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RELATIONS BETWEEN LAPLACE SPECTRA AND GEOMETRIC QUANTIZATION OF REIMANNIAN SYMMETRIC SPACES*

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Abstract. We consider a modified Kostant-Souriau geometric quantization scheme due to Czyz and Hess for Hamiltonian systems on the cotangent bundles of compact rank-one Riemannian symmetric spaces (CROSS). It is used, together with a symplectic reduction process, to relate its energy spectrum to the spectrum of the Laplace-Beltrami operator. Moreover, the corresponding eigenspaces have real dimension equal to the complex dimension of the space of the holomorphic sections of the quantum bundle which is obtained after the quantization. The relation between the two constructions was first noticed by Mladenov and Tsanov for the case of the spheres. In addition to the CROSS case, we announce preliminary results related to the case of compact Riemannian symmetric spaces of higher rank.

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Contents

*Dedicated to the memory of Professor Vasil V. Tsanov 1948-2017.

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1. Introduction

It is a well-known fact that on a Riemannian manifold (M, g) all of whose geodesics are closed there is a natural S^1-action and that this action extends to the cotangent bundle of M. The geodesic flow can be realized as solutions to an S^1-invariant Hamiltonian system on T^*M. For such systems, under mild conditions, there is a moment map and a symplectic reduction process, called also Marsden-Weinstein reduction. This reduction produces a reduced space $T^*M//S^1$ that can be identified with the space parametrizing all geodesics and that is equipped with an induced symplectic form. The induced symplectic form depends on a level set of the corresponding moment map which we call the energy level of the geodesic flow. In many examples the cotangent bundle has a "complex polarization" - a complex structure compatible with the symplectic form which becomes Kähler form. A natural question which has its origin in the relation between the Kepler's laws and the hydrogen atom is when such manifold could be "quantized". The geometric quantization is not a uniquely defined notion and there are various schemes which implement it - see, for example, [20, 21] and the references therein. One of them, originally due to [6, 14], is a twisted version of Kostant-Souriau scheme and assigns a holomorphic line bundle with first Chern class given by the induced Kähler form with an added extra term. This new term is half of the first Chern class of the canonical bundle of the manifold M. The quantum condition is the integrality of that corrected form, while the analog of the Hilbert space of quantum observables is the space of holomorphic sections of the line bundle.

One of the first examples that was quantized geometrically according to the scheme described above is the case when M is the n-dimensional sphere $\mathbb{S}^n$. This example was treated in [18] by Mladenov and Tsanov and it was observed that the quantum condition leads to an energy spectrum which is, up to an additive constant, equal to the spectrum of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$. Also, the (complex) dimension of the quantum Hilbert space equals the (real) dimension of the corresponding eigenspace of $\Delta_{\mathbb{S}^n}$. One explanation of this fact is that both spaces are natural irreducible representations of $\mathrm{SO}(n+1)$. A main purpose of the present note is to initiate the representation theory perspective of a similar relation between the geometric quantization and harmonic analysis on the compact Riemannian symmetric spaces. More precisely, we relate the spectrum of the Laplace-Beltrami operator to the quantized energy levels of a Hamiltonian system on the space of all tangents to maximal totally geodesic tori on a Riemannian symmetric space. Also, via symplectic reduction we relate the multiplicities of the eigenvalues to the dimensions of the spaces of holomorphic sections of the corresponding quantum bundles over the reduced space - which is also a generalized flag manifold.

This paper will follow an unorthodox approach - we present some detailed examples first, and then underline the general (representation) theory behind them. More specifically, in the next section we present the quantization procedure of the energy of the geodesic flow on $\mathbb{CP}^n$ and $\mathbb{HP}^n$, the complex and quaternionic projective spaces. We compare the energy spectra to the spectra of the Laplace-Beltrami operators and the multiplicities of the corresponding levels to the dimensions of the quantum bundles. We note that a quantization scheme for these projective spaces have been considered also in [7, 8, 19] and results similar to ours were independently obtained in [15, 16]. In Section 3 we explain the necessary preliminaries from the structure theory of simple Lie algebras and its relation to the Riemmanian symmetric spaces and generalized flag manifolds. Then in Section 4 we focus on and work out the rank-one case. This case includes the examples of the spheres and the projective spaces, and we treat these examples in the language of classical representation theory. In the following section we turn to the general case. We observe that we can substitute the space parametrizing all geodesics with the space of all maximal totally geodesic flat submanifolds. This is again a flag manifold and carries a natural "polarization" which could be used for the quantization - a Kähler complex structure. Since our aim is to underline the geometric approach through the Marsden-Weinstein reduction, we also need a Kähler space with a (multi-valued) Hamiltonian that, after the symplectic reduction, will become the generalized flag manifold with an appropriate reduced symplectic form, a form which is also integral and Kähler. This is done in [11] via construction of a Kähler structure on some open subset of the manifold of all tangent spaces of the maximal totally geodesic flat submanifold. Then we announce the main result in Section 5 - Theorem 15. We finish with one of the first examples of symmetric space of rank two - $M = \mathrm{SU}(3)/\mathrm{SO}(3)$. The flag manifold parametrizing the totally geodesic two-tori is the usual flag $F = \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1))$. We note that the eigenspaces of the Laplace-Beltrami Δ_M are not necessary irreducible $\mathrm{SU}(3)$-modules and compare them to the spaces of holomorphic sections of the sums of the corresponding line bundles on F.

Notation and conventions: We adopt the following notations. By $\mathbb{C}^n$ and $\mathbb{R}^n$ we denote the standard complex and real n-dimensional vector spaces, while $\mathbb{H}^n$ stands for the real vector space of n-tuples of quaternions on which $\mathbb{H}$ acts on the left. For a complex number z, $\Re(z)$ and $\Im(z)$ are the real and imaginary part of z, respectively. We set $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ when x and y are real and $\langle x, y\rangle_R = \langle x, y\rangle$ when $x, y \in \mathbb{R}^n$, $\langle x, y\rangle_C = 2\Re\langle x, \overline{y}\rangle$ when $x, y \in \mathbb{C}^n$, and $\langle x, y\rangle_H = 4\Re\langle x, \overline{y}\rangle$ when $x, y \in \mathbb{H}^n$ respectively. Similarly $||x||_R, ||x||_C, ||x||_H$ are the corresponding real norms. By $\mathbb{S}^n, \mathbb{CP}^n, \mathbb{HP}^n$ we denote the n-dimensional sphere, complex projective space and quaternionic projective space, respectively.

2. Symplectic Reduction and Quantization of the Geodesic Flow of the Complex and Quaternionic Projective Spaces

We first recall Marsden-Weinstein (or symplectic) reduction and the modified geometric quantization scheme due to Czyz [6] and Hess [14]. As indicated in the introduction, this scheme was used by Mladenov and Tsanov [18], who related the energy spectrum of the geodesic flow on a sphere with the eigenvalues of the Laplace-Beltrami operator. We refer to [18] for the details in the case of the sphere. In what follows we present a similar computations for two of the other compact rank-one Riemannian symmetric spaces (CROSSes), $\mathbb{CP}^n$ and $\mathbb{HP}^n$. For more details on the Marsden-Weinstein (or symplectic) reduction we refer the reader, for example, to [1].

If (M, ω) is a symplectic manifold and H is a function on M, then the vector field X_H defined as $\mathrm{d}H(Y) = \omega(X_H, Y)$ is called Hamiltonian vector field. We will call H a Hamiltonian function and the triple (M, ω, H) - a Hamiltonian system. If G is a group of symplectomorphisms then under mild conditions there is a map $\mu : M \to \mathfrak{g}^*$, defined by

$$\mathrm{d}\mu(X) = i_X\omega$$

where $\mathfrak{g}$ is the Lie algebra of G and $X \in \mathfrak{g}$ is identified with the induced vector field on M. When such μ exists, the action is called *Hamiltonian* and the space $N = \mu^{-1}(c)/G$ is called the Marsden-Weinstein reduction or the symplectic reduction, where c is a fixed element of the adjoint action of G on $\mathfrak{g}^*$. We denote N by $M//G$. The space $M//G$ inherits a natural symplectic form ω_{red} such that $i^*(\omega) = \pi^*(\omega_{\mathrm{red}})$ where $i : \mu^{-1}(c) \to M$ is the inclusion and $\pi : \mu^{-1}(c) \to N = \mu^{-1}(c)/G$ is the natural projection. The following result will be use repeatedly in the paper.

Proposition 1. *If $(N, \omega_{\mathrm{red}})$ is the symplectic reduction of (M, ω) under the action of a Lie group G and H is a G-invariant function on M, then there is a unique function H_{red} on N such that $\pi^*(H_{\mathrm{red}}) = i^*(H)$. Moreover the flow of the vector field X_H preserves $\mu^{-1}(c)$ and projects on N to the flow of the vector fields $X_{H_{\mathrm{red}}}$. Moreover, if we have a second Hamiltonian action of a Lie group G_1 on M which commutes with the action of G, then the level sets of its moment map μ_1 are $G-$invariant and $\mu_1|_{\mu^{-1}(c)} = \pi^*(\overline{\mu}_1)$ where $\overline{\mu}_1$ is the moment map associated to the action of G_1 on N.*

In [18] the space of oriented geodesics of $M = \mathbb{S}^n$ is explicitly identified with the complex quadric in $\mathbb{CP}^n$ via the Marsden-Weinstein reduction. It was noted that the energy levels of the moment map that satisfy a quantization condition coincide, up to an additive constant, with the eigenvalues of the Laplace-Beltrami operator and the their multiplicity are the same as the (complex) dimension of the holomorphic sections of the corresponding quantum bundle. Below we provide details on the next two of the classical examples - the complex and quaternionic projective space.

2.1. Complex Projective Space

For a point $[z] = [z_0, z_1, ..., z_n]$ in the complex projective space $\mathbb{CP}^n$, we identify the holomorphic cotangent space

$$T^*_{[u]}\mathbb{CP}^n \cong \{(u,v) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n=1};\ \langle u, v\rangle_C = 0\}.$$

To achieve a global description of the tangent bundle, we use the Hopf map $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ which is induced by the standard action of S^1 on $\mathbb{S}^{2n+1}$. This map is defined by $u \mapsto [u]$, where $u \in \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ with $||u|| = 1$. After identifying the tangent and cotangent bundles of the sphere via the canonical metric, we can identify the cotangent bundle as

$$T^*\mathbb{S}^{2n+1} = \{(u,v) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1};\ ||u|| = 1, \langle u, v\rangle_R = 0\}.$$

Then the S^1-action ρ for the Hopf projection π extends to $T^*\mathbb{S}^{2n+1}$ as

$$\rho(\mathrm{e}^{\mathrm{i}\theta})(u,v) = (\exp(\mathrm{i}\theta)u, \exp(\mathrm{i}\theta)v).$$

This action preserves the canonical symplectic form on $T^*\mathbb{S}^{2n+1}$, which is given by $i^*\Re\,(\mathrm{d}u \wedge \mathrm{d}\overline{v})$. The moment map for the action ρ can be used to show the following theorem. This theorem is first proven in [7], but for reader's convenience a short proof is presented.

Lemma 2. *The space $T^*\mathbb{CP}^n$ is biholomorphic to X_C and diffeomorphic to $\widetilde{X}_C$ where*

$$X_C \cong \{[u,v];\ \langle u,u\rangle_C = 1, \langle u,v\rangle_C = 0\}$$

with $[u,v]$ representing the class of (u,v) under $(u,v) \sim (\exp(\mathrm{i}\theta)u, \exp(\mathrm{i}\theta)v)$ and

$$\widetilde{X}_C \cong \{[[u,v]];\ \langle u,u\rangle_C = \langle v,v\rangle_C, \langle u,v\rangle = 0\}$$

with $[[u,v]]$ defined by the relation $(u,v) \sim (\exp(\mathrm{i}\theta)u, \exp(-\mathrm{i}\theta)v)$.

Proof: It is well-known that under the action ρ, $T^*\mathbb{CP}^n = T^*\mathbb{S}^{2n+1}//S^1$. The moment map Φ associated to the action ρ is simply $\Phi(u,v) = \Im\langle u,v\rangle_C$. Hence, $T^*\mathbb{S}^{2n+1}//S^1 = \Phi^{-1}(\mu)/S^1$, for a generic $\mu \in \mathbb{R} = \mathrm{i}\mathfrak{u}(1)$, is identified with X_C which gives the first diffeomorphism. The fact that it is a biholomorphism follows from the fact that the reduction is actually Kähler, provided that we consider the canonical form on $T^*\mathbb{S}^{2n+1}$ as a Kähler form for the complex structure induced from the embedding in $\mathbb{C}^{2n+2}$. Finally the diffeomorphism between X_C and $\widetilde{X}_C$ is given by the formulas

$$\tilde{u}_k = \frac{1}{\sqrt{2}}(\langle v, v\rangle_C u_k + \mathrm{i} v_k), \qquad \tilde{v}_k = \frac{1}{\sqrt{2}}(\overline{v}_k - \mathrm{i}\langle v, v\rangle_C \overline{u}_k).$$

■

The geodesic flow on a Riemannian manifold is represented as a Hamiltonian flow on its cotangent bundle. The cotangent bundle of each Riemannian manifold (M, g) has a canonical symplectic form given in local coordinates as $\sum \mathrm{d}x_i \wedge \mathrm{d}y_i$ where $(x_1, ..., x_n)$ are local coordinates of M and $(x_1, ..., x_n, y_1, ..., y_n)$ are the associated local coordinates of T^*M. Then the function $H(x, v) = \frac{1}{2}g(v, v)$ for $x \in M$ and $v \in T^*_x M$ has a Hamiltonian vector field X_H and its flow lines project on M to give the geodesics.

By Proposition 1, if a Lie group G of isometries acts on M, this action induces a Hamiltonian action on T^*M and the reduced space $T^*M//G$ becomes a (reduced) Hamiltonian system. Whenever $T^*M//G = T^*N$ for some Riemannian manifold N then the solutions of the new system is precisely the geodesic flow on N. In the particular case of $T^*\mathbb{S}^{2n+1}$ we obtain the following.

Proposition 3. *The canonical symplectic form Ω_C on $T^*\mathbb{CP}^n \cong X_C$ is*

$$\Omega_C = \frac{1}{2}(\mathrm{d}u \wedge \mathrm{d}\overline{v} + \mathrm{d}\overline{u} \wedge \mathrm{d}v)$$

and the Hamiltonian system $\mathcal{H}_{\mathbb{CP}^n} = (X_c, \Omega_C, H_C = \frac{||v||^2}{2})$ induces the geodesic flow on $\mathbb{CP}^n$. The system is equivalent to $(\widetilde{X}_C, \widetilde{\Omega}_C, \widetilde{H}_C)$ in view of the diffeomorphism in Lemma 2.

Since the orbits of $\mathcal{H}_{\mathbb{CP}^n}$ correspond precisely to the geodesics of $\mathbb{CP}^n$, we first identify the space parametrizing the geodesics. For this we first consider the geodesic flow on the sphere $\mathbb{S}^{2n+1}$. Since all of the geodesics on the sphere are closed, the flow of X_H in the cotangent space has also only closed trajectories. They define an S^1-action which is given by $(u, v) \to (\exp(\mathrm{i}\theta)u, \exp(-\mathrm{i}\theta)v)$. This action commutes with the action inducing the Hopf projection and is Hamiltonian. So it defines an action on $T^*\mathbb{CP}^n$ which has orbits - the flow lines of the Hamiltonian vector field defining the geodesics on $\mathbb{CP}^n$. We can identify a geodesic $c(t)$ in $\mathbb{CP}^n$ with the line $(c(t), c'(t))$ in $T\mathbb{CP}^n \simeq T^*\mathbb{CP}^n$ when t is a parameter such that c' has constant norm. From here we see that the space parametrizing the geodesics can be identified with the Marsden-Weisntein quotient. Let $N_c = \widetilde{H}_C^{-1}(c)/S^1$ be the reduced space. To identify N_c with a flag manifold, we use the Hamiltonian system $(\widetilde{X}_C, \widetilde{\Omega}_C, \widetilde{H}_C)$. Let

$$\begin{aligned}\mathbb{F} &= \{([x],[w]) \in \mathbb{CP}^n \times \mathbb{CP}^n;\ \langle z,w\rangle_C = 0\} \\ &= \{([x],[w]) \in \mathbb{CP}^n \times \mathbb{CP}^n;\ \langle z,z\rangle_C = \langle w,w\rangle_C = 1, \langle z,w\rangle = 0\}.\end{aligned}$$

One can see that $\mathbb{F}$ is biholomorphic to the (1,2)-flag in $\mathbb{C}^{n+1}$ with homogeneous representation $\mathbb{F} = \mathrm{U}(n+1)/\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(n-1)$. Denote by p_1 and p_2 the two projections on the corresponding factors of $\mathbb{CP}^n \times \mathbb{CP}^n$. Let α be the generator (the Fubini-Study form) of $H^2(\mathbb{CP}^n, \mathbb{Z})$. Then $\omega_1 = p_1^*\alpha$ and $\omega_2 = p_2^*\alpha$ are generators of $H^2(\mathbb{F}, \mathbb{Z})$. With this notation we have the following:

Proposition 4. *If $c \neq 0$ then the reduced manifold N_c is biholomorphic to the flag $\mathbb{F}$ and the reduced Kähler form is $\widetilde{\omega}_c = \pi\sqrt{2c}(\omega_1 + \omega_2)$.*

Proof: The S^1-action of the geodesic flow on $T^*\mathbb{CP}^n$ is induced from the one on $T^*\mathbb{S}^{2n+1}$. Hence this action is

$$\lambda(z,w) = (\lambda z, \lambda w)$$

for $(z,w) \in \widetilde{H}_C^{-1}(c)$. For the sphere $\mathbb{S}_R^{2n+1}$ of radius R the Hopf projection fits in the diagram $\mathbb{C}^{n+1} \xleftarrow{i} \mathbb{S}_R^{2n+1} \xrightarrow{h} \mathbb{CP}^n$ with $h^*\alpha = \frac{1}{\pi R^2} i^*\Omega$ (see [18]). If $\tilde{\pi}_c$ is the projection $\widetilde{H}_C^{-1}(c) \to N_c = \mathbb{F}$ then we have the following commutative diagram

$$\begin{tikzcd} \widetilde{H}_C^{-1}(c) \arrow[r, "\tilde{\pi}_c"] \arrow[d] & N_c \arrow[d, "\tilde{i}_c"] \\ \mathbb{S}^{2n+1} \times \mathbb{S}^{2n+1} \arrow[r, "h\times h"] & \mathbb{C}P^n \times \mathbb{C}P^n \end{tikzcd}$$

where the vertical arrows correspond to the natural embeddings. Therefore

$$\begin{aligned}\tilde{\pi}_c^*(\sqrt{2c}\pi(\omega_1 + \omega_2)) &= \pi\sqrt{2c}\frac{\mathrm{i}}{2\pi}\left(\frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{||z||^2} + \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{||w||^2}\right) \\ &= \frac{1}{\sqrt{2c}}\frac{\mathrm{i}}{2}(\mathrm{d}z \wedge \mathrm{d}\overline{z} + \mathrm{d}w \wedge \mathrm{d}\overline{w}) \\ &= \frac{1}{2}(\mathrm{d}u \wedge \mathrm{d}\overline{v} + \mathrm{d}\overline{u} \wedge \mathrm{d}v) = \tilde{i}_c^*(\tilde{\Omega}_c).\end{aligned}$$

■

In the above calculation we used that $\widetilde{H}_C(z,w) = c$, so $||z||^2 = ||w||^2 = 2c$.

Now recall some facts about the quantization scheme of Kostant and Souriau with the amends of Czyz and Hess.

Let X be a compact Kähler manifold with Kähler form λ. We say that the holomorphic line bundle L is a *quantum line bundle* if its first Chern class satisfies

$$c_1(L) = \frac{1}{2\pi}[\lambda] - \frac{1}{2}c_1(X).$$

Thus X will be quantizable if and only if $c_1(L) \in H^2(X, \mathbb{Z})$. The corresponding quantum Hilbert space is the (finite dimensional) linear space $H^0(X, \mathcal{O}(L))$.

Proposition 5. *We have $c_1(\mathbb{F}) = n(\omega_1 + \omega_2)$.*

Proof: We apply the adjunction formula for a hypersurface of degree (1,1) in $\mathbb{CP}^n \times \mathbb{CP}^n$ to obtain

$$\begin{aligned} c_1(\mathbb{F}) &= -(c_1(K_{\mathbb{CP}^n \times \mathbb{CP}^n}|_{\mathbb{F}}) + c_1([\mathbb{F}]|_{\mathbb{F}})) \\ &= c_1(\mathbb{CP}^n \times \mathbb{CP}^n)|_{\mathbb{F}} - c_1([\mathbb{F}]|_{\mathbb{F}}) \\ &= (n+1)(\omega_1 + \omega_2) - (\omega_1 + \omega_2) = n(\omega_1 + \omega_2). \end{aligned}$$

■

Theorem 6. *The energy spectrum of the geodesic flow on $\mathbb{CP}^n$ is*

$$E_k = \frac{1}{2}(n + 2k)^2, \qquad k \in \mathbb{N}$$

with corresponding multiplicities

$$m_k = \binom{n+k}{k}^2 - \binom{n+k-1}{k}^2.$$

Proof: For the exact cohomology sequence:

$$H^1(\mathbb{F}, \mathcal{O}) \to H^1(\mathbb{F}, \mathcal{O}^*) \to H^2(\mathbb{F}, \mathbb{Z}) \to H^2(\mathbb{F}, \mathcal{O})$$

and the identities $H^(\mathbb{F}, \mathcal{O}) = H^2(\mathbb{F}, \mathcal{O}) = 0$ follows that

$$c_1 : H^1(\mathbb{F}, \mathcal{O}^*) \xrightarrow{\cong} H^2(\mathbb{F}, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Therefore every holomorphic line bundle L on $\mathbb{F}$ is equivalent to $L_{k_1,k_2} = k_1\pi_1^*(H) + k_2\pi_2^*(H)$, where H is the hyperplane section on $\mathbb{CP}^n$.

The quantum condition on c is

$$\frac{1}{2\pi}[\omega_c] - \frac{1}{2}c_1(\mathbb{F}) = c_1(L_{k_1,k_2})$$

which implies

$$\frac{\sqrt{2c}}{2} - \frac{n}{2} = k$$

where $k = k_1 = k_2$ is a positive integer. In particular

$$c = \frac{1}{2}(2k+n)^2.$$

To count the multiplicities (i.e., $\dim H^0(\mathbb{F}, \mathcal{O}(L))$ we consider the exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{CP}^n \times \mathbb{CP}^n}(L_{k,k} \otimes L_{1,1}) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{CP}^n \times \mathbb{CP}^n}(L_{k,k}) \xrightarrow{r} \mathcal{O}|_{\mathbb{F}}(L_{k,k}) \to 0$$

where α is the multiplication of sections of $L_{k,k}$ by the polynomial $\sum_0^n z_i w_i$ which defines $\mathbb{F}$ in $\mathbb{CP}^n \times \mathbb{CP}^n$ and r is the restriction. The corresponding exact cohomology sequence gives

$$\begin{aligned} 0 &\to H^0(\mathbb{CP}^n \times \mathbb{CP}^n, \mathcal{O}(L_{k-1,k-1})) \to H^0(\mathbb{CP}^n \times \mathbb{CP}^n, \mathcal{O}(L_{k,k})) \\ &\to H^0(\mathbb{F}, \mathcal{O}(L_{k,k})) \to H^1(\mathbb{CP}^n \times \mathbb{CP}^n, \mathcal{O}(L_{k-1,k-1})) = 0 \end{aligned}$$

where the last term is zero by the Kodaira vanishing theorem. Thus we have

$$\begin{aligned} m_k &= \dim(H^0(\mathbb{F}, \mathcal{O}(L_{k,k})) \\ &= \dim(H^0(\mathbb{CP}^n \times \mathbb{CP}^n, \mathcal{O}(L_{k,k})) - \dim(H^0(\mathbb{CP}^n \times \mathbb{CP}^n, \mathcal{O}(L_{k-1,k-1}))) \\ &= \binom{n+k}{k}^2 - \binom{n+k-1}{k}^2. \end{aligned}$$

■

2.2. Quaternionic Projective Space

We first note that the results in this subsection were independently obtained in [15, 16]. The geodesic flow on $\mathbb{HP}^n$ can be described in a similar way as the one for $\mathbb{CP}^n$ but with the aid of the quaternionic Hopf map. For that we use three equivalent representations of $T^*\mathbb{S}^{4n+3}$

$$\begin{aligned} T^*\mathbb{S}^{4n+3} &= \{(x,y) \in \mathbb{R}^{4n+3} \times \mathbb{R}^{4n+3};\ ||x||_R = 1, \langle x, y\rangle_R = 0\} \\ &= \{(u,v) \in \mathbb{C}^{2n+2} \times \mathbb{C}^{2n+2};\ ||u||_C = 1, \Re\langle u, v\rangle_C = 0\} \\ &= \{(p,q) \in \mathbb{H}^{n+1} \times \mathbb{H}^{n+1};\ ||p||_H = 1, \langle p, q\rangle_R = 0\} \end{aligned}$$

where $p_k := u_{2k} + u_{2k+1}\mathrm{j}, q_k := v_{2k} + v_{2k+1}\mathrm{j}$ and $\langle p, q\rangle_H = \sum \overline{p}_k q_k$. The quaternionic Hopf map in this case is $\chi : \mathbb{S}^{4n+3} \to \mathbb{HP}^n$, $p \to [p]$ where $[p] = [p_0, p_1, ..., p_n]$ is the class of p for the relation $p \sim \sigma p, \sigma \in \mathrm{Sp}(1)$. The next lemma is again from [7].

Lemma 7. *The cotangent space $T^*\mathbb{HP}^n$ is diffeomorphic to both X_H and $\widetilde{X}_H$ defined as follows:*

$$X_H := \{\lfloor p, q\rfloor \in \mathbb{H}^{n+1} \times \mathbb{H}^{n+1};\ ||p||_H = 1, \langle p, q\rangle_H = 0\}$$

$$\widetilde{X}_H := \{\lfloor z, w\rfloor \in \mathbb{C}^{2n+2} \times \mathbb{C}^{2n+2};\ ||z||_C = ||w||_C, \langle z, w\rangle_C = 0, I(z, w) = 0\}$$

where $I(z, w) = z_0w_1 - z_1w_0 + ... + z_{2n}w_{2n+1} - z_{2n+1}w_{2n}$ and $\lfloor p, q\rfloor$ and $\lfloor z, w\rfloor$ denote the equivalence classes of (p, q) and (z, w) under $(p, q) \sim (\sigma p, \sigma q)$ and $(z, w) \sim (z, w)g$ for $\sigma \in \mathrm{Sp}(1)$ and $g \in \mathrm{SU}(2) \cong \mathrm{Sp}(1)$.

Proof: Consider the action of $\mathrm{SU}(2)$ on $\mathbb{S}^{4n+3}$ defined by

$$\Psi_g(p, q) := (p, q)g, \qquad g \in \mathrm{SU}(2). \tag{1}$$

This action has a moment map $G : T^*\mathbb{S}^{4n+3} \to \mathfrak{su}^*(2)$, given by the formulas $G(p, q) = (A(p, q), B(p, q), C(p, q))$, where

$$\langle p, q\rangle_H = \Re(\langle p, q\rangle_H) + A(p, q)\mathrm{i} + B(p, q)\mathrm{j} + C(p, q)\mathrm{k}$$

and the imaginary quaternions are identified with $\mathfrak{su}^*(2)$. Hence, $T^*\mathbb{S}^{4n+3}//\mathrm{SU}(2) = X_H \cong T^*\mathbb{HP}^n$.

To prove that X_H and $\widetilde{X}_H$ are diffeomorphic, consider the map $t_H : X_H \to \tilde{X}_H, (z, w) = t_H(p, q)$, where

$$z_{2k} := \frac{1}{\sqrt{2}}(||v||_C u_{2k} + \mathrm{i}v_{2k}), \qquad z_{2k+1} := \frac{1}{\sqrt{2}}(-||v||_C \overline{u}_{2k+1} - \mathrm{i}\overline{v}_{2k+1})$$
$$w_{2k} := \frac{1}{\sqrt{2}}(v_{2k+1} - \mathrm{i}||v||_C u_{2k+1}), \qquad w_{2k+1} := \frac{1}{\sqrt{2}}(\overline{v}_{2k+1} - \mathrm{i}||v||_C \overline{u}_{2k+1}).$$

■

The action Ψ defined in (1) commutes with the geodesic flow of $\mathbb{S}^{4n+3}$. Recall the diffeomorphism $t_H : X_H \to \widetilde{X}_H$ defined at the end of the last proof. Like in the previous subsection, we have the following.

Proposition 8. *Let $\Omega_H = \Omega_{T^*\mathbb{HP}^n}$ be the canonical symplectic form on $T^*\mathbb{HP}^n$. Then*

$$\Omega_H = \frac{1}{2}(\mathrm{d}u \wedge \mathrm{d}\overline{v} + \mathrm{d}\overline{u} \wedge \mathrm{d}v).$$

Moreover the geodesic flow of $\mathbb{HP}^n$ *is the flow of the equivalent Hamiltonian systems*

$$(X_H, \Omega_H, G_H) \cong (\widetilde{X}_H, \widetilde{\Omega}_H, \widetilde{G}_H)$$

where $G_H = \dfrac{||q||_H^2}{2} = \dfrac{||v||_C^2}{2}$, $\widetilde{\Omega}_H = t_H^* \Omega_H$ *and* $\widetilde{G}_H = t_H^*(G_H)$.

Next we compute the energy spectrum of the geodesic flow on $\mathbb{HP}^n$ in a similar way as in the case of $\mathbb{CP}^n$. We consider again the reduced space $\mathbb{O}_c = T^*\mathbb{HP}^n//S^1 = \widetilde{G}^{-1}(c)/S^1$ with the induced symplectic form ω_c obtained from $\tilde{i}_c^* \Omega_H = \tilde{\pi}_c^* \omega_c$, where $\tilde{i}_c : \widetilde{G}^{-1}(c) \to T^*\mathbb{HP}^n$ and $\tilde{\pi}_c : \widetilde{G}^{-1}(c) \to \mathbb{O}_c$. Denote by $\mathbb{F}_{is}$ the isotropic Grassmann manifold

$$\begin{aligned}
\mathbb{F}_{is} &= \{\Lambda \in \mathrm{Gr}_2(\mathbb{C}^{2n+2});\ I|_\Lambda = 0\} \\
&= \{[[z,w]] \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1};\ ||z||_C = ||w||_C = 1, \langle z, w\rangle_C = I(z,w) = 0\}
\end{aligned}$$

where $[[z,w]]$ is representative of (z,w) for $(z,w) \cong (\lambda z, \lambda w)g$, $\lambda \in S^1, g \in \mathrm{SU}(2)$ or equivalently $(z,w) \cong (z,w)g, g \in \mathrm{U}(2)$. Alternatively, $\mathbb{F}_{is}$ is a hyperplane in $\mathrm{Gr}_2(\mathbb{C}^{2n+2})$

$$\mathbb{F}_{is} \cong \{(\lambda_{ij}) \in \mathrm{Gr}_2(\mathbb{C}^{2n+2});\ \lambda_{01} + \lambda_{23} + ... + \lambda_{2n+1,2n+2} = 0\}$$

where (λ_{ij}) are the Plücker coordinates on $\mathrm{Gr}_2(\mathbb{C}^{2n+2})$, as well as a homogeneous space: $\mathbb{F}_{is} \cong \mathrm{Sp}(n+1)/\mathrm{U}(2)\mathrm{Sp}(n-1)$.

Proposition 9. *If* $c \neq 0$ *then the reduced space* $\mathbb{O}_c$ *is isomorphic to* $\mathbb{F}_{is}$ *equipped with the Kähler form* $\widetilde{\omega}_c = \pi\sqrt{2c}\omega$, *where* ω *is the restriction of the canonical Kähler form on* $\mathrm{Gr}_2(\mathbb{C}^{2n+2})$ *which generates* $H^2(\mathrm{Gr}_2(\mathbb{C}^{2n+2}), \mathbb{Z})$.

Proof: The S^1 action of the geodesic flow on $\widetilde{G}_H^{-1}(c) \subset T^*\mathbb{HP}^n \cong \widetilde{X}_H$ is

$$\lambda\lfloor z, w\rfloor = \lfloor \lambda z, \lambda w\rfloor$$

which commutes with the action of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ defining the quaternionic Hopf fibration. Now from $\widetilde{G}_H(z,w) = c$ we have $||z||^2 = ||w||^2 = 2c$. If $\lambda_{ij} = z_i w_j - z_j w_i$ are the Plücker coordinates on $\mathrm{Gr}_2(\mathbb{C}^{2n+2})$ then

$$\tilde{\pi}_c^*(\pi\sqrt{2c}\omega) = \pi\sqrt{2c}\frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}\lambda_{ij} \wedge \mathrm{d}\overline{\lambda}_{ij}}{\sum_{i,j} ||\lambda_{ij}||^2} = \frac{1}{\sqrt{2c}}\frac{\mathrm{i}}{2}(\mathrm{d}z \wedge \mathrm{d}\overline{z} + \mathrm{d}w \wedge \mathrm{d}\overline{w}) = \tilde{i}_c^*(\widetilde{\Omega}_H).$$

■

Proposition 10. *We have* $c_1(\mathbb{F}_{is}) = (2n+1)\omega$.

Proof: We note that $c_1(\mathrm{Gr}_2(\mathbb{C}^{2n+2})|_{\mathbb{F}_{is}} = (2n+2)\omega$ and then proceed with the adjunction formula as in Proposition 2.4 using the fact that $\mathbb{F}_{is}$ is a hypersurface in $\mathrm{Gr}_2(\mathbb{C}^{2n+2})$. ■

Theorem 11. *The energy spectrum of the geodesic flow on $\mathbb{HP}^n$ is*

$$E_k = \frac{1}{2}(2n+1+2k)^2, \qquad k \in \mathbb{N}$$

with corresponding multiplicities

$$m_k = \frac{\big((2+k(4+k)...(2n+k)\big)\big((1+k)(3+k)...(2n+1+k)\big)(2n+2k+1)}{(2n+1)!(2n-1)!}.$$

Proof: We only sketch the proof since it is similar to the $\mathbb{CP}^n$ case. We have $c_1 : H^1(\mathbb{F}_{is}, \mathcal{O}^*) \to H^2(\mathbb{F}_{is}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Therefore all holomorphic line bundles on $\mathbb{F}_{is}$ which arise from the quantization are $L_k := S^{\otimes k}$, where $S = \iota^*([H])$ and ι is the inclusion $\iota : \mathbb{F}_{is} \to \mathrm{Gr}_2(\mathbb{C}^{2N+2}))$. Hence

$$\frac{\sqrt{2c}}{2} - \frac{2n+1}{2} = k.$$

■

3. Riemannian Symmetric Spaces and Generalized Flag Manifolds

In this section, following the notations of [14], we collect some important facts about Riemannian symmetric spaces and generalized flag manifolds. Let G be a compact semisimple Lie group and K a Lie subgroup given by the fixed point set of an involution θ. Then $M = G/K$ is endowed with a Remannian metric which makes it a Riemannian symmetric space. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Consider the eigenspace decomposition of θ: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and identify $\mathfrak{p}$ with T_oM, the tangent space of M at $o = eK$. Then $[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. Denote by $\mathfrak{a}$ the maximal abelian subalgebra in $\mathfrak{p}$. Then dimension of $\mathfrak{a}$ is called the *rank* of M. It is known that $\mathfrak{g}_n = \mathfrak{k} + i\mathfrak{p}$ defines the non-compact dual Lie algebra of $\mathfrak{g}$ with respect to θ. Denote by G_n the simply-connected Lie group with Lie algebra $\mathfrak{g}_n$. Denote also by $\mathfrak{g}^c$, $\mathfrak{k}^c$, $\mathfrak{p}^c$ etc. the complexifications of $\mathfrak{g}, \mathfrak{k}, \mathfrak{p}$ etc. respectively. Note that $\mathfrak{g}^c$ is a complexification of $\mathfrak{g}_n$ and $i\mathfrak{a}$ is a maximal abelian subalgebra of $i\mathfrak{p}$.

The non-compact group G_n admits an Iwasawa decomposition $G_n = KAN$ where A is the simply-connected Lie group with algebra $i\mathfrak{a}$ and N is unipotent. There is a complex Iwasawa decomposition (see for example [5]) given by $G_0^c = K^c A^c N^c$ where K^c, A^c, N^c, and G^c are the complexifications of K, A, N, G, respectively, and G_0^c is some Zarisky open and dense subset of G^c.

It is known that K acts transitively on the set of all maximal abelian subalgebras in $\mathfrak{p}$. Denote by $\mathfrak{m}$ the Lie algebra of the stabilizer of $\mathfrak{a}$ in K. Then $\mathfrak{m} = \{X \in \mathfrak{k};\ [X, \mathfrak{a}] = 0\}$ and since $\mathfrak{a}$ is maximal, the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ is $\mathfrak{l} = \{X \in \mathfrak{g};\ [X, \mathfrak{a}] = 0\} = \mathfrak{m} + \mathfrak{a}$. If L is the corresponding subgroup in G, then L contains a maximal torus of G (as a centralizer of an abelian subgroup). Hence, the space G/L is a generalized flag manifold (also called a Kähler C-space and a rational homogeneous manifold) and carries a natural complex structure, as well as a Kähler metric. From geometric viewpoint a maximal abelian subalgebra of $\mathfrak{p}$ is tangent to maximal totally geodesic flat torus and every such torus is tangent at a point gK to a left translate of some $\mathfrak{a}$ from $o = eK$ to gK. In these terms the generalized flag manifold G/L parametrizes the set of all such tori.

We next provide more information on the roots and weights of the Lie algebra $\mathfrak{g}^c$ relative to $\mathfrak{k}^c$ and $\mathfrak{a}^c$. Choose a maximal abelian (Cartan) subalgebra $\mathfrak{h}^c$ of $\mathfrak{g}^c$ which contains $\mathfrak{a}^c$. Denote by $\mathfrak{h}_k^c$ the intersection $\mathfrak{h}^c \cap \mathfrak{k}^c$, so that $\mathfrak{h}^c = \mathfrak{h}_k^c + \mathfrak{a}^c$ and let $\Delta \subset (\mathfrak{h}^c)^*$ be the root system corresponding to $(\mathfrak{g}^c, \mathfrak{h}^c)$. There is a set of the so-called restricted roots $\Sigma = \Sigma(\mathfrak{g}^c, \mathfrak{a}^c) \subset (\mathfrak{a}^c)^* \cap \Delta$ and we can choose a basis $h_1, ..., h_k, h_{k+1}, ..., h_n$ of $(\mathfrak{h}^c)^*$, of basic roots, such that $h_1, ..., h_k$ (after restricting them via the projection $(\mathfrak{h}^c)^* \to (\mathfrak{a}^c)^*$) form a basis of $(\mathfrak{a}^c)^*$. We continue to use the same notation $h_1, ..., h_k$ for the restricted roots. After we choose an ordering of the basic roots , or equivalently, a positive Weyl chamber, every element of Δ (respectively, of Σ) is an integer linear combination of $h_1, ..., h_n$ ($h_1, ..., h_k$, respectively) with coefficients being all non-negative or all non-positive. Similarly we can choose a positive Weyl chamber in the restricted roots. One important observation is the following.

Lemma 12. *The center $\mathcal{Z}(\mathfrak{l}^c)$ of $\mathfrak{l}^c$ coincides with $\mathfrak{a}^c$.*

Proof: By definition $\mathfrak{a}^c$, is contained in $\mathcal{Z}(\mathfrak{l}^c)$, and $\mathfrak{l} = \mathfrak{m} + \mathfrak{a}$ where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, then $\mathfrak{l}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Because $\mathfrak{g}$ is simple, the statement follows. ■

With the aid of Lemma 12 we can compute the second cohomology of the flag manifold G/L. It is well known (see [3] for example) that there is an isomorphism $(\mathcal{Z}(\mathfrak{l}^c))^* \equiv H^2(G/L, \mathbb{C})$ sometimes called transgression, given by $\alpha \to \frac{\mathrm{i}}{2\pi}\mathrm{d}\alpha$. Moreover the first k elements $w_1, ..., w_k$ of the basis $w_1, ..., w_n$ of the fundamental weights dual to $h_1, ...h_n$ (with respect to the Killing form of $\mathfrak{g}^c$) define an integral basis of $H^2(G/L, \mathbb{Z})$.

Now denote by $\mathcal{L} = \mathcal{L}_{i_1,...,i_k}$ the holomorphic line bundle on G/L determined by $w = i_1 w_1 + ... + i_k w_k$, where $i_j \geq 0$. When all $i_1, i_2, ..., i_k$ are positive then $\mathcal{L}_{i_1,...,i_k}$ is positive and by the Kodaira vanishing theorem, the higher cohomology classes of $\mathcal{L}$ are zero. The space $H^0(G/L, \mathcal{O}(\mathcal{L}))$ is (complex) representation of G with highest weight w. The Borel-Weil-Bott theorem (which in case of positive line bundle is Borel-Weil theorem) shows that the representation is irreducible if w is dominant, and, in fact is the irreducible highest weight representation with highest weight w (see [17] for a short algebraic proof).

We finally recall some general facts for the Laplace-Beltrami spectrum Δ_M (or, simply, the Laplace spectrum on a symmetric space M. For proofs we refer the reader, for example to [12, Chapter 5.7] and [4]. The eigenvalues of Δ_M are $\lambda = ||w+\rho((\mathfrak{a}^c)^*)||^2 - ||\rho((\mathfrak{a}^c)^*)||^2$ where w is as before and $\rho((\mathfrak{a}^c)^*)$ is the half sum of positive restricted roots of $\mathfrak{a}^c$. Then $\rho((\mathfrak{a}^c)^*)$ represents one half of the first Chern class of G/L and $w + \rho((\mathfrak{a}^c)^*)$ represents the first Chern class of $\mathcal{L} \otimes K^{\frac{1}{2}}$. Since the dimension of a simple finite-dimensional representation can be computed by the Weyl dimension formula, we obtain a representation-theoretic interpretation of the multiplicity formulas given in Theorems 6 and 11.

4. Rank-One Case

In this section we explicitly formulate the representation-theoretic interpretation mentioned at the end of the last section for the compact rank-one symmetric spaces (CROSSes) considered earlier. These cases cover the two examples in Section 2, as well the case $M = \mathbb{S}^n$ treated in [18]. We first note that for all CROSSes, the Weyl chamber of the restricted roots is one-dimensional. The set of fundamental weights is

$$\Lambda^+ = \Lambda^+_{\mathfrak{a}^c} = \left\{ \lambda \in \mathfrak{a}^c;\ \frac{\langle \lambda, \psi \rangle}{\langle \psi, \psi \rangle} \in \mathbb{Z}^+, \quad \text{for all}\, \psi \in \Sigma \right\}$$

and in the rank-one case Λ^+ is generated by a single element θ. In particular, we can identify the set of (closed) geodesics $\mathrm{Geod}(M) = G/L$ in M with the Marsden-Weinstein reduced space of T^*M with the S^1-action defined by the geodesic flow. The reduced form Ω_c depends on the choice of the level set $\mu^{-1}(c)$ for the moment map of the action μ. We summarize the considerations of the previous section below.

Theorem 13. *Let $M = G/K$ be a* CROSS. *Assume that the metric on M is scaled as in* [2], *so that the curvature of $\mathbb{S}^n$ is one for example. Then the following hold.*

i) *Under the transgression, the reduced symplectic form* Ω_c *on* $\mathrm{Geod}(M) = G/L$ *corresponds to* $\pi\sqrt{2c}\theta$ *and with the choice of the positive Weyl chamber and complex structure as above,* $c_1(G/L)$ *corresponds to* $N_M\theta$ *for a positive integer* N_M.

ii) *The quantum condition on* $(\mathrm{Geod}(M), \Omega_c)$ *provides the following energy spectrum:* $c_k = \gamma/2(N_M + 2k)^2$, *where* $\gamma = \frac{1}{4}$ *for* $M = \mathbb{S}^n$ *and* $\gamma = 1$ *for all other* CROSSes.

iii) *The spectrum of the semi-Laplacian* $\frac{1}{2}\Delta_M$ *on* M *is given by* $\lambda_k = ||k\theta + \rho(\mathfrak{a}^c)||^2 - ||\rho(\mathfrak{a}^c)||^2$ *and* $c_k = ||k\theta + \rho(\mathfrak{a}^c)||^2$ *where* $\rho(\mathfrak{a}^c)$ *is the half-sum of the positive restricted roots of* $\mathfrak{a}^c$. *Here the norms are with respect to the re-scaled Killing form as in* i) *and* ii).

iv) *The multiplicities of* c_k *and* λ_k *coincide with the dimension of the finite-dimensional representation* $L(k\theta)$ *of* $\mathfrak{g}$ *with highest weight* $k\theta$. *Moreover the representation* $L(k\theta)$, *the complexified* λ_k*-eigenspace* $(\mathcal{L}^2(M)^c)^{\lambda_k}$ *of* Δ_M, *and the quantization space* $H^0(\mathrm{Geod}(M), \mathcal{O}(L^k))$ *coincide as complex vector spaces.*

Remark 14. *We note that the Laplace-Beltrami operator* Δ_M, *in particular, its spectrum, depends on the choice of the metric on* M. *For example, in the case of* $M = \mathbb{S}^n$ *we have two different metrics*: *the canonical one of constant curvature one, and the one arising from the corresponding Killing form* $B(X, Y) = 2(n-1)\mathrm{tr}(XY)$, $X, Y \in \mathfrak{so}(n)$. *This is valid also for the projective spaces and explains the difference in the spectra formulas given in* [2] *and in* [4]. *The difference of* γ *for the sphere and all other* CROSSes *could be explained with the fact that the length of the closed geodesics with respect to the chosen metric is* 2π *for the sphere and* π *otherwise.*

The canonical symplectic form on the cotangent bundle does not have a direct analog which could be used in the higher-rank case. On the other hand - not all simple compact Lie groups act transitively on some of the CROSSes, but every such group acts transitively on some Riemannian symmetric space, possibly of rank greater than one. So it is naturally to expect that a modification of the previous quantization scheme exists in the higher rank case.

In the next Section we announce a preliminary result about such correspondence in the general-rank symmetric spaces. The proof will appear in [11].

5. Symmetric Spaces of General Rank

We use similar setting as in Section 4, but for reader's convenience we repeat some of the definitions. Let again $M = G/K$ be a symmetric space with G being compact and semisimple. Also L is the connected subgroup of G with Lie algebra

$\mathfrak{l} = \mathfrak{m} + \mathfrak{a}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and as noted in Lemma 12, $\mathfrak{a}$ is the center of $\mathfrak{l}$. We can also write $L = MA$ where M (not to be confused with the space M) and A are the corresponding Lie groups (see, for example, [14], [9], [10]). Then G/L is a generalized flag manifold parametrizing the maximal totally geodesic tori of M. As such, it caries a natural complex structure, which depends on the choice of a Cartan subalgebra of $\mathfrak{g}^c$ and a partial order in it which determines a positive Weyl chamber which defines a positive Weyl chamber in $(\mathfrak{a}^c)^*$. The latter is dual to the cone of the restricted dominant weights. Then, as a complex manifold, G/L is equivalent to $G^c/M^cA^cN^c$ and has a principle A^c-bundle $G^c/M^cN^c \to G^c/M^cA^cN^c$ with a right action of A^c. The total space $\Theta = G^c/M^cN^c$ is called the horospherical manifold, [9]. Since $A^c = (\mathbb{C}-0)^r = (\mathbb{C}^*)^r$ is the complexification of the real torus $T^r = A$, then Θ can be identified with an open set of the (co)tangent vertical bundle of the principal bundle G/M over G/L with fiber A. This cotangent vertical bundle is also the set of all cotangent planes to all maximal totally geodesic tori in M. In the case when M has rank $r = 1$, this bundle is just T^*M. Since the characteristic classes of the bundle $G/M \to G/L$ are determined via transgression by the simple restricted roots of $\mathfrak{a}^*$, we obtain the following

Theorem 15 ([11]). *Let $M = G/K$ be a compact Riemannian symmetric space of rank k with G semisimple and let $\theta_1, ..., \theta_k$ be the basis of fundamental weights that is dual to the simple restricted roots of $\mathfrak{a}^c$. Let Θ be the associated horospherical manifold and $\Theta \to G/L$ be the corresponding principal $(\mathbb{C}^*)^k$-bundle, so G/L parametrizes the maximal totally geodesic tori in G/K. Let $\omega_M = \frac{\mathrm{i}}{2\pi}\mathrm{d}\rho$ be the two-form on G/L representing the half sum of the positive roots in $\mathfrak{a}^c$, in particular, the characteristic form representing $\frac{1}{2}c_1(G/L)$. Then there exists a symplectic form ω on Θ with the following properties*

i) *There are positive numbers n_i such that the reduced form $\tilde{\omega}$ on $\Theta//A$ corresponding to ω via the Marsden-Weinstein reduction is $\tilde{\omega} = \sum_{i=1}^k n_i \mathrm{d}\theta_i + \omega_M$, $n_i > 0$ on G/L.*

ii) *When $\alpha = n_1\theta_1 + n_2\theta_2 + ... + n_k\theta_k$ is integral (up to a factor of 2π) and determines a dominant weight, then the corresponding quantum bundle $\mathcal{L}$ determined by $\tilde{\omega} \in c_1(\mathcal{L})$ has the property that its space of holomorphic sections $H^0(G/L, \mathcal{O}(\mathcal{L}))$ is an irreducible unitary representation of G with highest weight α.*

iii) *The complexified eigenspaces of the Laplace-Beltrami operator* Δ_M *corresponding to the eigenvalue* $\lambda_\alpha = ||\alpha + \rho||^2 - ||\rho||^2$ *on* M *have dimension equal to the sum of dimensions of all* $H^0(G/L, \mathcal{O}(\mathcal{L}))$ *defined in ii) for which* $||\alpha + \rho||^2 = \lambda_\alpha + ||\rho||^2$. *All eigenvalues of* Δ_M *are equal to* λ_α *for some* α.

Remark 16. *The paper by Gindikin* [9] *is closely related to our considerations. In that paper the flag* $G/L = G^c/M^cA^cN^c$ *and the complex space* $\Theta = G^c/M^cN^c$ *play important role in a Cauchy-Radon transform. Also, considering the space* Θ *as (an open set of) the space parametrizing all tangent spaces to all totally geodesic tori in* G/K*, it is shown that its "symplectic quotient" by the* k*-tori (max totally geodesics) produces* G/L*. There seems to be a deeper relation between the correspondence above and the Cauchy-Radon transform, which we hope to explore in the future.*

Remark 17. *We should note that not all irreducible finite-dimensional representations of* G *appear as subspaces of the eigenspaces of the Laplace-Beltrami operator. The ones that appear are those corresponding to restricted roots in* $(\mathfrak{a}^c)^*$. *An important exception is the case of symmetric spaces of maximal rank, i.e., when* $\mathrm{rk}(M) = \mathrm{rk}(G)$. *In this case* $\mathfrak{a}^c$ *is maximal abelian in* $\mathfrak{g}^c$*, hence the restricted roots are all the roots. Note that in this case* L *is a maximal torus of* G*. By Borel-Weil-Bott theorem all irreducible finite-dimensional representations of* G *appear as spaces of sections of holomorphic line bundles over* G/L*. One such example is given below.*

Example: the space $M = \mathrm{SU}(3)/\mathrm{SO}(3)$. The space $\mathrm{SU}(3)/\mathrm{SO}(3)$ has rank two, the same as the rank of $\mathrm{SU}(3)$. Every Z in $\mathfrak{su}(3)$ has the form $Z = X + \mathrm{i}Y$ for some $X \in \mathfrak{so}(3)$. This gives a decomposition $\mathfrak{su}(3) = \mathfrak{so}(3) + \mathfrak{p}$ where $\mathfrak{p}$ is the space of purely imaginary matrices. In this case L is a maximal torus of $\mathrm{SU}(3)$ and $\mathfrak{a}^c$ consist of diagonal matrices. In particular the generalized flag manifold G/L is the standard manifold of full flags in $\mathbb{C}^3$ identified with $\mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1))$. We write the roots of $\mathfrak{a}^c$ as triples and choose the simple roots to be (up to a factor of $\sqrt{-1}$): $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (0, 1, -1)$. Then the half sum of the positive roots is $\frac{1}{2}\rho = \frac{1}{2}(2\alpha_1 + 2\alpha_2) = (1, 0, -1)$. Recall that Killing form $B(H, H')$ of two diagonal matrices $H = (h_1, h_2, h_3)$ $H' = (h_1', h_2', h_3')$ is $B(H, H') = \mathrm{tr}(\mathrm{ad}_H \mathrm{ad}_{H'}) = \sum_{i<j}(h_i - h_j)(h_i' - h_j')$.
The dominant weights are given by $k_1\alpha_1 + k_2\alpha_2$ with integers k_1, k_2 for which $k_1 < 2k_2 < 4k_1$. A straightforward computation gives $||k_1\alpha_1 + k_2\alpha_2 - \frac{1}{2}\rho||^2 = 6[(k_1-1)^2 - (k_1-1)(k_2-1) + (k_2-1)^2]$. In particular we see that the eigenspaces of the Laplacian on $\mathrm{SU}(3)/\mathrm{SO}(3)$ corresponding to $\lambda = ||k_1\alpha_1 + k_2\alpha_2 - \frac{1}{2}\rho||^2 -$

$||\frac{1}{2}\rho||^2$ split into irreducible representations subspaces when the equation $x^2 - xy + y^2 = Q$ where $Q = \lambda + ||\frac{1}{2}\rho||^2$ does not depend on x and y, have more than one integer solution with $x, y > 1, (x-1) < 2(y-1) < 4(x-1)$. The number of integer solutions of this Diophantine equation is a classical number theory question. In particular when $k_1 = k_2 = n^2 + 1$, so that $Q = n^2$ with all of the prime factors of n being of the type $3k + 2$, then the solution is unique and the corresponding eigenspace of the Laplacian is an irreducible $\mathrm{SU}(3)$-module. On the other hand, for example, if $Q = 8281 = 7^2 13^2$, then the corresponding eigenspace splits into a direct sum of 7 irreducible $\mathrm{SU}(3)$-modules.

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