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GEOMETRY OF TWISTED SASAKI METRIC

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Abstract. Let (M, g) be a n-dimensional smooth Riemannian manifold. In the present paper, we introduce a new class of natural metrics denoted by $G^{f,h}$ and called twisted Sasaki matric on the tangent bundle TM. We studied the geometry of $(TM, G^{f,h})$ by giving a relationships of the curvatures, Einstein structure, scalar and sectional curvatures between $(TM, G^{f,h})$ and (M, g).

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Contents

1. Preliminaries

We recall some basic facts about the geometry of the tangent bundle. In the present paper, we denote by $\Gamma(TM)$ the space of all vector fields of a Riemannian manifold (M, g). Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M)

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be its tangent bundle. A local chart $(U, x^i)_{i=1\dots n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1\dots n}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\mathcal{V}_{(x,u)} = \ker(\mathrm{d}\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}_{|(x,u)}; \ a^i \in \mathbb{R}\}$$
$$\mathcal{H}_{(x,u)} = \{\frac{\partial}{\partial x^i}_{|(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}_{|(x,u)}; \ a^i \in \mathbb{R}\}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.
Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \qquad X^H = X^i \frac{\delta}{\delta x^i} = X^i \Big\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \Big\} \tag{1}$$

Let us notice that we have $\left(\frac{\partial}{\partial x^i}\right)^H = \frac{\delta}{\delta x^i}$ and $\left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}$.
Then $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)_{i=1\dots n}$ is a local adapted frame in TTM. The tangent bundle TM of a Riemannian manifold (M, g) can be endowed in a natural way with a Riemannian metric g^s, the Sasaki metric, depending only on the Riemannian structure g of the base manifold M. It is uniquely determined by

$$\begin{aligned} g^s(X^H, Y^H) &= g(X, Y) \circ \pi \\ g^s(X^H, Y^V) &= 0 \\ g^s(X^V, Y^V) &= g(X, Y) \circ \pi \end{aligned} \tag{2}$$

for all vector fields X and Y on M. More intuitively, the metric g^s is constructed in such a way that the vertical and horizontal sub bundles are orthogonal and the bundle map $\pi : (TM, g^s) \longrightarrow (M, g)$ is a Riemannian submersion.
The geometry of the tangent bundle TM equiped with Sasaki metric has been studied by many authors starting with K. Yano and S. Ishihara (see [20]), A. Salimov, A. Gezer and N. Cengiz (see [2], [12], [13], [16]) etc. In [18] Vaisman has studied generalized Sasakian structures. The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM (see [17]). J. Cheeger and D. Gromoll have introduced the notion of Cheeger-Gromoll metric (see [3]). It is

uniquely determined by

$$
\begin{aligned}
g_{CG}(X^H, Y^H) &= g(X,Y) \circ \pi \\
g_{CG}(X^H, Y^V) &= 0 \\
g_{CG}(X^V, Y^V) &= \frac{1}{\alpha}\{g(X,Y) + g(X,u)g(Y,u)\} \circ \pi
\end{aligned} \tag{3}
$$

where $X, Y \in \Gamma(TM)$, $(x,u) \in TM$, $\alpha = 1 + g_x(u,u)$.
Zayatuev in [21] introduced a Riemannian metric on TM given by

$$
\begin{aligned}
g_f^s(X^H, Y^H) &= f(p)g_p(X,Y) \\
g_f^s(X^H, Y^V) &= 0 \\
g_f^s(X^V, Y^V) &= g_p(X,Y)
\end{aligned} \tag{4}
$$

for all vector fields X and Y on (M,g), where f is strictly positive smooth function on (M,g). In [19] Wang J and Wang Y called g_f^s the rescaled Sasaki metric and studied the geometry of TM endowed with g_f^s.
Cheeger and Gromoll in [3], we define a new class of naturally metric on TM given by

$$
\begin{aligned}
G^f_{(p,u)}(X^H, Y^H) &= g_p(X,Y) \\
G^f_{(p,u)}(X^H, Y^V) &= 0 \\
G^f_{(p,u)}(X^V, Y^V) &= f(p)g_p(X,Y)
\end{aligned} \tag{5}
$$

for some strictly positive smooth function f in (M,g) and any vector fields X and Y on M. We call G^f vertical rescaled metric.

Motivated by the above studies, we define a new class of naturally metric on TM given by

$$
\begin{aligned}
G^{f,h}_{(p,u)}(X^H, Y^H) &= f(p)g_p(X,Y) \\
G^{f,h}_{(p,u)}(X^V, Y^H) &= 0 \\
G^{f,h}_{(p,u)}(X^V, Y^V) &= h(p)g_p(X,Y)
\end{aligned} \tag{6}
$$

where f, h be strictly positive smooth functions on M and any vector fields X and Y on M. For $h = 1$ the metric $G^{f,h}$ is exactly the rescaled Sasaki metric. If $f = 1$ the metric $G^{f,h}$ is exactly the vertical rescaled metric. We call $G^{f,h}$ the twisted Sasaki metric.
In this paper, we introduce the twisted Sasaki metric on the tangent bundle TM as a new natural metric non-rigid on TM. First we investigate the geometry of the twisted Sasaki metrics and we characterize the Einstein structure (Theorem 8) and the sectional curvature (Theorem 9) and the scalar curvature (Theorem 11).

2. Twisted Sasaki Metric

Definition 1 *Let (M,g) be a Riemannian manifold and $f,h : M \longrightarrow \mathbb{R}$ be strictly positive smooth functions. On the tangent bundle TM, we define a twisted Sasaki metric noted $G^{f,h}$ by*

$$G^{f,h}_{(p,u)}(X^H,Y^H) = f(p)g_p(X,Y)$$
$$G^{f,h}_{(p,u)}(X^V,Y^H) = 0$$
$$G^{f,h}_{(p,u)}(X^V,Y^V) = h(p)g_p(X,Y)$$

for all vector fields $X,Y \in \Gamma(TM)$ and $(p,u) \in TM$.
Note that, if $f = h = 1$, then $G^{f,h}$ is the Sasaki metric. If $h = 1$, then $G^{f,h}$ is the rescaled Sasaki metric. If $f = 1$, then $G^{f,h}$ is the vertical rescaled metric.

2.1. Levi-Civita Connection of $G^{f,h}$

Lemma 2 *Let (M,g) be a Riemannian manifold and ∇(respectively $\nabla^{f,h}$) denote the Levi-Civita connection of (M,g) (respectively $(TM,G^{f,h})$), then we have*

$$G^{f,h}(\nabla^{f,h}_{X^H}Y^H,Z^H) = \Big(\nabla_X Y + \frac{1}{2f(p)}\big(X(f)Y + Y(f)X - g(X,Y)\nabla^M f\big)\Big)^H$$
$$G^{f,h}(\nabla^{f,h}_{X^H}Y^H,Z^V) = -\Big(\frac{1}{2}(R(X,Y)u\Big)^V$$
$$G^{f,h}(\nabla^{f,h}_{X^H}Y^V,Z^H) = \Big(\frac{h(p)}{2f(p)}R(u,Y)X\Big)^H$$
$$G^{f,h}(\nabla^{f,h}_{X^H}Y^V,Z^V) = \Big(\frac{X(h)}{2h(p)}Y + \nabla_X Y\Big)^V$$
$$G^{f,h}(\nabla^{f,h}_{X^V}Y^H,Z^H) = \Big(\frac{h(p)}{2f(p)}R(u,X)Y\Big)^H$$
$$G^{f,h}(\nabla^{f,h}_{X^V}Y^H,Z^V) = \Big(\frac{Y(h)}{2h(p)}X\Big)^V$$
$$G^{f,h}(\nabla^{f,h}_{X^V}Y^V,Z^H) = \Big(-\frac{1}{2f(p)}g(X,Y)\nabla^M h\Big)^H$$
$$G^{f,h}(\nabla^{f,h}_{X^V}Y^V,Z^V) = 0$$

where ∇^M is the gradient of the Riemannian manifold (M,g).

The proof of Lemma 2 follows directly from Koszul formula, Definition 1.
As a direct consequence of Lemma 2, we get the following theorem.

Theorem 3 *Let (M, g) be a Riemannian manifold and $\nabla^{f,h}$ be a Levi-Civita connection of the tangent bundle $(TM, G^{f,h})$. Then, we have*

$$
\begin{aligned}
&1) \quad (\nabla^{f,h}_{X^H} Y^H)_{(p,u)} = \Big(\nabla_X Y + A_f(X,Y)\Big)^H_{(p,u)} - \Big(\frac{1}{2}R_p(X,Y)u\Big)^V\\
&2) \quad (\nabla^{f,h}_{X^H} Y^V)_{(p,u)} = \Big(\frac{h(p)}{2f(p)}R(u,Y)X\Big)^H_{(p,u)} + \Big(\frac{X(h)}{2h(p)}Y + \nabla_X Y\Big)^V_{(p,u)}\\
&3) \quad (\nabla^{f,h}_{X^V} Y^H)_{(p,u)} = \Big(\frac{h(p)}{2f(p)}R(u,X)Y\Big)^H_{(p,u)} + \Big(\frac{Y(h)}{2h(p)}X\Big)^V_{(p,u)}\\
&4) \quad (\nabla^{f,h}_{X^V} Y^V)_{(p,u)} = \Big(-\frac{1}{2f(p)}g(X,Y)\nabla^M h\Big)^H_{(p,u)}
\end{aligned}
$$

for all vector fields $X, Y \in \Gamma(TM)$, $(p,u) \in TM$, where

$$A_f(X,Y) = \frac{1}{2f(p)}\big(X(f)Y + Y(f)X - g(X,Y)\nabla^M f\big).$$

Definition 4 *Let (M, g) be a Riemannian manifold and $K : TM \to TTM$ be a smooth bundle endomorphism of the tangent bundle TM. Then we define the vertical and horizontal lifts $K^V : TM \to TTM$, $K^H : TM \to TTM$ of K by*

$$K^V(\eta) = \sum_{i=1}^{n} \eta_i K(\partial i)^V \quad \text{and} \quad K^H(\eta) = \sum_{i=1}^{n} \eta_i K(\partial i)^H$$

where $\sum_{i=1}^{n} \eta_i \partial i \in \pi^{-1}(V)$ is a local representation of $\eta \in \mathcal{C}^\infty(TM)$.

From Definition 1 and Theorem 3, we have

Proposition 5 *Let (M, g) be a Riemannian manifold and $\nabla^{f,h}$ be a the Levi-Civita connection of the tangent bundle $(TM, G^{f,h})$. If K is a tensor field of type $(1,1)$ on M, then*

$$
\begin{aligned}
(\nabla^{f,h}_{X^H} K^H)_{(p,u)} &= \Big((\nabla_X K)_{(p,u)} + A_f(X, K(u))\Big)^H - \frac{1}{2}\Big(R(X, K(u))u\Big)^V\\
(\nabla^{f,h}_{X^H} K^V)_{(p,u)} &= \Big((\nabla_X K)_{(p,u)} + \frac{X(h)}{2h(p)}K(u)\Big)^V + \Big(\frac{h(p)}{2f(p)}R(u, K(u))X\Big)^H\\
(\nabla^{f,h}_{X^V} K^H)_{(p,u)} &= \Big((K(X) + \frac{h(p)}{2f(p)}R(u,X)K(u)\Big)^H + \Big(\frac{(K(u))(h)}{2h(p)}X\Big)^V\\
(\nabla^{f,h}_{X^V} K^V)_{(p,u)} &= \Big(K(X)\Big)^V - \frac{1}{2f}\Big(g(X, K(u))\nabla^M h\Big)^H
\end{aligned}
$$

for any $X \in \Gamma(TM)$ and $(p,u) \in TM$.

3. Curvatures of Twisted Sasaki Metric

Theorem 6 *Let (M, g) be a Riemannian manifold and $R^{f,h}$ be the Riemann curvature tensor of the tangent bundle $(TM, G^{f,h})$ equipped with the twisted Sasaki metric. Then the following formulae hold*

$$
\begin{aligned}
1)\quad R^{f,h}(X^H, Y^H)Z^H = \Big(& R(X,Y)Z + A_f(X, \nabla_Y Z) + \nabla_X A_f(Y,Z) \\
& + A_f(X, A_f(Y,Z)) - A_f(Y, \nabla_X Z) \\
& - A_f(Y, A_f(X,Z)) + \frac{h}{4f} R(u, R(X,Z)u)Y \\
& - \nabla_Y A_f(X,Z) - A_f([X,Y],Z) \\
& + \frac{h}{2f} R(u, R(X,Y)u)Z \Big)^H \\
& + \Big(-\frac{1}{2} R(X, \nabla_Y Z)u - \frac{1}{2} R(X, A_f(Y,Z))u \\
& - \frac{1}{2}\nabla_X (R(Y,Z)u) + \frac{1}{2} R([X,Y],Z)u \\
& - \frac{1}{4h} X(h) R(Y,Z)u + \frac{1}{2} R(Y, A_f(X,Z))u \\
& + \frac{1}{2} R(Y, \nabla_X Z)u + \frac{1}{2}\nabla_Y (R(X,Z)u) \\
& + \frac{1}{4h} Y(h) R(X,Z)u + \frac{1}{2h} Z(h) R(X,Y)u \Big)^V
\end{aligned}
$$

$$
\begin{aligned}
2)\quad R^{f,h}(X^H, Y^V)Z^H = \Big(& \frac{1}{2} X\left(\frac{h}{f}\right) R(u,Y)Z + \frac{h}{2f} \nabla_X R(u,Y)Z \\
& + \frac{h}{2f} A_f(X, R(u,Y)Z) - \frac{h}{2f} R(u,Y) A_f(X,Z) \\
& - \frac{1}{4f} g(Y, R(X,Z)u) \nabla^M h + \frac{1}{4f} Z(h) R(u,Y)X \\
& - \frac{h}{2f} R(u,Y)\nabla_X Z - \frac{h}{2f} R(u, \nabla_X Y)Z \Big)^H \\
& + \Big(-\frac{h}{4f} R(X, R(u,Y)Z)u + \frac{1}{2} X\left(\frac{Z(h)}{h}\right) Y \\
& + \frac{1}{4h^2} Z(h) X(h) Y - \frac{1}{2h} (\nabla_X Z)(h) Y \\
& - \frac{1}{2h} (A_f(X,Z))(h) Y + \frac{1}{2} R(X,Z)Y \Big)^V
\end{aligned}
$$

$$
\begin{aligned}
3)\quad R^{f,h}(X^V,Y^V)Z^H =& \Big(\frac{h}{f}R(X,Y)Z + \frac{h^2}{4f^2}\big(R(u,X)R(u,Y)Z \\
&- R(u,Y)R(u,X)Z\big)\Big)^H \\
&+ \frac{1}{4f}\Big((R(u,Y)Z)(h)X - (R(u,X)Z)(h)Y\Big)^V \\
4)\quad R^{f,h}(X^V,Y^V)Z^V =& \frac{h(p)}{4f^2(p)}\Big(g(X,Z)R(u,Y)\nabla^M h \\
&- g(Y,Z)R(u,X)\nabla^M h\Big)^H \\
&+ \frac{\|\nabla^M h\|^2}{4f(p)h(p)}\Big(g(X,Z)Y - g(Y,Z)X\Big)^V
\end{aligned}
$$

$$
\begin{aligned}
5)\quad R^{f,h}(X^H,Y^H)Z^V =& \Big(\frac{1}{2}X\left(\frac{h}{f}\right)R(u,Z)Y + \frac{h}{2f}\nabla_X R(u,Z)Y \\
&+ \frac{h}{2f}A_f(X,R(u,Z)Y) - \frac{1}{2}Y\left(\frac{h}{f}\right)R(u,Z)X \\
&- \frac{h}{2f}\nabla_Y R(u,Z)X - \frac{h}{2f}A_f(Y,R(u,Z)X) \\
&+ \frac{1}{4f}Y(h)R(u,Z)X - \frac{h}{2f}R(u,Z)\nabla_X Y \\
&- \frac{1}{4f}X(h)R(u,Z)Y + \frac{h}{2f}R(u,Z)\nabla_Y X \\
&+ \frac{h}{2f}R(u,\nabla_Y Z)X - \frac{h}{2f}R(u,\nabla_X Z)Y \\
&- \frac{1}{2f}g(Z,R(X,Y)u)\nabla^M h\Big)^H \\
&+ \Big(R(X,Y)Z - \frac{h}{4f}R(X,R(u,Z)Y)u \\
&+ \frac{1}{2}X\left(\frac{Y(h)}{h}\right)Z - \frac{1}{2h}(\nabla_X Y)(h)Z \\
&- \frac{1}{2}Y\left(\frac{X(h)}{h}\right)Z + \frac{h}{4f}R(Y,R(u,Z)X)u \\
&+ \frac{1}{2h}(\nabla_Y X)(h)Z\Big)^V
\end{aligned}
$$

$$
\begin{aligned}
6)\quad R^{f,h}(X^H,Y^V)Z^V = \Big(& -\frac{1}{2}X\left(\frac{1}{f}\right)g(Y,Z)\nabla^M h \\
& -\frac{1}{2f}g(Y,Z)(\nabla_X\nabla^M h + A_f(X,\nabla^M h)) \\
& -\frac{h}{2f}R(Y,Z)X - \frac{h^2}{4f^2}R(u,Y)R(u,Z)X \\
& +\frac{1}{4fh}X(h)g(Y,Z)\nabla^M h\Big)^H \\
& +\Big(\frac{1}{4f}g(Y,Z)R(X,\nabla^M h)u \\
& -\frac{1}{2h}(R(u,Z)X)Y\Big)^V
\end{aligned}
$$

for all vectors (u,X,Y,Z) *in the tangent space* T_xM *of Riemannian manifold* M.

Proof:

$$
\begin{aligned}
\nabla^{f,h}_{X^H}\nabla^{f,h}_{Y^H}Z^H &= \nabla^{f,h}_{X^H}(\nabla_Y Z)^H + \nabla^{f,h}_{X^H}(A_f(Y,Z))^H - \frac{1}{2}\nabla^{f,h}_{X^H}(R(Y,Z)u)^V \\
&= \Big(\nabla_X\nabla_Y Z + A_f(X,\nabla_Y Z) + \nabla_X A_f(Y,Z) \\
&\quad + A_f(X,A_f(Y,Z)) - \frac{h}{4f}R(u,R(Y,Z)u)X\Big)^H \\
&\quad + \Big(-\frac{1}{2}R(X,\nabla_Y Z)u - \frac{1}{2}R(X,A_f(Y,Z))u \\
&\quad - \frac{1}{2}\nabla_X(R(Y,Z)u) - \frac{1}{4h}X(h)R(Y,Z)u\Big)^V
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\nabla^{f,h}_{Y^H}\nabla^{f,h}_{X^H}Z^H &= \nabla^{f,h}_{Y^H}(\nabla_X Z)^H + \nabla^{f,h}_{Y^H}(A_f(X,Z))^H - \frac{1}{2}\nabla^{f,h}_{Y^H}(R(X,Z)u)^V \\
&= \Big(\nabla_Y\nabla_X Z + A_f(Y,\nabla_X Z) + \nabla_Y A_f(X,Z) \\
&\quad + A_f(Y,A_f(X,Z)) - \frac{h}{4f}R(u,R(X,Z)u)Y\Big)^H \\
&\quad + \Big(-\frac{1}{2}R(Y,\nabla_X Z)u - \frac{1}{2}R(Y,A_f(X,Z))u \\
&\quad - \frac{1}{2}\nabla_Y(R(X,Z)u) - \frac{1}{4h}Y(h)R(X,Z)u\Big)^V
\end{aligned} \tag{8}
$$

$$\nabla^{f,h}_{[X^H,Y^H]}Z^H = \left(\nabla_{[X,Y]}Z + A_f([X,Y],Z) - \frac{h}{2f}R(u,R(X,Y)u)Z\right)^H - \left(\frac{1}{2}R([X,Y],Z)u + \frac{1}{2h}Z(h)R(X,Y)u\right)^V. \tag{9}$$

From equations (7)-(9) and the second Bianchi identity

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0 \tag{10}$$

we obtain the formula (1). The other formulae are obtained by a similar calculation. ■

3.1. Curvature Properties

Theorem 7 *The tangent bundle $(TM, G^{f,h})$ is flat if and only if the following assertions hold*

1) (M,g) *is flat*

2) $\|\nabla^M h\| = 0$

3) *The function f satisfies the* PDE

$$\begin{aligned} T_f(X,Y,Z) = {} & A_f(X,\nabla_Y Z) + \nabla_X A_f(Y,Z) + A_f(X,A_f(Y,Z)) \\ & - A_f(Y,\nabla_X Z) - \nabla_Y A_f(X,Z) - A_f(Y,A_f(X,Z)) \\ & - A_f([X,Y],Z) = 0 \quad \text{for all} \quad X,Y,Z \in \Gamma(TM) \end{aligned}$$

and

$$A_f(X,Y) = \frac{X(f)Y + Y(f)X - g(X,Y)\nabla^M f}{2f(p)}.$$

Proof: If $\|\nabla^M h\| = 0$ and $T_f(X,Y,Z) = 0$, then $R \equiv 0$ implies $R^{f,h} \equiv 0$. If we assume that $R^{f,h} \equiv 0$ and calculate the Riemann curvature tensor at $(p,0)$ we have

$$\begin{aligned}
R^{f,h}_{(p,0)}(X^H,Y^H)Z^H &= R(X,Y)Z + A_f(X,\nabla_Y Z) + \nabla_X A_f(Y,Z) \\
&\quad + A_f(X,A_f(Y,Z)) - A_f(Y,\nabla_X Z) - A_f(Y,A_f(X,Z)) \\
&\quad - \nabla_Y A_f(X,Z) - A_f([X,Y],Z) = 0 \\
R^{f,h}_{(p,0)}(X^H,Y^H)Z^V &= R(X,Y)Z + \frac{1}{2}X\left(\frac{Y(h)}{h}\right)Z - \frac{1}{2h}(\nabla_X Y)(h)Z \\
&\quad - \frac{1}{2}Y\left(\frac{X(h)}{h}\right)Z + \frac{1}{2h}(\nabla_Y X)(h)Z = 0 \\
R^{f,h}_{(p,0)}(X^H,Y^V)Z^H &= \frac{1}{2}R(X,Z)Y + \frac{1}{2}X\Big(\frac{Z(h)}{h}\Big)Y + \frac{1}{4h^2}Z(h)X(h)Y \\
&\quad - \frac{1}{2h}(\nabla_X Z)(h)Y - \frac{1}{2h}(A_f(X,Z))(h)Y = 0 \\
R^{f,h}_{(p,0)}(X^H,Y^V)Z^V &= -\frac{1}{2f}g(Y,Z)\Big(\nabla_X\nabla^M h + A_f(X,\nabla^M h)\Big) \\
&\quad - \frac{1}{2}X\Big(\frac{1}{f}\Big)g(Y,Z)\nabla^M h - \frac{h}{2f}R(Y,Z)X \\
&\quad + \frac{1}{4fh}X(h)g(Y,Z)\nabla^M h = 0 \\
R^{f,h}_{(p,0)}(X^V,Y^V)Z^H &= \frac{h}{f}R(X,Y)Z = 0 \\
R^{f,h}_{(p,0)}(X^V,Y^V)Z^V &= \frac{\|\nabla^M h\|^2}{4f(p)h(p)}\Big(g(X,Z)Y - g(Y,Z)X\Big) = 0.
\end{aligned}$$

Then $R=0$, $\|\nabla^M h\|=0$ and $T_f(X,Y,Z)=0$. ∎

4. Einstein Structure

Let $\{e_1,..,e_n\}$ be an orthonormal basis of T_pM.
Then the family $\left\{e_1^H,..,e_n^H,e_1^V,..,e_n^V\right\}$ is an orthonormal basis of $T_{(p,u)}TM$. The Ricci curvature of $(TM,G^{f,h})$ is given by

$$\mathrm{Ric}^{f,h}(X^*,Y^*) = \sum_{i=1}^{n} G^{f,h}\Big(R^{f,h}(X^*,e_i^H)Y^*,e_i^H) + G^{f,h}(R^{f,h}(X^*,e_i^V)Y^*,e_i^V\Big).$$

Using the Theorem 6 and the above equation, we have

$$
\begin{aligned}
1)\quad \mathrm{Ric}^{f,h}(X^H,Y^H) &= \mathrm{Ric}(X,Y) - \frac{nh(p)}{2h}(A_f(X,Y))(h) \\
&+ f(p)\sum_{i=1}^{n} g\Big(A_f(X,\nabla_{e_i}Y) + \nabla_X A_f(e_i,Y) \\
&+ A_f(X,A_f(e_i,Y)) - \frac{h}{4f}R(u,R(e_i,Y)u)X \\
&- A_f(e_i,\nabla_X Y) - \nabla_{e_i}A_f(X,Y) - A_f(e_i,A_f(X,Y)) \\
&- A_f([X,e_i],Y) + \frac{h}{2f}R(u,R(X,e_i)u)Y, e_i\Big) \\
&+ h(p)\sum_{i=1}^{n} g\Big(-\frac{h}{4f}R(X,R(u,e_i)Y)u, e_i\Big) \\
&- \frac{nh(p)}{2h}(\nabla_X Y)(h) + \frac{nh(p)}{2}X\Big(\frac{Y(h)}{h}\Big) + \frac{n}{4h}Y(h)X(h)
\end{aligned}
$$

$$
\begin{aligned}
2)\quad \mathrm{Ric}^{f,h}(X^H,Y^V) &= \frac{h(p)}{4f}g(e_i,Y)\sum_{i=1}^{n} g(R(X,\nabla^M h)u,e_i) \\
&+ f(p)\sum_{i=1}^{n} g\Big(\frac{h}{2f}\nabla_X R(u,Y)e_i \\
&- \frac{1}{2}e_i\Big(\frac{h}{f}\Big)R(u,Y)X - \frac{h}{2f}\nabla_{e_i}R(u,Y)X \\
&- \frac{h}{2f}R(u,Y)\nabla_X e_i + \frac{h}{2f}R(u,Y)\nabla_{e_i}X \\
&- \frac{h}{2f}A_f(e_i,R(u,Y)X) + \frac{1}{4f}e_i(h)R(u,Y)X \\
&+ \frac{h}{2f}R(u,\nabla_{e_i}Y)X - \frac{1}{2f}g(Y,R(X,e_i)u)\nabla^M h \\
&+ \frac{h}{2f}A_f(X,R(u,Y)e_i),e_i\Big)
\end{aligned}
$$

$$
\begin{aligned}
3)\quad \mathrm{Ric}^{f,h}(X^V,Y^V) &= \frac{\|\nabla^M h\|^2}{4f(p)}g(X,Y)(n-1) \\
&+ \frac{1}{2}g(X,Y)\sum_{i=1}^{n} g\Big(\nabla_{e_i}\nabla^M h + A_f(e_i,\nabla^M h) \\
&+ \frac{1}{2}g(X,Y)e_i(h)\Big(e_i\Big(\frac{1}{f}\Big) - \frac{1}{2h}e_i(h)\Big) \\
&+ \frac{h^2}{4f}\mathrm{Tr}(R(u,X)R(u,Y)), e_i\Big).
\end{aligned}
$$

Theorem 8 *Let $(TM, G^{f,h})$ be an Einstein manifold. Then (M, g) is Einstein manifold if f, h satisfies the PDE*

$$\begin{aligned}
&f(p)\sum_{i=1}^{n} g\Big(A_f(X, \nabla_{e_i}Y) + \nabla_X A_f(e_i, Y) + A_f(X, A_f(e_i, Y)) \\
&\quad - A_f(e_i, \nabla_X Y) - \nabla_{e_i} A_f(X, Y) - A_f(e_i, A_f(X, Y)) - A_f([X, e_i], Y), e_i\Big) \\
&\quad + \frac{nh(p)}{2} X(\frac{Y(h)}{h}) + \frac{n}{4h} Y(h)X(h) - \frac{nh(p)}{2h}(\nabla_X Y)(h) \\
&\quad - \frac{nh(p)}{2h}(A_f(X, Y))(h) = 0.
\end{aligned}$$

Proof: We suppose that $(TM, G^{f,h})$ is Einstein and therefore

$$\mathrm{Ric}^{f,h}(X^*, Y^*) = \lambda G^{f,h}(X^*, Y^*). \tag{11}$$

If we take $u = 0$, we have

$$\begin{aligned}
\mathrm{Ric}^{f,h}(X^H, Y^H) = \mathrm{Ric}(X, Y) &+ f(p)\sum_{i=1}^{n} g\Big(A_f(X, \nabla_{e_i}Y) + \nabla_X A_f(e_i, Y) \\
&+ A_f(X, A_f(e_i, Y)) - A_f(e_i, \nabla_X Y) - \nabla_{e_i} A_f(X, Y) \\
&- A_f(e_i, A_f(X, Y)) - A_f([X, e_i], Y), e_i\Big) \\
&+ \frac{nh(p)}{2} X\Big(\frac{Y(h)}{h}\Big) + \frac{n}{4h} Y(h)X(h) - \frac{nh(p)}{2h}(\nabla_X Y)(h) \\
&- \frac{nh(p)}{2h}(A_f(X, Y))(h)
\end{aligned}$$

and in this way obtain

$$\begin{aligned}
\mathrm{Ric}(X, Y) = \lambda g(X, Y)& \\
&- \Big(f(p)\sum_{i=1}^{n} g\Big(A_f(X, \nabla_{e_i}Y) + \nabla_X A_f(e_i, Y) + A_f(X, A_f(e_i, Y) \\
&- A_f(e_i, \nabla_X Y) - \nabla_{e_i} A_f(X, Y) - A_f(e_i, A_f(X, Y)) \\
&- A_f([X, e_i], Y), e_i\Big) + \frac{nh(p)}{2} X\Big(\frac{Y(h)}{h}\Big) + \frac{n}{4h} Y(h)X(h) \\
&- \frac{nh(p)}{2h}(\nabla_X Y)(h) - \frac{nh(p)}{2h}(A_f(X, Y))(h)\Big).
\end{aligned}$$

■

4.1. Sectional Curvature

Theorem 9 *The sectional curvatures $K^{f,h}$ of the tangent bundle $(TM, G^{f,h})$ is given by the formulas*

$$1)\quad K^{f,h}_{(p,u)}(X^V, Y^V) = -\frac{\|\nabla^M h\|^2}{4fh^2}$$

$$2)\quad K^{f,h}_{(p,u)}(X^H, Y^V) = \frac{h}{4f^2}\|R(u,Y)X\|^2 + B_{(f,h)}(X,Y)$$

$$3)\quad K^{f,h}_{(p,u)}(X^H, Y^H) = \frac{1}{f}K(X,Y) - \frac{3h}{4f^2}\|R(X,Y)u\|^2 + L_{(f,h)}(X,Y)$$

where

$$\begin{aligned} B_{(f,h)}(X,Y) &= \big(-\frac{1}{2h}X\Big(\frac{1}{f}\Big) + \frac{1}{4fh^2}X(h)\big)g(\nabla^M h, X) \\ &\quad - \frac{1}{2fh}g\big(\nabla_X\nabla^M h + A_f(X,\nabla^M h), X\big) \end{aligned}$$

$$\begin{aligned} L_{(f,h)}(X,Y) = \frac{1}{f}\Big(& g\big(A(X,\nabla_Y Y),X\big) + g\big(\nabla_X A_f(Y,Y),X\big) \\ & - g\big(A(Y,\nabla_X Y),X\big) - g\big(\nabla_Y A(X,Y),X\big) \\ & + g\big(A(X,A(Y,Y),X\big) - g\big(A(Y,A(X,Y)),X\big) \\ & - g\big(A([X,Y],Y),X\big)\Big). \end{aligned}$$

Proof: Let $(p,u) \in TM$ and $X, Y \in T_pM$ be two orthonormal tangent vectors at the point p

$$\begin{aligned} 1)\quad K^{f,h}_{(p,u)}(X^V, Y^V) &= \frac{G^{f,h}(R(X^V,Y^V)Y^V, X^V)}{G^{f,h}(X^V,X^V)G^{f,h}(Y^V,Y^V) - G^{f,h}(X^V,Y^V)} \\ &= \frac{G^{f,h}(R(X^V,Y^V)Y^V, X^V)}{h^2} \\ &= \frac{\|\nabla^M h\|^2}{4fh^3}G^{f,h}(g(X,Y)Y^V, X^V) \\ &\quad - \frac{\|\nabla^M h\|^2}{4fh^3}G^{f,h}(g(Y,Y)X^V, X^V) \\ &= \frac{\|\nabla^M h\|^2}{4fh^3}\Big(hg(X,Y)g(X,Y) - hg(X,X)g(Y,Y)\Big) \\ &= -\frac{\|\nabla^M h\|^2}{4fh^2} \end{aligned}$$

$$
\begin{aligned}
2)\quad K^{f,h}_{(p,u)}(X^H,Y^V) &= \frac{G^{f,h}(R(X^H,Y^V)Y^V,X^H)}{G^{f,h}(X^H,X^H)G^{f,h}(Y^V,Y^V)-G^{f,h}(X^H,Y^V)}\\
&= \frac{1}{fh}G^{f,h}(R(X^H,Y^V)Y^V,X^H)\\
&= \frac{1}{fh}G^{f,h}\Big(\Big(-\frac{1}{2}X\left(\frac{1}{f}\right)g(Y,Y)\nabla^M h\\
&\quad -\frac{1}{2f}g(Y,Y)(\nabla_X\nabla^M h + A_f(X,\nabla^M h)\\
&\quad -\frac{h^2}{4f^2}R(u,Y)R(u,Y)X+\frac{1}{4fh}X(h)\nabla^M h\Big)^H,X^H\Big)\\
&= \frac{h}{4f^2}\|R(u,Y)X\|^2 + B_{(f,h)}(X,Y)
\end{aligned}
$$

where

$$
\begin{aligned}
B_{(f,h)}(X,Y) = &\big(-\frac{1}{2h}X\left(\frac{1}{f}\right)+\frac{1}{4fh^2}X(h)\big)g(\nabla^M h,X)\\
&-\frac{1}{2fh}g\big(\nabla_X\nabla^M h + A_f(X,\nabla^M h),X\big)
\end{aligned}
$$

$$
\begin{aligned}
3)\quad K^{f,h}_{(p,u)}(X^H,Y^H) &= \frac{G^{f,h}(R(X^H,Y^H)Y^H,X^H)}{G^{f,h}(X^H,X^H)G^{f,h}(Y^H,Y^H)-G^{f,h}(X^H,Y^H)}\\
&= \frac{1}{f^2}G^{f,h}(R(X^H,Y^H)Y^H,X^H)\\
&= \frac{1}{f^2}G^{f,h}\Big(\Big(R(X,Y)Y + A_f(X,\nabla_Y Z)\\
&\quad +\nabla_X A_f(Y,Y)+A_f(X,A_f(Y,Y))-A_f(Y,\nabla_X Y)\\
&\quad -\nabla_Y A_f(X,Y)-A_f(Y,A_f(X,Y))\\
&\quad +\frac{h}{4f}R(u,R(X,Y)u)Y - A_f([X,Y],Y)\\
&\quad +\frac{h}{2f}R(u,R(X,Y)u)Y\Big)^H,X^H\Big)\\
&= \frac{1}{f}K(X,Y)-\frac{3h}{4f^2}\|R(X,Y)u\|^2 + L_{(f,h)}(X,Y)
\end{aligned}
$$

where

$$
\begin{aligned}
L_{(f,h)}(X,Y) = \frac{1}{f}\Big(& g\big(A(X,\nabla_Y Y),X\big) + g\big(\nabla_X A_f(Y,Y),X\big) \\
& + g\big(A(X,A(Y,Y),X\big) - g\big(A(Y,\nabla_X Y),X\big) \\
& - g\big(\nabla_Y A(X,Y),X\big) - g\big(A(Y,A(X,Y)),X\big) \\
& - g\big(A([X,Y],Y),X\big)\Big).
\end{aligned} \tag{12}
$$

■

Corollary 10 *Let (M,g) be a n-dimensional Riemannian manifold of constant sectional curvature k. Then*

1) $K^{f,h}_{(p,u)}\Big(X^V,Y^V\Big) = -\dfrac{\|\nabla^M h\|^2}{4fh^2}$

2) $K^{f,h}_{(p,u)}\Big(X^H,Y^V\Big) = \dfrac{h}{4f^2}\lambda^2 g(u,X)^2 + B_{(f,h)}(X,Y)$

3) $K^{f,h}_{(p,u)}\Big(X^H,Y^H\Big) = \dfrac{1}{f}\lambda - \dfrac{3h}{4f^2}\lambda^2\Big(g(X,u)^2 + g(Y,u)^2\Big) + L_{(f,h)}(X,Y)$

where ∇^M is the gradient of the Riemannian manifold (M,g).

Proof: Using the Theorem 9 and the following equations

$$
\begin{aligned}
\|R(X,Y)u\|^2 &= \lambda^2(g(X,u)^2 + g(Y,u)^2) \\
\|R(u,Y)X\|^2 &= \lambda^2 g(u,X)^2
\end{aligned}
$$

we can immediately get the result of corollary. ■

4.2. Scalar Curvature

Theorem 11 *Let S (respectively $S^{f,h}$) be a scalar curvature of (M,g) (respectively $(TM,G^{f,h})$). Then*

$$
\begin{aligned}
S^{f,h}_{(p,u)} = \frac{1}{f^3}S_p + \sum_{i,j=1}^{n}\Bigg(& \frac{h}{2f^3}\big(1-\frac{3}{2f}\big)\|R(e_i,e_j)u\|^2 + \frac{1}{f^2}L_{(f,h)}(e_i,e_j) \\
& + \frac{2}{hf}B_{(f,h)}(e_i,e_j)\Bigg) - \frac{1}{4fh^4}\|\nabla^M h\|^2.
\end{aligned}
$$

Proof: Let $\{e_1,...,e_n\}$ orthonormal frame on M.
Then $\left\{\dfrac{1}{\sqrt{f}}e_1^H,...,\dfrac{1}{\sqrt{f}}e_n^H,\dfrac{1}{\sqrt{h}}e_1^V,...,\dfrac{1}{\sqrt{h}}e_n^V\right\}$ are a local orthonormal frame on

TM. Locally we obtain

$$\begin{aligned} S^{f,h}_{(p,u)} &= \sum_{i,j=1}^{n}\hat{K}\Big(\frac{1}{\sqrt{f}}e_i^H,\frac{1}{\sqrt{f}}e_j^H\Big)+\sum_{i,j=1}^{n}\hat{K}\Big(\frac{1}{\sqrt{f}}e_i^H,\frac{1}{\sqrt{h}}e_j^V\Big)\\ &\quad+\sum_{i,j=1}^{n}\hat{K}\Big(\frac{1}{\sqrt{h}}e_i^V,\frac{1}{\sqrt{h}}e_j^V\Big)\\ &= \frac{1}{f^2}\sum_{i,j=1}^{n}\hat{K}\Big(e_i^H,e_j^H\Big)+\frac{1}{fh}\sum_{i,j=1}^{n}\hat{K}\Big(e_i^H,e_j^V\Big)+\frac{1}{h^2}\sum_{i,j=1}^{n}\hat{K}\Big(e_i^V,e_j^V\Big)\\ &= \sum_{i,j=1}^{n}\Big(\frac{1}{f^3}K(e_i,e_j)-\frac{3h}{4f^4}\|R(e_i,e_j)u\|^2+\frac{1}{f^2}L_{(f,h)}(e_i,e_j)\\ &\quad+\frac{h}{2f^3}\|R(u,e_j)e_i\|^2+\frac{2}{hf}B_{(f,h)}(e_i,e_j)\Big)-\frac{1}{4fh^4}\|\nabla^M h\|^2. \end{aligned}$$

In order to simplify this last expression, we put $u=\sum_{i=1}^{n}u_iX_i$, and therefore $\sum_{i,j=1}^{n}\|R(e_i,u)e_j\|^2=\sum_{i,j=1}^{n}\|R(e_i,e_j)u\|^2$. ■

Proposition 12 $(TM,G^{f,h})$ *has constant scalar curvature if and only if* (M,g) *is flat and f,h satisfies the* PDE

$$\sum_{i,j=1}^{n}\Big(\frac{1}{f}L_{(f,h)}(e_i,e_j)+\frac{2}{h}B_{(f,h)}(e_i,e_j)\Big)-\frac{1}{4h^4}\|\nabla^M h\|^2-\frac{a}{f}=0. \qquad (13)$$

Proof: If $S^{f,h}_{(p,0)}=S^{f,h}_0$ is constant scalar curvature of $(TM,G^{f,h})$, then the function $p\to S^{f,h}_{(p,0)}=S^{f,h}_0$ is constant on (M,g) , equal to $S^{f,h}_0$

$$S^{f,h}_{(p,0)}=\frac{1}{f^3}S_p+\sum_{i,j=1}^{n}\Big(\frac{1}{f^2}L_{(f,h)}(e_i,e_j)+\frac{2}{hf}B_{(f,h)}(e_i,e_j)\Big)-\frac{1}{4fh^4}\|\nabla^M h\|^2.$$

Then

$$S^{f,h}_{(p,u)}=\frac{1}{f^3}S_0+\frac{h}{2f^3}(1-\frac{3}{2f})\sum_{i,j=1}^{n}\|R(e_i,e_j)u\|^2$$

ho give $\sum_{i,j=1}^{n}\|R(e_i,e_j)u\|^2=0$ and $R\equiv 0$ (i.e. (M,g) is flat). Then scalar curvature of (M,g) is $S_p=0$, then

$$f\left(\sum_{i,j=1}^{n}\Big(\frac{1}{f}L_{(f,h)}(e_i,e_j)+\frac{2}{h}B_{(f,h)}(e_i,e_j)\Big)-\frac{1}{4h^4}\|\nabla^M h\|^2\right)=a\,(\text{constant})$$

or

$$\sum_{i,j=1}^{n}\Big(\frac{1}{f}L_{(f,h)}(e_i,e_j)+\frac{2}{h}B_{(f,h)}(e_i,e_j)\Big)-\frac{1}{4h^4}\|\nabla^M h\|^2-\frac{a}{f}=0.$$

■

Conclusion

Several authors have studied the geometry of the tangent bundle TM endowed with different metrics, of a Riemannian manifold (M,g). We introduce a new kind of metrics denoted $G^{f,h}$ on TM as multiplication with strictly positive smooth functions in M, in the vertical and horizontal parts of Sasaki metric. We call this metric the twisted Sasaki metric. After, computing its Levi-Civita and the curvature tensor, we studied the geometry of $(TM,G^{f,h})$ by giving a relationships of the curvatures, Einstein structure, scalar and sectional curvatures between $(TM,G^{f,h})$ and (M,g). A non-flat metric $G^{f,h}$ has been obtained if the functions f,h does not satisfy the Eikonel equation and the metric $G^{f,h}$ can be Einstein metric on TM without the basic metric g being flat.

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