac{\psi+\varphi+\psi'-\varphi'}{2}}{\cos\frac{\psi+\varphi+\psi'+\varphi'}{2}-\tan\frac{\vartheta}{2}\tan\frac{\vartheta'}{2}\cos\frac{\psi-\varphi-\psi'+\varphi'}{2}} \\
&= \frac{1}{\cos\frac{\psi''+\varphi''}{2}}\tan\frac{\vartheta''}{2}\cos\frac{\psi''-\varphi''}{2} \\
c_3'' &= \frac{-\sin\frac{\psi+\varphi+\psi'+\varphi'}{2}+\tan\frac{\vartheta'}{2}\tan\frac{\vartheta'}{2}\sin\frac{\psi-\varphi+\psi'-\varphi'}{2}}{\cos\frac{\psi+\varphi+\psi'+\varphi'}{2}-\tan\frac{\vartheta}{2}\tan\frac{\vartheta'}{2}\cos\frac{\psi-\varphi-\psi'\varphi'}{2}} \\
&= -\frac{\sin\frac{\psi''+\varphi''}{2}}{\cos\frac{\psi''+\varphi''}{2}}.
\end{aligned}$$

In the case of Eulerian parameters $(q_o,\ q_1,\ q_2,\ q_3)$, the vector-parameter is

$$c=(-\frac{q_3}{q_o},\ -\frac{q_2}{q_o},\ \frac{q_1}{q_o}) \tag{72}$$

while, for the Bryant angles $(\varphi_1,\ \varphi_2,\ \varphi_3)$, we have the relations

$$c_1 = (\tan\frac{\varphi_1}{2},\ 0,\ 0),\ c_2 = (0,\ \tan\frac{\varphi_2}{2},\ 0),\ c_3 = (0,\ 0,\ \tan\frac{\varphi_3}{2}) \tag{73}$$

leading to

$$c = \langle c_3,\ c_2,\ c_1\rangle = \langle c_3,\ \langle c_2,\ c_1\rangle\rangle.$$

As a result

$$\begin{aligned} c = \frac{1}{1+\tan\frac{\varphi_1}{2}\tan\frac{\varphi_2}{2}\tan\frac{\varphi_3}{2}} & (\tan\frac{\varphi_1}{2} - \tan\frac{\varphi_2}{2}\tan\frac{\varphi_3}{2}, \\ & \tan\frac{\varphi_2}{2} + \tan\frac{\varphi_1}{2}\tan\frac{\varphi_3}{2}, \\ & \tan\frac{\varphi_3}{2} - \tan\frac{\varphi_1}{2}\tan\frac{\varphi_2}{2}). \end{aligned} \tag{74}$$

On the basis of the above considerations, we may conclude that the suggested approach of the vector-parameterization is applicable with any choice of coordinate frames and with different generalized parameters.

It is worth to note here that the composition laws of the other parameterizations are not used and that they have never been written in explicit form because of their complexity. As examples, the complex superpositions in the cases of Eulerian and Bryant angles are shown in this section.

Alternatively, the group SO(3) may be parameterized through a vector l, which is directed along the rotation axis and it is equal numerically to the angle of rotation α

$$l = \frac{\alpha c}{\|c\|} = 2\frac{c}{\|c\|}\arctan\|c\|$$

so that

$$c = \frac{l}{\|l\|}\tan\frac{\|l\|}{2}.$$

This parameterization is also linear and natural, but it has a very complex composition law.

2.6. Euclidean Motion

The Euclidean motions in $\mathbb{R}^3$, i.e., rotations about a fixed point and translations can be unified in the equation

$$r = Or_d + p \tag{75}$$

where $O \in$ SO(3), r_d and r are vectors in movable R_d and immovable R_o frames respectively and p is a vector of translation. If O and p are time dependent, the expression (75) gives a consequence of displacements. Introducing the skew-symmetric matrix of angular velocity $\Omega = \dot{O}\,O^T$, it follows that

$$\Omega^2 = -(\dot{O}\,O^T)(\dot{O}\,O^T)^T = -\dot{O}\,\dot{O}^T \quad \text{and} \quad \dot{\Omega} = \ddot{O}\,O^T + \dot{O}\,\dot{O}^T. \tag{76}$$

After differentiating (75), we obtain for the linear and acceleration velocity vectors respectively

$$\dot{r} = \dot{O}\,r_d + \dot{p} = \Omega\,(r - p) + \dot{p} \tag{77}$$

$$\ddot{r} = [\dot{\Omega} + \Omega^2]\,(r - p) + \ddot{p} \tag{78}$$

where $\dot{\Omega}(t)\,(r(t) - p(t))$ is the tangential acceleration, parallel to the velocity and $\Omega^2(t)(r(t) - p(t))$ is the normal acceleration. By using equation (65) and the relation $\Omega = \dot{O}\,O^T$ for the angular velocity vector, it can be verified that

$$\omega = \frac{2}{1 + c^2}\,(c \times \dot{c} + \dot{c}) \tag{79}$$

where c is the vector-parameter defining the rotation of the movable frame R_d with respect to the fixed one R_o. Then the vector of the angular acceleration is

$$\epsilon = \dot{\omega} = \frac{2}{1 + c^2}\,(c \times \ddot{c} + \ddot{c} - \omega(c\dot{c})). \tag{80}$$

The components of the vectors $\dot{c}$ and $\ddot{c}$ are respectively the first and second derivatives of the components of the vector-parameter c. The recurrent relations for the vectors of angular and linear velocities and accelerations of the links of a rigid-body system as well as their dynamical characteristics are presented in [61].

The **kinematic-differential equations** (KDE) relate the time derivatives of the angular position coordinates to the angular velocity vector. With the angular velocity vector in terms of vector-parameters (79), the KDE may be obtained directly from the formula

$$\dot{c} = \frac{[O(-c) + I]\,\omega}{1 + \mathrm{tr}(O)}. \tag{81}$$

It should be noted that the equation $\dot{O} = \Omega O$, for $O = O(c)$, is an alternative form of the KDE. Since the matrices O and the vectors c belong to a Lie group and Ω and ω are elements of the corresponding Lie algebra, the equations above give the relation between the algebra and the group. Lie groups and Lie algebras live in a closed relationship, with the exponential mapping being a key feature of this relationship. The configurational manifold of a rigid body motion with a fixed point in the ordinary Euclidean space $\mathbb{R}^3$ can be identified quite naturally with the special orthogonal group SO(3) and its phase space is nothing but the tangent bundle TSO(3). In classical mechanics, the usual description of a rigid-body state is through its pole and angular velocity. This is complemented by an identification of the tangent vectors at different points. A basic idea here is that the definition of angular velocity does not depend on the concrete body state. That is why one may speak about equality of the angular velocities independently of the rigid-body state. This is the essence of the **tangent bundle trivialization** [63]. There are at least two alternative trivializations of TSO(3):

Theorem 1. *When one trivializes the tangent bundle of the* SO(3) *group through vector-parameter* c *and angular velocity vector* ω *the group composition law is* $(c_1, \omega_1)\,(c_2, \omega_2) = (c_3,\, \omega_3)$, *where* $c_{_i} \in \mathrm{SO}(3)$, $\omega_{_i} = \omega(c_{_i}) \in \mathrm{T}_c\mathrm{SO}(3)$, $(i = 1, 2, 3)$, $c_3 = \langle c_1\,,\, c_2\rangle$ *and* ω_3 *is defined by* $\Omega_3 = \Omega_1 + O(c_1)\Omega_2 O^T(c_1)$.

Theorem 2. *Under the second alternative trivialization of the* SO(3) *tangent bundle through the vectors* c *and* $\dot{c}$ *the group law is* $(c_1\,,\, \dot{c}_1)\,(c_2\,,\, \dot{c}_2) = (c_3,\, \dot{c}_3)$.

On the basis of the theorems given above, if $x = [p \ \vdots\ o]^T$ is a 6-dimensional vector of position p and orientation o of a rigid body, then $\dot{o}$ in the velocity vector $\dot{x} = [\dot{p} \ \vdots\ \dot{o}]^T$ (treated before as an angular velocity vector, the derivatives of three independent elements of the rotation matrix, derivatives of Bryant angles, Eulerian parameters, etc.) may be considered either as a velocity of orientation change of a movable frame with respect to the fixed one $(\dot{o} = \dot{c}')$ or as an angular velocity vector $(\dot{o} = \omega)$.

Now we consider the bundle $\mathrm{T_1S^2}$ consisting of the unit tangents to $\mathbb{S}^2$ vectors, namely

$$\mathrm{T_1S^2} = \{(\mathrm{x}, \mathrm{y});\ \|\mathrm{x}\|^2 = 1,\ \mathrm{y}.\mathrm{x} = 0,\ \|\mathrm{dy}\|^2 = 1\}.$$

Theorem 3. *In spite of the fact that* $\mathbb{S}^2$ *is not a Lie group,* $\mathrm{T_1S^2}$ *has a group structure isomorphic to the one of* SO(3).

Proof: Let us consider $(x, y) \in \mathrm{T_1S^2}$ with $x(x_1, x_2, x_3),\ y(y_1, y_2, y_3)$. It may be immediately checked that the matrix $O(x, y)$, given by

$$O(x, y) = \begin{bmatrix} x_1 & y_1 & (x \times y)_1 \\ x_2 & y_2 & (x \times y)_2 \\ x_3 & y_3 & (x \times y)_3 \end{bmatrix} \tag{82}$$

is orthogonal. From the definition of $c^{\times}$ it follows that the components of the vector-parameter $c(c_1, c_2, c_3)$ corresponding to this rotation are

$$\begin{aligned} c_1 &= \frac{y.e_3 - (x \times y).e_2}{1 + x.e_1 + y.e_2 + (x \times y).e_3} \\ c_2 &= \frac{(x \times y).e_1 - x.e_3}{1 + x.e_1 + y.e_2 + (x \times y).e_3} \\ c_3 &= \frac{x.e_2 - y.e_1}{1 + x.e_1 + y.e_2 + (x \times y).e_3} \end{aligned} \tag{83}$$

where e_1, e_2, e_3 provide a standard basis in $\mathbb{R}^3$. Here the symbol "." means scalar product. The vector form of c is accordingly

$$c = \frac{J_1 x + J_2 y + J_3(x \times y)}{1 + e_1.x + e_2.y + e_3(x \times y)} \tag{84}$$

and the matrices $J_i = e_i^{\times}, \quad i = 1, 2, 3$

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad J_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are the standard infinitesimal generators of the rotation group. Since

$$1 + x.e_1 + y.e_2 + (x \times y).e_3 = 1 + \mathrm{tr}O(x, y) = 4/(1 + c^2)$$

if we denote $J_1 x + J_2 y + J_3(x \times y) = a$, from the equation $c = a/(1 + c^2)$ we get $4/(1 + c^2) = 2 - \sqrt{4 - a^2}$. Therefore, from equation (84), it follows that

$$c = \frac{a}{2 - \sqrt{4 - a^2}}. \tag{85}$$

Let us consider the couples: (x_1, y_1), $(x_2, y_2) \in \mathrm{T_1 S^2}$, whose components are three dimensional vectors, namely $x_i(x_{i1}, x_{i2}, x_{i3})$, $y_i(y_{i1}, y_{i2}, y_{i3})$, $i = 1, 2$. Let $a_i = J_1 x_i + J_2 y_i + J_3(x_i \times y_i)$ and $c_i(c_{i1}, c_{i2}, c_{i3})$ are the corresponding vector-parameters for the matrices $O(x_i, y_i)$. The resultant function is given by the vector c_3, which is a function of x_3 and y_3. If we denote by X_i the raw vector $[x_i\ y_i\ x_i \times y_i]$, $i = 1, 2, 3$, it may be proved that

$$X_3 = \frac{(1 - X_2 E)X_1 + (1 + X_1 E)X_2}{1 - (X_1 E)(X_2 E) - (X_1 \times E)(X_2 \times E)} \tag{86}$$

where $X_i E = \mathrm{tr}O(x_i, y_i) = 1 - \sqrt{4 - a_i^2}$. It may be also shown that

$$X_i \times E = [y_i.e_3 - (x_i \times y_i).e_2,\ -x_i.e_3 + (x_i \times y_i).e_1,\ x_i.e_2 - y_i.e_1]$$

where $E = [e_1, e_2, e_3]$, is exactly the vector a_i. Then the resultant couple (x_3, y_3) is

$$(x_3, y_3) = \frac{(2 - \sqrt{4 - a_2^2}(x_1, y_1) + (2 - \sqrt{4 - a_1^2}(x_2, y_2)}{\sqrt{4 - a_1^2} + \sqrt{4 - a_2^2} - \sqrt{4 - a_1^2}\sqrt{4 - a_2^2} - a_1 a_2}. \tag{87}$$

In explicit form we have

$$x_3 = Ax_1 + Bx_2, \qquad y_3 = Ay_1 + By_2 \tag{88}$$

where

$$\begin{aligned}
A &= b_2/c, \qquad B = b_1/c \qquad b_2 = 2 - \sqrt{4 - a_2^2}, \quad b_1 = 2 - \sqrt{4 - a_1^2} \\
c &= \sqrt{4 - a_1^2} + \sqrt{4 - a_2^2} - \sqrt{4 - a_1^2}\sqrt{4 - a_2^2} - a_1 a_2 \\
a_1 &= J_1 x_1 + J_2 y_1 + J_3(x_1 \times y_1), \quad a_2 = J_1 x_2 + J_2 y_2 + J_3(x_2 \times y_2).
\end{aligned}$$

The formulas (87) and (88) define the group operation inside $\mathrm{T_1S^2}$. Besides, the above theorem clarifies the topological structure of SO(3).

2.7. Dynamics and Control of a Rigid Body with Vector-Parameters

We consider now the dynamics of a rigid body in $\mathbb{R}^3$ which is subject to external torques M (for example, spacecraft attitude with gas-jet or momentum-exchange actuators [39], [43], [25], [8]). We treat a pure rotation of the body about its fixed point. According to the above notation, we denote by R a space frame and by R_b a frame fixed with respect to the body. The subindex "b" means that the corresponding vector is expressed in the body-frame R_b.

Let J_{b1}, J_{b2}, J_{b3} denote the principal moments of inertia (positive real numbers). It is well known that the angular momentum of the system in the body frame is

$$p_b = J_b\, \omega_b \tag{89}$$

with J_b = diag $[J_{b1},\ J_{b2},\ J_{b3}]$ which is a matrix of constants and is called the inertia matrix. The momentum balance condition in the space frame yields

$$\dot{p} = M \tag{90}$$

and $M = OM_b$. Then using the relation $p_b = O^T p$, we obtain

$$J_b\, \dot{\omega}_b = \dot{p}_b = \dot{O}^T p + O^T \dot{p} = \dot{O}^T O\, O^T p + M_b = -\,\Omega_b J_b\, \omega_b + M_b. \tag{91}$$

Here we use the relation $\dot{O}^T O = -\Omega_b$ which may be obtained from the time-differentiation of the matrix identity $OO^T = I$ and the definition $\Omega_b = O^T O$. The equations thus obtained

$$J_b\,\dot{\omega}_b = -\,\Omega_b\,J_b\,\omega_b + M_b \tag{92}$$

or

$$J_b\,\dot{\omega}_b = -\,\omega_b \times J_b\,\omega_b + M_b \tag{93}$$

are commonly known as Euler dynamic equations.

After introduction of the notation $S_b(\omega) = -\,\Omega_b$, the system which describes control of the body is expressed by means of equations of the form

$$J_b\,\dot{\omega}_b = S_b\,(\omega)\,J_b\,\omega_b + M_b \tag{94}$$

$$\dot{O} = O\,S_b(\omega), \qquad O = O(c) \tag{95}$$

with state $(\omega, O) \in \mathbb{R}^3\times$ SO(3) and input $M_b \in \mathbb{R}^3$. If we suppose the external torque M_b generated by a set of m independent pairs of gas jets (thrusters), we may set

$$M_b = \sum b_i\,u_i, \qquad i = 1, \ldots, m \tag{96}$$

where $b_1, \ldots, b_m \in \mathbb{R}^3$ represent the vectors of direction cosines, with respect to the body of the axes about which the control torques are applied, and $u_1, \ldots, u_m$ are the corresponding magnitudes. It is assumed that the vectors $b_1, \ldots, b_m$ are linearly independent. Setting $x = J_b\,\omega_b$ and using the property $S(w)v = -S(v)w$, we may rewrite the equation in question in the form

$$\dot{x} = S(x)J^{-1}x + Bu, \qquad S(x) = S_b(J_b\,\omega_b), \qquad B = [b_1\,b_2 \ldots b_n]. \tag{97}$$

Thus we obtain the state equations in the usual form

$$\dot{x} = f(x) + g_1(x)u_1 + \ldots + g_m(x)\,u_m \tag{98}$$

where

$$f(x) = S(x)\,J_b^{-1}x, \qquad g_i(x) = b_i. \tag{99}$$

The vector-parameterization of SO(3) and trivialization of TSO(3) through the vectors c and ω imply that the system which controls the rigid body motion is

$$J_b\,\dot{\omega}_b = S_b(\omega)\,J_b\,\omega_b + B\,u \tag{100}$$

$$\dot{c} = A\,O\,\omega_b, \qquad \omega = O\,\omega_b \tag{101}$$

where the matrix A is

$$A = \frac{1}{1 + \mathrm{tr}(O)}\,[O(-c) + I]. \tag{102}$$

In the state space $(\omega, c) \in \mathbb{R}^3 \times \mathbb{R}^3$, with the matrix operator A from $\dot{c} = A\,\omega$, realizing a transition from the space of the angular velocity vectors to the vectors $\dot{c}$, we have

$$\begin{aligned} \dot{x} &= S(x)\,J_b^{-1}x + B\,u && (103)\\ \dot{c} &= A\,O\,J_b^{-1}x. && (104) \end{aligned}$$

If the tangent bundle TSO(3) is trivialized by the vectors c and $\dot{c}$ using the substitution

$$y = J_b\,O^{-1}\,A^{-1}\dot{c} \tag{105}$$

the last system may be written in the form

$$\begin{aligned} \dot{y} &= S(y)\,J_b^{-1}y + \hat{B}\,\hat{u} && (106)\\ \dot{c} &= A\,O\,J_b^{-1}y. && (107) \end{aligned}$$

3. Kinematics and Dynamics of a Manipulator System with Vector-Parameters

3.1. Basic Kinematical Relations

As a basic example we consider here a MS with n degrees of freedom and links $B_o, B_1, B_2, \ldots, B_n$, where B_o is fixed (Fig. 1). A frame R_i centered in O_i is referred to the link B_i (and joint i) from the chain. The frame R_{n+1} is built in the gravity center G_n of the link B_n (gripper), which may be coincident with the gripper characteristic point C [75–77].

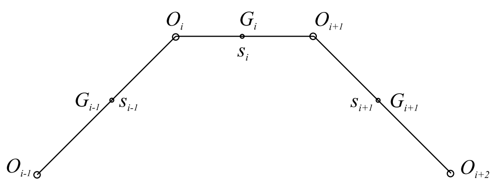

Figure 1. Links $i-1,\ i$ and $i+1$ of a manipulator

The vector $\overrightarrow{O_{i-1}\, O_i}$ between the respective centers of the frames R_{i-1} and R_i is denoted by s_{i-1}, the vectors $\overrightarrow{O_i\, G_i}$, $\overrightarrow{O_i\, G}_{i-1}$ and $\overrightarrow{G_{i-1}\, O_i}$ by $s_{ii}, s_{i,i-1}$ and $s_{i-1,i}$, respectively, and the vectors $\overrightarrow{O_o\, O_i}$, $\overrightarrow{O_o G_i}$ by p_i and p_{G_i}, respectively where

$$p_i = s_o + s_1 + \ldots + s_{i-1}, \qquad s_o = 0, \qquad p_{G_i} = p_i + s_{ii} \tag{108}$$

and $q_1, q_2, \ldots, q_n$ are the generalized coordinates (joint displacements) of the MS in the usual sense.

Let $(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)$ be the vectors describing the joint movements of the MS. They occupy the group configurational space

$$Q_{ct} = \{(c,\, t)^{\min} \leq (c,\, t) \leq (c,\, t)^{\max}\}. \tag{109}$$

The entries of the i-th pair $(c_i,\, t_i)$ coincide either with the vector-parameter c_i or translational vector t_i in dependence on the type of the i-th joint (revolute or prismatic, respectively).

The absolute gripper vector $x = [p \vdots o]^T$ gives the information for the position of the end–effector (through the vector $p \in \mathbb{R}^{m_1}$) and for the orientation (through the vector $o \equiv c' = \langle c_1, c_2, \ldots, c_n \rangle \in \mathbb{R}^{m_2}$) and $\dim(p) + \dim(o) = m_1 + m_2 = m$. Further on, for a more clear presentation, we shall consider MS with rotational pairs only, i.e., the system whose motion is described through the vector c

$$c = [c_1, c_2, \ldots, c_n]^T, \qquad c' = \langle c_1, c_2, \ldots, c_n \rangle \qquad \text{and} \qquad Q_{ct} \equiv Q_c.$$

The connection between x and c (DKP) is given by

$$x = F_V(c) \tag{110}$$

where $F_V: \;\; Q_c \to X$ is a smooth projection map over the target space

$$X = \{x;\, x = F_V(c),\; c \in Q_c \subset \mathbb{R}^m\} \tag{111}$$

which is the working space of the MS under consideration. After differentiation with respect to t of the equation (110), one obtains

$$\dot{x} = [\frac{\partial F_V}{\partial c}]\, \dot{c} = J_V(c)\, \dot{c} \tag{112}$$

where $J_V(c) \in \mathbb{R}^{m,3n}$ is the Jacobian matrix of the map F_V. As before all configurations for which the rank of $J_V(c) < m$ are called singular. One can use equation (112) to go to the standard joint variables q through the transformation

$$\dot{x} = [\frac{\partial F_V}{\partial c}]\,[\frac{\partial c}{\partial q}]\, \dot{q} = J_V(c)\,[\frac{\partial c}{\partial q}]\, \dot{q} = J(q)\, \dot{q} \tag{113}$$

where $\partial c/\partial q$ is $3n \times n$ block diagonal matrix with elements

$$[\frac{\partial c_{i1}}{\partial q_i}, \ \frac{\partial c_{i2}}{\partial q_i}, \ \frac{\partial c_{i3}}{\partial q_i}]^T, \qquad i = 1, 2, \ldots, n.$$

As it can be seen, the Jacobian of this transformation $J(q)$ may be presented as a product of two matrices which simplifies the computation while its inversion can be realized through parallel processors. In the case of a "pure" vector-parameter consideration of MS (i.e., in Q_c space), after measuring the joint displacements q_i by the transducers, the vectors c_i (as functions of $\tan(q_i/2)$, $i = 1, 2, \ldots, n$ are tabulated.

The vector of the end–effector angular velocity may be presented as

$$\omega = N_V \dot{c} \tag{114}$$

where N_V is the $3 \times 3n$ block matrix

$$N_V = [N_V^1 \, N_V^2 \, \ldots \, N_V^n] \tag{115}$$

with components

$$N_V^k = O(\langle c_1, c_2, \ldots, c_{k-1} \rangle) N(c_k) \ = \ O(g_{k-1}) N(c_k) \tag{116}$$

and

$$N(c_k) = \frac{1}{1 + c_k^2} [c_k^\times + I]. \tag{117}$$

By analogy, the gripper linear velocity vector may be written as

$$V = M_V \dot{c} \tag{118}$$

where M_V is also a $3 \times 3n$ matrix of the type

$$M_V = [M_V^1 \, M_V^2 \, \ldots \, M_V^n] \tag{119}$$

whose elements are the matrices

$$M_V^k = -s_{kc}^x O(g_{k-1}) N(c_k), \qquad s_{kc} = O_k G_n. \tag{120}$$

Hence, when $\dot{o} = \omega$, we have

$$\dot{x} = [V \vdots \omega]^T = [M_V \vdots N_V]^T \dot{c}. \tag{121}$$

The matrix $[M_V \vdots N_V]^T = J_V(c)$ is the Jacobian matrix of the map F_V. Now we use the kinematical differential equation in vector parameter terms. Denoting by Λ the matrix on the right hand side of the last equation, we see that, when $\dot{o} = \dot{c}'$, we have

$$\dot{x} = [\dot{p} \vdots \dot{c}']^T = [M_V \vdots N_{CV}]^T \dot{c} \tag{122}$$

where N_{CV} is $3 \times 3n$ block matrix

$$N_{CV} = [N_{CV}^1 \; N_{CV}^2 \; \dots \; N_{CV}^n] \tag{123}$$

with components 3×3 matrices of the type

$$N_{CV}^k = \Lambda N_V^k\,. \tag{124}$$

We may also write

$$N_{CV}^k = \frac{1}{1 + \mathrm{tr}O(c')}[O(\langle -c_n, -c_{n-1}, \dots, -c_k\rangle)N(c_k) + N_V^k]. \tag{125}$$

Finally, we have to note that the full information for the i-th joint motion $(i = 1, 2, \dots, n)$ is entirely encoded into just one of the components of the vector-parameter c_i.

3.2. Dynamics in Q_{ct}

Dynamical equations of MS are very important for simulation of motion and control of the manipulator systems. In design, they are used in motion simulation, where they furnish a powerful tool for the study of control strategies, optimization of the parameters, and for the testing of robot performance under various conditions. In connection with the robot operation, dynamical equations are used for the evaluation of nominal actuator torques and forces which drive the robot along a prescribed trajectory. Such calculations normally need to be performed on-line, and since the forces and torques need to be updated frequently, computational efficiency is of major concern.

In [61], dynamic models based on Lagrange's, Newton and Euler's, Tzenov's, as well Appel's and Nilsen's equations of motion are built. The final form of the equations for MS motion is expressed in the standard joint coordinates q on the configurational space Q.

Here a dynamic model in pure vector-parameter based on Lagrange's equations is presented. The linear velocity V_{G_i} of the center of gravity G_i of the i-th link will be written as $V_{G_i} = M_{V_i}\dot{c}$, where M_{V_i} are $3 \times 3n$ block matrices

$$M_{V_i} = [M^1_{V_i} \vdots M^2_{V_i} \vdots \ldots \vdots M^i_{V_i} \vdots \emptyset \vdots \emptyset \vdots \cdots \vdots \emptyset] \tag{126}$$

with components

$$M^k_{V_i} = \begin{cases} -p^{o\times}_{G_i} O(g_{k-1}) N(c_k) & k = 1, 2, ..., i-1 \\ (p_i - p^o_{G_i})^\times O(g_{k-1}) N(c_i) & k = i. \end{cases} \tag{127}$$

Since $\omega^i_i = O(-g_i)\,\omega^o_i$, it is appropriate to introduce the $3 \times 3n$ block matrix $\bar{N}_{V_i}$

$$\bar{N}_{V_i} = [\bar{N}^1_{V_i} \vdots \bar{N}^2_{V_i} \vdots \ldots \vdots \bar{N}^i_{V_i} \vdots \emptyset \vdots \emptyset \vdots \cdots \vdots \emptyset] \tag{128}$$

whose components are

$$\bar{N}^k_{V_i} = O(-g_i) O(c_{k-1}) N(c_k), \qquad k = 1, 2, \ldots, i \tag{129}$$

with $O(c_o) = I$. The kinetic energy of the i-th link is transformed in the following way

$$\begin{aligned} T_i &= \frac{1}{2} m_i \dot{c}^T M^T_{V_i} M_{V_i} \dot{c} + \dot{c}^T \bar{N}^T_{V_i} J_i \bar{N}_{V_i} \dot{c} && (130) \\ &= \frac{1}{2} \dot{c}^T [m_i M^T_{V_i} M_{V_i} + \bar{N}^T_{V_i} J_i \bar{N}_{V_i}] \dot{c} = \frac{1}{2} \dot{c}^T Z_{V_i} \dot{c}. && (131) \end{aligned}$$

The total kinetic energy is

$$T = \sum_{i=1}^{n} T_i = \frac{1}{2} \dot{c}^T \sum_{i=1}^{n} Z_{V_i} \dot{c} = \frac{1}{2} \dot{c}^T H_V \dot{c}. \tag{132}$$

After a substitution into the second order Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial T}{\partial \dot{c}} - \frac{\partial T}{\partial c} = Q_V \tag{133}$$

we get

$$H_V \ddot{c} + \dot{H}_V \dot{c} - \frac{\partial T}{\partial c} = Q_V \tag{134}$$

where the column matrix

$$Q_V = [Q^M_{V_1}/Q^F_{V_1} \vdots Q^M_{V2}/Q^F_{V2} \vdots ... \vdots Q^M_{Vn}/Q^F_{Vn}]^T \tag{135}$$

represents the generalized moments and forces. The generalized forces are derived by means of the vertical displacement method. The vector of generalized forces can be represented as a sum

$$Q_V = P_V + Y_V \tag{136}$$

where

$$P_V = [P^M_{V1}/P^F_{V1} \vdots P^M_{V2}/P^F_{V2} \vdots ... \vdots P^M_{Vn}/P^F_{Vn}]^T \tag{137}$$

is the column matrix of driving torques and forces, and Y_V can be calculated independently of P_V

$$Y_{V_j} = \begin{cases} \sum_{k=j}^{n} m_k[g, e_j, r_k^{(j)}] & \mu_j = 0 \text{ rotation} \\ \sum_{k=j}^{n} e_j g m_k & \mu_j = 1 \text{ translation} \end{cases} \tag{138}$$

In the equation above, $r_k^{(j)} = \sum_{p=j}^{k-1}(s'_{pp} - s_{p,p+1}) + s'_{kk}$, $s'_{pp} = s_{pp} + q_p e_p \mu_p$ and $g = [0,\ 0\ , -9.81]^T$.
Introducing

$$h_V = Y_V + \frac{\partial T}{\partial c} - \dot{H}_V \dot{c} \tag{139}$$

the equations (134) may be reduced to the simple form

$$H_V \ddot{c} = P_V + h_V \tag{140}$$

where $H_V = H_V(c)$ is the $3n \times 3n$ inertial matrix. The $3n \times 1$ matrix $h = h(c, \dot{c})$ takes into account Coriolis, centrifugal and gravitational forces, and P_V is the $3n \times 1$ matrix of the driving torques and forces. However, the real dimension of the differential equations is n because only one of the components of the vectors c_i is essential and informative.

4. Screw Considerations

Because of the fact that every Euclidean motion may be presented as a screw motion, this section proposes an useful interplay of screw geometry, dual algebra and vector and matrix transformations. The special structure of a manipulator as a series of coupled bodies allows the specialization of the general line coordinate transformation matrix in the form called dual orthogonal matrix and defining of the notion of dual vector-parameter. The geometrical and kinematical models of a manipulator are expressed in a closed form using dual orthogonal matrices and dual vector-parameters. The angular velocity vector of the end-effector takes a particularly simple form leading to a simple geometric interpretation of the Jacobian matrix. After describing of the velocity-screw, the force-screw and momentum-screw are introduced and Newton-Euler equations in screw-matrix form are presented.

4.1. Screw Kinematics of a Rigid Body

As it is known every movement $g \in E(3)$ may be presented as **a screw motion** [12], [34]: a translation of a point along an axis l and a rotation about this axis, i.e., g depends on the parameters $c_1, c_2, c_3, p_1, p_2, p_3$. Concerning the screw axis l of (75): since it is parallel to the rotational axis c for the transformation $O(c)$, its Plücker vectors L and L' are

$$L = 2c, \qquad L' = c \times p - p + \nu c, \qquad \nu = (c.p)/c^2, \qquad \nu = \text{const} \quad (141)$$

which, together with the angle of rotation α, defined by $\|c\| = \tan(\alpha/2)$ and the translation distance $(c.p)/c^2$, gives a complete description of the screw displacement of the six parameters c_i and p_i $(i = 1, 2, 3)$.

It is interesting to find out how these requirements look on a velocity level. The basic equations which we use now are (75) and (77). It may be proved that the velocity distribution is identical with that of a screw motion, with l as an axis, with angular velocity ω and translational velocity $\sigma\omega$. The scalar σ being a ratio of the linear and the angular velocity, it is known as the pitch of the screw motion. The Plücker vectors of the screw axis now are

$$L = \omega, \qquad L' = \dot{p} - \omega \times p - \sigma\omega \quad (142)$$

which, in vector-parameter form look like

$$\sigma = \frac{1 + c^2}{2} \frac{(c \times \dot{c} + \dot{c}).p}{c^2\dot{c}^2 - (c\dot{c})^2} \quad (143)$$

and

$$L = \frac{2}{1+c^2}(c \times \dot{c} + \dot{c}) \tag{144}$$

$$L' = \dot{p} - \frac{2}{1+c^2}(c \times \dot{c} + \dot{c}) \times p - \frac{(c \times \dot{c} + \dot{c}).\dot{p}(c \times \dot{c} + \dot{c})}{c^2\dot{c}^2 - (c\dot{c})^2} \tag{145}$$

4.2. Dual Vector-Parameter in Manipulator Kinematics

4.2.1. Manipulator Skeleton and Line Geometry

The system of coupled rigid bodies $B_i,\ i = 0, 1, 2, \ldots, n$ (B_o is the fixed body while B_n may be treated as the end-effector) that make up a manipulator, can be reduced to a skeleton consisting of the lines forming the axes of the cylindric joints and the lines defined by the common normals to consecutive pairs of these axes forming the rigid links of the manipulator. We denote by R_o the base frame (fixed) and R_{n+1} is connected with a characteristic point C of the gripper (usually it coincides with the mass center). We define the coordinate frames $R_i(x_i, y_i, z_i)$ and $R_{i-1}(x_{i-1}, y_{i-1}, z_{i-1})$ for bodies B_i and B_{i-1} by choosing the z-axis of each reference frame along the axis of the joint and the x-axis of each frame along the common normals of succeeding axes [69] (see Fig. 2 and Fig.3).

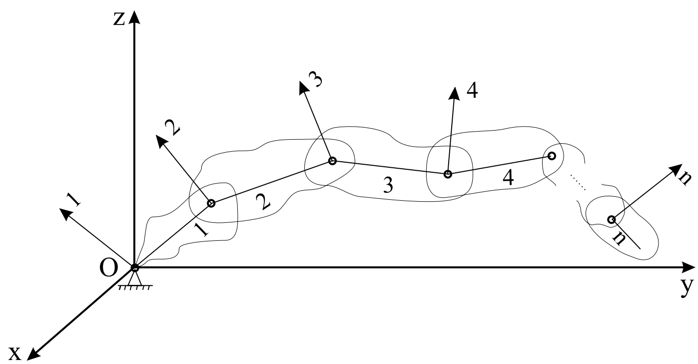

Figure 2. Scheme of a manipulator with general geometry

The displacement of the body B_i relative to B_{i-1} is specified by the rotational angle ϑ_i about the line l_{i-1} coinciding with its z-axis, i.e., with the axis z_{i-1} of the frame R_{i-1}, and the translation by the amount d_i along this line. The position of l_i in the reference frame R_i of B_i relative to the joint axis l_{i-1} is given by the angle α_i about the common normal between the axes x_i) and the distance a_i

along it. The parameters d_i and a_i are the dimensions of the link making up B_{i-1}, whereas the parameters ϑ_i and α_i are the joint variables that prescribe the position of B_i relative to B_{i-1}. The coordinate transformation relating the position of B_i to B_{i-1} is made up by a pair of transformations, each of which consists of a rotation about and a translation along a given line, which is just a screw displacement.

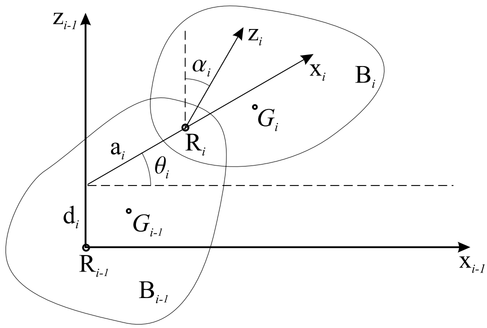

Figure 3. The standard reference frame

Since $\bar{a}(a,0,0)$ and $\bar{d}(0,0,d)$, the group of Euclidean motions – E(3) in $\mathbb{R}^3$, can be defined as a set of pairs with the following composition low

$$(O(c_\vartheta),\ \bar{d})\ (O(c_\alpha),\ \bar{a}) = (O(\langle c_\vartheta,\ c_\alpha\rangle),\ O(c_\vartheta)\bar{a}+\bar{d}) = (O(c_{\vartheta\alpha}),\ p). \quad (146)$$

Hence, in terms of homogeneous transformations, (75) converts into

$$\begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} O(c_{\vartheta\alpha}) & p \\ \bar{0} & 1 \end{bmatrix} \begin{bmatrix} r_o \\ 1 \end{bmatrix} \quad (147)$$

where $\bar{0}$ is a 1×3 zero vector. The analog of 4×4 matrix in equation (147) for the transformations of line coordinates looks like

$$\begin{bmatrix} E \\ X\times E \end{bmatrix} = \begin{bmatrix} O(c) & \emptyset \\ p^\times O(c) & O(c) \end{bmatrix} \begin{bmatrix} e \\ x\times e \end{bmatrix} \quad (148)$$

where the six dimensional **Plücker** coordinate vector of the line l is: $l=(E, X\times E)$, $E=Y-X$, $p^\times$ is a skew-symmetric matrix obtained from the components of $p(p_1,p_2,p_3)$, $\emptyset$ is the 3×3 zero matrix, x,y and X,Y are point coordinate vectors of points on l measured in R_b and R_o, respectively and

$$E = Y - X = O(c)(y-x) = O(c)e \quad (149)$$

$$X\times E = O(c)(x\times e) + p^\times O(c)e. \quad (150)$$

4.2.2. Dual Orthogonal Matrices and Dual Vector–Parameters

In the language of the **dual algebra** [57], [142], equation (148) has the compact form

$$\hat{E} = \hat{O}(c)\hat{e} \tag{151}$$

where $\hat{E} = E + \mu X \times E$, $\hat{O}(c) = O(c) + \mu p^{\times} O(c)$, $\hat{e} = e + \mu x \times e$, and $\mu^2 = 0$. $\hat{O}(c)$ is 3×3 **dual orthogonal matrix**, whose dual number elements represent a general line coordinate transformation.

We now introduce the **dual angle** between two lines as the dual number $\hat{\varphi} = \varphi + \mu\varphi_o$, where φ is the angle between the lines and φ_o is the distance between them about the common normal. Then $\hat{\vartheta}$ and $\hat{\alpha}$ are expressed in the following way

$$\hat{\vartheta} = \vartheta + \mu d, \qquad \hat{\alpha} = \alpha + \mu a. \tag{152}$$

Having in mind that $O(c_\vartheta)$ represents a rotation about the z-axis, its **dual matrix** is $O(c_{\hat{\vartheta}})$ and the corresponding **dual vector-parameters** $\hat{c}_\vartheta$ and $\hat{c}_\alpha$ are

$$\hat{c}_\vartheta = (0,\ 0,\ \tan\frac{\hat{\vartheta}}{2}), \qquad \hat{c}_\alpha = (\tan\frac{\hat{\alpha}}{2},\ 0,\ 0). \tag{153}$$

Let c_i be the vector-parameter giving the orientation of R_i referred to R_{i-1} and $\hat{c}_i$– its dual one [64]. The corresponding orthogonal and dual orthogonal matrices are $O(c_i)$ and $\hat{O}(c_i) = O(\hat{c}_i)$, where $c_i = \langle c_{\vartheta_i}, c_{\alpha_i}\rangle$, $\hat{c}_i = \langle \hat{c}_{\vartheta_i},\ \hat{c}_{\alpha_i}\rangle$ and $\hat{c}' = \langle \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n\rangle$. We shall denote further on $O(c_i)$ as $O_{i-1,i}$. Given the structure of the manipulator, we can determine the sequence of dual matrices $\hat{O}_{o1}$, $\hat{O}_{12}$, $\hat{O}_{23}$, $\ldots$, $\hat{O}_{n-1,n}$. Thus, we have the pair of transformation equations relating line coordinates measured in R_n to those measured in R_o

$$\hat{E} = \hat{O}_{on}\hat{e}, \qquad \hat{E} = \hat{O}(c')\hat{e} \tag{154}$$

$$\hat{E} = \hat{O}_{o1}\hat{O}_{12}\ldots\hat{O}_{n-1,n}\hat{e}, \qquad \hat{E} = \hat{O}(c_1)\hat{O}(c_2)\ldots\hat{O}(c_n)\hat{e}. \tag{155}$$

From these relations we obtain the matrix identity

$$\hat{O}_{on} = \hat{O}_{o1}\hat{O}_{12}\ldots\hat{O}_{n-1,n}. \tag{156}$$

The corresponding relation in dual vector-parameters is $\hat{c}' = \langle \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n\rangle$. The matrix equation (156) or its vector counterpart (154) relates the position and orientation of the end-effector of a manipulator to its joint parameters. Considering the dual angular velocity vector $\hat{\omega}$

$$\hat{\omega} = \omega(\hat{c}), \qquad \hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3) \tag{157}$$

associated with the matrix $O(\hat{c})$, where $\hat{c} = (\hat{c}_1, \hat{c}_2, \hat{c}_3)$, the dual matrix of angular velocity $\hat{\Omega}$, associated with the dual orthogonal matrix $\hat{O}(c_\vartheta)$ depends on $\hat{\omega}_3$. Since $\mathrm{d}\hat{\vartheta}/\mathrm{d}t = \dot{\vartheta} + \mu\dot{d}$, we obtain that $\hat{\omega}_3 = \mathrm{d}\hat{\vartheta}/\mathrm{d}t$, and the dual angular velocity vector is

$$\hat{\omega} = (\mathrm{d}\hat{\vartheta}/\mathrm{d}t)\hat{\mathrm{k}}. \tag{158}$$

$\hat{\mathrm{k}} = [0\ \ 0\ \ 1]^T$ is the dual vector representing the z-axis (its dual part is equal to zero). The dual angular velocity vector $\hat{\omega}_{i-1,i}$ of B_i relating to B_{i-1} is transformed to the frame R_o by the equation

$$\hat{\omega}^o_{i-1,i} = \hat{O}_{o1}\hat{O}_{12}\ldots\hat{O}_{i-1,i}\hat{\omega}_{i-1,i}. \tag{159}$$

Using the notation $\hat{K}_{i-1}$ as the image of $\hat{k}_{i-1}$ in R_o, we have

$$\begin{aligned} \hat{\omega}^o_{i-1,i} &= (\mathrm{d}\hat{\vartheta}_i/\mathrm{d}t)\hat{O}_{o1}\hat{O}_{12}\ldots\hat{O}_{i-2,i-1}\hat{\mathrm{k}}_{i-1} = (\mathrm{d}\hat{\vartheta}_i/\mathrm{d}t)\hat{K}_{i-1} && (160)\\ \hat{\omega}^o_{on} &= (\mathrm{d}\hat{\vartheta}_1/\mathrm{d}t)\hat{K}_1 + (\mathrm{d}\hat{\vartheta}_2/\mathrm{d}t)\hat{K}_2 + \ldots + (\mathrm{d}\hat{\vartheta}_n/\mathrm{d}t)\hat{K}_n && (161)\end{aligned}$$

which may be written in turn as the following matrix equation

$$\hat{\omega}^o_{on} = [\hat{K}_1\hat{K}_2\ldots\hat{K}_n][(\mathrm{d}\hat{\vartheta}_1/\mathrm{d}t)(\mathrm{d}\hat{\vartheta}_2/\mathrm{d}t)\ldots(\mathrm{d}\hat{\vartheta}_n/\mathrm{d}t)]^T. \tag{162}$$

For a typical manipulator, $n = 6$ and $[\hat{K}_1\hat{K}_2\ldots\hat{K}_6]$ is 3×6 matrix with dual number elements which is called the **dual Jacobian matrix** of the manipulator [29]. The columns of this matrix are just the dual vectors of the lines corresponding to the manipulator's joint axes, measured in the base reference frame R_o. Equating the real and dual parts of equation (162) and introducing the notation ω^* and K^*_i for the dual part of $\hat{\omega}$ and $\hat{K}_i$, there follows that

$$\begin{bmatrix} \omega \\ \omega^* \end{bmatrix} = \begin{bmatrix} [K_1\ldots K_6] & [0\ldots 0] \\ [K^*_1\ldots K^*_6] & [K_1\ldots K_6] \end{bmatrix} \begin{bmatrix} \dot{\vartheta} \\ \dot{d} \end{bmatrix} \tag{163}$$

where $[\dot{\vartheta}\quad \dot{d}]^T = [\dot{\vartheta}_1\ldots\dot{\vartheta}_6\ \ \dot{d}_1\ldots\dot{d}_6]^T$. Twelve unknowns, the real and dual parts of the six dual angular velocities, appear in (163) because single-degree-of-freedom revolute and prismatic joints are being modelled in this general case by two-degrees-of-freedom cylindric joints. Actually, for a six-degree-of-freedom manipulator only six of these unknowns are non-zero and the matrix in equation (163) becomes of 6×6. This matrix is just the Jacobian of the manipulator.

4.3. Dynamics of Multi-Body Systems in Screw Form

There are some vector pairs in mechanics, such as the resultant force f and couple n reduced by a system of forces about an arbitrary point R, the instantaneous angular velocity ω of a rigid body and the velocity V of a point R of the body, the linear momentum F and angular momentum M about a point R of a system of particles, which have different physical meanings, but they all satisfy the definition of the screw and can be defined as force-screw, velocity-screw and momentum-screw, respectively (see [143], [99].

We denote by $V_i^i, \omega_i^i, F_i^i, M_i^i, f_i^i, n_i^i$ the coordinate columns of the following vectors in the body fixed base $e^{(i)} = (\mathrm{i}_i, \mathrm{j}_i, \mathrm{k}_i)$ with origin R_i

V_i, ω_i – vectors of the linear velocity of R_i and the angular velocity of B_i respectively;
F_i, M_i – linear momentum and angular momentum about R_i of B_i;
f_i, n_i – resultant force and couple of the external forces on B_i about R_i.

The corresponding dual base is $\hat{e}^{(i)} = (\hat{\mathrm{i}}_i, \hat{\mathrm{j}}_i, \hat{\mathrm{k}}_i)$ and according to previous notation $\hat{E}^{(i)} = \hat{O}_{oi}\hat{e}^{(i)}$. We define the **velocity-screw** $\hat{V}_i$, **momentum-screw** $\hat{F}_i$ and **force-screw** $\hat{f}_i$ as follows [143]

$$\hat{V}_i = [V_i^{iT}\ \omega_i^{iT}]^T \qquad \hat{F}_i = [F_i^{iT}\ M_i^{iT}]^T \qquad \hat{f}_i = [f_i^{iT}\ n_i^{iT}]^T. \tag{164}$$

Compose 6×6 square matrix $\hat{V}_i^{\times}$ from 3×3 skew-symmetric matrices $V_i^{\times}$ and $\omega_i^{\times}$ of vectors V_i^i and ω_i^i on the base $e^{(i)}$

$$\hat{V}_i^{\times} = \begin{bmatrix} \omega_i^{\times} & \emptyset \\ V_i^{\times} & \omega_i^{\times} \end{bmatrix}. \tag{165}$$

The recurrent relations for the position vectors as well for vectors of linear and angular velocities and accelerations may be found in [69].

Let m_i and J_i be the mass and inertia matrix of B_i about R_i, $s_{ii}^{\times}$ be the skew-symmetric matrix of radial vector s_{ii}^i from R_i to the mass center G_i of the body B_i, on base $e^{(i)}$. The formulas of linear and angular momenta, can be expressed by two compact screw equations

$$\hat{F}_i = \hat{\phi}_i \hat{V}_i, \qquad \dot{\hat{F}}_i + \hat{V}_i^{\times}\hat{F}_i = \hat{f}_i \tag{166}$$

where 6×6 matrix $\hat{\phi}_i$ is called the **generalized inertia matrix** of B_i on base $e^{(i)}$

$$\hat{\phi}_i = \begin{bmatrix} m_i I & m_i s_{ii}^{\times T} \\ m_i s_{ii}^{\times} & J_i \end{bmatrix}. \tag{167}$$

Equation (167) is just the Newton-Euler equations written in screw-matrix form. Through external iterative procedure for kinematical analysis and internal iterative procedure for synthesis of forces and moments, the dynamics of an open-loop mechanical system may be described analogically as it is done in [61].

5. An Approach for Diagonalization of Real 3 x 3 Symmetric Matrix and Mechanical Applications

5.1. Reasons for Investigation of the Problem

Many classical and quantum mechanical analysis lead to considerations of the spectrum and eigenvectors of either 3×3 or 4×4 real symmetric matrices as the lowest realistic cases of the problem at hand. The inertia tensor in rigid body mechanics given in [140], the determination of the point groups of symmetry in crystallography [100] and the gradient flows on the space of orthogonal matrices [14] are just a few of the numerous cases.

The story starts with the Jacobi's method for solving the eigenproblem for a concrete 8×8 symmetric matrix that arises in his study on dynamics. Jacobi diagonalizes the above matrix by performing a sequence of orthogonal similarity transformations and his method is relevant and effective in all dimensions (cf. [27] for numerical counterpart). Each transformation is a plane rotation, chosen so that the induced similarity diagonalizes some 2×2 principal submatrix, moving the weight of the annihilated elements onto the diagonal. Performing the same procedure in lower dimensions has a lot of specificity. E.g., using the isomorphism between 4×4 orthogonal matrices and algebra of quaternions [3] and [52] present a construction of an orthogonal similarity that acts directly on 2×2 blocks and diagonalizes a 4×4 symmetric matrix.

A straightforward analysis of the procedure of finding eigenvalues of a symmetric 3×3 matrix can be found in [119], while the authors in [11] provide explicit formulas for eigenvectors. This was done using the standard result about rotational matrices in the real three-dimensional space, namely that any of them can be represented as a product of three plane rotations.

In this section an easily tractable analytical method for casting a real three dimensional symmetric matrix into its diagonal form along with explicit formulas for the corresponding eigenvectors is given. This is achieved by two steps relying on a nice geometrical parameterization of the rotational group SO(3) and the fundamental algebraic theorem about solutions of the polynomial equations. Results for the mass inertia matrix in rigid body mechanics and special systems of linear differential equations are examined in some detail. Another aim is the approach to be applied for obtaining a new form of dynamical equations of multi-body mechanical systems which is quite convenient for the control process.

Here we follow the already mentioned procedure based on the vector parameterization, which is quite different from the standard Euler's parameterization of the rotational group SO(3). Contrary to other known coordinatizations of SO(3) the vector-parameterization is a symmetrical and a natural one. Our aim is to find the vector-parameter defining the orthogonal matrix which diagonalizes the given symmetric matrix. This process will be realized by two steps. At each level we will obtain the corresponding vector-parameter and the resultant vector-parameter will be their composition. The advantage of closed, albeit relatively involved expressions for eigenvalues and eigenvectors is that they can be examined for the effect of any kind of variations of the original matrix elements.

Last but not least, this can help in identifying the hidden symmetries which might manifest themselves through various special combination of the matrix entries.

Why we have chosen the vector-parameter to reach the above aim?

The well known parameters like Eulerian angles [11], Bryant angles, Eulerian parameters or quaternions [2], [52] as well as Euler–Rodrigues parameters or Gibbs vectors [125] are standard in the literature. Specially the last two are classical parameterizations which play an important role in the geometrical and kinematical descriptions of motion, especially in the dynamics of spacecrafts and aircrafts (see also [107]). The absence of trigonometric functions in the kinematical differential equations (giving the connections between the angular velocity vector and time derivatives of these parameters) makes these differential equations more attractive for many applications. But this is not the case in the problem when quaternions are used for interpolation of SO(3), which is important in motion planning. The vector-parameterization is the best among others coordinatizations for all these purposes. On the base of vector-parameterization we have developed an unified numerically efficient approach for kinematical and dynamical modelling and control of a rigid body and mechanical systems of rigid and elastic bodies [61], [62], [69]. Because of the decoupling of the differential equations of motion, the problem of diagonalization of the inertia matrix is of a great importance. And since vector-parameters are convenient also for motion planing, we consider the problem of diagonalization using the same parameters. This method may be also successfully used in dynamics of flexible multi-body systems and could be developed for example as a separate module within the computer program system DynaFlex [111], that generates and operates with the equations of motion symbolically.

As examples of the proposed method, the case of the mass inertia matrix in rigid body mechanics is worked out in detail and an effective procedure for computing the exponential mapping generated by three–dimensional symmetric matrices is presented. At the end our aim is reached considering the dynamics of multi-body mechanical systems and using the suggested approach for obtaining a new form of the differential equations of motion which is quite appropriate for real time control.

5.2. Setting the Problem

Our objective is: given a symmetric 3×3 matrix A, construct a diagonalizing rotation matrix $O(c)$, where c is the vector-parameter of resultant rotation, such that

$$O^T(c)A\,O(c) = \Lambda = \text{Diag}\,[\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)}] \tag{168}$$

where T means the transposed matrix, $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the eigenvalues of the matrix A and σ is an element of the group Σ_3 of permutations of the three element set $1, 2, 3$.

5.3. First Level of Diagonalization

Let $A = [a_{ij}],\ \ i, j = 1, 2, 3$ be a real, symmetric 3×3 matrix. Consider the vector-parameter $c = (x, 0, z)$. The corresponding orthogonal matrix is $O(c) = O(x, z)$ and $O^T(c) = O^T(x, z)$. We denote by $B = [b_{ij}]$ the real symmetric 3×3 matrix

$$B = O^T(x, z)\,A\,O(x, z). \tag{169}$$

We are going to eliminate the elements b_{12} and b_{13}. For this purpose we have to solve a pair of coupled highly nonlinear equations. Several methods how to deal with such problems exist in computational kinematics. Most promising in our case seem to be the elimination methods. Here we shall use the so called **Dialytic elimination method** [106]. The basic steps in this method are

1. Rewrite equations with one variable suppressed.
2. Define the remaining power products as new linear, homogeneous unknowns.
3. Obtain new linear equations so as to have as many linearly independent homogeneous equations as the linear unknowns are.

4. Set the determinant of the coefficient matrix to zero, and obtain a polynomial in the suppressed variable. (If one is interested in numerical solutions, this is omitted as we can go directly to the next step).
5. Determine the roots of the characteristic polynomial or the eigenvalues of the matrix. This yields all possible values of the suppressed variable.
6. Substitute (one of the roots or eigenvalues) of the suppressed variable and solve the linear system for the remaining unknowns. Repeat this for each value of the suppressed variable.

Guided by this strategy we present the elements b_{12} and b_{13} as polynomials of z which coefficients depend on x. In this form we denote them as B_{12} and B_{13} respectively, i.e.,

$$B_{12} = F(z) = \text{Pol}\,[b_{12}, z], \qquad B_{13} = G(z) = \text{Pol}\,[b_{13}, z]. \tag{170}$$

The degree of the polynomials B_{12} and B_{13} in the variable z is four. With the letters given below we denote the coefficients of the polynomials B_{12} and B_{13} in front of the corresponding power of z

$$\begin{array}{lll} C_0 = \text{Coeff}\,[B_{12}, z^0], & & D_0 = \text{Coeff}\,[B_{13}, z^0] \\ C_1 = \text{Coeff}\,[B_{12}, z^1], & & D_1 = \text{Coeff}\,[B_{13}, z^1] \\ C_2 = \text{Coeff}\,[B_{12}, z^2], & & D_2 = \text{Coeff}\,[B_{13}, z^2] \\ C_3 = \text{Coeff}\,[B_{12}, z^3], & & D_3 = \text{Coeff}\,[B_{13}, z^3] \\ C_4 = \text{Coeff}\,[B_{12}, z^4], & & D_4 = \text{Coeff}\,[B_{13}, z^4] \end{array} \tag{171}$$

where

$$\begin{aligned} C_0 &= -a_{12}x^4 + 2a_{13}x^3 + 2a_{13}x + a_{12} \\ C_1 &= -2a_{23}x^3 + 4a_{33}x^2 - 2a_{11}x^2 - 2a_{22}x^2 + 6a_{23}x + 2a_{2}2 - 2a_{11} \\ C_2 &= -6a_{13}x - 6a_{12}, \quad C_3 = -2a_{23}x + 2a_{11} - 2a_{22} \quad C_4 = a_{12} \end{aligned} \tag{172}$$

and

$$\begin{aligned} D_0 &= -a_{13}x^4 - 2a_{12}x^3 - 2a_{12}x + a_{13} \\ D_1 &= -2a_{33}x^3 + 2a_{11}x^3 - 6a_{23}x^2 + 2a_{11}x - 4a_{22}x + 2a_{33}x + 2a_{23} \\ D_2 &= \quad 6a_{13}x^2 + 6a_{12}x \\ D_3 &= -2a_{11}x + 2a_{33}x + 2a_{23} \\ D_4 &= -a_{13}. \end{aligned} \tag{173}$$

So we have

$$
\begin{aligned}
B_{12} &= C_4 z^4 + C_3 z^3 + C_2 z^2 + C_1 z + C_0 \\
B_{13} &= D_4 z^4 + D_3 z^3 + D_2 z^2 + D_1 z + D_0
\end{aligned}
\tag{174}
$$

where one should take into account that the new coefficients $C_0, \ldots, D_0$ contain the suppressed variables x. In step two we consider each power of z as separate independent linear indeterminate. We have to note that the number one is counted as a variable as well since it is always convenient to have homogeneous equations and it provides a rationale to discard trivial solutions. The coefficient of the "variable" 1 is the constant term. Having in mind all these arguments we rewrite the equations (174) as the following linear set

$$
\begin{aligned}
C_4 Z_1 + C_3 Z_2 + C_2 Z_3 + C_1 Z_4 + C_0 Z_5 &= 0 \\
D_4 Z_1 + D_3 Z_2 + D_2 Z_3 + D_1 Z_4 + D_0 Z_5 &= 0.
\end{aligned}
\tag{175}
$$

Since we have two equations with five unknowns, we need additional equations. In our case this can be accomplished by multiplying equations (175) first by z, after that by z^2 and at the end by z^3. So we obtain eight equations with eight unknowns since three new power products appear. Using the concept of step two, we invoke new independent variables $Z_6 = z^5,\ Z_7 = z^6,\ Z_8 = z^7$. As a result we obtain a system of eight homogeneous linear equations in eight unknowns

$$
\begin{aligned}
C_4 Z_1 + C_3 Z_2 + C_2 Z_3 + C_1 Z_4 + C_0 Z_5 &= 0 \\
D_4 Z_1 + D_3 Z_2 + D_2 Z_3 + D_1 Z_4 + D_0 Z_5 &= 0 \\
\\
C_4 Z_6 + C_3 Z_1 + C_2 Z_2 + C_1 Z_3 + C_0 Z_4 &= 0 \\
D_4 Z_6 + D_3 Z_1 + D_2 Z_2 + D_1 Z_3 + D_0 Z_4 &= 0 \\
\\
C_4 Z_7 + C_3 Z_6 + C_2 Z_1 + C_1 Z_2 + C_0 Z_3 &= 0 \\
D_4 Z_7 + D_3 Z_6 + D_2 Z_1 + D_1 Z_2 + D_0 Z_3 &= 0 \\
\\
C_4 Z_8 + C_3 Z_7 + C_2 Z_6 + C_1 Z_1 + C_0 Z_2 &= 0 \\
D_4 Z_8 + D_3 Z_7 + D_2 Z_6 + D_1 Z_1 + D_0 Z_2 &= 0.
\end{aligned}
\tag{176}
$$

This is the main idea in the dialytic elimination method, namely that even though the new equations are dependent on the original equations their dependence is not linear but encoded into a linear system. We go to step four where we obtain a single polynomial equation in the suppressed variable x. We rewrite the system (176) in a matrix form

$$\begin{bmatrix} C_4 & C_3 & C_2 & C_1 & C_0 & 0 & 0 & 0 \\ D_4 & D_3 & D_2 & D_1 & D_0 & 0 & 0 & 0 \\ C_3 & C_2 & C_1 & C_0 & 0 & C_4 & 0 & 0 \\ D_3 & D_2 & D_1 & D_0 & 0 & D_4 & 0 & 0 \\ C_2 & C_1 & C_0 & 0 & 0 & C_3 & C_4 & 0 \\ D_2 & D_1 & D_0 & 0 & 0 & D_3 & D_4 & 0 \\ C_1 & C_0 & 0 & 0 & 0 & C_2 & C_3 & C_4 \\ D_1 & D_0 & 0 & 0 & 0 & D_2 & D_3 & D_4 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \end{bmatrix} = 0 \tag{177}$$

and in more compact notation we have $U\,Z = 0$. Since we know that $Z_5 = 1$, then the trivial solution $Z_i \equiv 0$, $(i = 1, 2, ..., 8)$ is not admissible and therefore the determinant of the coefficient matrix U must be equal to zero, i.e., $\det U \equiv 0$. Having in mind that the entries of the matrix U contain the suppressed variable x, expansion of its determinant produces polynomial of sixteen degree which factorizes into the form

$$\det U = 256\,(1 + x^2)^3\, P^2(x)\, Q(x) \tag{178}$$

where

$$\begin{aligned} P(x) \;=\; & (a_{11} a_{12} a_{23} - a_{12}^2 a_{13} + a_{13} a_{23}^2 - a_{12} a_{23} a_{33})\, x^3 + (a_{11} a_{12} a_{22} - a_{12}^3 \\ & + 2\, a_{12} a_{13}^2 + 2\, a_{13} a_{22} a_{23} - a_{11} a_{13} a_{23} - a_{12} a_{23}^2 - a_{11} a_{12} a_{33} \\ & - a_{12} a_{22} a_{33} - a_{13} a_{23} a_{33} + a_{12} a_{33}^2)\, x^2 + (2\, a_{12}^2 a_{13} - a_{13}^3 \\ & - a_{11} a_{13} a_{22} + a_{13} a_{22}^2 - a_{11} a_{12} a_{23} - a_{12} a_{22} a_{23} - a_{13} a_{23}^2 \\ & - a_{13} a_{22} a_{33} + a_{11} a_{13} a_{33} + 2\, a_{12} a_{23} a_{33})\, x + a_{11} a_{13} a_{23} - a_{12} a_{13}^2 \\ & - a_{13} a_{22} a_{23} + a_{12} a_{23}^2 \end{aligned} \tag{179}$$

and

$$\begin{aligned} Q(x) = & (a_{11}^2 + 4\, a_{13}^2 - 2\, a_{11} a_{33} + a_{33}^2)\, x^4 + (8\, a_{12} a_{13} - 4\, a_{11} a_{23} \\ & + 4\, a_{23} a_{33})\, x^3 + \; (2\, a_{11}^2 + 4\, a_{12}^2 + 4\, a_{13}^2 - 2\, a_{11} a_{22} + 4\, a_{23}^2 \\ & - 2\, a_{11} a_{33} + 2\, a_{22} a_{33})\, x^2 + \; (8\, a_{12} a_{13} - 4\, a_{11} a_{23} + 4\, a_{22} a_{23})\, x \\ & + a_{11}^2 + 4\, a_{12}^2 - 2\, a_{11} a_{22} + a_{22}^2 . \end{aligned} \tag{180}$$

According to Abel's fundamental theorem it is always possible to write down the solutions of polynomial equations up to fourth degree in analytical form using rational operations and radicals and this means that our equations

$$P(x) = 0 \qquad \text{and} \qquad Q(x) = 0 \tag{181}$$

can be solved explicitly in any concrete case. In both analytical and numerical cases, we are only interested in real roots (at least one coming from the first of the above equations always exists). Therefore, any complex or purely imaginary roots which meets the determinant condition (as those coming from the multiplier $(1+x^2)^3$) reduce the number of the admissible solutions to the maximal possible value of seven.

Finally, in step six, we substitute for the variable x into the linear set of equations, and solve them for the other original variables, which in this case is z. Substituting any of the real roots of x into equation (177) and setting $Z_5 = 1$ we obtain the corresponding variable z. Since the system is linear, this yields just one z for each x (when the rank of the matrix U in (177) is maximal).

It is worth to be noted here, that the introduction of new power products and the so obtained additional equations is optimized in our approach and the proof of this fact is just the form of both polynomials which we have obtained. In spite of the fact that U is 8×8 matrix we manage to derive analytically solvable equations. Our experience shows that if we start the procedure of diagonalization with another vector-parameter (e.g. $\check{c} = (x, y, 0)$) the elimination procedure does not give polynomials of such low degrees.

So, in principle one can associate with the seven couples $(x_i,\ z_i)$ the corresponding seven vector-parameters $c_i,\ (i = 1,\ 2, ...,\ 7)$. We substitute their values in the matrix B and we continue the procedure towards elimination of the third non–zero (in the general case) element b_{23} of the new symmetric matrix B. If we exchange the suppressed variable x with z the relevant polynomials $\tilde{P}(z)$ and $\tilde{Q}(z)$ are of degree four and six respectively and there is no guarantee that the equations $\tilde{P}(z) = 0$ and $\tilde{Q}(z) = 0$ allow any real root. Up to now we have tastefully assumed that both a_{12} and a_{13} are non-zero elements. If this is not the case and one of them vanishes this simplifies considerably the foregoing procedure. If both are zero one simply goes directly to the next stage described in the section to follow.

5.4. Second Level of Diagonalization

After actualization of the matrix B, the resultant matrix will be denoted by $C = [C_{ij}]$, i.e., $C = \text{Actual}[B]$ which is again 3×3 matrix. Now we continue the process of diagonalization keeping

$$C_{12} = 0 \quad \text{and} \quad C_{13} = 0. \tag{182}$$

Consider the vector-parameter $\tilde{c} = (u, 0, 0)$. There exists an orthogonal matrix $O_{\tilde{c}} = O(\tilde{c})$ and $O^T(\tilde{c}) = O^T_{\tilde{c}}$. Now we form the matrix $S = [S_{ij}]$ as follows

$$S \;=\; O^T_{\tilde{c}}\, C\, O_{\tilde{c}} \;=\; O^T_{\tilde{c}}\, O^T_c\, A\, O_c\, O_{\tilde{c}} \tag{183}$$

and set $S_{23} = 0$. In this way we obtain a polynomial in u which power is not greater then four. Assuming that we are in the generic case this gives

$$u^4 - 2au^3 - 6u^2 + 2au + 1 = 0, \qquad a = (C_{33} - C_{22})/C_{23}. \tag{184}$$

Solving this equation we find the following solutions

$$u_1 \;=\; (a + M - \sqrt{2(M^2 + aM)})/2 \tag{185}$$

$$u_2 \;=\; (a - M + \sqrt{2(M^2 - aM)})/2 \tag{186}$$

$$u_3 \;=\; (a - M - \sqrt{2(M^2 - aM)})/2 \tag{187}$$

$$u_4 \;=\; (a + M + \sqrt{2(M^2 + aM)})/2 \tag{188}$$

and all of them are real because $M = \sqrt{a^2 + 4} > |a|$. Choosing any of these roots we actually fix the vector $\tilde{c}$. Composing the so obtained vector-parameters $(x, 0, z)$ and $(u, 0, 0)$ we get the vector-parameter of the resultant rotation

$$c'' = \langle c,\, \tilde{c} \rangle \tag{189}$$

and actually we have proved

Theorem 4. *Let* $A = [a_{ij}]$, $i, j = 1,\, 2,\, 3$ *be a real symmetric* 3×3 *matrix. Then there exist a vector* $c'' \in \mathbb{R}^3$ *given in* equation (189) *and an uniquely associated with it via* (66) *orthogonal matrix O such that*

$$O^T A\, O \;=\; \Lambda \;=\; \mathrm{Diag}\,[\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)}], \qquad \sigma \in \Sigma_3 \tag{190}$$

where $O = O(c'')$ *and* $O^T = O^T(c'')$.

5.5. Mechanical Applications

5.5.1. The Mass Inertia Matrix

As a first illustration of the above procedure we will consider the diagonalization of the inertia matrix

$$I(\xi, \eta, \zeta) = \begin{bmatrix} m(\eta^2 + \zeta^2) & -m\xi\eta & -m\xi\zeta \\ -m\xi\eta & m(\xi^2 + \zeta^2) & -m\eta\zeta \\ -m\xi\zeta & -m\eta\zeta & m(\xi^2 + \eta^2) \end{bmatrix}. \tag{191}$$

Using the homogeneity of the matrix elements we can forget at the moment about the mass parameter m and restore its presence at the end. So, we assume that $m \equiv 1$ in $I(\xi, \eta, \zeta)$ and continue with determination of x and z which appear in equation (169). At this stage we recognize immediately that any real x is a root of $P(x)$ as this polynomial is identically zero. This situation is just a manifestation of the hidden symmetries of our matrix and means that we are free to choose x arbitrarily. As we are working in the analytical setting the most convenient choice is to fix x to be zero. Thus we are left at the first level only with the problem of finding z. This can be done either by solving the linear system given through equation (177) or identifying the common roots of (170). The result of the second procedure is

$$z_{+} = \frac{\eta + \sqrt{\xi^2 + \eta^2}}{\xi}, \qquad z_{-} = \frac{\eta - \sqrt{\xi^2 + \eta^2}}{\xi}. \tag{192}$$

Rotation around vectors $c_{\pm} = (\,0,\ 0,\ z_{\pm})$ produces respectively

$$I_{\pm} = \begin{bmatrix} \xi^2 + \eta^2 + \zeta^2 & 0 & 0 \\ 0 & \zeta^2 & \pm\,\zeta\sqrt{\xi^2 + \eta^2} \\ 0 & \pm\,\zeta\sqrt{\xi^2 + \eta^2} & \xi^2 + \eta^2 \end{bmatrix}. \tag{193}$$

Now the roots of equation (184) are

$$\begin{aligned} u_1^{\pm} &= \mp\,\frac{\zeta - \sqrt{\xi^2 + \eta^2 + \zeta^2}}{\sqrt{\xi^2 + \eta^2}}, \qquad u_2^{\pm} = \pm\,\frac{\sqrt{\xi^2 + \eta^2} + \sqrt{\xi^2 + \eta^2 + \zeta^2}}{\zeta} \\ u_3^{\pm} &= \mp\,\frac{\zeta + \sqrt{\xi^2 + \eta^2 + \zeta^2}}{\sqrt{\xi^2 + \eta^2}}, \qquad u_4^{\pm} = \pm\,\frac{\sqrt{\xi^2 + \eta^2} - \sqrt{\xi^2 + \eta^2 + \zeta^2}}{\zeta} \end{aligned} \tag{194}$$

and the rotations generated by odd, respectively even numbered vectors $\tilde{c}_i^{\pm} = (u_i^{\pm},\ 0,\ 0)$, $(i = 1,\ 2,\ 3,\ 4)$ brings I_{-}, accordingly I_{+} into the form

$$I^{odd} = \begin{bmatrix} \xi^2 + \eta^2 + \zeta^2 & 0 & 0 \\ 0 & \xi^2 + \eta^2 + \zeta^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{195}$$

$$I^{even} = \begin{bmatrix} \xi^2 + \eta^2 + \zeta^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi^2 + \eta^2 + \zeta^2 \end{bmatrix}. \tag{196}$$

Restoration of the mass parameter amount simply to multiplication by m, i.e.,

$$\Lambda^{odd} = m.\,I^{odd}, \qquad \Lambda^{even} = m.\,I^{even}, \tag{197}$$

and the rotational matrices which furnish these diagonalizations can be explicitly built using (66) from

$$c_i^{odd} = (u_i^{\pm},\, u_i^{\pm}.\, z_{\pm},\, z_{\pm}), \quad i \in \{1,3\},$$

and

$$c_j^{even} = (u_j^{\pm},\, u_j^{\pm}.\, z_{\pm},\, z_{\pm}), \quad j \in \{2,4\}.$$

5.5.2. Explicit Formulae for Solutions of Some Differential Equations

Let A be a given symmetric 3×3 matrix and let $X(t)$ denote a three-dimensional column vector function of the real variable t. It is well known that the system of first-order linear differential equations

$$X'(t) = A\, X(t) \tag{198}$$

with prescribed initial value $X(0)$ has the unique solution

$$X(t) = \mathrm{Exp}[tA]\, X(0) \tag{199}$$

on the entire interval $(-\infty, +\infty)$ and the only problem here is the effective calculation of the matrix exponential (for more details see [80]).

Below we shall describe a simple procedure for its explicit evaluation using our method of diagonalization.

So, let us take any rotational matrix $O(c)$ which brings A into its diagonal form, i.e.,

$$O^T(c)\, A\, O(c) = \mathrm{Diag}[\mu_1, \mu_2, \mu_3]. \tag{200}$$

Because it is non-degenerate we can unambiguously introduce new state variables

$$\tilde{X} = O^T(c)\, X \quad \Longleftrightarrow \quad X = O(c)\, \tilde{X} \tag{201}$$

and transform our initial problem as follows

$$\begin{aligned} \tilde{X}'(t) &= O^T(c)\, X'(t) = O^T(c)\, A\, X(t) = O^T(c)\, A\, O(c)\, \tilde{X}(t) \\ &= \mathrm{Diag}[\mu_1, \mu_2, \mu_3]\, \tilde{X}(t). \end{aligned} \tag{202}$$

The first and final members of the above chain of equalities tell us that in the new state variables $\tilde{X}$ our system of differential equations decouples completely and we can write its solution into the form

$$\begin{aligned} \tilde{X}(t) &= \mathrm{Exp}[t\ \mathrm{Diag}[\mu_1, \mu_2, \mu_3]]\, \tilde{X}(0) \\ &= \mathrm{Diag}[\mathrm{e}^{\mu_1 t}, \mathrm{e}^{\mu_2 t}, \mathrm{e}^{\mu_3 t}]\, \tilde{X}(0). \end{aligned} \tag{203}$$

Going back to the original state variables X we have

$$\begin{aligned} X(t) &= O(c)\,\tilde{X}(t) = O(c)\,\mathrm{Diag}[\mathrm{e}^{\mu_1 t}, \mathrm{e}^{\mu_2 t}, \mathrm{e}^{\mu_3 t}]\,\tilde{X}(0) \\ &= O(c)\,\mathrm{Diag}[\mathrm{e}^{\mu_1 t}, \mathrm{e}^{\mu_2 t}, \mathrm{e}^{\mu_3 t}]\,O^T(c)\,X(0) \end{aligned} \tag{204}$$

and this means that we have evaluated the matrix exponential $\mathrm{Exp}[tA]$ as well. It is obvious that the foregoing method works also for systems of differential equations of order s

$$X^{(s)}(t) = A\,X(t) \tag{205}$$

with prescribed initial values $X(0), X'(0), \ldots, X^{(s-1)}(0)$ and their matrix variants when the vector X is exchanged for a $3 \times m$ matrix function of the real variable t.

5.5.3. Dynamics of a Three Degrees–of–Freedom Manipulator System

We consider an open loop mechanical system with 3 degrees of freedom (for example a manipulator system) . We introduce the following notation:
$q := [q(1) \ldots q(3)]^T$ is the 3×1 matrix of the generalized coordinates (joint displacements) of the manipulator in the usual sense, $q \in Q \subset \mathbb{R}^3$ and Q is the configurational manifold. The dynamic equations are expressed again in the known form (5), namely

$$H(q)\ddot{q} + h(q, \dot{q}) = P$$

where $H := H(q)$ is the 3×3 inertia matrix, the 3×1 matrix $h := h(q, \dot{q})$ takes into account Coriolis, centrifugal and gravitational forces, and P is the 3×1 matrix of the generalized forces and moments. After multiplying the above equation from the left side with the matrix $O^T(c)$ and using the following substitutions

$$\bar{q} = O(c)^T q, \qquad q = O\bar{q} = O(c)\bar{q} \tag{206}$$

$$\dot{q} = \dot{O}\bar{q} + O\dot{\bar{q}}, \qquad \ddot{q} = \ddot{O}\bar{q} + 2\dot{O}\dot{\bar{q}} + O\ddot{\bar{q}} \tag{207}$$

$$H_{diag} = O^T(c)HO(c) \tag{208}$$

$$\bar{H} = H(\bar{q}), \qquad \bar{h} = h(\bar{q}, \dot{\bar{q}}), \qquad \bar{P} = P(\bar{q}) \tag{209}$$

$$O^T H(\bar{q}) O O^T [\ddot{O}^T \bar{q} + 2\dot{O}\dot{\bar{q}} + O\ddot{\bar{q}}] + O^T h(\bar{q}, \dot{\bar{q}}) = O^T P(\bar{q}) \tag{210}$$

we obtain

$$H_{\rm diag}[O^T \ddot{O}^T \bar{q} + 2O^T \dot{O}\dot{\bar{q}} + \ddot{\bar{q}}] + O^T \bar{h} = O^T \bar{P} \tag{211}$$

or

$$H_{\rm diag}\ddot{\bar{q}} + H_{\rm diag}O^T \ddot{O}^T \bar{q} + 2H_{\rm diag}O^T \dot{O}^T \dot{\bar{q}} + +O^T \bar{h} = O^T \bar{P}. \tag{212}$$

Denoting all members except the first one in the left part of the last equation by $\bar{\bar{h}}$ and the right part by $\bar{\bar{P}}$ the final form of the dynamical equations is

$$H_{\rm diag}\ddot{\bar{q}} + \bar{\bar{h}} = \bar{\bar{P}}. \tag{213}$$

The higher dimensions are treated in a similar way but using the dynamical equations over the configurational space in vector-parameters (see [69]). This form of the dynamical equations is quite convenient for decoupled control of manipulators.

The purely algebraic feature of the presented approach allows us to find eigenvalues and eigenvectors of an arbitrary real 3×3 symmetric matrix in a closed analytical form following a straightforward algorithm. It offers a means of studying in full details various models of theoretical and experimental relevance. Contrary to Jacobi's method which is based on three consecutive plane rotations, our method is based upon general two-parameter rotation (first level of diagonalization) followed by one-parameter plane rotation (second level of diagonalization). This is the first main point in the diagonalization procedure. The second interesting moment is that the information about these rotations is encoded in a vector form, their composition is expressed by simple vector operations and any use of transcendental functions is avoided. Besides, one will be able to examine the behaviour of critical parameters as functions of input data. And the third essential element is that in the first level of the diagonalization procedure a pair of coupled highly nonlinear equations has to be solved which is realized by using the so called dialytic elimination method. Having in mind the abundance of contexts in physics, mechanics, crystallography, elasticity, hydromechanics, robotics, etc., where symmetric matrices appear, we hope that the potential users will find this method useful in any concrete situation.

Finally, it is worth to note here that our analytical algorithm is realized as *Mathematica*® package for symbolic calculations [141].

6. Modeling and Control of Nonholonomic Mechanical Systems

6.1. Theoretical Background and History

This section aims to provide some tools for analyzing, modeling and control of nonholonomic mechanical systems. This classical subject has received the attention of many authors because nonholonomic constraints arise in many robotic structures like mobile robots, space manipulators, multifingered robot hands. The nonholonomic constraints are nonintegrable motion constraints that occure mainly in rolling motion. The number of system coordinates needed to identify the system's configuration is usually greater than the number of the instantaneous degrees of freedom of motion or with other words – the nonholonomic behavior implies that the mechanism can be completely controlled with a reduced number of actuators. Both planning and control of such systems are more difficult than in the case of holonomic systems. The term nonholonomic is also used to describe certain types of quasi-velocities described in some details below. It is important also to mention that the nonholonomy of the kinematic constraints in mechanical systems is equivalent to the controllability of the associated control systems and takes some concepts of the nonlinear control theory may be applied. The dynamics of nonholonomic systems is introduced here on the base of Hamel's equations of motion and a method of obtaining the reaction forces is given.

Nonholonomic constraints exist in the rolling problems due to the direct contact mechanism based on friction forces. Another, rather new domain of nonholonomic dynamics are navigation and automatic control system based on feedback phenomena. In such systems one deals with servo-constraints, i.e., holonomic or nonholonomic constraints of non-contact origin. The nonholonomic constraints appear in the literature at the end of XIX century in the papers of Ferrers, Appell, Voronetz, Hamel and many other scholars. More details concerning the topic may be found in [23] or [85]. At present time the modern algebraic, differential-algebraic and differential-geometric approaches are applied and in this field found real applications as exemplified in [22], [116], [117], [118], [37], [115], [15], [56], [9], [10] [135], [126], [109], [44], [45], [7], [50], [68], [70], [71] and many others. Some of these authors use successfully the theory of Lie group and Lie algebras in wheel vehicles and mobile robots control. When a rigid body motion is studied a rotation operator and operators for coordinate transfer are naturally involved. The velocity of a given body point is defined through them. It is found that the velocity of a rigid body point is obtained by the action of a skew–symmetric operator over the radius-vector of the point. This skew-symmetric operator is called angular velocity matrix (operator) (see [12], [1], [6], [57], [31]). A vector which corresponds in one-to-one manner to the angular velocity operator is known as a vector of angular velocity. Its components are peculiar kinematical characteristics of the rotated body. They are not equal to the time derivatives of some Lagrangian coordinates. The so-called nonholonomic velocities (pseudo or quasi-velocities) correspond to them.

In this section we consider Hamel's equations of motion of nonholonomic systems in terms of pseudo-coordinates and the efficient method (thereafter called Hamel's method) for writing down the equations for reactions. The group of the nonholonomic operators is defined and its group structure constants are determined.

6.2. Equations of Motion and Reaction Forces in Pseudo-Coordinates

Let us consider the motion of a mechanical system whose position is described by n generalized coordinates q_s $(s = 1, 2, \ldots, n)$. Let us assume also that some constraints of arbitrary order are imposed on the sytem. Among the nonholonomic constraints (differential-nonintegrable) may also occur holonomic constraints. In our further considerations we shall assume the presence only of $n - m$ linear nonholonomic constraints

$$\sum_{s=1}^{n} a_{m+i,s}\dot{q}_s = 0, \qquad \det \mid a_{kj} \mid \neq 0, \qquad i = 1, 2, \ldots n - m. \tag{214}$$

The presence of nonholonomic constraints limits the system mobility in a completely different way if compared to holonomic constraints. To illustrate this point, consider a single Pfaffian constraint

$$a^T(q)\dot{q} = 0. \tag{215}$$

If constraint (215) is holonomic, then it can be integrated as

$$h(q) = c \tag{216}$$

where $\partial h/\partial q = a^T(q)$ and c is the integration constant. In this case, the system motion is confined to a particular **level surface** of h, which depends on the initial conditions through the value of $c = h(q_o)$.

Assume instead that constraint (215) is nonholonomic. Then, even if the instantaneous mobility of the system is restricted to $n - 1$ dimensional space, it is still possible to reach any configutation in the configurational space. Correspondingly, the number of degrees of freedom is reduced to $(n - 1)$, but the number of the generalized coordinates cannot be reduced. This conclusion is general: for a mechanical system with n generalized coordinates and k nonholonomic constraints, although the generalized velocities at each point are confined to $(n-k)$ - dimensional subspace, the accessibility of the whole configuration space is preserved. Some useful results about the integrability of the constraints using tools from nonlinear control theory may be found in [50].

In nonholonomic mechanics the equations of the motion with undetermined multipliers, built in the space of the generalized coordinates, are known as **Routh's equations**

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s + \sum_{i=1}^{n-m} \lambda_{m+i} a_{m+i,s}, \qquad s = 1, 2, \ldots, n. \tag{217}$$

Let us introduce n expressions $\dot{\pi}_s$ in the following manner

$$\begin{gathered} \dot{\pi}_k = \sum_{s=1}^{n} a_{k,s}\dot{q}_s, \quad \dot{\pi}_{m+i} = \sum_{s=1}^{n} a_{m+i,s}\dot{q}_s = 0 \\ k = 1, 2, \ldots, m, \qquad i = 1, 2, \ldots, n-m \end{gathered} \tag{218}$$

where $A = (a_{ij}(q))$ is $n \times n$ matrix, $q \in U \subset \mathbb{R}^n$ and $\det A \neq 0$, $\dot{\pi}$ does not denote the full derivative with respect to t of any q. The variations $\delta\pi_i$ and δq_i are related through

$$\begin{gathered} \delta\pi_{m+i} = \sum_{s=1}^{n} a_{m+i,s}\delta q_s, \qquad \delta\pi_k = \sum_{s=1}^{n} a_{ks}\delta q_s \\ i = 1, 2, \ldots, n-m, \qquad k = 1, 2, \ldots, m \end{gathered} \tag{219}$$

and according to the nonholonomic constraints (214), it is valid that

$$\delta\pi_{m+i} = 0, \qquad i = 1, 2, \ldots, n-m. \tag{220}$$

Consider a Cartan moving frame on U

$$\Pi_j = \partial/\partial\pi_j = \sum_i b_{ij}\partial/\partial q_i, \qquad i, j = 1, 2, \ldots, n \tag{221}$$

$$\sum \dot{\pi}_j \Pi_j = \sum \dot{q}_i\, \partial/\partial q_i, \qquad \text{or} \qquad \dot{\pi} = A(q)\dot{q}. \tag{222}$$

The variables $\dot{\pi}_1, \dot{\pi}_2, \ldots, \dot{\pi}_n$ are called nonholonomic velocities or pseudo(quasi)-velocities and the symbolic quantities $\pi_1, \pi_2, \ldots, \pi_n$ – nonholonomic coordinates which in fact have no meaning of real coordinates. The motivation for the notation $\partial/\partial\pi_j$ is as follows: if π_j were true coordinates then

$$\partial f/\partial\pi = \sum_i (\partial f/\partial q_i)(\partial q_i/\partial\pi_j) = \sum_i (\partial f/\partial q_i) b_{ij} = \Pi_i(f). \tag{223}$$

According to the constraints free principle we may without changing the motion or stability of the mechanical system forget about the constraints replacing them by corresponding reaction forces. The so obtained free system is dynamically equivalent to the initial one but not kinematically and d'Alembert-Lagrange's equations in this case look like

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} - Q_s - D_s\right)\delta q_s = 0, \qquad s = 1, 2, \ldots, n \tag{224}$$

where D_s are unknown reaction forces satisfying $D_s \delta q_s = 0$, while T and Q_s are the kinetic energy and the generalized forces respectively. In pseudo-coordinates the equations (224) take the form

$$(B_k - P_k - R_k)\delta\pi_k = 0, \qquad k = 1, 2, \ldots, n \tag{225}$$

and

$$B_k = \left(\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial T^\star}{\partial \dot{\pi}_k} - \frac{\partial T^\star}{\partial \pi_k}\right) + \sum_{i,j} \gamma_{kij}\frac{\partial T^\star}{\partial \dot{\pi}_i}\dot{\pi}_j, \qquad i,\ j,\ k = 1, 2, \ldots, n \tag{226}$$

$$\dot{\pi}_k = \sum_s a_{ks}\dot{q}_s, \qquad \dot{q}_s = \sum_l b_{sl}\dot{\pi}_l, \quad \mid b_{sl} \mid = \mid a_{ks} \mid^{-1} \tag{227}$$

$$T(q_1, \ldots, q_n;\ b_{1k}\dot{\pi}_k, \ldots, b_{nk}\dot{\pi}_k,\ t) = T^*(q_1, \ldots, q_n;\ \dot{\pi}_1, \ldots, \dot{\pi}_n,\ t) \tag{228}$$

$$T^*(q_1, \ldots, q_n; a_{1s}\dot{q}_s, \ldots, a_{ns}\dot{q}_s,\ t) = T(q_1, \ldots, q_n;\ \dot{q}_1, \ldots, \dot{q}_n,\ t) \tag{229}$$

$$P_k = \sum Q_s b_{sk}, \qquad R_k = \sum D_s b_{sk} \quad s, k = 1, 2, \ldots, n \tag{230}$$

$$\gamma_{kij} = \sum_{i,j} b_{sk}b_{lj}\left(\frac{\partial a_{is}}{\partial q_l} - \frac{\partial a_{il}}{\partial q_s}\right), \qquad k, i, j, s, l = 1, 2, \ldots, n \tag{231}$$

$$\frac{\partial T^*}{\partial \pi_k} = \sum_s b_{sk}\frac{\partial T^*}{\partial q_s}, \qquad s, k = 1, 2, \ldots, n. \tag{232}$$

Since we consider the system till now as a free one, and since the $\delta\pi_k$ are independent, we have

$$B_k - P_k - R_k = 0, \qquad k = 1, 2, \ldots, n. \tag{233}$$

According to the well known theorem in mechanics a necessary and sufficient condition that the sum of the elementary works of the force system $D_i(t), \quad i = 1, 2, \ldots, n$ at every virtual displacement compatible with the constraints be equal to zero, i.e., $\sum_{i=1}^{n} a_{ri} \partial q_i = 0$, $r = 1, 2, \ldots, m$, is that the forces D_i obey the constraints $D_i = \sum_{r=1}^{n} \lambda_r a_{ri} \quad i = 1, 2, \ldots, n$. Therefore we obtain for the reaction forces

$$R_k = \sum_i \lambda_{m+i} \sum_s a_{m+i,s}\, b_{sk}, \quad i = 1, 2, \ldots, n-m, \quad s, k = 1, 2, \ldots, n. \tag{234}$$

The motion is defined by the second order differential equations

$$\begin{aligned} P_k &= \left(\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial T^\star}{\partial \dot{\pi}_k} - \frac{\partial T^\star}{\partial \pi_k} \right) + \sum_{i,j} \gamma_{kij} \frac{\partial T^\star}{\partial \dot{\pi}_i} \dot{\pi}_j \qquad (235) \\ i, j &= 1, 2, \ldots, n, \qquad k = 1, 2, \ldots, m \end{aligned}$$

and the reaction forces are obtained from the following algebraic equations

$$\begin{aligned} R_k &= P_k + \left(\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial T^\star}{\partial \dot{\pi}_k} - \frac{\partial T^\star}{\partial \pi_k} \right) + \sum_{i,j} \gamma_{kij} \frac{\partial T^\star}{\partial \dot{\pi}_i} \dot{\pi}_j \qquad (236) \\ i, j &= 1, 2, \ldots, n, \qquad k = m+1, \ldots, n. \end{aligned}$$

The systems (235) and (236) define a direct and an inverse dynamic problem, respectively. After we have established both, their solution in the language of nonholonomic coordinates is simpler than the solution of the Routh equations in Lagrangian coordinates with multipliers as given by (217).

6.3. Matrix Representations

To make our presentation more compact, we shall apply further matrix notation. First we introduce the vectors (in sense of $n \times 1$ matrices) of the generalized coordinates, velocities and accelerations and the vectors of the nonholonomic coordinates and their first derivatives with respect to time

$$q = [q_1 \ldots q_n]^T, \qquad \dot{q} = [\dot{q}_1 \ldots \dot{q}_n]^T, \qquad \ddot{q} = [\ddot{q}_1 \ldots \ddot{q}_n]^T \tag{237}$$

$$\dot{\pi} = [\dot{\pi}_1 \ldots \dot{\pi}_n]^T, \qquad \ddot{\pi} = [\ddot{\pi}_1 \ldots \ddot{\pi}_n]^T. \tag{238}$$

Let us denote by B the inverse matrix of A, namely $B = \mid b_{sl} \mid = \mid a_{ks} \mid = A^{-1}$. The matrices A and B have the block forms

$$A = [A_1 \vdots A_2]^T, \qquad [B_1 \vdots B_2]^T \tag{239}$$

where A_1 is $(m \times n)$, A_2 is $(n - m \times n)$ dimensional matrices. We also introduce the following matrices

$$\frac{\partial T}{\partial \dot{q}} = [\frac{\partial T_1}{\partial \dot{q_1}} \dots \frac{\partial T_n}{\partial \dot{q_n}}]^T = h_1(q, \dot{q}), \qquad \frac{\partial T}{\partial q} = [\frac{\partial T_1}{\partial q_1} \dots \frac{\partial T_n}{\partial q_n}]^T \tag{240}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}(\frac{\partial T}{\partial \dot{q}}) = H(q)\ddot{q} + h_2(q, \dot{q}). \tag{241}$$

Here $H(q)$ is the $n \times n$ inertial matrix, $h_1(q, \dot{q}),\ h_2(q, \dot{q})$ and $h = -(h_1 + h_2)$ are vectors $(n \times 1)$ of Coriolis, centrifugal and gravitational forces. Since Q, D, P and R are also $n \times 1$ vectors and $\Lambda = [\lambda_{m+1} \dots \lambda_n]^T$, the equations of motion become

$$H(q)\ddot{q} + h(q, \dot{q}) = Q + D \tag{242}$$

where $D = A_2^T \Lambda$. Let us introduce also

$$\dot{\pi}_{1m} = [\dot{\pi}_1 \dots \dot{\pi}_m]^T, \qquad \dot{\pi}_{m+1,n} = [\dot{\pi}_{m+1} \dots \dot{\pi}_n]^T. \tag{243}$$

The following relations are quite useful

$$\dot{\pi}_{1m} = A_1\dot{q}, \qquad \dot{\pi}_{m+1,n} = 0 = A_2\dot{q} \tag{244}$$

$$\dot{q} = B_1\dot{\pi}_{1m} + B_2\dot{\pi}_{m+1,n} \qquad \text{or only} \qquad \dot{q} = B_1\dot{\pi}_{1m} \tag{245}$$

where $A_1 B_1 = I_m$, $A_2 B_1 = O$, I_m is the $m \times m$ identity matrix, $A_1 B_2 = O$, $A_2 B_2 = I_{n-m}$, O is the corresponding zero matrix. The second set of equations in (244) is just the constraint equation. By differentiation we obtain

$$\ddot{q} = B_1\ddot{\pi}_{1m} - (B_1\dot{A}_1 + B_2\dot{A}_2)B_1\dot{\pi}_{1m}. \tag{246}$$

When A_1 is a constant matrix the pseudo-coordinates are linear combinations of the generalized velocities, $\dot{A}_1$ is a zero matrix which simplifies the model. When A_1 coincides with the Jacobian matrix, the nonholonomic coordinates are just the task-velocities. After multiplying the both sides of equation (242) by $B = [B_1, B_2]^T$ – two decoupled systems for the motion and reactions, respectively, can be extracted: m scalar differential equations of motion

$$B_1^T H\ddot{q} = B_1^T(Q - h) \tag{247}$$

and $n - m$ equations for the Lagrange multipliers

$$\Lambda = B_2^T(H\ddot{q} + h - Q). \tag{248}$$

Having in mind the expressions for $\dot{q}$ and $\ddot{q}$ from equations (245) and (246) we obtain the following relation for $\ddot{\pi}$

$$\ddot{\pi}_{1m} = (B_1^T H B_1)^{-1} B_1^T [H(B_1 \dot{A}_1 + B_2 \dot{A}_2) B_1 \dot{\pi}_{1m} + Q - h]. \tag{249}$$

After that the reaction forces may be obtained from

$$R = A_2^T B_2^T (H\ddot{q} + h - Q) \tag{250}$$

where $\ddot{q}$ comes just from equation (246). By the principle of the compatibility of the equations of motion, we may write them also in the following way

$$H\ddot{q} + h = Q + U \tag{251}$$

where $U = [u_1 \dots u_n]^T$ is the control vector assuring the performance of the constraints from (214). Relying on the fact that the constraint equations (214) could be written as an algebraic system $GD + g = 0$, where G and g are given later, we may formulate the following

Theorem 5. *A solution of equation* (251) *is compatible with the constraint equations* (214) (*task program*) *and conversely a set of functions satisfying the task conditions* (214) *is a solution of* (251) *if and only if the control vector U satisfies the algebraic equation*

$$GU + g = O \tag{252}$$

$$\text{where} \quad G = A_2 H^{-1} \quad \text{and} \quad A_2 H^{-1}(Q - h) + A_2 \ddot{q} = g. \tag{253}$$

After some matrix transformations it follows that

$$A_2 H^{-1} U + A_2 H^{-1}(Q - h) + \dot{A}_2 B_1 \dot{\pi}_{1m} = 0. \tag{254}$$

It is easy to see that this is just the equation (252), i.e., the theorem is valid also in terms of pseudo-coordinates.

Reviewing our considerations we may generalize that the basic feature of the pseudo-coordinate approach for motion and reaction description suggested in the present paper, is that until a definite moment we can assume the mechanical system to be free. Otherwise additional unknown forces compatible with the constraints should be applied to it. It is proved also that these forces are just the reactions. In fact a system with a definite kinetic energy and generalized forces is constructed in such a way that its motion is compatible with the imposed constraints.

Looking at the Newtonian mechanics of any mechanical system from a geometric point of view the following conclusions may be extracted:

The configurational space is a differentiable manifold. The system dynamics is formulated on the tangent $(\dot{q}/\dot{\pi})$ and second tangent space $(\ddot{q}/\ddot{\pi})$, while the forces may be defined as elements of the cotangent bundle. After specifying a Riemannian metric on the configurational space, the dynamical properties of the system may be studied. In addition the equations of motion, written in a coordinate non-invariant form, may be interpreted as follows: H is the matrix of the metric tensor (kinetic energy) and h may be considered as the matrix product $h(q,\dot{q}) = \hat{h}(q,\dot{q})\dot{q}$, where $\hat{h}(q,q)$ is an $n \times n$ matrix composed of the covariant Christoffel symbols of the connection induced by the metric. Within the motion equation in pseudo-coordinates the structural constants γ_{kij} of the group of the nonholonomic operators play the same role. The generalized reaction forces Q and D (respectively P and R in the equations in pseudo-coordinates) are elements of the cotangent bundle of the configurational Lagrangian space and they may serve as two different types of control parameters.

It is worth here to note that descriptor systems are very much used in identification and control of constrained mechanical systems [82], [115]. The descriptor form of the differential–algebraic equations of motion (235) and reaction forces (236) in the language of pseudo–coordinates has the following matrix form

$$\begin{aligned}\begin{bmatrix} A_1 & 0 & 0 \\ 0 & H^* & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q} \\ \ddot{\pi}_{1m} \\ \dot{\Lambda} \end{bmatrix} &= \begin{bmatrix} 0 & I_{mn} & 0 \\ 0 & 0 & A_2^T \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{q} \\ \dot{\pi}_{1m} \\ \Lambda \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ -h^* \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ P \\ \dot{\pi}_{m+1,n} \end{bmatrix}\end{aligned} \tag{255}$$

where $H^*(q)$ and $h^*(q,\pi)$ have the meaning of H and h but expressed in terms of q and π. Here the descriptor vector is $[q\ \pi_{1m}\ \Lambda]$ with real dimension $2n$. In the descriptor form of the system of equations (235) and (236) but in generalized variables, the descriptor vector $[q\ \dot{q}\ \Lambda]$ is $3n - m$ dimensional. Since $m \leq n$, it is clear that the number of the computations is reduced when pseudo-coordinates are used. In a contemporary geometrical language, it means that the problem can be easily solved in the tangent (cotangent) space, which is just the idea of this section.

7. Dynamics and Control of Elastic Joint Manipulators Through Vector-Parameter

Kinematical and dynamical equations of a manipulator system play an extremely important role as for motion simulation so in control. Because so many authors have approached this problem, it is impossible even to list them here and that is why we refer to the books [131], [19] and references therein. The dynamical equations are used in motion simulation, where they furnish a powerful tool for the study of control strategies, optimization of the parameters, and for testing of robot performance under various conditions. In connection with the robot operation, dynamical equations are used for the evaluation of nominal actuator torques and forces which drive the robot along a prescribed trajectory.

7.1. The Main Idea of the Model

The investigations presented in this section are based again on our previous studies relying on vector-parameterization of the SO(3) group. The statement that this parameterization has the nice property of a Lie group which simplifies drastically some considerations, reduces the computational burden, and all this is valid for models built through vector-parameter, becomes stronger in pure vector-parameter considerations. It is proved additionally that the computational effectiveness of the vector-parameter approach increases with the increasing number of the revolute degrees of freedom. Here we show that this can be used successfully in the problems of elastic joint manipulators, where except the real n links, n fictious links are included and n additional revolute degrees of freedom are involved. Dynamic models 'through' vector-parameter and in 'pure' vector-parameter form are developed and the inverse dynamic problem is discussed.

The basic equations that characterize the first approach (through vector-parameter) are (1) – (5), where the vector-parametrization of SO(3) group is used to facilitate some statements and proofs and to reduce the calculation burden. The computational efficiency is proved in solving direct and inverse kinematic problems, both in dynamic modeling and full simulation of manipulators system motion. After that all the models are transferred again in the configurational manifold of the standard joint displacements.

The idea of the second approach (in pure vector-parameter) is that the geometry, kinematics and dynamics of an open loop kinematic chain should be described in a new extended configurational space with a group structure – the space of vectors describing the joints displacements (vector-parameters c and translational vectors tr) as it is presented in equations (256) – (260). In this manner the transition operations from vector-parameters to generalized coordinates are saved. The computational efficiency that is proved in the first approach becomes here even a stronger one. Here the kinematic and dynamic equations are pure algebraic and the differential equations of motion are over a Lie group. In this frame the merge of the powerful theory of Lie groups and Lie algebras with fundamental problems of controllability and observability of manipulator systems is quite natural and promising.

In this section is shown also that the nonlinear equations of motion are globally linearizable by smooth invertable coordinate transformation and nonlinear state feedback (see also [73], [79], [120], [20]).

7.2. Basic Notation

We consider a flexible joint manipulator with n rotational degrees of freedom, and links $B(0)$ (fixed), $B(1), B(2), \ldots, B(n)$ (Fig. 4). A frame $R(i)$ centered in $C(i)$ is referred to the link $B(i)$ (and joint i) from the chain. The frame $R(n+1)$ is built in the gravity center $G(n)$ of the link $B(n)$ (gripper), which may be coincident with the gripper characteristic point C. We denote by $Rm(i)$ the frame referred to the motor $M(i)$ that moves the link $B(i)$, with $Cm(i)$ as its origin.

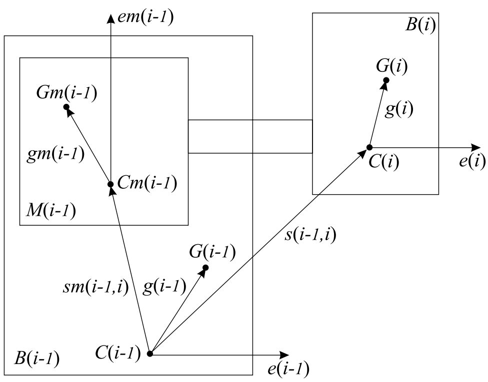

Figure 4. Links $i-1$ and i of a flexible joint manipulator

$q := [q(1) \ldots q(n)]^T$ is $n \times 1$ vector of the generalized coordinates (joint displacements) of the manipulator system (MS) in the usual sense, $q \in Q \subset \mathbb{R}^n$ and Q is the configurational manifold; $qm := [qm(1) \ldots qm(n)]^T$ is $n \times 1$ vector of the motor rotation angles, $qm \in Qm \subset \mathbb{R}^n$ and Qm is the space of motor angles; x is m-dimensional vector of end-effector position and orientation. We follow the basic notation given in the first section and equations (1)–(5). Further we denote by:

$\times$ – the cross product of vectors, $a \times b := a^{\times} b,\ a^{\times}$ is the skew-symmetric matrix whose components are the components of vector a;
$G(i)$ – the mass center of the link $B(i)$ and $Gm(i)$ – the mass centre of motor $M(i)$;
$s(i, i+1) - 3 \times 1$ vector from $C(i)$ to $C(i+1)$, written in $R(i)$;
$sm(i-1, i) - 3 \times 1$ vector from $C(i-1)$ to $Cm(i)$, written in $R(i-1)$;
$g(i)$ – the 3×1 vector from $C(i)$ to the mass center $G(i)$, written in the frame $R(i)$, which is constant in this frame;
$gm(i)$ – the 3×1 vector from $Cm(i)$ to the mass center $Gm(i)$, written in the frame $Rm(i)$, which is constant in this frame;
$e(i) := [0\ 0\ 1]^T$ is 3×1 vector aligned with the i-th rotation axis;
$em(i) := [0\ 0\ 1]^T$ is 3×1 vector of rotation for the motor $M(i)$ written in the frame $Rm(i)$;
$p(i) := s(0,1) + s(1,2) + \ldots + s(i-1, i), \qquad (s(0,1) := 0)$;
$pg(i) = p(i) + g(i)$;
$c(i)$ – the vector-parameter of rotation of $R(i-1)$ according to $R(i)$;
$cc(i)$ is a composition vector, i.e., $cc(i) := \langle c(1), c(2), \ldots, c(i)\rangle$;
$\mathrm{tr}(i)$ – the translation vector along the axis of i-th prismatic joint;
$O(i, i+1) = O(c(i+1))$ is the rotation matrix that relates the orientation of frame i to frame $i+1$ and we use the following identities:

$$O(1, i) = O(cc(i)), O(i, k) = O(cc(k))O(-cc(i));$$

$\omega(i) - 3 \times 1$ vector of angular velocity of link i at $C(i)$ written in the frame $R(i)$;
$\omega m(i)$ – the 3×1 vector of angular velocity of motor i at $C(i)$ written in the frame $R(i)$;
$v(i)$ – the 3×1 vector of linear velocity of link $B(i)$ written in the frame $R(i)$;
$vm(i)$ – the 3×1 vector of linear velocity of the motor $M(i)$ at $Cm(i)$ written in $Rm(i)$;
$V(i) - 6 \times 1$ vector of spatial velocity of link i written in the frame $R(i)$;
$V(i) := [\omega(i)^T\ v(i)^T]^T$;
$Vm(i) - 6 \times 1$ vector of spatial velocity of the motor $M(i)$ written in $Rm(i)$;
$Vm(i) := [\omega m(i)^T\ vm(i)^T]^T$;
$W(i) - 6 \times 1$ vector of spatial acceleration of $C(i)$ written in the frame $R(i)$:

$$W(i) := [\dot{\omega}(i)^T\ \dot{v}(i)^T]^T;$$

$Wm(i)$ – 6×1 vector of spatial acceleration of $Cm(i)$ written in the frame $Rm(i)$:
$Wm(i) := [\dot{\omega m}(i)^T \ \dot{vm}(i)^T]^T$;
$F(i)$ – 3×1 vector of inertial forces acting upon link i at point $G(i)$;
$N(i)$ – 3×1 vector of inertial moments acting upon link i at $G(i)$;
$f(i)$ – 3×1 vectors of forces acting upon link i at point $C(i)$ written in the frame $R(i)$;
$n(i)$ – 3×1 vectors of moments acting upon link i at point $C(i)$, written in $R(i)$;
$t(i)$ – 6×1 vector of spatial force acting upon link i at point $C(i)$ written in $Rm(i)$;
$t(i) := [f(i)^T \ n(i)^T]^T$;
$P(i)$ – the torque acting on the link i, and $Pm(i)$ – the i-th motor torque.

Let $(c(1), \mathrm{tr}(1)),\ \ldots,\ (c(n), \mathrm{tr}(n))$ are couples of vectors describing the link movements of the MS. They define the group configurational space Q_{ct}

$$Q_{ct} := \{(c, \mathrm{tr})^{\min} \ \leq \ (c, \mathrm{tr}) \ \leq \ (c, \mathrm{tr})^{\max}\}. \tag{256}$$

The entries of the i-th pair $((c(i),\ \mathrm{tr}(i))$ coincide either with the vector-parameter $c(i)$ or translational vector $\mathrm{tr}(i)$ in dependence on the type of the i-th joint (revolute or prismatic respectively). We assume that all joints are rotational and
$c := [c(1) \ldots c(n)]^T$, $\quad c' := g_c(n) := \langle c(1), \ldots, c(n) \rangle$ and $Q_{ct} \equiv Q_c$
where $c(i) = (cx(i),\ cy(i),\ cz(i))$, $\quad g_c(i) = (g_c x(i),\ g_c y(i),\ g_c z(i))$.

Remark 6. *According to the results in* [59] *only one of the components of the vector-parameters contains the whole information for the joint rotation, and the superfluous coordinates are discarded by proper geometrical considerations. So in reality the dimensions of the kinematic and dynamic models do not increase.*

The connection between x and c, i.e., the direct kinematic problem (DKP) in this case is given by

$$x = F_V(c) \tag{257}$$

where $F_V : Q_c \to X$ is a smooth projection map over the target space

$$X = \{x\ ;\ x = F_V(c),\ \ c \in Q_c \subset \mathbb{R}^{3n}\} \tag{258}$$

which is the working space of the MS under consideration. After differentiation of (257) with respect to t one obtains

$$\dot{x} = \left[\frac{\partial F}{\partial c}\right] \dot{c} = J_V(c)\dot{c} \tag{259}$$

where $J_V(c) \in R^{m,3n}$ is the Jacobian matrix of the map F_V. Again all configurations for which the rank of $J_V(c) < m$ are called singular.

The dynamic equations for a rigid body MS on the group configurational space Q_c look like

$$H_V(c)\ddot{c} + h_V(c, \dot{c}) = P_V \tag{260}$$

where $H_V := H_V(c)$ is $3n \times 3n$ inertia matrix, the $3n \times 1$ vector $h_V := h_V(\dot{c}, c)$ takes into account Coriolis, centrifugal and gravitational forces, and P_V is $3n \times 1$ matrix of the generalized forces and moments. Analogically the vector cm with components $cm(i)$, $i = 1, \dots n$ which describes the motor rotations is introduced, $cm \in Q_{cm} \subset \mathbb{R}^{3n}$.

We continue with the appropriate notation for spatial transformations [81], [36]:

$\Phi(i+1, i) - 6 \times 6$ matrix which translates a spatial velocity at point $C(i)$ written in the frame $R(i)$ to a spatial velocity at $C(i+1)$ written in $R(i+1)$

$$\Phi(i+1, i) := \begin{bmatrix} O(i, i+1) & \emptyset \\ -O(i, i+1)s^{\times}(i, i+1) & O(i, i+1) \end{bmatrix}$$

where $\emptyset$ means 3×3 zero matrix.

$\Phi^T(i+1, i)$ – transpose of $\Phi(i+1, i)$ matrix. Φ^T translates the spatial force acting at $C(i+1)$ written in the frame $R(i+1)$ to a spatial force acting at $C(i)$ written in $R(i)$

$$\Phi^T(i+1, i) := \begin{bmatrix} O(i+1, i) & s(i+1, 1)O(i+1, 1) \\ \emptyset & O(i+1, i) \end{bmatrix};$$

$Om(i-1, i) \; - \; 3 \times 3$ rotation matrix which relates the orientation of the frame $Rm(i)$ at $Cm(i)$ to the frame $R(i-1)$. The corresponding vector-parameter is $cm(i)$ and

$Om(i-1, i) := O(cm(i))$;

$\Phi m(i, i-1)$ - 6×6 matrix which translates a spatial velocity at point $C(i-1)$ written in the frame $R(i-1)$ to a spatial velocity at $Cm(i)$ written in $Rm(i)$

$$\Phi m(i, i-1) := \begin{bmatrix} O(i-1, i) & \emptyset \\ Om(i-1, i)sm^{\times}(i-1, i) & Om(i-1, i) \end{bmatrix};$$

$mb(i)$ – mass of link i, $mm(i)$ – mass of motor i, that drives link i,
$I(i) - 3 \times 3$ inertia matrix of the link i at $C(i)$ written in $R(i)$,

$Im(i) - 3 \times 3$ inertia matrix of the motor/gear at $Cm(i)$ written in the frame $Rm(i)$;
$Mb(i) - 6 \times 6$ spatial mass matrix of link i at $C(i)$ written the in frame $R(i)$. It connects the inertia matrix of link i, link mass and the location of the mass centre $g(i)$

$$Mb(i) := \begin{bmatrix} I(i) & m(i)g^{\times}(i) \\ -m(i)g^{\times}(i) & m(i)I \end{bmatrix}$$

where I is 3×3 identity matrix,
$Mm(i) - 6 \times 6$ spatial mass matrix of the motor and gear at $Cm(i)$ written in $R(i)$

$$Mm(i) := \begin{bmatrix} Im(i) & mm(i)gm^{\times}(i) \\ -mm(i)g^{\times}(i) & mm(i)I \end{bmatrix};$$

$a(i) - 6 \times 1$ vector of spatial accelerations of link i, written in frame $R(i)$, which in case of a rotational joint

$$a(i) := \begin{bmatrix} O(i-1,i)\omega(i-1) \times \omega(i) \\ O(i-1,i)(\omega(i-1) \times (\omega(i-1) \times s(i-1,i))) \end{bmatrix};$$

$am(i)$ - 6×1 vector of spatial accelerations of the motor(gear) driving link i written in frame $Rm(i)$

$$am(i) := \begin{bmatrix} Om(i-1,i)\omega m(i-1) \times \omega m(i) \\ Om(i-1,i)(\omega m(i-1) \times (\omega m(i-1) \times sm(i-1,i))) \end{bmatrix};$$

$b(i)$ - 6×1 vector of spatial forces acting on link i at point $C(i)$, written in frame $R(i)$, which for a rotational joint look as

$$b(i) := \begin{bmatrix} \omega(i) \times I(i)\omega(i) \\ mb(i)\omega(i) \times (\omega(i) \times g(i)) \end{bmatrix};$$

$bm(i)$ - 6×1 vector of spatial forces acting on the motor/gear at point $C(i)$, written in frame $Rm(i)$ as

$$bm(i) := \begin{bmatrix} \omega m(i) \times Im(i)\omega m(i) \\ mm(i)\omega m(i) \times (\omega m(i) \times gm(i)) \end{bmatrix};$$

$E(i)$ – 6×1 vector for the spatial axis of motion for link i, written in $R(i)$, which for a rotational joint is $E(i) := [e^T(i)\ \ 0\ \ 0]^T$, or $E(i) := [0\ 0\ 1\ 0\ 0\ 0]^T$;
$E^T(i)$ – transpose of $E(i)$;
$Em(i) - 6 \times 1$ vector for the spatial axis of motion for the motor driving link i written in $R(i)$, $Em(i) := [em^T(i)\ 0\ 0\ 0]^T$, or $Em(i) := [0\ 0\ 1\ 0\ 0\ 0]^T$;
$Em^T(i)$ – transpose of $Em(i)$.

7.3. Inverse Dynamic Problem (IDP) for Elastic Joint Manipulators

The future implementation of robot manipulators, especially in space, will require extensive modeling, simulation and analysis during manipulator construction, development and verification of controllers, training of the operators and actual operation of the manipulators. The performance of real time simulations, control and analysis requires also the development of computationally effective methods and approaches. The Inverse Dynamic Problem (IDP) plays an important role in real time simulation and control. Its solution on the base of Newton-Euler equations is extremely efficient. This approach is appropriate also for parallel computations in full manipulator simulation. A detailed treatment from this perspective of the rigid body manipulators can be found in [62].

Further on, the above notation are used so that IDP relationships can be outlined just in case of flexible joint manipulator. At this stage the problem is treated "through" vector parameter and spatial operator technique as described in [81] is also used.

7.3.1. External Iterative Procedure (Forward Recursion)

This procedure is realized from the first link to the last one. It contains the propagation of the velocities and the accelerations. The velocity of each link is the sum of the velocity of the previous link and the link's own relative rotational velocity. The velocity of each motor is the velocity of the link upon which it is mounted plus the local rotational velocity of the motor. For the flexible joint, the motor velocity is independent of the link velocity. For a geared joint, the motor velocity and link velocity are related through the gear ratio, Γ.

Similar equations hold for the accelerations but with additional terms added to account the centrifugal accelerations. The accelerations are contained in $a(i)$ and $am(i)$ terms. The base is assumed to be fixed with respect to an inertial frame. The recursive equations in an algorithmic form seem like

$$\begin{aligned} V(0) &:= \text{Base Spatial Velocity} \\ W(0) &:= \text{Base Spatial Acceleration.} \end{aligned} \tag{261}$$

For $i = 1, 2, \ldots, n$ we have

$$\begin{aligned} \text{link}: \ V(i) &:= \Phi(i, i-1)V(i-1) + E(i)\dot{q}(i) \\ W(i) &:= \Phi(i, i-1)W(i-1) + E(i)\ddot{q}(i) + a(i) \\ \text{motor}: \ Vm(i) &:= \Phi m(i, i-1)Vm(i-1) + E(i)\dot{q}(i) \\ Wm(i) &:= \Phi m(i, i-1)Wm(i-1) + E(i)q\ddot{m}(i) + am(i) \\ \text{Geared}: \ q\dot{m}(i) &:= \Gamma(i)\dot{q}(i), \qquad q\ddot{m}(i) = \Gamma(i)\ddot{q}(i). \end{aligned} \tag{262}$$

7.3.2. Internal Iterative Procedure (Backward Recursion)

It is realized from the end-effector to the base. Independent of the geared or flexible joint case, the force acting over every link is a sum of the link inertial forces, the forces from the outer link and the motor inertial forces

$$\begin{aligned} t(n+1) &:= \text{vector of external torques(forces)} \\ &\quad\;\; \text{applied on the end-effector} \\ tm(n+1) &:= 0. \end{aligned} \qquad \begin{aligned} (263) \\ \\ (264) \end{aligned}$$

Besides, for $i = n, \ldots, 1$ we have that

$$\begin{aligned} \text{link}: \; t(i) &:= \Phi^T(i+1,i)t(i+1) + \Phi m^T(i+1,i)tm(i+1) \\ &\quad +M(i)W(i) + b(i) \\ \text{motor}: tm(i) &:= Mm(i)Wm(i) + bm(i). \end{aligned} \qquad (265)$$

The next step aims obtaining of motor and link torques, namely: The torque acting on the joint axis i is the projection of the force acting on the link i along the axis of the joint

$$P(i) = E^T(i)t(i). \qquad (266)$$

This torque is also a function of the state-dependent coupling between the motor and the link. For example, for a simple torsional spring model, the torque is a function of the angular displacement between the motor and the link

$$P(i) = K(i)(qm(i) - q(i)) \qquad (267)$$

where $K(i)$ is the torsional spring constant for i-th axis. To account the internal damping, the torque may be written

$$P(i) = K(i)(qm(i) - q(i)) + Z(i)(q\dot{m}(i) - \dot{q}(i)) \qquad (268)$$

where $Z(i)$ characterizes the damping coefficient. We may generalize that the torque can be expressed as a function of the state of the manipulator

$$P(i) = f(q,\ \dot{q},\ qm,\ q\dot{m}). \qquad (269)$$

Through a force balance on the motor and the link, the motor torque can be expressed as the sum of the projection of the forces along the motor axis and the link torque

$$Pm(i) = E^T(i)tm(i) + P(i) \qquad (270)$$

In case of a point contact at interactions between gears, the torque of the motor can be expressed as the link torque reduced by the gear ratio plus the torque due to the acceleration of the motor

$$Tm(i) = E^T(i)t(i)/\Gamma(i) + Em^T(i)tm(i). \tag{271}$$

Equations (261) – (267) define the complete inverse dynamics of both flexibly jointed and geared manipulators.

7.4. Dynamic Problem and Control

7.4.1. Direct Dynamic Problem

The goal of simulation, or forward dynamics, is to find the motor and link accelerations when the motor (link) angular positions, velocities and motor torques are given. A lot of algorithms exist. For example, the recursive sweep method [81] begins by assuming that the state (force vector) $t(i)$ and the costate (acceleration vector) are related by

$$t(i) = \hat{H}(i)W(i) + \hat{h}(i) \tag{272}$$

where $\hat{H}(i)$ and $\hat{h}(i)$ are the articulated body inertias and bias forces respectively for body i and they are defined recursively from link n.

We refer to our previous works, mentioned above, concerning rigid body dynamics where elegant recursive procedures of building equation (5) on the base of different mechanical equations are given. We present equation (5) as

$$H(q)\ddot{q} + h_1(q,\dot{q})\dot{q} + h_2(q) = P \tag{273}$$

where in $h_1(q,\dot{q})\dot{q}$ enter centrifugal terms (these which contain $\dot{q}^2$) and Coriolis terms (these which contain $\dot{q}\dot{q}$), and $h_2(q)$ is the gravity vector, i.e.,

$$H(q)\ddot{q} + h_1 \text{ centr}(\dot{q}^2) + h_1 \text{ Coriolis}(\dot{q}\dot{q}) + h_2(q) = P.$$

From [63] it may be seen that

$$h_1 = \dot{H}\dot{q} - \partial T/\partial q \qquad \text{and} \qquad h_2 = \partial U/\partial q \tag{274}$$

where T and U are the kinetic and potential energy of the system respectively, as well all recursions for the matrix H and vectors h_1 and h_2.

We consider now a model of an n-link manipulator with joint flexibility. For simplicity we assume (as it is done in [121]) that the joints are revolute, they are actuated by DC motors and the flexibility of i-th joint is modeled as a linear torsional spring with spring constants $K(i)$ for $i = 1, \dots, n$ as it is shown on Fig.3. We note that due to the joint flexibility, there are now twice as many degrees of freedoms as compared with the rigid joint case. We denote by

$$\bar{q} := [q_1 \dots q_{2n}]^T \tag{275}$$

the set of generalized coordinates of the system, where

$$\begin{aligned} q_{2i-1} &:= q(i) \quad \text{(the angle of link } i), \qquad i = 1, \dots, n \\ q_{2i} &:= -\frac{1}{\Gamma(i)} qm(i), \qquad i = 1, \dots, n \end{aligned} \tag{276}$$

In this case $q_{2i} - q_{2i-1}$ is the elastic displacement of joint i. If we define the n-dimensional vectors

$$\bar{q}_1 := [q_1 \ q_3 \ \dots \ q_{2n-1}]^T \qquad \bar{q}_2 := [q_2 \ q_4 \ \dots \ q_{2n}]^T \tag{277}$$

the kinetic T and potential U energies of this system can be expressed as

$$T = \frac{1}{2}\dot{\bar{q}}_1^T H(\bar{q}_1)\dot{\bar{q}}_1 + \frac{1}{2}\dot{\bar{q}}_2^T J_m \dot{\bar{q}}_2, \qquad U = U_1(\bar{q}_1) + U_2(\bar{q}_1 - \bar{q}_2) \tag{278}$$

where $H(\bar{q}_1)$ and $U_1(\bar{q}_1)$ are respectively the inertia matrix and potential energy of the rigid manipulator, Jm is a $n \times n$ diagonal matrix

$$Jm := \text{Diag}\ [Imzz(1)/\Gamma^2(1), \dots, Imzz(n)/\Gamma^2(n)] \tag{279}$$

which components contain the inertia $Imzz(i)$ of the i-th rotor about its principal axes of rotation. From the previous subsection we have

$$Im(i) := \text{Diag}[Imxx(i)\ Imyy(i)\ Imzz(i)]$$

where the diagonal elements are the moments of inertia of the rotor about the principal axis. U_2 is the elastic potential energy of the joints

$$U_2 = \frac{1}{2}(\bar{q}_1 - \bar{q}_2)^T K(\bar{q}_1 - \bar{q}_2). \tag{280}$$

Then the dynamics of a manipulator with joint elasticity can be described by

$$H(\bar{q}_1)\ddot{\bar{q}}_1 + h_1(\bar{q}_1, \dot{\bar{q}}_1) + h_2(\bar{q}_1) + K(\bar{q}_1 - \bar{q}_2) = 0 \tag{281}$$

$$Jm\ddot{\bar{q}}_2 - K(\bar{q}_1 - \bar{q}_2) = u. \tag{282}$$

The input vector u (input torque applied to the motor shaft) which has units of torque is a function of the generalized forces and moments of the rigid mechanical system and some motor characteristics and constants. For more details concerning the last system, as well for general theory of nonlinear control we refer to [122], [82], [83], [20], [37] and references quoted therein. The both equations may be combined in the form

$$\begin{bmatrix} H(\bar{q}) & 0 \\ 0 & Jm \end{bmatrix} \ddot{\bar{q}} + \begin{bmatrix} h_1(\bar{q}_1, \dot{\bar{q}}_1) \\ 0 \end{bmatrix} + \begin{bmatrix} h_2(\bar{q}_1) \\ 0 \end{bmatrix} + \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \bar{q} = \begin{bmatrix} 0 \\ I \end{bmatrix} u. \quad (283)$$

Viscous friction terms acting both on link and on the motor sides of the elastic joints could be easily included in the dynamic model. This model describes system dynamics "through" vector-parameter coordinates.

How does the dynamic model looks in "pure" vector-parameter notation?

We introduce the following system of vectors

$$\bar{c} := [c_1 \ \dots \ c_{2n}]^T \quad (284)$$

as vector-parameters of the system, in which

$$c_{2i-1} := c(i), \qquad c_{2i} := -\frac{1}{\Gamma(i)} cm(i), \qquad i = 1, \dots, n \quad (285)$$

and $cm(i)$ is the vector-parameter describing the rotation of i-th rotor. Analogically we can define the $3n$-dimensional vectors

$$\bar{c}_1 := [c_1 \ c_3 \ \dots \ c_{2n-1}]^T, \qquad \bar{c}_2 := [c_2 \ c_4 \ \dots \ c_{2n}]^T \quad (286)$$

and using the formalism already described in [62], to obtain

$$H_V(\bar{c}_1)\ddot{\bar{c}}_1 + h_{V1}(\dot{\bar{c}}_1, \bar{c}_1) + h_{V2}(\bar{c}_1) + K(\bar{c}_1 - \bar{c}_2) = 0 \quad (287)$$

$$Jm_V \ddot{\bar{c}}_2 - K(\bar{c}_1 - \bar{c}_2) = u_V \quad (288)$$

$$\begin{bmatrix} H_V(\bar{c}) & 0 \\ 0 & Jm_V \end{bmatrix} \ddot{\bar{c}} + \begin{bmatrix} h_{V_1}(\bar{c}_1, \dot{\bar{c}}_1) \\ 0 \end{bmatrix} + \begin{bmatrix} h_2(\bar{c}_1) \\ 0 \end{bmatrix} + \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \bar{c} = \begin{bmatrix} 0 \\ I \end{bmatrix} u_V. \quad (289)$$

However, the real dimension of the differential equations is just $2n$ because only one of the components of the vectors $c(i)$ and $cm(i)$ is essential and informative. Here the subindex V means again that the model is considered in "pure" vector-parameter approach. We note that the gyroscopic forces between each rotor and the other links are not included in the last equations since we assume [121] that the kinetic energy of the rotors is due mainly to its own rotation, equivalently, the motion of the rotor is a pure rotation with respect to an inertial frame. The matrix H and the vectors h and u from the dynamic model over the generalized manifold Q given above as well as the corresponding H_V, h_V and u_V in "pure" vector-parameter dynamics are defined also recursively.

7.4.2. Inverse Dynamics Control and Feedback Linearization

It is well known that the rigid robot equations (283) may be globally linearized and decoupled by nonlinear feedback. This is just the scheme of the inverse dynamic control. The key idea of the inverse dynamics is to seek a nonlinear feedback control law

$$u = f(\bar{q}, \dot{\bar{q}}), \qquad u_V = f_V(\bar{c}, \dot{\bar{c}}) \tag{290}$$

which after entering in the equations (283) or (289) results in a linear closed loop system. For example applying the nonlinear control law to the system described by (283)

$$u = A(\bar{q})v + B(\bar{q}, \dot{\bar{q}}) \tag{291}$$

which is also called the inverse dynamic control and

$$A(\bar{q}) = H(\bar{q}) + Jm, \qquad B(\bar{q}, \dot{\bar{q}}) = h_1(\bar{q}, \dot{\bar{q}}) + h_2(\bar{q}) \tag{292}$$

the so called double integrator system is obtained

$$\ddot{\bar{q}} = v. \tag{293}$$

The new system is linear and decoupled and it can be controlled by adding an "outer loop" control [122]. We have to note that here v does not mean velocity vector but control variable. The technique of the inverse dynamic control may be considered as a special case of a more general procedure for transforming a nonlinear system to a linear one, known as external or Feedback Linearization [35], [87], [86]. The basic idea of the feedback linearization is to find a nonlinear control law (inner control law) which linearizes the nonlinear system exactly after an appropriate change of the coordinates in the state space. Then a second stage of outer loop control in the new coordinates is designed which satisfies some control specifications. In the case of rigid manipulators the inverse dynamic problem and feedback linearizations are covered each other. The power of the techniques of the feedback linearization may be really seen when manipulator systems with elastic joints are considered.

In the general case of n-link manipulator the dynamics equations represent a multi-input nonlinear system. We consider the system (287) and (289). We define in the state space $\mathbb{R}^{4n}$ the state variables in block form

$$x_1 = 2\arctan c_{13}, \qquad x_2 = \frac{2\dot{c}_{13}}{1 + c_{13}^2} \tag{294}$$

$$x_3 = 2\arctan c_{23}, \qquad x_4 = \frac{2\dot{c}_{23}}{1 + c_{23}^2} \tag{295}$$

where $\bar{c}_1 = (0,\ 0,\ c_{13}),\ \bar{c}_2 = (0,\ 0,\ c_{23})$. Then (294), (287) and (289) are equivalent to the system

$$\begin{aligned}
\dot{x}_1 &= x_2 &(296)\\
\dot{x}_2 &= -H^{-1}(x_1)(h_1(x_1,x_2)+K(x_1-x_3)) &(297)\\
\dot{x}_3 &= x_4 &(298)\\
\dot{x}_4 &= Jm^{-1}K(x_1-x_3)+Jm^{-1}u &(299)
\end{aligned}$$

whose general form is

$$\dot{x} = f(x) + g(x)u. \tag{300}$$

Having in mind the theorem mentioned in the Appendix, the feedback linearization algorithm for multi-input systems may be used. On the base of the nonlinear change of the coordinates, namely

$$\begin{aligned}
y_1 &= D_1(x) = x_1 &(301)\\
y_2 &= D_2(x) = \dot{y}_1 = x_2 &(302)\\
y_3 &= D_3(x) = \dot{y}_2 = \dot{x}_2 \\
&= -H^{-1}(x_1)(h_1(x_1,x_2)+K(x_1-x_3)) &(303)\\
y_4 &= D_4(x) = \dot{y}_3 \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}(H^{-1}(x_1))(h_1(x_1,x_2)+K(x_1-x_3)) \\
&\quad -H^{-1}(x_1)(\frac{\partial h_1}{\partial x_1}x_2 + \frac{\partial h_1}{\partial x_2}(-H^{-1}(x_1)(h_1(x_1,x_2) &(304)\\
&\quad +K(x_1-x_3))) + K(x_2-x_4)) \\
&= a_4(x_1,x_2,x_3) + H^{-1}(x_1)Kx_4
\end{aligned}$$

and a nonlinear feedback, the transformed system has the known linear block form given in the Appendix. The function a_4 contains everything from y_4 except the last term.

7.5. Single-Link Flexible-Joint Robot Arm

Consider a single-link flexible-joint robot arm as shown in Fig. 5. We first rewrite the equations of motion in state variables and then verify that the resulting nonlinear system satisfies some necessary and sufficient conditions in order to be linearized globally by a nonlinear feedback. We have to notice that this analysis is

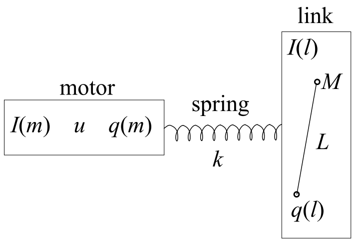

Figure 5. Single-link flexible-joint robot arm

standard in nonlinear systems control theory and for this particular case of robot it has been done in [55].

For simplicity, the damping will be ignored in this system. The joint is assumed to be of revolute type, and the link is assumed to be rigid with inertia $I(l)$ about the axis of rotation. Let $q(l)$ be the link-angular variable with the vertical axis as its reference and $q(m)$ is the motor-shaft angle. We suppose that the rotor inertia of the motor is $I(m)$. Assume also that the flexible joint is modeled as a linear spring of stiffness k. The corresponding vector-parameters are: $c_1 = (0,\ 0,\ c(l))$ and $c_2 = (0,\ 0,\ c(m))$, where $c(l) = \tan(q(l)/2)$, $c(m) = \tan(q(m)/2)$. The motion equations are

$$\begin{aligned} \frac{2I(l)}{1+c^2(l)}(\ddot{c}(l) &+ \frac{2c(l)\dot{c}^2(l)}{1+c^2(l)}) + 2Mg\frac{c(l)}{1+c^2(l)} \\ &+ 2K(\arctan c(l) - \arctan c(m)) = 0 \\ \frac{2I(m)}{1+c^2(m)}(\ddot{c}(m) &+ \frac{2c(m)\dot{c}^2(m)}{1+c^2(m)}) \\ &- 2K(\arctan c(l) - \arctan c(m)) = u \end{aligned} \tag{305}$$

where M is the total mass of the link, L is the distance from the link mass center to the axis of rotation, g – the acceleration constant of gravity, and u – the (generalized) force-input to the shaft by the actuator. Let

$$x_1 = 2\arctan c(l), \qquad x_2 = \frac{2\dot{c}(l)}{1+c^2(l)}$$

$$x_3 = 2\arctan c(m), \qquad x_4 = \frac{2\dot{c}(m)}{1+c^2(m)}.$$

Then equation (305) can be rewritten as

$$\dot{x} = f(x) + g(x)u \tag{306}$$

where $x := [x_1\ x_2\ x_3\ x_4]^T$ and

$$f(x) := \begin{bmatrix} x_2 \\ -\frac{MgL}{I(l)}\sin(x_1) - \frac{K}{I(l)}(x_1 - x_3) \\ x_4 \\ \frac{K}{I(m)}(x_1 - x_3) \end{bmatrix}, \qquad g(x) := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I(m)} \end{bmatrix}. \tag{307}$$

It is easy to be verified that the vector fields $g,\ [f,g],\ [f,[f,g]],\ [f,[f,[f,g]]]$ are constant and so they form an involutive set. Moreover, since these vector fields are linearly independent for all $k > 0$, and $I(l),\ I(m) < \infty$, then the nonlinear system (306) is globally feedback linearizable in the sense that an equivalent linear feedback system with a nonlinear transform exists and it may be found in the well known manner [123]. The nonlinear system (306) may be linearized as

$$\dot{y} = Ay + Bu \tag{308}$$

where $A(4 \times 4)$ and $B(4 \times 1)$ are matrices from the type given in the Appendix and the physical meaning of $y_1,\ y_2,\ y_3,\ y_4$ are position, velocity, acceleration and jerk of the link respectively.

Appendix. Lie Groups, Lie Algebras and Nonlinear Control Process

A Lie group is a set G such that:

1) G is a group;
2) G is a smooth manifold;
3) the group operations of composition and inversion are smooth maps of G into itself relative to the manifold structure defined in 2).

A Lie algebra over the real numbers $\mathbb{R}$ is a real vector space $\mathcal{G}$ equipped with the bilinear operation called a Lie bracket [,]: $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ with the properties:

For $X, Y, Z \in \mathcal{G}$

$$[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z], \qquad \alpha, \beta \in \mathbb{R}$$

$$[X,Y] = -[Y,X]$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad - \quad \text{Jacobi identity.}$$

If f and g are two vector fields on $\mathbb{R}^n$, the Lie bracket of f and g denoted by $[f,g]$ is a third vector field defined by

$$[f,\ g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

where $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial f}{\partial x}$ are Jacobian matrices of f and g respectively. Sometimes another notation for the Lie bracket is used, namely

$$[f, g] = L_f(g)$$

and by induction argument we have as well

$$L_f^k(g) = [f, L_f^{k-1}(g)].$$

There is no general theory for nonlinear control. There are methods based on the analysis of linearized models. Another approach consists in using Lie-algebraic and Lie-group techniques [35], [87], [13], [90].

Given a control system

$$\dot{x} := f(x) + \sum_i^m u_i g_i(x) = f(x) + g(x)u.$$

where $f(x)$ and $g_i(x)$ are smooth vector fields on $\mathbb{R}^n$ we can form the respective Lie algebra $\{f, g_1, \ldots, g_m\}_{LA}$ generated by the vector fields $f, g_1, \ldots, g_m$. It is called a **controllability Lie algebra**. A linearly independent set of vector fields $X_1, \ldots, X_m$ is said to be involutive if and only if there are scalar functions $\alpha_{ijk} : \mathbb{R}^n \to \mathbb{R}$ such that

$$[X_i, X_j] = \sum_k \alpha_{ijk} X_k \text{ for all } i, j, k.$$

Involutivity simply means that if one forms the Lie brackets of any pair of vector fields from the set $X_1, \ldots, X_m$, then the resulting vector field can be expressed as a linear combination of the original vector fields $X_1, \ldots, X_m$. The coefficients of this linear combination are allowed to be smooth functions on $\mathbb{R}^n$.

The necessary conditions for the existence of solutions to certain systems of first order partial differential equations are provided by the Frobenius' theorem which states:

The set of vector fields $X_1, \ldots, X_m$ that are linear independent at each point is completely integrable if and only if it is involutive.

The nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{$*$}$$

is said to be feedback linearizable in a neighborhood U_o of the origin if there exists a diffeomorphism $D : U_o \to \mathbb{R}^n$ and nonlinear feedback $u = \alpha(x) + \beta(x)v$ such that the transformed state $y = D(x)$ satisfies the linear system $\dot{y} = Ay + Bv$, where (A, B) is a controllable linear system with

$$A = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & I \\ 0 & 0 & . & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ I \end{bmatrix}$$

and I is $n \times n$ identity matrix, 0 is $n \times n$ zero matrix, $y \in \mathbb{R}^{4n}$, $v \in \mathbb{R}$. A **diffeomorphism** is simply a differentiable map whose inverse exists and is also differentiable. We have to look at the diffeomorphism D as a nonlinear change of coordinates in the state space. The idea of the feedback linearization is that if we first change the coordinate system $y = D(x)$, then there exists a nonlinear control law which cancels the system nonlinearities. If the region U is all of R^n, then the feedback linearization is said to be global.

Theorem 7. (*See* [123]) *The nonlinear system* $(*)$, *where* $f(x)$ *and* $g(x)$ *are smooth vector fields and* $f(0) = 0$ *is feedback linearizable if and only if there exists a region* U *containing the origin in* $\mathbb{R}^n$, *where the following conditions are satisfied*:

i) *The vector fields* $g, L_f(g), \ldots, L_f^{n-1}(g)$ *are linearly independent in* U.

ii) *The set* $g, L_f(g), \ldots, L_f^{n-2}(g)$ *is involutive in* U.

8. An Approach to Automatic Generation of Dynamic Equations in Vector-Parameters of Elastic Joint Manipulators in Symbolic Language

The present section describes an approach for automatic generation of the equations of motion of elastic joint manipulators in symbolic language [67], [101], [88], [89]. It is based on a vector-parameterization of the Lie group SO(3) and uses the Lagrange's formalism to derive the dynamical equations, whose final forms are like in the recursive Newton-Eulerian algorithm. Both characteristics together, the first one – on kinematic level and the second one – on dynamic level, increase the computational efficiency. This makes the suggested algorithm quite appropriate as for the purposes of modeling and identification, so for real time simulations and control.

The automatic symbolic generation of dynamic equations of manipulators in a closed form is a topic that has received a very extensive research and is therefore very well developed. Most algorithms to this effect are based on either the Newton-Eulerian formalism [113], [108], [30], [138], [134] [19], [51], [130], [102], [105], [114], [47], [48], [46], [144] or the Lagrangian dynamics [127], [128], [40], [136], [32], [17], [137] [58]. Almost all these algorithms concentrate on the development of rigid body models. This is inspite of the considerable attention that control of elastic joint manipulators has been given in the last two decades or more, where it has generally been confirmed that elastic joint manipulators can not be controlled by a procedure that is designed for robots with rigid joints. Some works in the area of control of elastic joint robots therefore use, under special assumptions, a simple model of the actuator, neglecting the coupling moments betweens the actuators and the links [121]. According to the authors knowledge, the only program, that has also been used in this area, and that can generate the full dynamic model of manipulators with elastic joints, is the DYMIR [16], [54].

The model presented in this section is based on the methods given in [51] and [17], but has the specific and original feature, that it is built for elastic joint manipulators, although it can be also used for rigid manipulators. In addition, the modified Denavit-Hartenberg convention [19] to describe the Euclidean motions is used in this section, while the above mentioned works apply the approach suggested by Denavit and Hartenberg [21]. Finally, this section differs in the important item, that it uses the so called vector-parameters. Both the computational efficiency of the vector-parameterization and the effective generation of the dynamic equations make the suggested approach quite appropriate for on–line simulation and control. In order to improve further the computational efficiency, the positive characteristics of the symbolic–numeric programming, as pointed out in [130], have been used during the realization of this algorithm.

8.1. Geometrical Description

We consider a general structure of an industrial robot with N serial links. If the joints are elsatic, then the position of the i-th actuator and that of the i-th link will be different (Fig.4.) and each joint-actuator combination can be seen as two different subsystems coupled by a torsional spring. The effective number of degrees of freedom for a robot with N links and N elastic joints is then $2N$. To derive the equations of motion, the modified Denavit-Hartenberg convention [19] is used here as to assign a coordinate frame to each body, so to determine the position and orientation of the actuators. The following general assumptions are added to this convention in order to take the elasticity of each joint into account:

- The links are marked with the indices $1, .., i, ..N$, and the movable frames and the variables with $1, ..., s, ..., 2N$ (including the position of the end-effector).
- The inertial frame has index 0.
- The i-th actuator has index $s = 2i - 1$ and the i-th link index $s = 2i$ for $i = 1, ..., N$.
- The index $l = 2i - 2$ is used, when $s = 2i - 1$ and $s = 2i$.
- The z-axes of the frames $2i - 1$ and $2i$ coincide.
- Each link has its own actuator whose rotor is modelled as an uniform body of revolution with the axis of symmetry along the axis of rotation (z-axis).
- The joint elasticity is modelled as a linear torsional spring with a spring constant k_i.
- For the variables, the generalized coordinate q_s is used for each degree of freedom. The variable for the motion of the rotor of the i-th actuator is q_{2i-1} with respect to the l-th frame and that of the i-th link is q_{2i} also with respect to the l-th frame. If the link i has a revolute joint, then q_{2i} is the rotational angle of the frame $2i$ around its z-axis and if the link i is prismatic, then q_{2i} is d_{2i} along the z-axis. As mentioned above, all rotors of the actuators have revolute degrees of freedom. According to the convention mentioned above, $q_s = \vartheta_s$, when the joint s is rotational and $q_s = d_s$, if it is translational.

8.2. Kinematics of Elastic Joint Manipulators

For describing the Euclidean motions, we use a modified Denavit-Hartenberg convention, in which, the position, presented by 3×1 vector p and the orientation, presented by 3×3 matrix R, of the frame s with respect to the frame l and the frame 0, are given by the following 4×4 transformation matrices

$$ {}^{l}A_s = \begin{bmatrix} c\vartheta_s & -s\vartheta_s & 0 & a_l \\ s\vartheta_s c\alpha_l & c\vartheta_s c\alpha_l & -s\alpha_l & -s\alpha_l d_s \\ s\vartheta_s s\alpha_l & c\vartheta_s \alpha_l & c\alpha_l & c\alpha_l d_s \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{l}R_s & {}^{l}p_s \\ \bar{0} & 1 \end{bmatrix} \tag{309} $$

where $c = \cos$ and $s = \sin$, $\bar{0}$ is 1×3 zero vector and

$$ {}^{0}T_s = {}^{0}A_2 {}^{2}A_4 \ldots {}^{l-2}A_l {}^{l}A_s = \begin{bmatrix} {}^{0}R_s & {}^{0}p_s \\ \bar{0} & 1 \end{bmatrix} \tag{310} $$

is the resultant transformation matrix. Note that a_l and α_l are the same for both the actuator and the link and

$$^0R_s = {}^lR_s{}^0R_l, \qquad {}^0p_s = {}^0p_l + {}^0R_l{}^lp_s. \tag{311}$$

As mentioned before, the orientation matrix ${}^lR_s$ is expressed here in terms of vector parameters as in (77), that is

$$^lR_s = O(c_s), \qquad c_s = (\tan\frac{\alpha_l}{2},\ -\tan\frac{\alpha_l}{2}\tan\frac{\vartheta_s}{2},\ \tan\frac{\vartheta_s}{2}). \tag{312}$$

Then the multiplication of the orientation matrices is replaced by the composition of the vector parameters as follows

$$\begin{aligned} g_1 &\equiv c_1, \qquad g_2 \equiv c_2, \\ g_3 &= \langle g_2, c_3 \rangle, \qquad g_4 = \langle g_2, c_4 \rangle, \qquad g_s = \langle g_l, c_s \rangle \end{aligned} \tag{313}$$

in which $g_l = \langle c_1, c_s, ..., c_l \rangle$ for $s = 2i$.

The angular and linear velocities of a point p with a position vector ${}^sr_s$ (fixed) on link s expressed with respect to frame l are

$$^l\omega_s = {}^lR_s[0\ 0\ \dot{q}_s]^T \tag{314}$$

$$^lv_s = \frac{\partial^l A_s(q_s)}{\partial q_s}\dot{q}_s{}^sr_s = \begin{bmatrix} {}^l\tilde{\omega}_s{}^lR_s & {}^l\dot{p}_s \\ \bar{0} & 0 \end{bmatrix}\begin{bmatrix} {}^sr_s \\ 1 \end{bmatrix} = {}^l\tilde{\omega}_s{}^sr_s + {}^l\dot{p}_s. \tag{315}$$

Since the transformation matrices A and T, as defined above, belong to the Euclidean Lie group, that describes the motion of a rigid body in the three-dimensional space, (315) can be rewritten as

$$\frac{\partial^l A_s}{\partial q_s}\dot{q}_s = \begin{bmatrix} {}^l\tilde{\omega}_s{}^lR_s & {}^l\dot{p}_s \\ \bar{0} & 0 \end{bmatrix} = \begin{bmatrix} {}^l\tilde{\delta}_s{}^lR_s & {}^l\tau_s \\ \bar{0} & 0 \end{bmatrix}\dot{q}_s \tag{316}$$

where

$$^l\tilde{\delta}_s = {}^lR_s{}^s\tilde{\delta}_s{}^lR_s^T \qquad \text{or} \qquad {}^l\tilde{\delta}_s = {}^lR_s{}^s\delta_s \qquad \text{and} \qquad {}^l\tau_s = {}^lR_s{}^s\tau_s. \tag{317}$$

The matrices

$$^s\tilde{\delta}_s = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad {}^s\tau_s = [0\ 0\ 0]^T \qquad \text{if } s \text{ has a rotational joint} \tag{318}$$

$$^s\tilde{\delta}_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad {}^s\tau_s = [0\ 0\ 1]^T \qquad \text{if } s \text{ has a prismatic joint} \tag{319}$$

are the infinitesimal generators of such a group of motions consisting of rotations about and translations along the z-axis. From (316) it can be seen that

$$\frac{\partial^l A_s}{\partial q_s} = {}^l\Delta_s = \begin{bmatrix} {}^l\tilde{\delta}_s & {}^l\tau_s \\ \bar{0} & 0 \end{bmatrix} = \begin{bmatrix} {}^l\tilde{\delta}_s & {}^l\lambda_s \\ \bar{0} & 0 \end{bmatrix} {}^l A_s \tag{320}$$

in which

$${}^l\lambda_s = -{}^l\tilde{\delta}_s{}^l p_s + {}^l\tau_s = -{}^s\tilde{\delta}_s{}^l R_s{}^{Tl} p_s + {}^s\tau_s. \tag{321}$$

It is worth noting here, that because of the properties of the orthogonal matrices as elements of the group $\mathrm{SO}(3)$, namely

$${}^l R_s^T = {}^l R_s^{-1} = {}^s R_l \tag{322}$$

as well as the vector-parameterization of $\mathrm{SO}(3)$ and equation (312), it is valid that

$${}^s R_l = O(-c_s). \tag{323}$$

Using the above equations and the inverse of the transformation matrix ${}^0T_s$

$$[{}^0T_s]^{-1} = \begin{bmatrix} {}^0R_s^T & -{}^0R_s^{T0} p_s \\ \bar{0} & 1 \end{bmatrix}$$

the linear velocity of frame s with respect to the base frame 0 is

$$\begin{aligned} {}^0 v_s &= {}^0\dot{T}_s = \sum_{k=1}^{i} \frac{\partial^0 T_s}{\partial q_{2k}} \dot{q}_{2k} + \frac{\mathrm{d}^0 T_s}{\mathrm{d}q_s} \dot{q}_s = (\sum_{k=1}^{i} {}^0\Delta_{2k}\dot{q}_{2k} + {}^0\Delta_s \dot{q}_s)^0 T_s \\ &= \left(\sum_{k=1}^{i} \begin{bmatrix} {}^0\tilde{\delta}_{2k}\dot{q}_{2k} & {}^0\lambda_{2k}\dot{q}_{2k} \\ \bar{0} & 0 \end{bmatrix} + \begin{bmatrix} {}^0\tilde{\delta}_s \dot{q}_s & 0\lambda_s \dot{q}_s \\ \bar{0} & 0 \end{bmatrix} \right) {}^0 T_s \\ &= \begin{bmatrix} {}^0\tilde{\omega}_s & {}^0 u_s \\ \bar{0} & 0 \end{bmatrix} {}^0 T_s, \qquad i = l/2 \end{aligned} \tag{324}$$

where

$${}^0\tilde{\delta}_{2k} = {}^0R_{2k}{}^{2k}\tilde{\delta}_{2k}{}^0R_{2k}^T \tag{325}$$

$${}^0\lambda_{2k} = {}^0R_{2k}{}^{2k}\lambda_{2k} = {}^0\tilde{\delta}_{2k}{}^0 p_{2k} + {}^0\tau_{2k} \tag{326}$$

$${}^0 u_s = \sum_{k=1}^{i} {}^0\lambda_{2k} + {}^0\lambda_s \dot{q}_s = {}^0 u_{s-1} + {}^0\lambda_s \dot{q}_s, \qquad i = l/2 \tag{327}$$

$${}^0\tilde{\omega}_s = \sum_{k=1}^{i} {}^0\tilde{\delta}_{2k}\dot{q}_{2k} + {}^0\tilde{\delta}_s \dot{q}_s = {}^0\tilde{\omega}_l + {}^0\tilde{\delta}_s \dot{q}_s, \qquad i = l/2. \tag{328}$$

The second derivative of ${}^0T_s$ is obtained from (324) as

$$ {}^0\ddot{T}_s = \begin{bmatrix} {}^0\dot{\tilde{\omega}}_s + {}^0\tilde{\omega}_s^2 & {}^0\dot{u}_s + {}^0\tilde{\omega}_s {}^0u_s \\ \bar{0} & 0 \end{bmatrix} {}^0T_s. \tag{329} $$

According to the above relations we have

$$ {}^0\dot{u}_s = {}^0\dot{u}_l + {}^0\dot{\lambda}_s\dot{q}_s + {}^0\lambda_s\ddot{q}_s \tag{330} $$

$$ {}^0\dot{\omega}_s = {}^0\dot{\omega}_l + {}^0\dot{\delta}_s\dot{q}_s + {}^0\delta_s\ddot{q}_s \tag{331} $$

respectively, the expressions for ${}^0\dot{\delta}_s$ and ${}^0\dot{\lambda}_s$ are

$$ {}^0\dot{\delta}_s = {}^0\omega_l \times {}^0\delta_s \tag{332} $$

$$ {}^0\dot{\lambda}_s = -({}^0\dot{\tilde{\delta}}_s {}^0p_s + {}^0\tilde{\delta}_s {}^0\dot{p}_s) + {}^0\omega_s \times {}^0\tau_s \tag{333} $$

and ${}^0\dot{p}_s$ is determined from

$$ {}^0\dot{T}_s = \begin{bmatrix} {}^0\tilde{\omega}_s {}^0R_s & {}^0\dot{p}_s \\ \bar{0} & 0 \end{bmatrix} = \begin{bmatrix} {}^0\tilde{\omega}_s + {}^0R_s & {}^0\tilde{\omega}_s {}^op_s + {}^ou_s \\ \bar{0} & 0 \end{bmatrix} \tag{334} $$

as

$$ {}^0\dot{p}_s = {}^0\omega_l \times {}^0p_s + {}^0u_s. \tag{335} $$

The substitution of (335) in (332) and using the vector triple product identity $A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0$ yields

$$ {}^0\dot{\lambda}_s = {}^0\omega_l \times {}^0\lambda_s - {}^0\delta_s \times {}^0u_s. \tag{336} $$

Finally, the following expressions for ${}^0\dot{\omega}_s$ and ${}^0\dot{u}_s$ are obtained

$$ {}^0\dot{\omega}_s = {}^0\dot{\omega}_l + {}^0\omega_l \times {}^0\delta_s\dot{q}_s + {}^0\delta_s\ddot{q}_s \tag{337} $$

$$ {}^0\dot{u}_s = {}^0\dot{u}_l + {}^0\omega_s \times {}^0\lambda_s\dot{q}_s + {}^0\lambda_s\ddot{q}_s \tag{338} $$

In case $\ddot{q}_s = 0$, it is valid that

$$ {}^0\dot{\omega}_s^* = {}^0\dot{\omega}_l^* + {}^0\omega_l \times {}^0\delta_s\dot{q}_s \tag{339} $$

$$ {}^0\dot{u}_s^* = {}^0\dot{u}_l^* + {}^0\omega_s \times {}^0\lambda_s\dot{q}_s + {}^0u_s \times {}^0\delta_s\dot{q}_s. \tag{340} $$

The equations given above in this section are the kinematic equations of frame s with respect to the base frame 0. They are recursively determined from frame 1 to frame $2N$. Computationally more efficient versions of these equations are obtained by expressing them with respect to their own coordinate frames. This is given in the following.

Outward Iterations (in joint coordinates) $\quad s = 1, \ldots, 2N$

The main relations here are:

$$
\begin{aligned}
&{}^0R_s = {}^0R_l{}^lR_s, && {}^sp_s = {}^lR_s({}^lp_l + {}^lp_s) && (341)\\
&{}^s\lambda_s = \begin{cases} -{}^s\delta_s{}^lR_s^{T\,l}p_s \\ {}^s\tau_s \end{cases} && \begin{matrix}\text{if joint } s \text{ is rotational} \\ \text{if joint } s \text{ is prismatic}\end{matrix} && (342)\\
&{}^su_s = \begin{cases} {}^lR_s^T({}^lu_l + {}^lp_s \times {}^s\delta_s\dot{q}_s) \\ {}^lR_s^{T\,l}u_l + {}^s\tau_s\dot{q}_s \end{cases} && \begin{matrix}\text{if joint } s \text{ is rotational} \\ \text{if joint } s \text{ is prismatic}\end{matrix} && (343)\\
&{}^s\omega_s = \begin{cases} {}^lR_s^{T\,l}\omega_l + {}^s\delta_s\dot{q}_s \\ {}^lR_s^{T\,l}\omega_l \end{cases} && \begin{matrix}\text{if joint } s \text{ is rotational} \\ \text{if joint } s \text{ is prismatic}\end{matrix} && (344)\\
&{}^sv_s = \begin{cases} {}^lR_s^T({}^lv_l + {}^l\omega_l \times {}^lp_s) \\ {}^lR_s^T({}^lv_l + {}^l\omega_l \times {}^lp_s) + s\tau_s\dot{q}_s \end{cases} && \begin{matrix}\text{if joint } s \text{ is rotational} \\ \text{if joint } s \text{ is prismatic.}\end{matrix} && (345)
\end{aligned}
$$

Finally, it follows that

$$
\begin{aligned}
{}^s\dot{\omega}_s^* &= {}^lR_s^{T\,l}\omega_l^* + {}^lR_s^{T\,l}\omega_l \times {}^s\delta_s\dot{q}_s && (346)\\
{}^s\dot{u}_s^* &= {}^lR_s^{T\,l}\dot{u}_l^* + {}^s\omega_s \times {}^s\lambda_s\dot{q}_s + {}^su_s \times {}^s\delta_s\dot{q}_s. && (347)
\end{aligned}
$$

8.3. Dynamics of Elastic Joint Manipulators

The dynamic equations of an elastic joint manipulator are derived on the basis of the Lagrange's equations of second order

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial W_k}{\partial \dot{q}_s}\right) - \frac{\partial W_k - W_p}{\partial q_s} = F_{qs} \qquad (348)$$

where W_k and W_p are the kinetic and potential energy of the body and q_s and F_{qs} are the generalized coordinate and applied force corresponding to the s-th degree of freedom respectively. For a robot with elastic joints, the potential energy consist of the elastic energy W_{pe} and the gravitational energy W_{pg}. The elasticity of each joint i is modeled here by a linear torsional spring of a constant stiffness k_i so that

$$W_{pe} = \frac{1}{2}k_i\,(q_{2i} - q_{2i-1}/N_i). \qquad (349)$$

Here N_i is the gear ratio for the i-th set of the reduction gear. The presence of elasticity at each joint of a manipulator with N-links, doubles the number of degrees of freedom of the manipulator to $2N$. The s-th dynamic equation of motion of an elastic joint robot with N-links, derived from the above equations is

$$\sum_{k=1}^{2N} B_{sk}\ddot{q}_k + C_s(q,\dot{q}) - G_s(q) + E_s(q) \;=\; U_s \tag{350}$$

where B_{sk} is the coupling inertia matrix between bodies s and l and it seems like

$$B_{sk} \;=\; \sum_{j=k}^{2N} \mathrm{tr}\left[\frac{\partial^0 T_j}{\partial q_s} J_j \frac{\partial^0 T_j^T}{\partial q_k}\right], \qquad k \geq s = 1, ..., 2N \tag{351}$$

and C_s is a vector which includes the Coriolis's and the centrifugal forces on body s as expressed according to [32]

$$C_s = \sum_{j=s}^{2N} \mathrm{tr}\left[\frac{\partial^0 T_j}{\partial q_s} J_j \frac{\mathrm{d}^{20} T_j^T}{\mathrm{d}t^2}\right]_{\ddot{q}=0}, \qquad 1 \leq s \leq 2N \tag{352}$$

G_s is a vector of the gravitational force acting on body s

$$G_s \;=- \sum_{j=s}^{2N} m_j g^T \frac{\partial^0 T_j}{\partial q_s} {}^s r_{sg}, \qquad 1 \leq s \leq 2N \tag{353}$$

and E_s are the elastic forces at either side of the gear of joint i

$$E_s = \begin{cases} -\frac{k_i}{N_i}(q_{2i} - q_{2i-1}/N_i), & \text{if } s = 2i-1, \qquad (i = 1, \ldots, N) \\ k_i(q_{2i} - q_{2i-1}/N_i), & \text{if } s = 2i. \end{cases} \tag{354}$$

The inertia matrix

$$J_j \;=\; \begin{bmatrix} {}^j I_{jg} + m_j {}^j r_{jg} {}^j r_{jg}^T & m_j {}^j r_{jg} \\ m_j {}^j r_{jg}^T & m_j \end{bmatrix} \tag{355}$$

consists of the position vector ${}^j r_{jg}$ of the center of mass, the mass m_j of link or actuator s and the matrix

$${}^j I_{jg} \;=\; \begin{bmatrix} a & {}^j I_{jxy} & {}^j I_{jxz} \\ {}^j I_{jxy} & b & {}^j I_{yz} \\ {}^j I_{xz} & {}^j I_{yz} & c \end{bmatrix} \tag{356}$$

where

$$\begin{aligned} a &= \frac{-{}^j I_{jxx} + {}^s I_{jyy} + {}^s I_{jzz}}{2} \\ b &= \frac{{}^j I_{jxx} - {}^j I_{jyy} + {}^j I_{jzz}}{2} \\ c &= \frac{{}^j I_{jxx} + {}^j I_{jyy} - {}^j I_{jzz}}{2}. \end{aligned} \tag{357}$$

Here ${}^{j}I_{jxx}$, ${}^{j}I_{jyy}$, ${}^{j}I_{jzz}$ and ${}^{j}I_{jxy}$, ${}^{j}I_{jxz}$, ${}^{j}I_{jyz}$ are the moments and products of inertia of the body s determined about a rotational axis through the mass center of the body. All the terms in J_j are expressed with respect to their own frame. In order to determine the elements B_{sk}, C_s and G_s in (350), in view of the presence of joint elasticity, (351) is representatively analyzed using Table 4.

Table 4. The elements B_{sk} of the inertia matrix B for $s = 1$

	$k = 1$	$k = 2$	$k = 3$	...
$s = 1, j = 1:$	$\mathrm{tr}\frac{\partial^0 T_1}{\partial q_1} J_1 \frac{\partial^0 T_1^T}{\partial q_1}$			
$s = 1, j = 2:$	$\mathrm{tr}\frac{\partial^0 T_2}{\partial q_1} J_2 \frac{\partial^0 T_2^T}{\partial q_1}$	$\mathrm{tr}\frac{\partial^0 T_2}{\partial q_1} J_2 \frac{\partial^0 T_2^T}{\partial q_2}$		
$s = 1, j = 3:$	$\mathrm{tr}\frac{\partial^0 T_3}{\partial q_1} J_3 \frac{\partial^0 T_3^T}{\partial q_1}$	$\mathrm{tr}\frac{\partial^0 T_3}{\partial q_1} J_3 \frac{\partial^0 T_3^T}{\partial q_2}$	$\mathrm{tr}\frac{\partial^0 T_3}{\partial q_1} J_3 \frac{\partial^0 T_3^T}{\partial q_3}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$s = 1, j = 2N$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B_{sk}:	B_{11}	B_{12}	B_{13}	$\cdots$

Since index l in the transformation matrix ${}^{l}A_s$ is always even, because it refers to the previous link, the following points should be noted:

1) if j is even, then ${}^{0}T_j$ is not a function of any odd variable $q_s, \quad s = 2i - 1$.

2) if j is odd, then in ${}^{0}T_j$ only ${}^{l}A_j$ is a function of the odd variable q_j. All its other variables are even. This implies that,

3) if k is even, $\; k \geq s = 1, \cdots, 2N$, then

$$\left[\frac{\partial^o T_j}{\partial q_s} J_j \frac{\partial^0 {T_j}^T}{\partial q_k}\right] = 0 \qquad \text{if } s \text{ is odd.} \tag{358}$$

4) if k is odd, $\; k \geq s = 1, \ldots, 2N$, then

$$B_{sk} = \begin{cases} \mathrm{tr}\left[\frac{\partial^0 T_j}{\partial q_s} J_j \frac{\partial^0 {T_j}^T}{\partial q_k}\right] & \text{if } s = j = k, \\ & \text{or } \; (j = k \text{ and } s \text{ is even}) \\ 0 \quad \text{if } s \text{ is odd and } s < k. & \end{cases} \tag{359}$$

Considering a six degree-of freedom manipulator, the dynamical system (350) contains 12 nonlinear differential equations. After multiplying (350) with a permutation matrix P, where $P^{-1} = P^T$, the first six equations are for the actuator dynamics and the second for the dynamics of the links. The resultant matrix B

consists of 6×6 block matrices ${B^*}_{11},\ {B^*}_{12},\ {B^*}_{21},\ {B^*}_{22}$ and it has the following structure

$$B = \left[\begin{array}{cccccccccccccc} b_{11} & 0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 & 0 & . & b_{27} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 & 0 & . & b_{37} & b_{38} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 & 0 & . & b_{47} & b_{48} & b_{49} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} & 0 & . & b_{57} & b_{58} & b_{59} & b_{510} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{66} & . & b_{67} & b_{68} & b_{69} & b_{610} & b_{611} & 0 \\ \multicolumn{13}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ 0 & b_{27} & b_{37} & b_{47} & b_{57} & b_{67} & . & b_{77} & b_{78} & b_{79} & b_{710} & b_{711} & b_{712} \\ 0 & 0 & b_{38} & b_{48} & b_{58} & b_{68} & . & b_{87} & b_{88} & b_{89} & b_{810} & b_{811} & b_{812} \\ 0 & 0 & 0 & b_{49} & b_{59} & b_{69} & . & b_{97} & b_{98} & b_{99} & b_{910} & b_{911} & b_{912} \\ 0 & 0 & 0 & 0 & b_{510} & b_{610} & . & b_{107} & b_{108} & b_{109} & b_{1010} & b_{1011} & b_{1012} \\ 0 & 0 & 0 & 0 & 0 & b_{611} & . & b_{117} & b_{118} & b_{119} & b_{1110} & b_{1111} & b_{1112} \\ 0 & 0 & 0 & 0 & 0 & 0 & . & b_{127} & b_{128} & b_{129} & b_{1210} & b_{1211} & b_{1212} \end{array}\right] \tag{360}$$

in which the symmetry is maintained. A similar analysis is done for the vectors C_s und G_s.

Algorithms for B, C and G

Using the kinematic equations from the last section and taking (358) and (359) into account, an algorithm for an efficient computation of the terms B_{sk}, C_s and G_s given in equations (351)-(353) for manipulators with elastic joints is derived in this section. With the help of (324), (351) can be rewritten as

$$\begin{aligned} B_{sk} &= \mathrm{tr} \sum_{j=k}^{2N} {}^0\Delta_s {}^0T_j J_j {}^0{T_j}^T {}^0{\Delta_k}^T \;=\; {}^0\Delta_s \begin{bmatrix} L_k & {}^0H_k \\ {}^0{H_k}^T & M_k \end{bmatrix} {}^0{\Delta_k}^T \\ &= \mathrm{tr}\Big\{ {}^0\tilde{\delta}_s \big(L_k {}^0{\tilde{\delta}_k}^T +^0 H_k {}^0{\lambda_k}^T \big) +^0 \lambda_s \big({}^0H_k {}^0{\tilde{\delta}_k}^T + M_k {}^0{\lambda_k}^T \big) \Big\} \\ & \quad k \geq s = 1, \dots, 2N \end{aligned} \tag{361}$$

where, from (355) we get

$$L_k = \sum_{j=k}^{2N} \Big\{ {}^0{R_j}^j I_{jg} {}^0{R_j}^T + m_j \big({}^0{R_j}^j r_{jg} +^0 p_j \big) \big({}^0{R_j}^j r_{jg} +^0 p_j \big)^T \Big\} \tag{362}$$

$${}^0H_k = \sum_{j=k}^{2N} m_j \big({}^0{R_j}^j r_{jg} +^0 p_j \big), \qquad M_k = \sum_{j=k}^{2N} m_j. \tag{363}$$

Using the relationship between vector arrays and dyadic arrays, (361) may be presented in the following form

$$B_{sk} = {}^0\delta_s{}^T\left({}^0K_k{}^0\delta_k +{}^0 H_k \times {}^0\lambda_k{}^T\right) + {}^0\lambda_s{}^T\left({}^0\delta_k \times^0 H_k + M_k{}^0\lambda_k\right) \quad (364)$$

in which the elements

$${}^0K_k = \sum_{j=k}^{2N}\left({}^0I^*{}_{jg} + m_j{}^0\tilde{r^*}_{jg}{}^0\tilde{r^*}_{jg}{}^T\right), \qquad {}^0r^*{}_{jg} = {}^0R_j{}^j r_{jg} +{}^0 p_j \quad (365)$$

are the effective inertia matrix and

$${}^0I^*{}_{jg} = {}^0R_j{}^jI^*{}_{jg}{}^0R_j{}^T = \sum_{i=1}^{\infty}{}^0\tilde{r}_{ij}{}^0\tilde{r}_{ij}{}^T m_i \quad (366)$$

is the inertia tensor of the body j, where r_{ij} is the position vector of a particle i with a mass m_i on the body j. The vector C_s, for the Coriolis's and centrifugal forces, can also be rewritten using the same method. After substituting (320) and (329) in (352), it follows

$$C_s = \operatorname{tr}\sum_{j=s}^{2N}{}^0\Delta_s{}^0T_jJ_j{}^0T_j{}^T\begin{bmatrix} {}^0\dot{\tilde{\omega}}{}^*_j + {}^0\tilde{\omega}_j{}^2 & {}^0\dot{u}^*{}_j +{}^0 \tilde{\omega}_j{}^0u_j \\ \bar{0} & 0 \end{bmatrix}^T, \ 1 \le s \le 2N. \quad (367)$$

Because of the same relationship between vector arrays and dyadic arrays, the vectors C_s may be expressed as

$$C_s = {}^0\delta_s{}^{T0}N_s + {}^0\lambda_s{}^{T0}D_s \quad (368)$$

where

$${}^0N_s = \sum_{j=s}^{2N}\left\{{}^0I^*{}_{jg}{}^0\dot{\omega}^*{}_j +{}^0 \tilde{\omega}_j\left({}^0I^*{}_{jg}{}^0\omega_j\right)\right\} + {}^0D^*{}_s \quad (369)$$

with

$${}^0D^*{}_s = \sum_{j=s}^{2N} m_j{}^0r^*{}_{jg}\times\left[{}^0\dot{u}^*{}_j+{}^0\dot{\omega}^*{}_j\times{}^0r^*{}_{jg}+{}^0\omega_j\times\left({}^0u_j+{}^0\omega_j\times{}^0r^*{}_{jg}\right)\right] \quad (370)$$

and

$${}^0D_s = \sum_{j=s}^{2N} m_j\left[{}^0\dot{u}^*{}_j +{}^0 \dot{\omega}^*{}_j \times {}^0r^*{}_{jg} +{}^0 \omega_j \times \left({}^0u_j +{}^0 \omega_j \times {}^0r^*{}_{jg}\right)\right]. \quad (371)$$

The gravity force G_s is similarly obtained by substituting (318) in (353)

$$\begin{aligned} G_s &= -\big[g^T \;\; 0\big]{}^0\Delta_s \sum_{j=s}^{2N} \begin{bmatrix} m_j{}^0r^*{}_{jg} +{}^0 p_j \\ 1 \end{bmatrix} \\ &= -g^T\big({}^0\delta_s \times {}^0H_s + M_s{}^0\lambda_s\big), \qquad 1 \le s \le 2N. \end{aligned} \tag{372}$$

By expressing the kinematic variables and functions in the equations (364)–(372) in joint coordinates, the following set of recursive relations for the dynamical matrices in joint coordinates can be easily derived:

Inward Iterations (in joint coordinates) $k \geq s = 1, \ldots, 2N$

$$ {}^kK_k = \begin{cases} {}^kI^*{}_{kg} + m_k\big({}^kr^*{}_{kg}{}^T{}^kr^*{}_{kg}I - {}^kr^*{}_{kg}{}^kr^*{}_{kg}{}^T\big) & \text{if } k \text{ is odd} \\ {}^kK_{k+2} + {}^kI^*{}_{kg} + m_k\big({}^kr^*{}_{kg}{}^T{}^kr^*{}_{kg}I - {}^kr^*{}_{kg}{}^kr^*{}_{kg}{}^T\big) & \text{if } k \text{ is even} \end{cases} \tag{373}$$

$$ {}^{k-2}K_k = {}^{k-2}R_k{}^kK_k{}^{k-2}R_k{}^T + {}^{k-2}R_{k+1}{}^{k+1}K_{k+1}{}^{k-2}R_{k+1}{}^T \qquad \text{if } k \text{ is even} \tag{374}$$

$\big({}^{2N+2}K_{2N+2},\, {}^{2N+1}K_{2N+1} = 0\big)$

$$ {}^kH_k = \begin{cases} m_k\big({}^kr_{kg} +{}^k p_k\big) & \text{if } k \text{ is odd} \\ {}^kH_{k+2} + m_k\big({}^kr_{kg} +{}^k p_k\big), & \text{if } k \text{ is even} \end{cases} \tag{375}$$

$$ {}^{k-2}H_k = {}^{k-2}R_k{}^kH_k + {}^{k-2}R_{k+1}{}^{k+1}H_{k+1} \qquad \text{if } k \text{ is even} \tag{376}$$

$\big({}^{2N+2}H_{2N+2},\, {}^{2N+1}H_{2N+1} = 0\big)$

$$ M_k = \begin{cases} m_k & \text{if } k \text{ is odd} \\ M_{k+2} + m_k + m_{k+1}, \;\; (M_{2N+2} = 0,\, m_{2N+1} = 0) & \text{if } k \text{ is even} \end{cases} \tag{377}$$

$$ {}^kX_k = {}^kK_k{}^k\delta_k +{}^k H_k \times{}^k \lambda_k \;\; {}^kY_k ={}^k \delta_k \times{}^k H_k + M_k{}^k\lambda_k \tag{378}$$

$$ B_{kk} = {}^k\delta_k{}^T{}^kX_k + {}^k\lambda_k{}^T{}^kY_k \tag{379}$$

$$ {}^kN_k = \begin{cases} {}^kI^*{}_{kg}{}^k\dot{\omega}^*{}_k +^k \omega_k \times \left({}^kI^*{}_{kg}{}^k\omega_k\right) + m_k{}^kr^*{}_{kg}\times \\ \left[{}^k\dot{u}^*{}_k +^k \dot{\omega}^*{}_k \times {}^kr^*{}_{kg} +^k \omega_k \times \left({}^ku_k +^k \omega_k \times {}^kr^*{}_{kg}\right)\right] & \text{if } k \text{ is odd} \\ \\ {}^kN_{k+2} + \left({}^kI^*{}_{kg}{}^k\dot{\omega}^*{}_k +^k \omega_k \times \left({}^kI^*{}_{kg}{}^k\omega_k\right) + m_k{}^kr^*{}_{kg}\times \right. \\ \left[{}^k\dot{u}^*{}_k +^k \dot{\omega}^*{}_k \times {}^kr^*{}_{kg} +^k \omega_k \times \left({}^ku_k +^k \omega_k \times {}^kr^*{}_{kg}\right)\right] & \text{if } k \text{ is even} \end{cases} \tag{380} $$

$$ {}^{k-2}N_k = {}^{k-2}R_k{}^kN_k + {}^{k-2}R_{k+1}{}^{k+1}N_{k+1} \qquad \text{if } k \text{ is even} \tag{381} $$

$\left({}^{2N+2}N_{2N+2},\ {}^{2N+1}N_{2N+1} = 0\right)$

$$ {}^kD_k = \begin{cases} m_k\left[{}^k\dot{u}^*{}_k +^k \dot{\omega}^*{}_k \times {}^kr^*{}_{kg} +^k \omega_k \times \left({}^0u_k + \right.\right. \\ \left.\left.{}^k\omega_k \times {}^kr^*{}_{kg}\right)\right] & \text{if } k \text{ is odd} \\ \\ {}^kD_{k+2} + m_k\left[{}^k\dot{u}^*{}_k +^k \dot{\omega}^*{}_k \times {}^kr^*{}_{kg} +^k \omega_k\times \right. \\ \left.\left({}^0u_k +^k \omega_k \times {}^kr^*{}_{kg}\right)\right] & \text{if } k \text{ is even} \end{cases} \tag{382} $$

$$ {}^{k-2}D_k = {}^{k-2}R_k{}^kD_k + {}^{k-2}R_{k+1}{}^{k+1}DN_{k+1} \qquad \text{if } k \text{ is even} \tag{383} $$

$\left({}^{2N+2}D_{2N+2},\ {}^{2N+1}D_{2N+1} = 0\right)$

$$ C_k = {}^k\delta_k{}^{T\,k}N_k + {}^k\lambda_k{}^{T\,k}D_k \tag{384} $$

$$ G_s = -{}^kg^{T\,k}Y_k \tag{385} $$

$$ {}^{k-2}g = {}^{k-2}R_k{}^{T\,k}g,\ ({}^0g = g), \qquad \text{if } k \text{ is even} \tag{386} $$

$$ {}^sX_k = {}^lR_s \cdots {}^{k-2}R_k{}^kX_k, \qquad {}^sY_k = {}^lR_s \cdots {}^{k-2}R_k{}^kY_k \tag{387} $$

$$ B_{sk} = {}^s\delta_s{}^{T\,s}X_k + {}^s\lambda_s{}^{T\,s}Y_k. \tag{388} $$

The algorithm presented in this section is derived using Lagrange's equations of motion, although the final form of its equations is as those given in [19] and [113], where the Newton-Euler formalism is used. In this way, a good insight into the dynamics of the mechanical system is achieved and at the same time terms like the inertia matrix are explicitly obtained. This algorithm is therefore a very suitable starting point for the identification of these terms. Differentiations of functions are avoided and the multiplications of skew-symmetric matrices are substituted by vector products. It is worth mentioning also, that, in order to avoid multiplications with and additions to zero, explicit expressions for prismatic and revolute joints are used during the implementation of the procedure, where the symbolic language REDUCE has been employed that gives an output program in FORTRAN codes. The applications of the output program for control design have given very satisfying results.

9. Control Problems of the System of Mobile Platform and Manipulator

9.1. Specification of the Task

The present section considers the problems arising in the control of a mechanical system with a holonomic and a nonholonomic part as the combination of a manipulator and a mobile platform respectively looks like. We show the common and the special moments in it and reduce the problem in the motion planning, i.e., how we have to follow the interpolation curve of the generalized coordinates (the curve of vector-parameters) so that the concrete task to be fulfilled in a most efficient way.

The synthesis of control algorithms is a main problem in robot devices. The combination of a mobile platform and a manipulator is a couple of bodies which are quite different as controllable objects. The manipulator may be considered as a controllable system of interconnected bodies with holonomic constraints. Without having in mind the deformation of the wheels, the mobile platform (or so called transportation robot) is a system with nonholonomic constraints. The common of the control approaches of such objects is: first – the deriving of the dynamical models and the second – on the base of these models the control laws are built so that the motion and the dynamical characteristics of the system to be satisfied.

There are a lot of control algorithms depending on the concrete task. For example, for manipulator control may be used: linear algorithms for systems with changeable structure, nonlinear and adaptive algorithms, methods from the theory of nonlinear chains, etc. There are also a lot of algorithms for control of mobile platforms.

In the problem of program control of manipulators, methods in which the trajectories are described with polynomial approximation in the space of the generalized coordinates are used, as well as methods in which the output trajectory is approximated with circle arcs and closed intervals of lines. In control of mobile robots such an approximation allows the problem to be limited in two types of motions – along a line and along a circle. These regimes are realized with the most simple control law, namely fixing of the position of the leading wheel. In these problems, the parameterization of the program trajectories and the synthesis of the system for stabilization are of a great importance.

How we shall follow the given program trajectory on a generalized coordinates level or on a level of Cartesian coordinates – this depend on the task, our aim and the technical devices we have.

For the program control of the system of mobile platform and manipulator it is important the following:

1. The mobile platform (nonholonomic mechanical system) has the functions of a transfer system, i.e., a system through which the manipulator under consideration is localized to the place, where it has to work.
2. The manipulator (holonomic mechanical system) realizes a definite task according to the general problem.

In the both problems the trajectory is given in Cartesian coordinates and from the task requirements, constraints for velocities, accelerations or for forces or moments of the manipulator gripper are imposed.

In dependence of the concrete task, the joint parameters (i.e., the corresponding control functions), which have to be realized are obtained. In the theory which we follow the joint parameters are the translation parameters and the vector-parameters describing the rotation motions.

The problem is how the rotational motions to be parameterized. The interpolation of the translational motions is a standard problem and we will not discuss here.

9.2. An Approach for Motion Planning

Our further considerations are based again on the vector-parameterization of the rotation group SO(3) [78]. In principle, motion planning in our consideration is the generation of the generalized coordinates which have to perform some task, i.e., in the sense of our parameterization – the obtaining of the vector-parameter polynomial time functions. This problem depends on the type of control – pure kinematical or dynamical. In the first case the inverse kinematical problem is solved, in the second – the direct dynamical problem is solved.

We present an algorithm generating differentiable curve on the rotation group SO(3) (see also [93] and [95], [94]) that interpolates a set of rotation matrices at their specified time knots. The problem of interpolation of smooth curves on the rotation group SO(3) arising in computer graphics, animation, robot motion planning and machine vision is a subject of many papers at recent time. The curve depends on the choice of local coordinates. Using the standard parameterizations of SO(3) group as Eulerian and Bryant angles, quaternions, Caley-Klein parameters, etc. the resulting trajectories have the “multiple winding” effect. But this is not the case with the vector-parameterization of the group SO(3) which is used in our approach. The resulting algorithms are computationally efficient and do not require transcendental functions.

9.2.1. Problem Statement

A common problem that arises not only in computer graphics and animation, but also in robot motion planning and machine vision is the interpolation of smooth curves on the rotation group SO(3). We address the following **problem**:

Given an ordered set of $n+1$ rotation matrices $\{O_o, O_1, ..., O_n\}$, and a set of $n+1$ scalars $t_o < t_1 < \ldots < t_n$, **find** a twice-differentiable curve $O(t)$ on the rotation group such that $O(t_i) = O_i, \;\; i = 0, 1, \ldots, n$.

Our goal is to find a computationally efficient, coordinate-invariant method of interpolating smooth curves on the rotation group that produces smooth, bi–invariant orientation trajectories. The crux of our approach is to parameterize the rotation group in terms of the vector-parameters. Doing so leads to a particularly simple and efficient set of expressions for the angular velocity and acceleration that does not require the evaluation of any transcendental functions. Also, the resulting trajectory can be viewed as an approximation to a minimum angular acceleration curve in the same sense that Euclidean cubic splines can be viewed as an approximation to minimum curvature curves, and does not suffer from the winding effect.

9.2.2. Two Point Interpolation

We are considering now the problem of interpolating between two elements of SO(3), where two specified values for the angular velocities at the two endpoints are given. Mathematical problem statement is:

Find a curve $O(t) \in$ SO(3) with boundary conditions: $O(0) = O_o$, $O(1) = O_1$; $O^T(0)\dot{O}(0) = \omega_o^{\times} O^T(1)\dot{O}(1) = \omega_1^{\times}$, where $O_o, O_1 \in$ SO(3), such that $\mathrm{tr}(O_o^T O_1) \neq -1, \;\; \omega_o, \;\; \omega_1 \in \mathbb{R}^3$ (angular velocities in body - fixed frames).

The class of admissible curves that we consider are left-invariant, i.e., in the form

$$O(t) = O_o(I - c^{\times}(t))\,(I + c^{\times}(t))^{-1}. \tag{389}$$

Furthermore, we require that $c(t)$ is a three-dimensional cubic polynomial with zero constant term, namely

$$c(t) = x\,t^3 + y\,t^2 + z\,t \tag{390}$$

with coefficients x, y, z to be determined. From the definition of the matrix O

and the skew matrix $c^{\times}$ we have

$$\begin{aligned} c(0) &= 0 && (391)\\ c^{\times}(1) &= \frac{1}{1+\mathrm{tr}(O_o^T O_1)}(O_o^T O_1 - O_1^T O_o) && (392)\\ \dot{c}(0) &= \frac{\omega_o}{2} && (393)\\ \dot{c}(1) &= \frac{1+\parallel c(1) \parallel^2}{2}(I - c^{\times}(1))^{-1}\omega_1. && (394) \end{aligned}$$

The boundary condition for $c(0)$ is automatically satisfied. From the boundary condition for $\dot{c}(0)$ follows

$$\begin{aligned} z &= \frac{\omega_o}{2} && (395)\\ x+y &= c(1) - \frac{\omega_o}{2} && (396)\\ 3x+2y &= \frac{1+\parallel c(1) \parallel^2}{2}(I - c^{\times}(1))^{-1}\omega_1 - \frac{\omega_o}{2}. && (397) \end{aligned}$$

$c(1)$ is determined from the equation for $c_1^{\times}$. Once x, y, z (and hence $c(t)$) are obtained, the interpolating curve $O(t)$ can be determined from the closed form of the equation for the matrix O given in the previous section.

Bi-invariance reflects the property that the actual rotation trajectory generated in physical space should be independent of how one chooses the inertial and body – fixed reference frames. Our interpolating trajectory is bi-invariant but for a brief exposition we will not present the proof of this fact here. Because of the same reasons we will not show here that the generated cubic splines on SO(3) can be viewed as an approximation to a minimum angular acceleration trajectory.

9.2.3. Multiple Point Interpolation Algorithm

Given:

$\{O_o, \ldots, O_n\}$ – $n+1$ rotation matrices satisfying $\mathrm{tr}(O_{i-1}^T O_i) \neq -1$, $i = 1, \ldots, n$

$\{t_o, \ldots, t_n\}$ – $n+1$ knot times

ω_o – angular velocity at t_o in body fixed cord

$\dot{\omega}_o$ – angular acceleration at t_o in body fixed cord.

Preprocessing: for $i = 1$ to n do

$$f_i^{\times} = \frac{O_{i-1}^T O_i - O_i^T O_{i-1}}{1 + \mathrm{tr}(O_{i-1}^T O_i)}.$$

Initialization:

$$\begin{aligned} z_1 &= \frac{\omega_o}{2} \\ y_1 &= \frac{\dot{\omega}_o}{2} \\ x_1 &= f_1 - z_1 - y_1. \end{aligned}$$

Recursion: for $i = 1$ to n do

$$\begin{aligned} w &= f_i \\ v &= 3x_{i-1} + 2y_{i-1} + z_{i-1} \\ u &= 6x_{i-1} + 2y_{i-1} \\ z_i &= \frac{-w \times v + v}{1 + w^T w} \\ y_i &= \frac{-w \times u + u - 2w^T v z_i}{2(1 + w^T w)} \\ x_i &= f_i - z_i - y_i. \end{aligned}$$

Once all the coefficients x_i, y_i, z_i, $i = 1, \ldots, n$ have been found, then for a given value of t, where $t_{i-1} \leq t \leq t_i$, the corresponding orientation $O(t)$ for the interpolating trajectory is found as follows:

Results:

$$\begin{aligned} \tau &= \frac{t - t_{i-1}}{t_i - t_{i-1}} \\ c &= x_i \tau^3 + y_i \tau^2 + z_i \tau \\ O(t) &= O_{i-1} \frac{(1 - c^T c)I + 2cc^T + 2c^{\times}}{1 + c^T c}. \end{aligned}$$

On the base of vector-parameterization of the rotation group, the program control of our mechanical system which is a combination of holonomic and nonholonomic part, is concluded in planning and realization of the joint motions, i.e., following some interpolation curves in a way, that the program trajectory to be followed most effectively. This efficiency comes from the fundamental new theoretical statement of the considerations still on the kinematical level. This does not exclude some standard approaches known in this field to be also used.

10. Motion Planning of Robot Locomotion

Since the treatment of robot locomotion is quite closed with this one of human locomotion, here we treat the problems of modelling, simulation and motion planning of robot locomotion on the base of knowledge of the skeletal system, as well as on the base of multi-body system modelling, simulation and control using some ideas from Lie group theory and differential geometry.

10.1. Theoretical Background

The term "locomotion" indicates the act of going from one place to another. The human walking is a locomotion with stride character with has four walking phases. The basic problems of locomotion dynamics are: identification of the external forces and moments acting on the body and its links and obtaining of the joint moments which are connected with the motion control.

In principle, motion planning is the generating of the generalized coordinates which have to perform some task. We are using an algorithm generating differentiable curve on the rotation group SO(3) that interpolates a set of rotation matrices at their specified time knots, i.e., in the sense of our parameterization – the obtaining of the vector-parameter polynomial time functions. This problem depends on the type of control – pure kinematical or dynamical. In the first case the inverse kinematical problem is solved, in the second – the direct dynamical problem is solved. The resulting algorithms are computationally efficient and do not require transcendental functions.

10.2. Case Study

We consider a planar bipedal device (in the sagittal plane XZ) which consists of a body and legs (see Fig. 6). The weightless and uninertial legs play a fundamental role. They allow the problem to be divided in two parts: calculating the control moments and ground reaction forces. In this aspect the biped walking may be considered analytically.

In our case we consider a biped device (see Fig. 6) with five weight inertial elements: a trunk-balance and two identical legs (lower limbs) every with thigh length $2a$ and shank (calf) length $2b$. O is the center of the hip joint which connects the trunk with the legs and it coincides with the gravity center of the pelvis. This point is modeled as a material point with mass m_o. The point O is defined in the space by two cartesian coordinates x and z, and the legs and trunk position –

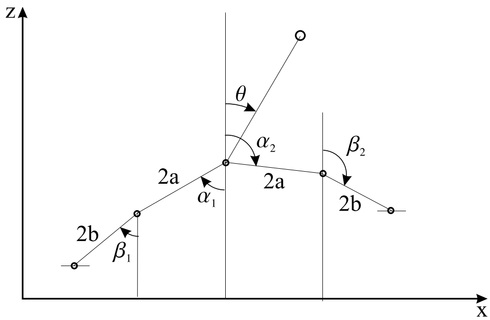

Figure 6. A Scheme of a Biped Device

by the angle coordinates $\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \theta$. The angle coordinates are measured clockwise from the vertical lines given in the Fig. 1 to the corresponding links. All these seven quantities are the system generalized coordinates. The corresponding vector parameters are : $C_1 = (0, c_1, 0)$, $D_1 = (0, d_1, 0)$, $C_2 = (0, c_2, 0)$, $D_2 = (0, d_2, 0)$, $C = (0, c, 0)$, where $c_1 = \tan\alpha_1/2$, $d_1 = \tan\beta_1/2$, $c_2 = \tan\alpha_2/2$, $d_2 = \tan\beta_2/2$, $c = \tan\theta/2$.

We denote by: u_1 and u_2 – the control moments in the knee joints; q_1 and q_2 – the control moments between the thighs and the trunk; $R_i,\ i = 1, 2$ are the ground reaction forces. In the general case there are ankle control moments $-p_1, -p_2$ in the ankle joints. We suppose here that $p_i = 0$. Further we introduce: M_1 – the mass of the trunk-balance; ρ – the distance from point O to the mass center C of the trunk; I – the inertial moment of the trunk with respect to the axis Z' which is parallel to Z and passes through the point O; m_a – thigh mass; a is the distance from point O to the thigh mass center; I_a^o is the inertial moment of the thigh with respect to Z'; m_b is the shank mass; b – the distance from knee joint center to the shank mass center; I_b – the inertial moment with respect to an axis through the knee joint and parallel to Z. Also

$$M = m_o + 2m_a + 2m_b + M_1, \qquad I_a = I_a^o + 4m_b a^2$$
$$k_a = a(m_a + 2m_b), \qquad k_b = bm_b, \qquad I_{ab} = 2m_b ab, \qquad k_\rho = M_1\rho.$$

We denote by $\sin(*) = s(*)$ and $\cos(*) = c(*)$ and after the substitutions

$$\begin{aligned} s\theta &= \frac{2c}{1+c^2}, \qquad c\theta = \frac{1-c^2}{1+c^2} \\ s\alpha_i &= \frac{2c_i}{1+c_i^2}, \qquad c\alpha_i = \frac{1-c_i^2}{1+c_i^2} \\ s\beta_i &= \frac{2d_i}{1+d_i^2}, \qquad c\beta_i = \frac{1-d_i^2}{1+d_i^2}. \end{aligned} \tag{398}$$

the equations of motion look like

$$M\ddot{x} + k_\rho(\mathrm{dd}\theta\, c\theta - (\mathrm{d}\theta)^2\, s\theta) \\ - \sum_{i=1}^{2}[k_a(\mathrm{dd}\alpha_i\, c\alpha_i - (\mathrm{d}\alpha_i)^2\, s\alpha_i) + k_b(\mathrm{dd}\beta_i\, c\beta_i - (\mathrm{d}\beta_i)^2\, s\beta_i)] = Q_x$$

$$M\ddot{z} + k_\rho(\mathrm{dd}\theta\, s\theta + (\mathrm{d}\theta)^2\, c\theta) \\ + \sum_{i=1}^{2}[k_a(\mathrm{dd}\alpha_i\, s\alpha_i + (\mathrm{d}\alpha_i)^2\, c\alpha_i) + k_b(\mathrm{dd}\beta_i\, s\beta_i + (\mathrm{d}\beta_i)^2\, c\beta_i)] = Q_z - Mg$$

$$I\mathrm{dd}\theta + k_\rho(\ddot{x}\, c\theta - \ddot{z}\, s\theta) - gk_\rho\, s\theta = Q_\theta$$

$$I_a\, \mathrm{dd}\alpha_i + I_{ab}\, \mathrm{dd}\beta_i\, c(\alpha_i - \beta_i) - k_a(\ddot{x}\, c\alpha_i - \ddot{z}\, s\alpha_i) \\ + I_{ab}\, (\mathrm{d}\beta_i)^2\, s(\alpha_i - \beta_i) + gk_a\, s\alpha_i = Q_{\alpha i}$$

$$I_b\, \mathrm{dd}\beta_i + I_{ab}\, \mathrm{dd}\alpha_i\, c(\alpha_i - \beta_i) - k_b(\ddot{x}\, c\beta_i - \ddot{z}\, s\beta_i) \\ - I_{ab}\, (\mathrm{d}\alpha_i)^2\, s(\alpha_i - \beta_i) + gk_b\, s\beta_i = Q_{\beta i}, \qquad i = 1, 2$$

where the symbols d and dd mean the first and the second derivatives with respect to time of the angles θ, α, β, and

$$\begin{aligned} Q_x &= \sum_{i=1}^{2} R_{ix}, \qquad Q_z = \sum_{i=1}^{2} R_{iz} \qquad Q_\theta = -\sum_{i=1}^{2} q_i \\ Q_{\alpha i} &= -u_i + q_i - 2a(R_{ix}\, c\alpha_i - R_{iz}\, s\alpha_i) \\ Q_{\beta i} &= u_i - 2b(R_{ix}\, c\beta_i - R_{iz}\, s\beta_i) - p_i \qquad i = 1, 2. \end{aligned} \tag{399}$$

Here we denote by
$c(\alpha_i - \beta_i) = (c\,\alpha_i)\,(c\,\beta_i) + (s\,\alpha_i)\,(s\,\beta_i), \quad s(\alpha_i - \beta_i) = (s\,\alpha_i)\,(c\,\beta_i) - (c\,\alpha_i)\,(s\,\beta_i)$.
It may be seen that in the equations written above all trigonometric functions are ignored and they are pure algebraic. After obtaining c, c_i, d_i we get the joint angles from the following equations: $\theta = 2\arctan c$; $\alpha_i = 2\arctan c_i$; $\beta_i = 2\arctan d_i$.

In the above equations: index "1" is related to the support leg (in case of single-support phase) or to the foreleg (in case of double-support phase), index "2" – to the swing leg (single-support phase) or to the back leg (double-support phase).

We shall apply the semi-inverse method of partially prescribed coordinates as time functions. From some sensible kinematical reasons the most expedient way is the leg motion to be given, i.e., $\alpha_i,\ \beta_i$ as explicit functions of time. Then the functions that have to be found are : $x, z, \theta, u_i, q_i, R_{ix}, R_{iz}$. They depend on the character of the motion: swing, single-support and double-support.

In case of swing phase by definition it is fulfilled: $R_{ix} = 0,\ R_{iz} = 0,\ i = 1, 2$. Then only $x,\ z,\ \theta,\ u_i,\ q_i,\ i = 1, 2$ have to be obtained and since we have seven equations the problem is determined. In single-support phase, $\alpha_i,\ \beta_i$ as well as x, z are given and by definition we have $R_{2x} = 0,\ R_{2z} = 0$. Seven functions have to be defined, namely θ, u_i, q_i, R_{1x}, R_{2z}, so that again the problem is determined. In the double-support phase we have to obtain all nine functions x, z, θ, u_i, q_i, R_{ix}, R_{iz}, so that our problem is undetermined. The problem has to be determined with additional equations to the equations of motion like some constraints equations.

11. Conclusion

The present paper makes a review of our long standing studies for creating an unified approach for modeling and control of open-loop mechanical rigid body systems by making use of efficient vector-parameterization of the rotation group SO(3). Additionaly, on the same language some examples and problems with nice practical application are solved.

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