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FERMI-WALKER PARALLEL TRANSPORT ACCORDING TO QUASI FRAME IN THREE DIMENSIONAL MINKOWSKI SPACE

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Abstract. In this paper, we present the Fermi-Walker parallel transport and the generalized Fermi-Walker parallel transport according to quasi frame in three dimensional Minkowski space.

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1. Introduction

A relativistic observer ξ needs reference frames in order to measure the movement and position of a object. If ξ is free falling, its restspaces are transported with Levi-Civita parallelism. For accelerated observers, the restspaces are not transported by the Levi-Civita parallelism. In this case Fermi-Walker parallelism is used to define constant directions. Fermi-Walker parallelism is an isometry between the tangent spaces along relativistic observer ξ. [6, 11].

Balakrishnan *et al* investigated time evolutions of the space curve associated with a geometric phase using Fermi-Walker parallel transport in three dimensional Euclidean space [2]. Gürbüz had introduced new geometric phases according three classes of a curve evolution in Minkowski space [7, 8].

Usual Fermi-Walker parallel derivative for any vector field A is given with respect to Frenet frame $\{t, n, b\}$ in three dimensional Euclidean space as following (cf. [9])

$$\frac{\mathcal{D}^f A}{\mathcal{D}^f s} = \frac{\mathrm{d}A}{\mathrm{d}s} - \langle t, A\rangle \frac{\mathrm{d}t}{\mathrm{d}s} - \langle \frac{\mathrm{d}t}{\mathrm{d}s}, A\rangle t.$$

Dandoloff and Zakrzewski [4] introduced the modified Fermi-Walker derivative of the vector field A according to Frenet frame in three dimensional Euclidean space as

$$\frac{\mathcal{D}^f A}{\mathcal{D}^f s} = \frac{\mathrm{d}A}{\mathrm{d}s} - \langle b, A\rangle \frac{\mathrm{d}b}{\mathrm{d}s} - \langle \frac{\mathrm{d}b}{\mathrm{d}s}, A\rangle n.$$

Generalized Fermi–Walker parallelism is used by both accelerated observers and not accelerated observers and it offers better choice of reference systems than the

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classical one [11]. Recently many authors studied Fermi-Walker and generalized Fermi-Walker derivative for various spaces [10, 12, 13].

Coquillart introduced the quasi-normal vector of a space curve to construct the 3D curve offset [3]. The quasi frame has many advantages according to Frenet and Bishop frames. It is defined even when the curvature $(\kappa = 0)$ vanishes. The construction of the quasi frame does not depend on the fact if the space curve has an unit speed or not.

In this paper we study Fermi-Walker derivative, generalized Fermi-Walker derivative and the modified Fermi-Walker derivative according to a quasi frame along a non-null curve in three dimensional Minkowski space.

2. Preliminaries

The three dimensional Minkowski space $\mathbb{R}^3_1$ is the real vector space $\mathbb{R}^3$ with the indefinite metric defined as

$$\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3_1$. A vector x of $\mathbb{R}^3_1$ is said to be spacelike if $\langle x, x\rangle > 0$ or $x = 0$, timelike if $\langle x, x\rangle < 0$ and lightlike or null if $\langle x, x\rangle = 0$ and $x \neq 0$ [1].

Let $\alpha : I \to \mathbb{R}^3_1$ be a timelike curve with an arc-length parameter s in three dimensional Minkowski space $\mathbb{R}^3_1$. The derivative formulas of the Frenet frame $\{t, n, b\}$ of a timelike curve are given by

$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix} \tag{1}$$

where t is unit tangent vector, n is unit normal vector and b is unit binormal vector. κ and τ are the curvature and the torsion of a timelike curve in $\mathbb{R}^3_1$.

A quasi frame (or q-frame) is defined as follows (cf. [5])

$$t_q = t = \frac{\alpha'}{\|\alpha'\|}, \qquad n_q = \frac{t \wedge k}{\|t \wedge k\|}, \qquad b_q = -t_q \wedge n_q$$

where t_q is the unit tangent vector, n_q is the quasi normal vector, b_q is the quasi binormal vector and k is the projection vector.

If t and k are parallel, then the q-frame is singular and $t \wedge k$ vanishes. For this reason we will not study the case $t = k$. In this paper, for simplicity we choose the projection vector as $k = (0, 1, 0)$.

The derivative formulas of the quasi frame $\{t_q, n_q, b_q, k\}$ of a timelike curve are given by

$$\begin{pmatrix} t_q' \\ n_q' \\ b_q' \end{pmatrix} = \begin{pmatrix} 0 & \xi_1 & \xi_2 \\ \xi_1 & 0 & \xi_3 \\ \xi_2 & -\xi_3 & 0 \end{pmatrix} \begin{pmatrix} t_q \\ n_q \\ b_q \end{pmatrix} \tag{2}$$

$$\begin{aligned} \langle t_q, t_q \rangle &= -1, \qquad \langle n_q, n_q \rangle = 1, \qquad \langle b_q, b_q \rangle = 1 \\ n_q \wedge t_q &= b_q, \qquad b_q \wedge t_q = -n_q, \qquad n_q \wedge b_q = t_q \\ \xi_1 &= \kappa \cos\theta, \qquad \xi_2 = -\kappa \sin\theta, \qquad \xi_3 = \tau + \theta'. \end{aligned}$$

Here ξ_1, ξ_2, ξ_3 are the q-frame curvatures of a timelike curve along quasi frame and θ is the pseudo-Euclidean angle between the vectors the principal normal vector n and the quasi-normal vector n_q. The relationship between quasi frame and Frenet frame is given by

$$n_q = \cos\theta n + \sin\theta b, \qquad b_q = -\sin\theta n + \cos\theta b.$$

Let $\alpha : I \to \mathbb{R}^3_1$ be a spacelike curve with a timelike binormal vector b in $\mathbb{R}^3_1$. The derivative formulas of the Frenet frame $\{t, n, b\}$ along a spacelike curve are given by

$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}. \tag{3}$$

The derivative formulas of the quasi frame $\{t_q, n_q, b_q, k\}$ along a spacelike curve with a projection vector $k = (0, 1, 0)$ are given by (cf. [5])

$$\begin{pmatrix} t_q' \\ n_q' \\ b_q' \end{pmatrix} = \begin{pmatrix} 0 & \xi_1 & -\xi_2 \\ -\xi_1 & 0 & -\xi_3 \\ -\xi_2 & -\xi_3 & 0 \end{pmatrix} \begin{pmatrix} t_q \\ n_q \\ b_q \end{pmatrix} \tag{4}$$

$$\begin{aligned} \langle t_q, t_q \rangle &= 1, \qquad \langle n_q, n_q \rangle = 1, \qquad \langle b_q, b_q \rangle = -1 \\ n_q \wedge t_q &= -b_q, \qquad b_q \wedge t_q = -n_q, \qquad n_q \wedge b_q = -t_q \\ \xi_1 &= \kappa \cosh\theta, \qquad \xi_2 = -\kappa \sinh\theta, \qquad \xi_3 = -\tau - \theta' \end{aligned}$$

where θ is defined as the pseudo-angle between the binormal b (timelike) and quasi-normal b_q (timelike) vectors and ξ_1, ξ_2, ξ_3 are the quasi frame curvatures of a spacelike curve for the quasi frame in the three dimensional Minkowski space.

3. Fermi-Walker Parallel Transport with Quasi Frame

3.1. The curve α is spacelike

Definition 1. *The Fermi-Walker derivative of a vector field of V along a spacelike curve according to quasi frame $\{t_q, n_q, b_q\}$ is defined as*

$$\frac{D^f V}{\mathcal{D}^f s} = \frac{\mathrm{d}V}{\mathrm{d}s} - (\frac{\mathrm{d}t_q}{\mathrm{d}s} \wedge t_q) \wedge V. \tag{5}$$

Definition 2. *Let $V = \eta_1 t_q + \eta_2 n_q + \eta_3 b_q$ be a vector field according to quasi frame $\{t_q, n_q, b_q\}$ along a spacelike curve α. If*

$$\frac{D^f V}{\mathcal{D}^f s} = 0 \tag{6}$$

is satisfied, the vector field V is called the Fermi-Walker parallel according to quasi frame $\{t_q, n_q, b_q\}$ along a spacelike curve.

Lemma 3. *The Fermi-Walker derivative of a vector field $V = \eta_1 t_q + \eta_2 n_q + \eta_3 b_q$ according to quasi frame $\{t_q, n_q, b_q\}$ along a spacelike curve is given by*

$$\frac{D^f V}{\mathcal{D}^f s} = \frac{\mathrm{d}V}{\mathrm{d}s} - (\xi_2 n_q - \xi_1 b_q) \wedge V. \tag{7}$$

Proof: Using (5), we obtain (7). ■

Theorem 4. *The vector field $V = \eta_1 t_q + \eta_2 n_q + \eta_3 b_q$ is the Fermi-Walker parallel along a spacelike curve according to quasi frame if and only if*

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}\eta_2}{\mathrm{d}s} = \eta_3 \xi_3, \qquad \frac{\mathrm{d}\eta_3}{\mathrm{d}s} = \eta_2 \xi_3. \tag{8}$$

Here η_1, η_2 and η_3 are smooth functions with respect to s.

Proof: Using (7) we have

$$\frac{D^f V}{\mathcal{D}^f s} = \frac{\mathrm{d}\eta_1}{\mathrm{d}s} t_q + (\frac{\mathrm{d}\eta_2}{\mathrm{d}s} - \eta_3 \xi_3) n_q + (\frac{\mathrm{d}\eta_3}{\mathrm{d}s} - \eta_2 \xi_3) b_q. \tag{9}$$

If V is the Fermi-Walker parallel along a spacelike curve α according to quasi frame in $\mathbb{R}^3_1$, then

$$\frac{D^f V}{\mathcal{D}^f s} = 0. \tag{10}$$

Thus (9) and (10) give (8). The other part is trivial. ■

Theorem 5. *The Fermi-Walker derivative of the vector field V coincides with derivative of V according to quasi frame along a spacelike curve in $\mathbb{R}_1^3$ if and only if*

$$V = c(\xi_1 b_q - \xi_2 n_q), \qquad c = \text{constant}. \tag{11}$$

Proof:

$$\frac{D^f V}{\mathcal{D}^f s} = \frac{\mathrm{d}V}{\mathrm{d}s} - (\xi_2 n_q - \xi_1 b_q) \wedge V = \frac{\mathrm{d}V}{\mathrm{d}s}$$

if and only if

$$V = c(\xi_2 n_q - \xi_1 b_q), \qquad c = \text{constant}.$$

■

Theorem 6. *If η_1, η_2, η_3 are constants and $\xi_3 = 0$, then the vector field V is the Fermi-Walker parallel according to quasi frame along a spacelike curve α in $\mathbb{R}_1^3$.*

Proof: If η_1, η_2, η_3 are constants and $\xi_3 = 0$, (9) implies

$$\frac{\mathcal{D}^f V}{\mathcal{D}^f s} = 0.$$

Thus, V is the Fermi-Walker parallel according to quasi frame. ■

3.2. The curve α is timelike

Definition 7. *The Fermi-Walker derivative of a vector field U along a timelike curve according to quasi frame $\{t_q, n_q, b_q\}$ is defined as*

$$\frac{D^f U}{\mathcal{D}^f s} = \frac{\mathrm{d}U}{\mathrm{d}s} + (\frac{\mathrm{d}t_q}{\mathrm{d}s} \wedge t_q) \wedge U. \tag{12}$$

Definition 8. *Let $U = \mu_1 t_q + \mu_2 n_q + \mu_3 b_q$ be a vector field according to quasi frame $\{t_q, n_q, b_q\}$ of a timelike curve α. If*

$$\frac{D^f U}{\mathcal{D}^f s} = 0$$

is satisfied, the vector field U is called the Fermi-Walker parallel according to quasi frame $\{t_q, n_q, b_q\}$ of a timelike curve.

Lemma 9. *The Fermi-Walker derivative of a vector field $U = \mu_1 t_q + \mu_2 n_q + \mu_3 b_q$ according to quasi frame $\{t_q, n_q, b_q\}$ along a timelike curve α is given by*

$$\frac{D^f U}{\mathcal{D}^f s} = \frac{\mathrm{d}U}{\mathrm{d}s} + (\xi_1 b_q - \xi_2 n_q) \wedge U. \tag{13}$$

Proof: Using (12) we obtain (13). ■

Theorem 10. *The vector field $U = \mu_1 t_q + \mu_2 n_q + \mu_3 b_q$ is the Fermi-Walker parallel along a timelike curve according to quasi frame if and only if*

$$\frac{\mathrm{d}\mu_1}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}\mu_2}{\mathrm{d}s} = \mu_3\xi_3, \qquad \frac{\mathrm{d}\mu_3}{\mathrm{d}s} = -\mu_2\xi_3. \tag{14}$$

Here μ_1, μ_2 and μ_3 are smooth functions with respect to s.

Proof: From (13) we have

$$\frac{D^f U}{\mathcal{D}^f s} = \frac{\mathrm{d}U}{\mathrm{d}s} + (\xi_1 b_q - \xi_2 n_q) \wedge U \tag{15}$$

$$= \frac{\mathrm{d}\mu_1}{\mathrm{d}s} t_q + (\frac{\mathrm{d}\mu_2}{\mathrm{d}s} - \mu_3\xi_3) n_q + (\frac{\mathrm{d}\mu_3}{\mathrm{d}s} + \mu_2\xi_3) b_q. \tag{16}$$

If U is the Fermi-Walker parallel along a timelike curve α according to q-frame, then

$$\frac{D^f U}{\mathcal{D}^f s} = 0 \tag{17}$$

which implies (14). The rest part is obvious. ■

Theorem 11. *The Fermi-Walker derivative of the vector field U coincides with derivative of U according to quasi frame along a timelike curve in $\mathbb{R}^3_1$ if and only if*

$$U = c(\xi_1 b_q - \xi_2 n_q), \qquad c = \text{constant}. \tag{18}$$

Proof:

$$\frac{\mathcal{D}^f U}{\mathcal{D}^f s} = \frac{\mathrm{d}U}{\mathrm{d}s} + (\xi_1 b_q - \xi_2 n_q) \wedge U = \frac{\mathrm{d}U}{\mathrm{d}s}$$

if and only if

$$U = c(\xi_1 b_q - \xi_2 n_q), \qquad c = \text{constant}.$$

■

Theorem 12. *If μ_1, μ_2, μ_3 are constants and $\xi_3 = 0$, then the vector field U is the Fermi-Walker parallel according to quasi frame along a timelike curve α in $\mathbb{R}^3_1$.*

Proof: If μ_1, μ_2, μ_3 are constants and $\xi_3 = 0$ with the help of (16), we obtain

$$\frac{\mathcal{D}^f U}{\mathcal{D}^f s} = 0.$$

Thus U is the Fermi-Walker parallel according to quasi frame. ■

Definition 13. *The generalized Fermi-Walker derivative of a vector field U according to quasi frame along a timelike curve is defined as*

$$\frac{\mathcal{D}^G U}{\mathcal{D}^G s} = \frac{\mathcal{D}^f U}{\mathcal{D}^f s} + A(U), \qquad \langle A(I), t_q \rangle = 0 \tag{19}$$

where A is a $(1,1)$-tensor field and $I \in \chi^{\perp}$.

Definition 14. *Let $U = \mu_1 t_q + \mu_2 n_q + \mu_3 b_q$ be a vector field according to quasi frame of a timelike curve in $\mathbb{R}^3_1$. If*

$$\frac{\mathcal{D}^G U}{\mathcal{D}^G s} = 0$$

is satisfied, U is called the generalized Fermi-Walker parallel transport with respect to quasi frame with a timelike curve α in $\mathbb{R}^3_1$.

Theorem 15. *The generalized Fermi-Walker derivative of the vector field U according to quasi frame along a timelike curve is given by*

$$\frac{\mathcal{D}^G U}{\mathcal{D}^G s} = \frac{\mathrm{d}\mu_1}{\mathrm{d}s} t_q + (\frac{\mathrm{d}\mu_2}{\mathrm{d}s} - \mu_3 \xi_3) n_q + (\frac{\mathrm{d}\mu_3}{\mathrm{d}s} + \mu_2 \xi_3) b_q + A(U). \tag{20}$$

Proof: Using (9) and (19), we obtain (20). ■

Theorem 16. *The vector field $U = \mu_1 t_q + \mu_2 n_q + \mu_3 b_q$ is the generalized Fermi-Walker parallel according to quasi frame along a timelike curve α if and only if*

$$A(U) = -\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}s} t_q + (\frac{\mathrm{d}\mu_2}{\mathrm{d}s} - \mu_3 \xi_3) n_q + (\frac{\mathrm{d}\mu_3}{\mathrm{d}s} + \mu_2 \xi_3) b_q \right) \tag{21}$$

where μ_1, μ_2, μ_3 are smooth functions with respect to s.

Proof: If U is the generalized Fermi-Walker parallel transport with respect to quasi frame along a timelike curve α, then

$$\frac{\mathcal{D}^G U}{\mathcal{D}^G s} = 0. \tag{22}$$

Thus, (20) and (22) imply (21). ■

4. Modified Fermi-Walker Parallel Transport with Quasi Frame

Definition 17. *Let $W = \lambda_1 t_q + \lambda_2 n_q + \lambda_3 b_q$ be a vector field. The modified Fermi-Walker derivative along a timelike curve α with respect to quasi frame is defined by*

$$\frac{D^f W}{\mathcal{D}^f s} = \frac{\mathrm{d}W}{\mathrm{d}s} - (\frac{\mathrm{d}n_q}{\mathrm{d}s} \wedge n_q) \wedge W. \tag{23}$$

Definition 18. *Let $W = \lambda_1 t_q + \lambda_2 n_q + \lambda_3 b_q$ be a vector field according to quasi frame $\{t_q, n_q, b_q\}$ along a timelike curve α. If*

$$\frac{D^f W}{\mathcal{D}^f s} = 0$$

is satisfied, the vector field W is called the modified Fermi-Walker parallel according to quasi frame $\{t_q, n_q, b_q\}$ along a timelike curve with a projection k.

Lemma 19. *The modified Fermi-Walker derivative of a vector field $W = \lambda_1 t_q + \lambda_2 n_q + \lambda_3 b_q$ according to quasi frame $\{t_q, n_q, b_q\}$ along a timelike curve is given by*

$$\frac{D^f W}{\mathcal{D}^f s} = \frac{\mathrm{d}W}{\mathrm{d}s} + (\xi_1 b_q + \xi_3 t_q) \wedge W. \tag{24}$$

Proof: Using (23) we obtain (24). ■

Theorem 20. *The vector field $W = \lambda_1 t_q + \lambda_2 n_q + \lambda_3 b_q$ is the modified Fermi-Walker parallel along a timelike curve according to quasi frame if and only if*

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}s} = -\lambda_3 \xi_2, \qquad \frac{\mathrm{d}\lambda_2}{\mathrm{d}s} = 0, \qquad \frac{\mathrm{d}\lambda_3}{\mathrm{d}s} = -\lambda_1 \xi_2 \tag{25}$$

where λ_1, λ_2 and λ_3 are smooth functions with respect to s.

Proof: Equation (24) implies

$$\frac{D^f W}{\mathcal{D}^f s} = (\frac{\mathrm{d}\lambda_1}{\mathrm{d}s} + \lambda_3 \xi_2) t_q + (\frac{\mathrm{d}\lambda_2}{\mathrm{d}s} - \lambda_3 \xi_3) n_q + (\frac{\mathrm{d}\lambda_3}{\mathrm{d}s} + \lambda_1 \xi_2) b_q. \tag{26}$$

If W is the modified Fermi-Walker parallel along a timelike curve α according to quasi frame, then

$$\frac{D^f W}{\mathcal{D}^f s} = 0 \tag{27}$$

and this implies (25). By this the claim of the Theorem is obvious. ■

Theorem 21. *The modified Fermi-Walker derivative of the vector field W coincides with derivative of W according to quasi frame of a timelike curve in $\mathbb{R}^3_1$ if and only if*

$$W = \varrho(\xi_1 b_q + \xi_3 t_q), \qquad \varrho = \text{constant}. \tag{28}$$

Proof:

$$\frac{D^f W}{\mathcal{D}^f s} = \frac{\mathrm{d}W}{\mathrm{d}s} + (\xi_1 b_q + \xi_3 t_q) \wedge W = \frac{\mathrm{d}W}{\mathrm{d}s}$$

if and only if

$$W = \varrho(\xi_1 b_q + \xi_3 t_q), \qquad \varrho = \text{constant}.$$

■

Theorem 22. *If $\lambda_1, \lambda_2, \lambda_3$ are constants and $\xi_2 = 0, \xi_3 = 0$, then the vector field W is the modified Fermi-Walker parallel according to quasi frame with a timelike curve in $\mathbb{R}^3_1$.*

Proof: If $\lambda_1, \lambda_2, \lambda_3$ are constants and $\xi_2 = \xi_3 = 0$ equation (26) implies

$$\frac{\mathcal{D}^f W}{\mathcal{D}^f s} = 0. \tag{29}$$

Therefore, W is the modified Fermi-Walker parallel according to quasi frame. ■

Definition 23. *The generalized modified Fermi-Walker derivative of a vector field W along a timelike curve with respect quasi frame is defined by*

$$\frac{\mathcal{D}^G W}{\mathcal{D}^G s} = \frac{\mathcal{D}^f W}{\mathcal{D}^f s} + A(W), \qquad \langle A(I), n_q \rangle = 0. \tag{30}$$

Here A is a $(1,1)$- tensor field and $I \in \chi^{\perp}$.

Definition 24. *Let W be a vector field according to quasi frame in $\mathbb{R}^3_1$. If*

$$\frac{\mathcal{D}^G W}{\mathcal{D}^G s} = 0$$

is satisfied, W is called the generalized modified Fermi-Walker parallel along a timelike curve with respect to quasi frame.

Theorem 25. *The generalized modified Fermi-Walker derivative of a vector field $W = \lambda_1 t_q + \lambda_2 n_q + \lambda_3 b_q$ along a timelike curve with respect to q-frame is given by*

$$\frac{\mathcal{D}^G W}{\mathcal{D}^G s} = (\frac{\mathrm{d}\lambda_1}{\mathrm{d}s} + \lambda_3 \xi_2) t_q + (\frac{\mathrm{d}\lambda_2}{\mathrm{d}s} - \lambda_3 \xi_3) n_q + (\frac{\mathrm{d}\lambda_3}{\mathrm{d}s} + \lambda_1 \xi_2) b_q + A(W). \tag{31}$$

Proof: By combining (26) and (30), we obtain (31). ■

Theorem 26. *The vector field $W = \lambda_1 t_q + \lambda_2 n_q + \lambda_3 b_q$ is the generalized modified Fermi-Walker parallel along a timelike curve with respect to q-frame in $\mathbb{R}^3_1$ if and only if*

$$A(W) = -\left((\frac{\mathrm{d}\lambda_1}{\mathrm{d}s} + \lambda_3 \xi_2) t_q + (\frac{\mathrm{d}\lambda_2}{\mathrm{d}s} - \lambda_3 \xi_3) n_q + (\frac{\mathrm{d}\lambda_3}{\mathrm{d}s} + \lambda_1 \xi_2) b_q \right). \quad (32)$$

Proof: If W is the generalized modified Fermi-Walker parallel along a timelike curve with respect to q-frame in $\mathbb{R}^3_1$, then

$$\frac{\mathcal{D}^G W}{\mathcal{D}^G s} = 0.$$

From this, we obtain (32). ■

5. Concluding Remarks

In the present work we have focused our attention on

- The study of the Fermi-Walker parallel transport along a spacelike and timelike curve with respect to quasi frames in the three dimensional Minkowskian space.
- The investigation of the properties the modified and generalized Fermi-Walker parallel vector fields.

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