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SPACES REALIZED AND NON-REALIZED AS DOLD-LASHOF CLASSIFYING SPACES

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Abstract. Let X be a simply connected CW-complex of finite type. Denote by $\mathrm{Baut}_1(X)$ the Dold-Lashof classifying space of fibrations with fiber X. This paper is a survey about the problem of realizing Dold-Lashof classifying spaces. We will also present some new results: we show that not all rank-two rational H-spaces can be realized as $\mathrm{Baut}_1(X)$ for simply connected, rational elliptic space X. Moreover, we construct an infinite family of rational spaces X, such that $\mathrm{Baut}_1(X)$ is rationally a finite H-space of rank-two (up to rational homotopy type).

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Contents

1. Introduction

In this paper, all spaces are simply connected CW-complex and are of finite type over $\mathbb{Q}$, i.e., have finite-dimensional rational homology in each degree.

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In 1959, Dold and Lashof [4] generalized Milnor's construction from the setting of topological groups to that of topological monoids obtaining a universal principal H-quasifibration, where H is a topological monoid

$$H \to EH \to BH.$$

In this period, the monoid aut(X) of all homotopy self-equivalences of X emerged as a central object for the theory of fibrations. In general, aut(X) is a disconnected space. J. Stasheff [28] constructed the universal X-fibration

$$X \to UE \to \mathrm{Baut}\,(X)$$

for X a finite CW-complex, building on work of Dold-Lashof. His result implied that the universal X-fibration is obtained, up to homotopy, by applying the Dold-Lashof classifying space functor to the evaluation fibration ev : aut$(X) \to X$. One outcome of this work is the correspondence

$$[B, \mathrm{Baut}\,(X)] \cong \{\text{fibre homotopy equivalence classes of } X\text{-fibrations over } B\}.$$

The extension of Stasheff's result to any CW-complex was obtained by Dold [3]. See also [1]. Let aut$_1(X)$ the space of self-homotopy equivalences of X that are homotopic to the identity. This is a monoid with multiplication given by composition of maps and topologized as a sub-space of $\mathrm{Map}(X, X)$; the space of all continuous functions with the compact-open topology. By a proposition of Félix, Lupton and Smith [7, Proposition 2.2], $\mathrm{aut}_1(X)$ has the homotopy type of a CW-complex and H-homotopy type of a loop space. So, by applying the Dold-Lashof construction to the monoid aut$_1(X)$, we obtain $\mathrm{Baut}_1(X)$, which may be identified as the universal cover of $\mathrm{Baut}\,(X)$ (cf [5]), and so classifies orientable X-fibrations.

The identity $\Omega\mathrm{Baut}_1(X) \simeq \mathrm{aut}_1(X)$ [3, satz 7.3] gives the isomorphism

$$\pi_{*+1}\left(\mathrm{Baut}_1(X); *\right) \cong \pi_*\left(\mathrm{aut}_1(X); 1\right).$$

Then, the space $\mathrm{Baut}_1(X)$ is, in turn, a simply connected CW-complex and so admits a rationalization $\mathrm{Baut}_1(X)_{\mathbb{Q}}$. We observe that, as the space $\mathrm{Baut}_1(X)$ is quite complicated, calculations and other descriptions will be difficult to obtain.

The calculations of $H^*\left(\mathrm{Baut}_1(X), \mathbb{Q}\right)$ and $\pi_*\left(\mathrm{Baut}_1(X)\right) \otimes \mathbb{Q}$ are the subject of a long line of celebrated structure theorems. In Appendix 1 of [20], Milnor computed $H^*\left(\mathrm{Baut}_1(\mathbb{S}^n), \mathbb{Q}\right)$ to be a polynomial algebra with a single positive degree generator. Work of Federer [9], Thom [31] and then others gave calculations of these homotopy groups for certain space X. Milnor showed that, when $X = \mathbb{S}^n$, we have $\mathrm{Baut}_1(X) \simeq_{\mathbb{Q}} K\left(\mathbb{Q}, 2n\right)$ if n is even and $\mathrm{Baut}_1(X) \simeq_{\mathbb{Q}} K\left(\mathbb{Q}, 2n+1\right)$

if n is odd. Here $\simeq_{\mathbb{Q}}$ means having the same rational homotopy type (cf [6, Proposition 9.8]). In general, the determination of the homotopy type of the Dold-Lashof classifying space is a very hard problem.

Realizing rational homotopy types as $\mathrm{Baut}_1(X)$ is the subject of the following long-standing problem in rational homotopy theory (see [6, p.519]).

Problem 1. *Which simply connected spaces can be realized as Dold-Lashof classifying spaces?*

This question is often interpreted as a conjecture to the effect that: every simply connected CW-complex Y is realized, up to rational homotopy, as a $\mathrm{Baut}_1(X)$. That is, there is a simply connected CW-complex X with

$$Y \simeq_{\mathbb{Q}} \mathrm{Baut}_1(X).$$

At this time, however, problem 1 remains very much open. In this survey paper we consider a particular problem suggested by Smith in [27].

Problem 2. *Realize rank-two rational H-spaces as Dold-Lashof classifying spaces.*

Our contributions to this problem are as follows.

Theorem 3. 1) *There is no simply connected, rational pure space X for which*:

- $K(\mathbb{Q}, 2p) \times K(\mathbb{Q}, 2q) \simeq_{\mathbb{Q}} \mathrm{Baut}_1(X)$ *for* $(p, q) \in (\mathbb{Z} - 2\mathbb{Z}) \times (\mathbb{Z} - 3\mathbb{Z})$ *and* $p \neq q$
- $K(\mathbb{Q}, 2p+1) \times K(\mathbb{Q}, 2q) \simeq_{\mathbb{Q}} \mathrm{Baut}_1(X)$ *for* $p \geq 1$ *and* $q \geq p+1$.

2) *There is no simply connected, rational elliptic space X for which*:

- $K(\mathbb{Q}, 2p) \times K(\mathbb{Q}, 2q+1) \simeq_{\mathbb{Q}} \mathrm{Baut}_1(X)$ *for* $p \geq 1$ *and* $q \geq p$
- $K(\mathbb{Q}, 2p+1) \times K(\mathbb{Q}, 2q+1) \simeq_{\mathbb{Q}} \mathrm{Baut}_1(X)$ *for* $p, q \geq 1$ *and* $p \neq q$.

Theorem 4. *The following spaces are rationally realizable as* $\mathrm{Baut}_1(X)$ *for some simply connected, rational space* X:

- $K\mathbb{Q}, 2m) \times K(\mathbb{Q}, 3m)$ *for each* $m \geq 2$ *and* m *is even*
- $K(\mathbb{Q}, 4m+1) \times K(\mathbb{Q}, 4m+2n+2)$ *for each* $n \geq 1$ *and* $m \geq n$
- $K(\mathbb{Q}, 4r-1) \times K(\mathbb{Q}, 4r+2s)$ *for each* $r \geq 1$ *and* $s \geq 1$
- $K(\mathbb{Q}, 4r-1) \times K(\mathbb{Q}, 4r+2s-1)$ *for each* $r \geq 1$ *and* $s \geq 2r-1$.

Here, $K(\mathbb{Q}, n)$ is the Eilenberg-Mac Lane space. In general, let G be an abelian group and $n \geq 2$. The Eilenberg-Mac Lane space $K(G, n)$ is a CW-complex such that $\pi_n(K(G, n)) = G$ and the other homotopy groups are zero.

We will obtain these results working with the theory elaborated by D. Sullivan [29], which asserts that the homotopy of one-connected rational spaces is equivalent to the homotopy of one-connected minimal cochain commutative algebras over the rationales.

Now we briefly summarize the contents of the paper. Section 2 establishes our notations and basic conventions. In Section 3, we recall some previous results of G. Lupton and S. Smith. In Section 4, we give an infinite family of rank-two H-spaces that cannot be realized as $\mathrm{Baut}_1(X)$, under some restrictions on X. This result is proved using standard tools familiar from rational homotopy theory: minimal Sullivan model, Euler-Poincaré characteristic, ect. Finally, in Section 5, we prove theorem 4, which gives many examples of rank-two H-spaces that are realized as $\mathrm{Baut}_1(X)$ for some simply connected, rational space X with finite-dimensional homotopy groups. The general approach and many of the details follow exactly those of [16], and we conclude this section with some comments.

2. Preliminaries in Rational Homotopy Theory

We begin this introductory section with a brief review of some ideas from rational homotopy theory. All results of this paper are proved using standard tools of the subject. We refer to [6], [8] and [12] for a general introduction to these techniques. We recall some of the notations here. By a vector space we mean a graded vector space over the field of rational numbers $\mathbb{Q}$,i.e., a collection

$V = \left\{V^k;\, k \text{ an integer} \geq 0\right\}$, such that each V^k is a vector space over $\mathbb{Q}$. If $v_1, ..., v_r, ...$ is a basis of V, that is $v_1, ..., v_{i_0}$ is a basis of V^0, $v_{i_0+1}, ..., v_{i_1}$is a basis of V^1, etc., then we write $V = \langle v_1, ..., v_r, ...\rangle$. If the set of basis vectors of V is finite, we say that V is finite-dimensional.

We consider a differential graded commutative algebra (A, d) over $\mathbb{Q}$, called DGCA for a short, with differential d of degree $+1$. We write $x \in A$ to indicate that $x \in A^n$ for some $n \geq 0$ and $|x| = n$ be the degree of x. We use similar notation for a vector space V. We denote the cohomology algebra of A by $H^*(A)$ and let $[x] \in H^*(A)$ stand for the cohomology class of the cocycle $x \in A$.

The free graded commutative algebra generated by the vector space V is denoted ΛV. A basis for V is then called a set of algebra generators for ΛV.

If $V = \langle v_1, ..., v_r, ...\rangle$, we write $\Lambda V = \Lambda(v_1, ..., v_r, ...)$. A DGCA (A, d) is a Sullivan algebra if $A \cong \Lambda V$ and if V admits a basis $v_1, ..., v_r, ...$ such that $d(v_r) \in$

$\Lambda(v_1, ..., v_{r-1})$ for each r and $|v_1| \leq |v_2| \leq ....$ If the differential d has image in the decomposables of ΛV i.e.,

$$d\left(V^i\right) \subset \left(\wedge^+ V. \wedge^+ V\right)^{i+1}$$

then we say (A, d) is minimal. Here $\wedge^+ V$ is the ideal of ΛV generated by elements of positive degree.

In [29], D. Sullivan defined a functor $A_{PL}()$ from topological spaces to DGCA. The functor $A_{PL}()$ is connected to the cochain algebra functor $C^*(-;\mathbb{Q})$ by a sequence of natural quasi-isomorphisms. A DGCA (A, d) is a Sullivan model for X if (A, d) is a Sullivan algebra i.e., $(A, d) \cong (\Lambda V, d)$ and there is a quasi-isomorphism $(\Lambda V, d) \to A_{PL}(X)$ (induces isomorphisms on homology). If $(\Lambda V, d)$ is minimal, then it is the minimal Sullivan model of X.

Note that $(\Lambda V, d)$ determines the rational homotopy type $X_{\mathbb{Q}}$ of X. In particular there are isomorphisms:

$$H^*(X, \mathbb{Q}) \cong H^*(\Lambda V, d) \text{ as graded commutative algebras}$$
$$\pi_*(X) \otimes \mathbb{Q} \cong \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q}) \text{ as vector spaces.}$$

Next, we focus on certain spaces X that satisfy the following conditions:

1) $H^*(X, \mathbb{Q})$ is finite-dimensional
2) $\pi_*(X) \otimes \mathbb{Q}$ is finite-dimensional.

For example, spheres $\mathbb{S}^n$ for $n \geq 2$ and Eilenberg-Mac Lane spaces $K(\mathbb{Q}, 2n+1)$ for $n \geq 1$ satisfy these conditions. A space that satisfies 1) and 2) is called rationally elliptic. See [6, Chapter 32] or [13] for a discussion of these spaces. Condition 1) above implies that the minimal Sullivan model has finite-dimensional cohomology, as the cohomology of the minimal model is identified with that of the space. Condition 2) translates into the condition that the minimal Sullivan model be finitely-generated as a free graded algebra, since the algebra generators of the minimal Sullivan model are identified, as a graded vector space, with the rational homotopy groups of the space.

In [13], S. Halperin shows that, for an elliptic space X, $\chi_\pi(X) \leq 0$ where $\chi_\pi(X)$ denotes the homotopy Euler-Poincaré characteristic of X. This is a number defined for any simply connected space that has finite-dimensional rational homotopy groups by

$$\chi_\pi(X) = \textstyle\sum_i (-1)^i \dim\ (\pi_i(X) \otimes \mathbb{Q}).$$

In particular, from the isomorphism above we have

$$\chi_\pi(X) = \dim V^{\text{even}} - \dim V^{\text{odd}}.$$

Furthermore, an elliptic minimal Sullivan model $(\Lambda V, d)$ with $dV^{\text{even}} = 0$ and $dV^{\text{odd}} \subset \Lambda V^{\text{even}}$ is said to be pure, similarly X is said to be pure when its minimal Sullivan model $(\Lambda V, d)$ is pure. There are many examples of such spaces: finite products of even dimensional spheres, finite products of complex projective spaces and homogeneous spaces G/H, where H is a closed connected sub-group of a compact connected Lie group G.

The space $\mathrm{Baut}_1(X)$ was among the first geometric objects described in rational homotopy theory. In his foundational paper [29], Sullivan gave a model for this simply connected space in terms of the derivations of a minimal Sullivan model. For the convenience of the reader, we recall the construction of models of $\mathrm{Baut}_1(X)$. First, we give the definition of differential graded Lie algebra.

Definition 5. *A differential graded Lie algebra* (DGLA *for short*) *is the data of a differential graded vector space* (L, δ) *together a with bilinear map*

$$[-,-] : L \times L \to L, \qquad x \otimes y \to [x, y]$$

(*called bracket*) *of degree* 0 *such that*:

1) (*graded skewsymmetry*) $[x, y] = -(-1)^{|x||y|}[y, x]$
2) (*graded Jacobi identity*) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$
3) (*graded Leibniz rule*) $\delta[x, y] = [\delta x, y] + (-1)^{|x|}[x, \delta y]$.

The Leibniz rule implies in particular that the bracket of a DGLA L induces a structure of graded Lie algebra on its cohomology $H^*(L) = \oplus_i H^i(L)$.

A derivation of degree i of a DGCA $(\Lambda V, d)$ will mean a linear map lowering degrees by i and satisfying the product law

$$\theta\left(V^j\right) \subseteq (\Lambda V)^{j-i} \text{ and } \theta(xy) = \theta(x)\, y + (-1)^{i|x|} x\theta(y) \text{ for } x, y \in \Lambda V.$$

We write $\mathrm{Der}_i(\Lambda V, d)$ for the vector space of all degree i derivations. The boundary operator $D : \mathrm{Der}_i(\Lambda V, d) \to \mathrm{Der}_{i-1}(\Lambda V, d)$ is defined by

$$D(\theta) = d\theta - (-1)^i \theta d.$$

Let $\mathrm{Der}_*(\Lambda V, d)$ be the set of all positive degree derivations with the restriction that $\mathrm{Der}_1(\Lambda V, d)$ is the vector space of derivations of degree one which commute with the differential d. Then $(\mathrm{Der}_*(\Lambda V, d), D)$ has the structure of a DGLA with the commutator bracket

$$[\theta_1, \theta_2] = \theta_1\theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2\theta_1$$

for $\theta_1, \theta_2 \in \mathrm{Der}_*(\Lambda V, d)$. We often denote $(\mathrm{Der}_*(\Lambda V, d), D)$ by $\mathrm{Der}_*(\Lambda V)$.

We use the following theorem to prove our results.

Theorem 6 (D. Sullivan) *Let X be a simply connected CW-complex of finite type with minimal Sullivan model $(\Lambda V, d)$, then*

$$(\mathrm{Der}_*(\Lambda V, d), D)$$

is a DGLA *model for* $\mathrm{Baut}_1(X)$.

In particular, we have

$$\pi_*(\Omega \mathrm{Baut}_1(X)) \otimes \mathbb{Q} \cong H_*(\mathrm{Der}(\Lambda V, d), D)$$

as graded Lie algebras, in which the left hand side has the Samelson bracket.

We elaborate on the meaning of a DGLA model for a space. In [22], Quillen constructed a minimal DGLA $(\mathcal{L}(V), \delta)$, called the Quillen minimal model for simply connected CW-complex of finite type. Here $\mathcal{L}(V)$ is the free graded Lie algebra on a connected vector space $V_i \cong H_{i+1}(X; \mathbb{Q})$ and δ has degree -1. Minimality is satisfied if the differential δ is decomposable. Quillen constructed the commutative cochains functor which takes a connected DGLA (L, δ) and returns a differential graded algebra $C^*(L, \delta)$. If L is of finite type, $C^*(L, \delta)$ is a Sullivan algebra [6, Lemma 23.1]. When $(\mathcal{L}(V), \delta)$ is the Quillen model for a simply connected CW-complex X of finite type, $C^*(\mathcal{L}(V), \delta)$ is a Sullivan model for X (in general it is not minimal).

We say that a connected DGLA (L, δ) of finite type is DGLA model for a space X if $C^*(L, \delta)$ is a Sullivan model for X. For more details, we refer the reader to the standard book [30].

Schlessinger and Stasheff [23] constructed a second equivalent model in terms of derivations of a Quillen model. In the same way, we define a DGLA $\mathrm{Der}_*\mathcal{L}(V) = \oplus_{i\geq 1}\mathrm{Der}_i(\mathcal{L}(V))$, where $\mathrm{Der}_i(\mathcal{L}(V))$ is the vector space of derivations which increase the degree by i with the restriction that $\mathrm{Der}_1(\mathcal{L}(V))$ is the vector space of derivations of degree one which commute with the differential δ. Define the differential graded Lie algebra

$$(s\mathcal{L}(V) \oplus \mathrm{Der}_*(\mathcal{L}(V)), \widetilde{D})$$

as follows

- If $\theta, \theta_1, \theta_2 \in \mathrm{Der}_*\mathcal{L}(V)$ and $sx, sy \in s\mathcal{L}(V)$, then

  $$[\theta_1, \theta_2] = \theta_1\theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2\theta_1, \quad [\theta, sx] = (-1)^{|\theta|} s\theta(x) \text{ and } [sx, sy] = 0.$$

- $\widetilde{D}(\theta) = [\delta, \theta]$ and $\widetilde{D}(sx) = -s\delta x + ad(x)$, where $ad(x)$ is the adjoint derivation defined by $ad(x)(y) = [x, y]$.

We recall that the suspension of a graded vector space V is the graded vector space sV defined by $(sV)_i = V_{i-1}$. If $v \in V_{i-1}$ the corresponding element in $(sV)_i$ is denoted by sv.

Theorem 7 (Schlessinger-Stasheff). *Let X be a simply connected CW-complex of finite type with Quillen minimal model $(\mathcal{L}(V), \delta)$. Then*

$$(s\mathcal{L}(V) \oplus \mathrm{Der}_*(\mathcal{L}(V)), \widetilde{D})$$

is a DGLA *model for* $\mathrm{Baut}_1(X)$.

Both Theorems 6 and 7 are great illustrations of the power of rational homotopy theory for modeling a complex geometric construction in relatively simple terms.

Notation 8. i) *In the following, we assume all spaces X appearing in this paper are rational spaces. That is, all spaces satisfy $X = X_{\mathbb{Q}}$.*

ii) *The symbol $(v, w) \in \mathrm{Der}_{|v|-|w|}$ means the derivation sending an element $v \in V$ to $w \in \Lambda V$ and the other generators to zero.*

iii) *Denote by $\mathbb{Q}\{e\}$ the $\mathbb{Q}$-graded vector space of basis e.*

Calculations of the rational homotopy type of $\mathrm{Baut}_1(X)$ follow the process:

$$\begin{array}{cccc} X \rightarrow & (\Lambda V, d) & \rightarrow (\mathrm{Der}_*(\Lambda V, d), D) \rightarrow & \mathrm{Baut}_1(X) \\ \text{space} \rightarrow & \text{minimal model} & \rightarrow \text{DGLA} \quad \rightarrow & \text{classifying space.} \end{array}$$

To well illustrate this process, we propose the following examples.

Example 9. *For $n \geq 1$, we have*

$$\mathrm{Baut}_1(\mathbb{S}^{2n}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4n).$$

Indeed, let $(\Lambda V, d)$ be the minimal Sullivan model of $\mathbb{S}^{2n}$. First, we give an explicit description of the model $(\Lambda V, d)$, write $(\Lambda V, d)$ as $(\Lambda(x, y), d)$ with $|x| = 2n$ and $|y| = 4n - 1$. The differential is defined as follows: $dx = 0$ and $dy = x^2$.

A vector space basis for $(\mathrm{Der}_*(\Lambda(x, y), d), D)$ is given by

$*$	$\mathrm{Der}_*(\Lambda(x, y))$
$4n-1$	$(y, 1)$
$2n$	$(x, 1)$
$2n-1$	(y, x)

We have $D(y,1) = d \circ (y,1) + (y,1) \circ d$*. When this is evaluated on* y *and* x*, we find that* $D(y,1) = 0$*. Furthermore, by direct computation we obtain*

$$D(x,1) = -(y,x) \quad \text{and} \quad D(y,x) = 0.$$

Then $(\mathrm{Der}_*(\Lambda(x,y), d), D)$ *has homology of rank* 1 *in degree* $4n-1$ *and zero otherwise. Applying the* Theorem 6*, it follows that* $\pi_i\left(\mathrm{Baut}_1(\mathbb{S}^{2n})\right) \cong \mathbb{Q}$ *for* $i = 4n$ *and zero otherwise. So, we must have*

$$\mathrm{Baut}_1(\mathbb{S}^{2n}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4n).$$

Example 10. *For* $n \geq 1$*, we have*

$$\mathrm{Baut}_1(\mathbb{S}^{2n+1}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2n+2).$$

Write the minimal Sullivan model of $\mathbb{S}^{2n+1}$*as* $(\Lambda(x), 0)$ *with* $|x| = 2n+1$*. Then the* DGLA *of derivations* $(\mathrm{Der}_*(\Lambda(x), 0), D)$ *is the abelian Lie algebra* $\langle x^* \rangle$ *where* x^* *denotes the dual derivation to* x *with* $D = 0$*. So* $(\mathrm{Der}_*(\Lambda(x), 0), 0)$ *has homology isomorphic, as a graded vector space, to* $\langle x^* \rangle$*. This is exactly a* DGLA *model for* $K(\mathbb{Q}, 2n+2)$*.*

We apply similar arguments to prove

$$\mathrm{Baut}_1(K(\mathbb{Q}, p)) \simeq_{\mathbb{Q}} K(\mathbb{Q}, p+1) \text{ for } p \geq 2.$$

Note that, many of the computations and examples introduced in this section have been obtained by various mathematicians with a great amount of work [9], [20] and [31]. We have tried to arrange these results into a coherent form.

3. Results of G. Lupton and S. Smith

A variety of results has been published about the realizability of the Dold-Lashof classifying space. The aim of this section is devoted to recall some results about this. We first give the definition of H-space, or Hopf space.

Definition 11. *An* H*-space is a based topological space* $(X, *)$ *together with a continuous map* $\mu : X \times X \to X$ *such that the self maps* $x \to \mu(x, *)$ *and* $x \to \mu(*, x)$ *of* X *are homotopic to the identity.*

The rationalization of such a space X has the homotopy type of generalized Eilenberg-Mac Lane space (see [24, Corollary 1])

$$X_{\mathbb{Q}} \simeq_{\mathbb{Q}} \prod_{j \geq 2} K(\pi_j(X), j)$$

Then H-spaces have the minimal Sullivan models of the form $(\Lambda V, 0)$.

Now, we can translate Problem 2 in terms of minimal Sullivan model to the following: let $M(Y) = (\Lambda(a,b), 0)$ be the minimal Sullivan model of Y with $|a| \neq |b|$ and look for a minimal Sullivan model $(\Lambda V, d)$ with $C^*(\mathrm{Der}((\Lambda V, d)), D)$ quasi-isomorphic to $M(Y)$.

In [33], Yamaguchi gave a necessary and sufficient condition of a rationally elliptic space X such that the Dold-Lashof classifying space $\mathrm{Baut}_1(X)$ is rank-one H-space. He proved the following result

Theorem 12. *For an elliptic space X, rank $\pi_*(\mathrm{Baut}_1(X))$ is one if and only if $X \simeq_{\mathbb{Q}} \mathbb{S}^m$ or X has a minimal Sullivan model of the form*

$$(\Lambda(x_1, ..., x_n, y_1, ..., y_n, v), d)$$

where $|x_i| = |y_i|$ is odd for $1 \leq i \leq n$. The differential is defined as follows: $dx_i = dy_i = 0$ for $1 \leq i \leq n$ and $dv = \sum_{i=1}^{n} x_i y_i$.

We turn next to rank-two H-spaces. G. Lupton and S. Smith were the first ones who were interested in realizing rank-two H-spaces as Dold-Lashof classifying spaces. In [16], the authors realized a family of rank-two H-spaces as $\mathrm{Baut}_1(X)$.

Theorem 13. *The following spaces occur as $\mathrm{Baut}_1(X)$ for some simply connected space X*

i) $K(\mathbb{Q}, 2n+1) \times K(\mathbb{Q}, 4n+1)$ *for $n \geq 1$ and n odd*

ii) $K(\mathbb{Q}, r) \times K(\mathbb{Q}, r+4m+1)$ *for $r \geq 2$ and $m \geq 1$.*

If we allow X to be nilpotent, we may also take $m = 0$ in ii).

Nilpotent spaces. If $(X, *)$ is a based space then its higher homotopy groups $\pi_n(X, *)$ come equipped with an action of the fundamental group $\pi = \pi_1(X, *)$. If X is also a connected CW-complex, then we say that X is nilpotent if π is a nilpotent group and also the action of π on the higher homotopy groups is nilpotent. The latter condition is equivalent to the statement that each $\pi_n(X, *)$ possesses a finite filtration of π-modules $M_n(i) \subset M_n(i+1) \subset ...$ such that the action on the associated graded $M_n(i+1)/M_n(i)$ is trivial. More generally, if X is any based connected space, we call X nilpotent if X has the homotopy type of a nilpotent CW-complex. Topological groups having the homotopy type of a connected CW-complex are nilpotent, since the action of π_1 in this case is trivial.

In the same paper, the authors proved the following non-realization result

Theorem 14. *The spaces $\mathbb{C}P^2$ and $\mathbb{S}^4$ are not realized as the Dold-Lashof classifying space* $\mathrm{Baut}_1(X)$ *for any simply connected, π-finite space X.*

We say that a space X is π-finite if X it has only finitely many non-zero homotopy groups. The authors gave an example of a simply connected space of dimension five that does not satisfy a certain structural condition that cannot be realized as the Dold-Lashof classifying space of any simply connected π-finite space.

Theorem 15. *Suppose Y is the space with minimal Sullivan model*

$$(\Lambda\left(a_2, b_2, c_2, x_3, y_3, z_3, t_4, u_4, v_5\right), d)$$

where subscripts denote degrees. The differential, where non-zero is given by

$$\begin{aligned} dx &= a^2 + ac, & dy &= ab, & dz &= bc \\ dt &= xb - ay - az, & du &= cy - az & & \\ dv &= ta + xy + ua + c^3 + b^3. & & & & \end{aligned}$$

Then Y cannot be of the rational homotopy type of $\mathrm{Baut}_1(X)$, for any X a simply connected, π-finite, rational space.

The following lemma plays a key role in the sequel

Lemma 16. [16, Proposition 2.2] *Suppose X is a simply connected space and π-finite with*

$$\pi_i(X) = \begin{cases} 0 & \textit{for } i > N \\ \mathbb{Q}^r \textit{ some } r \geq 1 & \textit{for } i = N \end{cases}$$

Then

$$\pi_i(\mathrm{Baut}_1(X)) = \begin{cases} 0 & \textit{for } i > N+1 \\ \mathbb{Q}^r & \textit{for } i = N+1. \end{cases}$$

Proof: Denote by $(\Lambda V, d)$ the minimal Sullivan model of X with V non-zero only in degrees $\leq N$ and $\dim\left(V^N\right) = r$. Then the DGLA of derivations $(\mathrm{Der}_*(\Lambda V, d), D)$ is non-zero only in degrees $\leq N$. Therefore, we have

$$H_i(\mathrm{Der}(\Lambda V, d), D) = 0 \quad \text{for} \quad i > N.$$

By Theorem 6, it follows that

$$\pi_i(\mathrm{Baut}_1(X)) = 0 \quad \text{for} \quad i > N+1.$$

Furthermore, in degree N, for each $\theta \in \mathrm{Hom}_{\mathbb{Q}}\left(V^N, \mathbb{Q}\right)$, we obtain a derivation in $(\mathrm{Der}_*(\Lambda V, d), D)$ of degree N by setting $\theta\left(V^N\right) = 1$ and extending as a derivation. Any such derivation is a D-cycle, since the elements of V^N do not occur in the differential of any other generators. There are no non-zero boundaries of degree N, since $(\mathrm{Der}_*(\Lambda V, d), D)$ is zero in degree $N+1$ and higher. So, we have

$$H_N(\mathrm{Der}(\Lambda V, d), D) = \mathrm{Hom}_{\mathbb{Q}}\left(V^N, \mathbb{Q}\right).$$

It follows that by using Theorem 6

$$\pi_{N+1}(\mathrm{Baut}_1(X)) = \mathrm{Hom}_{\mathbb{Q}}\left(V^N, \mathbb{Q}\right).$$

■

4. Non-Realization of H-spaces as Dold-Lashof Classifying Spaces

In this section, we show that by using the theory of minimal Sullivan models and other invariants in rational homotopy theory not all rank-two H-spaces can be realized as $\mathrm{Baut}_1(X)$ for some simply connected, elliptic space X. We begin by the following:

Lemma 17. *Let us denote by* $(\Lambda V, d) = (\Lambda(x_1, x_2, y_1, y_2), d)$ *the minimal Sullivan model of a simply connected space* X, *where* $|x_i|$ *are even with* $|x_1| \leq |x_2|$, $|y_i|$ *are odd with* $|y_1| < |y_2|$. *The differential is defined as follows*: $dx_1 = dx_2 = 0$ *and* $dy_i = P_i \in \mathbb{Q}[x_1, x_2]$ *for* $i = 1, 2$, *then* $K(\mathbb{Q}, 2p) \times K(\mathbb{Q}, 2q)$ *cannot be realized as* $\mathrm{Baut}_1(X)$ *for* $p \geq 1$ *and* $p < q$.

Proof: We argue by contradiction, suppose that for $p \geq 1$ and $p < q$ we have

$$\pi_i(\mathrm{Baut}_1(X)) \cong \begin{cases} \mathbb{Q} & \text{for } i = 2q \\ \mathbb{Q} & \text{for } i = 2p \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 6 this condition is equivalent to

$$H_i(\mathrm{Der}(\Lambda V)) \cong \begin{cases} \mathbb{Q} & \text{for } i = 2q-1 \\ \mathbb{Q} & \text{for } i = 2p-1 \\ 0 & \text{otherwise.} \end{cases}$$

We see that the differential D in $\mathrm{Der}_*(\Lambda V)$ satisfies

$$D(y, x^\alpha)(e) = d((y, x^\alpha)(e)) + (y, x^\alpha) d(e) = 0 \tag{1}$$

for a certain $\alpha \geq 0$ and e in V with $y = y_1$, y_2 and $x = x_1$, x_2. On the other hand, since the both derivations $(y_1, 1)$ and $(y_2, 1)$ are non-exact D-cycles, we have

$$[(y_1, 1)] \in H_{|y_1|}(\mathrm{Der}(\Lambda V)) \text{ and } [(y_2, 1)] \in H_{|y_2|}(\mathrm{Der}(\Lambda V)).$$

Hence, by using the starting hypothesis, we must have $|y_1| = 2p-1$ and $|y_2| = 2q-1$ for $p \geq 1$ and $p < q$. For degree reasons, we have two cases to check:
First case: let $(\Lambda V, d) = (\Lambda(x_1, x_2, y_1, y_2), d)$ with $|x_1| \leq |x_2| < |y_1| < |y_2|$ and furthermore that the differential, where non-zero is $dy_i \in \mathbb{Q}[x_1, x_2]$ for $i = 1, 2$. From 1, it follows that the derivations (y_1, x_1) and (y_1, x_2) are D-cycles, which must be boundaries. Further we can say that $D(\theta)(y_1) = -\theta d(y_1)$ with θ is a derivation of even-degree. So, since $(\Lambda V, d)$ is supposed elliptic and $|y_1| < |y_2|$ we see directly that

$$\theta d(y_1) = x_1 \text{ if and only if } \begin{cases} d(y_1) = x_1^2 \\ \theta = (x_1\ , 1) \end{cases}$$

and

$$\theta d(y_1) = x_2 \text{ if and only if } \begin{cases} d(y_1) = x_2^2 \\ \theta = (x_2\ , 1). \end{cases}$$

Thus summarizing the analysis above, we infer that one of the derivations (y_1, x_1) and (y_1, x_2) is non-exact D-cycle. Therefore, we have

$$\dim H_*(\mathrm{Der}(\Lambda V)) \geq 3.$$

Second case: let $(\Lambda V, d) = (\Lambda(x_1, y_1, x_2, y_2), d)$ with $|x_1| < |y_1| < |x_2| < |y_2|$ and the differential is given by: $dx_1 = dx_2 = 0$, $dy_1 \in \mathbb{Q}[x_1]$ and $dy_2 \in \mathbb{Q}[x_1, x_2]$. From 1, it follows that the derivation (y_2, x_1) is a cycle, but it is easy to see that this derivation cannot bound. Therefore, the homology of $\mathrm{Der}_*(\Lambda V)$ contains at least the following elements: $[(y_1, 1)]$, $[(y_2, 1)]$ and $[(y_2, x_1)]$.
As a consequence, in both cases we get

$$\dim H_*(\mathrm{Der}(\Lambda V)) \geq 3.$$

This contradicts our assumption.

■

Gottlieb group. Recall that the n-th Gottlieb group $G_n(X)$ of a space X is the sub-group of the n-th homotopy group $\pi_n(X)$ of X consisting of homotopy classes of maps $\alpha : \mathbb{S}^n \to X$ such that the wedge $[\alpha, id] : \mathbb{S}^n \vee X \to X$ extends to a map $\widetilde{\alpha} : \mathbb{S}^n \times X \to X$ [11]. If $(\Lambda V, d)$ is the minimal Sullivan model of X, then an element $v \in V^n \cong \mathrm{Hom}_{\mathbb{Q}}(\pi_n(X), \mathbb{Q})$ represents a Gottlieb element of $\pi_n(X)$ if and only if there is a derivation of ΛV verifying $\theta(v) = 1$ and such that $[d, \theta] = 0$. Such a derivation represents a non-zero homology class in $(\mathrm{Der}_*(\Lambda V), D)$.

Remark 18. *The proof of* Lemma 17 *above may also be argued by considering the Gottlieb group $G_{2n+1}(X)$. If X is pure, then every odd-degree generator* *is a Gottlieb element, and hence* $\pi_*(\mathrm{Baut}_1(X)) \cong \pi_{*-1}(\mathrm{aut}_1(X))$ *has rank at least that of the odd-dimensional rational homotopy of* X*. This reduces* X *to being essentially of the form in our* Lemma 17*, remaining possibilities are eliminated by direct calculation.*

Proposition 19. *There is no simply connected, pure space* X *for which*

$$\mathrm{Baut}_1(X) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2p) \times K(\mathbb{Q}, 2q)$$

for $(p, q) \in (\mathbb{Z}\mathbb{Z}) \times (\mathbb{Z} - 3\mathbb{Z})$ *and* $p \neq q$*.*

Proof: Suppose that there is a simply connected, pure space X such as

$$\mathrm{Baut}_1(X) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2p) \times K(\mathbb{Q}, 2q)$$

for $(p, q) \in (\mathbb{Z} - 2\mathbb{Z}) \times (\mathbb{Z} - 3\mathbb{Z})$ and $p \neq q$. Since X is π-finite in particular and $\pi_*(\mathrm{Baut}_1(X))$ is zero above degree $2q$, we can conclude that $\pi_*(X)$ is concentrated in degrees $\leq 2q - 1$. Further, since we have $\pi_{2q}(\mathrm{Baut}_1(X)) \cong \mathbb{Q}$, we must have $\pi_{2q-1}(X) \cong \mathbb{Q}$ by Lemma 16. Thus the minimal Sullivan model for X takes the form

$$(\Lambda V, d) = (\Lambda(x_{i,k}), d)_{k \in I_i}$$

where $V^i = \langle x_{i,k} \rangle_{k \in I_i}$ for $i \in \{2, ..., 2q - 1\}$ and furthermore V^{2q-1} is of dimension one. The differential satisfies $dV^{\text{even}} = 0$ and $dV^{\text{odd}} \subseteq \Lambda V^{\text{even}}$. so, we have

$$D(x_{2j+1,k}, 1) = 0 \tag{2}$$

for each $j \in \{1, ..., q - 1\}$ and $k \in I_i$. Otherwise

$$D(\theta) \neq (x_{2j+1,k}, 1) \tag{3}$$

where θ is a derivation of degree $|x_{2j+1,k}| + 1$. One sees from this that the derivations $(x_{2j+1,k}, 1)$ are non-exact D-cycles. Therefore according to the starting hypothesis and from Theorem 6, we have necessarily $V^{2i+1} = 0$, for $i \in \{1, 2, ..., q - 2\} - (p - 1)$ and V^{2p-1} is of dimension at most one. Recalling again that for an elliptic space X, $\chi_\pi(X) \leq 0$, then, there are two cases, which we handle slightly differently

First case: $\dim V^{2p-1} = 0$. In this case, the minimal Sullivan model for X takes the form $(\Lambda V, d) = (\Lambda(x_{\text{even}}, z_{2q-1}), d)$ with subscripts denoting degrees and differential $dx = 0$ and $dz = x^\alpha$ with $\alpha = \dfrac{2q-2}{|x|}\cdot$ The cases $\alpha = 0$ and $\alpha = 1$ are not taken in consideration here because we suppose that the model of X is elliptic and minimal.

If $\alpha = 2$, without loss of generality the DGLA $\mathrm{Der}_*(\Lambda V)$ is generated by $(x,1)$, (z,x) and $(z,1)$. Since $D(x,1) = (z,x)$ and $D(z,1) = 0$, it follows that $H_*(\mathrm{Der}(\Lambda V)) = [(z,1)]$. Then we conclude that

$$\dim H_*(\mathrm{Der}(\Lambda V, d)) \neq 2.$$

If $\alpha \neq 3$, it is easy to show that the derivations $(x,1)$, $(z,1)$ and (z,x) are non-exact D-cycles (possibly other derivations). This implies that

$$\dim H_*(\mathrm{Der}(\Lambda V, d)) \geq 3.$$

As a consequence, we get $\dim \pi_*(\mathrm{Baut}_1(X)) \neq 2$, so this is a contradiction.
The other possibility is that $\alpha = 3$ (cf Proposition 27 below).

Second case: dim $V^{2p-1} = 1$. Since $\chi_\pi(X) \leq 0$, we will discuss the following cases:

i) Write $(\Lambda V, d)$ as $(\Lambda(x_{\text{even}}, y_{2p-1}, z_{2q-1}), d)$ with subscripts indicating degrees and differential $dx = 0$, $dy = x^{2m}$ and $dz = x^{2n}$ for $m, n \in \mathbb{N}$. Note that we can choose m and n so as to have $(\Lambda V, d)$ is elliptic and minimal. From 2 and 3, we conclude that

$$[(y,1)] \in H_{2p-1}(\mathrm{Der}(\Lambda V, d)) \text{ and } [(z,1)] \in H_{2q-1}(\mathrm{Der}(\Lambda V, d)).$$

Furthermore, we can observe that the derivation (z,x) or (z,y) are non-exact D-cycles (possibly both). Consequently, we have

$$\dim H_*(\mathrm{Der}(\Lambda V, d)) \geq 3.$$

ii) Write $(\Lambda V, d)$ as $(\Lambda(x_{\text{even}}, t_{\text{even}}, y_{2p-1}, z_{2q-1}), d)$ with $|x| \leq |t|$ and $|y| < |z|$ and differential $dx = dt = 0$, $dy \in \mathbb{Q}[x,t]$ and $dz \in \mathbb{Q}[x,t]$. Now apply the preceding Lemma 17 to complete this case and assembling these two cases completes the proof. ■

Lemma 20. *For each $p \geq 1$ and $q \geq p+1$, there is no simply connected, pure space X for which $K(\mathbb{Q}, 2p+1) \times K(\mathbb{Q}, 2q)$ has the rational homotopy type of* $\mathrm{Baut}_1(X)$.

Proof: We argue by contradiction. For each $p \geq 1$ and $q \geq p+1$, suppose there is a simply connected, pure space X with

$$\pi_i(\mathrm{Baut}_1(X)) = \begin{cases} \mathbb{Q} & \text{for } i = 2q \\ \mathbb{Q} & \text{for } i = 2p+1 \\ 0 & \text{otherwise} \end{cases}$$

and let $(\Lambda V, d)$ be the minimal Sullivan model of X . Now introduce the following notation: max $V :=$ max$\{i;\ V^i \neq 0\}$, which is finite since X is pure. By using Lemma 16 we deduce there is an element y of V such that $|y| = \max V = 2q - 1$. It is well-known that $(y, 1)$ is a non-exact D-cycle. Then, the minimal Sullivan model for X must be of the form $(\Lambda\left(x_{\text{even}}, y_{2q-1}\right), d)$ where $dx = 0$ and $dy = x^{\alpha}$ with $\alpha = \dfrac{2q-2}{|x|}\cdot$ Note that α cannot be equal to 0 and 1 because we suppose that $(\Lambda V, d)$ is elliptic and minimal. Now if $\alpha \geq 2$, we see directly that

$$Z\left(\mathrm{Der}_*\left(\Lambda V\right)\right) = \left\langle \left(y, x^{\beta}\right)\ ;\ 0 \leq \beta < \alpha \right\rangle \ \text{ and } B\left(\mathrm{Der}_*\left(\Lambda V\right)\right) = \left\langle \left(y, x^{\alpha-1}\right)\right\rangle .$$

Here, we have used the notations $Z\left(\mathrm{Der}_*\left(\Lambda V\right)\right)$ and $B\left(\mathrm{Der}_*\left(\Lambda V\right)\right)$ to denote respectively the sub-vector space of cycles and boundaries in $\mathrm{Der}_*\left(\Lambda V\right)$. As a consequence, we obtain

$$\dim H_*\left(\mathrm{Der}\left(\Lambda V\right)\right) = \dim H_{\text{odd}}\left(\mathrm{Der}\left(\Lambda V\right)\right) = \alpha - 1.$$

It is automatic that $\pi_{\text{odd}}\left(\mathrm{Baut}_1\left(X\right)\right) = 0$, and so this is a contradiction. ■

By the same manner we have the following

Lemma 21. *There is no simply connected, elliptic space X for which*

$$K\left(\mathbb{Q}, 2p\right) \times K\left(\mathbb{Q}, 2q+1\right) \simeq_{\mathbb{Q}} \mathrm{Baut}_1(X) \quad \textit{for} \quad p \geq 1 \quad \textit{and} \quad q \geq p.$$

Proof: As in previous arguments, we proceed by contradiction. Suppose there is a simply connected, elliptic space X such that for $p \geq 1$ and $q \geq p$, we have

$$\mathrm{Baut}_1(X) \simeq_{\mathbb{Q}} K\left(\mathbb{Q}, 2p\right) \times K\left(\mathbb{Q}, 2q+1\right).$$

Without loss of generality, from Lemma 16 we may assume that X has a minimal Sullivan model of the form $(\Lambda V, d) = (\Lambda\left(W \oplus x\right), d)$ with W is a sub-space of V and $|x| = 2q$. Now, we appeal to some results of Halperin concerning elliptic minimal Sullivan models. First of all, to any elliptic minimal Sullivan model $(\Lambda V, d)$, there is an associated pure model, denoted $(\Lambda V, d_\sigma)$, which is defined by adjusting the differential d to d_σ as follows: We set $d_\sigma = 0$ on each even degree generator of V, and on each odd degree generator $v \in V$, we set $d_\sigma\left(v\right)$ equal to the part of $d\left(v\right)$ contained in $\Lambda\left(V^{\text{even}}\right)$. One checks that this defines a differential d_σ on ΛV, and thus we obtain a pure minimal Sullivan model $(\Lambda V, d_\sigma)$. Then by [13, Proposition 1] (cf also [6, Proposition 32.4]) $\dim H_*\left(\Lambda V, d\right)$ is finite-dimensional if and only if $\dim H_*\left(\Lambda V, d_\sigma\right)$ is finite-dimensional. Applying all this to the minimal Sullivan model $(\Lambda\left(W \oplus x\right), d)$, we obtain for $n \geq 1$, $[x^n] \in H_*\left(\Lambda V, d_\sigma\right)$, so we have $\dim H_*\left(\Lambda V, d\right)$ is infinite, contradicting the assumption that X is elliptic. ■

We next recall an interesting invariant of a simply connected space X. The L-S category of X as introduced by Lusternik-Schnirelmann [17], is defined as follows. The L-S category of X, $\operatorname{cat}(X)$, is the least integer n such that X can be covered by $(n+1)$ open subsets contractible in X and is ∞ if no such n exists.
The rational category of a simply connected space X, $\operatorname{cat}_0(X)$, is defined by $\operatorname{cat}_0(X) = \operatorname{cat}(X_{\mathbb{Q}})$. For example $\operatorname{cat}_0(\mathbb{S}^n) = 1$ and $\operatorname{cat}_0(\mathbb{C}P^n) = n$ for $n \geq 2$, and moreover if Y and Z are simply connected topological spaces with rational homology of finite type then $\operatorname{cat}_0(Y \times Z) = \operatorname{cat}_0(Y) + \operatorname{cat}_0(Z)$. For more details see [6]. For the next lemma we need to recall the following:

Theorem 22. [10, Theorem 1] *Let X be an elliptic space. Then* $\operatorname{Baut}_1(X)$ *has infinite rational L-S category.*

We are now in a position to prove the following lemma

Lemma 23. *For $p, q \geq 1$ and $p \neq q$, $K(\mathbb{Q}, 2p+1) \times K(\mathbb{Q}, 2q+1)$ cannot be of the rational homotopy type of* $\operatorname{Baut}_1(X)$ *for any X a simply connected, elliptic space.*

Proof: Gatsinzi's result imply that the rational category of $\operatorname{Baut}_1(X)$ is infinite when X is an elliptic space. In the other words we have

$$\operatorname{cat}_0(K(\mathbb{Q}, 2p+1) \times K(\mathbb{Q}, 2q+1)) = 2 \quad \text{for } p, q \geq 1.$$

This means that

$$\operatorname{Baut}_1(X) \neq K(\mathbb{Q}, 2p+1) \times K(\mathbb{Q}, 2q+1).$$

■

Gathering Proposition 19, Lemma 20, Lemma 21 and Lemma 23, we have proved the Theorem 3. The following result is an immediate consequence of Theorem 3.

Corollary 24. *For $(p, q) \in (\mathbb{Z} - 4\mathbb{Z}) \times (\mathbb{Z} - 6\mathbb{Z})$, $K(\mathbb{Q}, p) \times K(\mathbb{Q}, q)$ cannot be of the rational homotopy type of* $\operatorname{Baut}_1(X)$ *for any X a simply connected, pure space.*

5. Realization of H-Spaces as Dold-Lashof Classifying Spaces

In this section we give the proof of Theorem 4. We break this proof into four results. We begin by the following observations: we may always realize $K(\mathbb{Q}, p) \times$

$K(\mathbb{Q},p)$ as $\mathrm{Baut}_1(X)$ for $p \geq 3$. Indeed; we define the space X in terms of minimal Sullivan model $(\Lambda(x_{p-1}, y_{p-1}), 0)$ with subscripts indicating degrees. Then the DGLA $\mathrm{Der}_*((\Lambda(x_{p-1}, y_{p-1}), 0))$ has a vector space basis $\{(x,1),(y,1)\}$. Since $d=0$, we have $D=0$. To describe the rational homotopy type of $\mathrm{Baut}_1(X)$ we may apply the commutative cochains functor $C^*()$ to this DGLA (c.f [30], I.1(3)). As differential graded algebras, we have

$$C^*(\mathrm{Der}_*(\Lambda(x_{p-1},y_{p-1}))) = \Lambda\big(s^{-1}(\mathrm{Hom}(\mathrm{Der}_*(\Lambda(x_{p-1},y_{p-1}))),\mathbb{Q})\big) = \Lambda(u,v)$$

where $|u|=|v|=p$. The differential d decomposes as $d=d_1+d_2$ with d_1 dual to the Lie model differential D and d_2 is dual to the Lie bracket. Since $D=0$ we conclude that $d_1=0$. From the simple bracket structure of $\mathrm{Der}_*((\Lambda(x_{p-1}, y_{p-1}), 0))$, we conclude that $\mathrm{Baut}_1(X)$ has minimal Sullivan model $(\Lambda(u,v),0)$. Clearly, this gives that $\mathrm{Baut}_1(X)$ has the rational homotopy type of $K(\mathbb{Q},p)\times K(\mathbb{Q},p)$. Note that, we may also take $p\geq 2$, if we allow X to be nilpotent.

Based on the previous argument, we conclude that for $m\geq 1$ and $p\geq 2$

$$\mathrm{Baut}_1(K(\mathbb{Q}^m,p)) \simeq_{\mathbb{Q}} \overbrace{K(\mathbb{Q},p+1)\times K(\mathbb{Q},p+1)\times \ldots \times K(\mathbb{Q},p+1)}^{\text{m times}}.$$

The following example shows that the situation changes dramatically when the input H-space X has generators in distinct degrees.

Example 25. *Consider the space $X=K(\mathbb{Q},2)\times K(\mathbb{Q},4)$. The minimal Sullivan model of $\mathrm{Baut}_1(X)$ is $(\Lambda(x_3,y_3,z_5),d)$ with subscripts denoting degrees and differential $dx=dy=0$ and $dz=xy$. This means that*

$$\mathrm{Baut}_1(K(\mathbb{Q},2)\times K(\mathbb{Q},4)) \neq \mathrm{Baut}_1(K(\mathbb{Q},3))\times \mathrm{Baut}_1(K(\mathbb{Q},5)).$$

Given a simply connected space X, let min $\pi_*(X)$):= min$\{n \mid \pi_n(X)\neq 0\}$ and max $\pi_*(X)$):= max$\{n;\ \pi_n(X)\neq 0\}$. S. Smith showed the following

Theorem 26. [26, Theorem 3] *Let us suppose that X is an F_0-space with $\mathrm{Der}_+(H^*(X;\mathbb{Q}))$ equal to zero and Y is a rational H-space. If min $\pi_*(X)+$ min $\pi_*(Y)\geq$ max $\pi_*(X\times Y)$ and max $\pi_*(X)\leq$ min $\pi_*(Y)$, then*

$$\mathrm{Baut}_1(X\times Y)\simeq_{\mathbb{Q}} \mathrm{Baut}_1(X)\times \|\mathrm{Der}(H^*(Y;\mathbb{Q}),H^*(X\times Y;\mathbb{Q}))\|$$

where $\|L\|$ is the spatial realization of a differential graded Lie algebra L (cf [22]).

Finally, it is not trivial to realize a product of Eilenberg-Mac Lane spaces with nonzero homotopy groups in two distinct degrees. Next, we examine various issues related to the realizability of Dold-Lashof classifying spaces. Our first result is the following

Proposition 27. *For each $m \geq 2$ and m is even, there exists a simply connected, pure space X_m with*

$$K(\mathbb{Q}, 2m) \times K(\mathbb{Q}, 3m) \simeq_{\mathbb{Q}} \mathrm{Baut}_1(X_m).$$

Proof: For every natural $m \geq 2$ and m is even, let us consider the following 1-connected minimal Sullivan model

$$\Lambda V = \Lambda(x, y)$$

with $|x| = m$ and $|y| = 3m - 1$. The differential is as follows: $dx = 0$ and $dy = x^3$. Then the generating derivations are given by

$*$	$\mathrm{Der}_*(\Lambda(x,y), d)$
$3m-1$	$(y, 1)$
$2m-1$	(y, x)
m	$(x, 1)$
$m-1$	(y, x^2)

It is a straightforward computation to verify that

$$D(y,1) = D(y,x) = 0,\ D(x,1) = -3\left(y, x^2\right) \text{ and } D\left(y, x^2\right) = 0.$$

Therefore, the derivations $(y, 1)$ and (y, x) are cycles, which are not boundaries. As a consequence, we obtain

$$H_*(\mathrm{Der}(\Lambda(x,y), d), D) = \mathbb{Q}\{[(y,1)]\} \oplus \mathbb{Q}\{[(y,x)]\}.$$

Denote the class $[(y, 1)]$ in $H_{3m-1}(\mathrm{Der}(\Lambda(x, y), d), D)$ by τ and the class $[(y, x)]$ in $H_{2m-1}(\mathrm{Der}(\Lambda(x, y), d), D)$ by ρ. We can now deduce from theorem 6 that $\pi_*(\mathrm{Baut}_1(X_m))$ is concentrated in degree $3m$ and $2m$. Thus $\mathrm{Baut}_1(X_m)$ has a minimal Sullivan model of the form

$$(\Lambda(\rho, \tau), d)$$

with $|\rho| = 2m$ and $|\tau| = 3m$. For degree reasons we must have $d = 0$. Therefore, it is exactly the minimal Sullivan model of $K(\mathbb{Q}, 2m) \times K(\mathbb{Q}, 3m)$, as needed. ■

This result, in turn, translates to the special case of a famous conjecture in rational homotopy theory due to Halperin, as we discuss now. First, let X be a simply connected elliptic CW-complex with evenly graded rational cohomology. We refer to such spaces as F_0-spaces. The class includes products of spheres, products of complex projective spaces and homogeneous spaces G/H with G a compact connected Lie group and $H \subset G$ a closed sub-group of maximal rank. Motivated by this last case, Halperin [14] conjectured that the rational Serre spectral sequence collapses at the E_2 term for all orientable fibrations with fibre an F_0-space. Thomas [32] and Meier [19] independently proved that Halperin's conjecture is equivalent to the condition $H_{\text{even}}\left(\text{Der}(M\left(X\right))\right) = 0$, with $M\left(X\right)$ the minimal Sullivan model for an F_0-space X. Thus by theorem 6 of Sullivan Haleprin's conjecture holds for an F_0-space X if and only if

$$\text{Baut}_1\left(X\right) \simeq_{\mathbb{Q}} \Pi_{i=1}^{s} K\left(\mathbb{Q}, 2n_i\right) \qquad \text{for some } n_i.$$

The Halperin conjecture has been affirmed in several special cases including Kähler manifolds by Meier [19], for homogeneous spaces of maximal rank pairs by Shiga-Tezuka [25] and for F_0-spaces with rational cohomology generated by ≤ 3 generators by Lupton [15].

Next, denote by $(\Lambda V, d) = \left(\Lambda\left(x, y, u, v\right), d\right)$ the minimal Sullivan model of the space $X_{m,n}$ with the degrees of these generators given by $|x| = 2n + 1$, $|y| = 2m + 1$, $|u| = 2m + 2n + 1$ and $|v| = 4m + 2n + 1$. The differential is defined as follows: $dx = 0$, $dy = 0$, $du = xy$ and $dv = uy$. The following result extends the example 2.4 in [16], here we prove that

Theorem 28. *For each $n \geq 1$ and $m \geq n$, we have*

$$K\left(\mathbb{Q}, 4m+1\right) \times K\left(\mathbb{Q}, 4m+2n+2\right) \simeq_{\mathbb{Q}} \text{Baut}_1(X_{m,n}).$$

Proof: We give the proof by dividing m into three cases.

When $m \geq 2n + 1$.

Without loss of generality, we may write a vector space basis for $\left(\text{Der}_*\left(\Lambda V\right), D\right)$ in positive degrees as follows

$*$	$\text{Der}_*(\Lambda V)$
$4m+2n+1$	$(v,1)$
$4m$	(v,x)
$2m+2n+1$	$(u,1)$
$2m+2n$	(v,y)
$2m+1$	$(y,1)$
$2m$	$(v,u),(u,x)$
$2m-1$	(v,xy)
$2m-2n$	(y,x)
$2m-2n-1$	(v,xu)
$2n+1$	$(x,1)$
$2n$	(u,y)

Computation of $H_*(\mathrm{Der}(\Lambda V), D)$

It is clear that $(v,1)$ and (v,x) are non-bounding D-cycles. Then

$$H_{4m+2n+1}(\mathrm{Der}(\Lambda V)) = \mathbb{Q}\{[(v,1)]\} \text{ and } H_{4m}(\mathrm{Der}(\Lambda V)) = \mathbb{Q}\{[(v,x)]\}. \quad (4)$$

On the other hand we have

$$D(u,1) = (v,y) \quad \text{and} \quad D(v,y) = 0.$$

This means that

$$H_{2m+2n+1}(\mathrm{Der}(\Lambda V)) = H_{2m+2n}(\mathrm{Der}(\Lambda V)) = 0.$$

Next, using the fact

$$\begin{aligned} D(y,1) &= -(v,u) - (u,x), & D(v,u) &= (v,xy) \\ D(u,x) &= -(v,xy), & D(v,xy) &= 0. \end{aligned}$$

We deduce that

$$0 \to \langle (y,1) \rangle \xrightarrow{D} \langle (v,u),(u,x) \rangle \xrightarrow{D} \langle (v,xy) \rangle \to 0$$

is a short exact sequence, which implies that

$$H_{2m+1}(\mathrm{Der}(\Lambda V)) = H_{2m}(\mathrm{Der}(\Lambda V)) = H_{2m-1}(\mathrm{Der}(\Lambda V)) = 0.$$

Furthermore, we have

$$D(y,x) = -(v,xu) \qquad \text{and} \qquad D(v,xu) = 0$$

Therefore

$$D(x,1) = (u,y) \qquad \text{and} \qquad D(u,y) = 0.$$

We conclude that

$$H_i(\mathrm{Der}(\Lambda V)) = 0 \text{ for } i = 2m-2n,\ 2m-2n-1,\ 2n+1 \text{ and } 2n.$$

Now, applying Theorem 6 in conjunction with 4 we obtain the following

$$\pi_i(\mathrm{Baut}_1(X_{m,n})) \simeq_{\mathbb{Q}} \mathbb{Q} \text{ for } i = 4m+1,\ 4m+2n+2 \text{ and zero otherwise.}$$

Hence, we must have

$$\mathrm{Baut}_1(X_{m,n}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4m+1) \times K(\mathbb{Q}, 4m+2n+2).$$

When $n < m < 2n+1$.

Observe that we have $|(x,1)| > |(u,y)| \geq |(y,x)| > |(v,xu)|$. So we come back to the table above and we change the derivations (y,x) by $(x,1)$ and (v,xu) by (u,y) and vice versa and the other generators will not be changed. Note that, in this case the differential D will not be changed, then we have the same result as above.

When $m = n$.

Here there is more coalescing of degrees of the various terms. We have $4m+1 = |(u,1)| > |(v,x)| = |(v,y)| = 4m$, $|(y,1)| = |(x,1)| = 2m+1$ and $|(u,x)| = |(v,u)| = |(u,y)| = 2m$, on the other hand we have $2m-2n = 0$ and $2m-2n-1 < 0$, so both derivations (y,x) and (v,xu) will not be considered. This makes no difference to our calculation. ■

Remark 29. *Note that the restriction $m \geq n$ is necessary. Indeed, if $n > m$, in this case we have another derivation cycle (x,y) , which does not bound. It follows that $\pi_* \left(\mathrm{Baut}_1 (X)\right)$ is of dimension ≥ 3.*

For our last results, we will need to the following

Claim 30. *Let X be a simply connected space of finite type with minimal Sullivan model $(\Lambda V, d) = (\Lambda (W \oplus y), d)$ and let $x \in \Lambda W$, then the derivation (y,x) is a D-cycle if x is a d-cycle.*

Proof: We have $D(y,x) = d \circ (y,x) \pm (y,x) \circ d$. When this is evaluated on y, we find that $D(y,x) = dx = 0$. ■

It would be interesting to know whether there are other examples which are realized as Dold-Lashof classifying spaces. The last goal of this sub-section is to extend the Lupton-Smith Theorem 3.3 [16], see also Theorem 13 i).

Theorem 31. *For every $r \geq 1$ and $s \geq 1$, there exists a simply connected, π-finite space $X_{r,s}$ with*

$$K(\mathbb{Q}, 4r-1) \times K(\mathbb{Q}, 4r+2s) \simeq_{\mathbb{Q}} \mathrm{Baut}_1(X_{r,s}).$$

Proof: We will discuss two cases:
First case: $\mathbf{r \geq s+1}$. We define the minimal Sullivan model associated to $X_{r,s}$ as $(\Lambda V, d) = (\Lambda (x,y,z,t,u,v), d)$ where degrees and differential are described by

$$\begin{array}{lll}
|x| = 2s+1, & dx = 0 & \\
|y| = |z| = 2r, & dy = dz = 0 & \\
|t| = |u| = 2r+2s, & dt = xy, & du = xz \\
|v| = 4r+2s-1, & dv = yu - zt. &
\end{array}$$

We assume that $s+1 < r \leq 2s$ and we do our reasoning in this case. For $r \geq 2s+1$ and $r = s+1$, as in the proof of Theorem 28 we can have some minor differences due to the fact that, for these low end cases, the degrees of some of the generators or the differences between the degrees of some of the terms coincide. This makes no difference to our calculation.

In the style of the above examples, we may write a basis for $\mathrm{Der}_*(\Lambda V)$ as follows

$*$	$\mathrm{Der}_*(\Lambda V)$
$4r+2s-1$	$(v,1)$
$4r-2$	(v,x)
$2r+2s$	$(t,1),(u,1)$
$2r+2s-1$	$(v,z),(v,y)$
$2r$	$(y,1),(z,1)$
$2r-1$	$(t,x),(v,u),(v,t),(u,x)$
$2r-2$	$(v,xz),(v,xy)$
$2s+1$	$(x,1)$
$2s$	$(u,z),(u,y),(t,y),(t,z)$
$2s-1$	$(v,yz),\left(v,y^2\right),\left(v,z^2\right)$
$2r-2s-1$	$(z,x),(y,x)$
$2r-2s-2$	$(v,xt),(v,xu)$

Let us calculate $H_*(\mathrm{Der}(\Lambda V))$

It is easily verified that $(v,1)$ and (v,x) are non-bounding D-cycles. Then

$$H_{4r+2s-1}(\mathrm{Der}(\Lambda V)) = \mathbb{Q}\{[(v,1)]\} \text{ and } H_{4r-2}(\mathrm{Der}(\Lambda V)) = \mathbb{Q}\{[(v,x)]\}.$$

Since

$$D(t,1) = (v,z),\ D(u,1) = -(v,y) \text{ and } D(v,z) = D(v,y) = 0.$$

It follows that

$$H_{2r+2s}(\mathrm{Der}(\Lambda V)) = H_{2r+2s-1}(\mathrm{Der}(\Lambda V)) = 0.$$

Furthermore, we have

$$\begin{aligned} &D(z,1) = (v,t) - (u,x) \quad \text{and} \quad D(y,1) = -(t,x) - (v,u) \\ &D(v,t) = (v,xy), \quad D(u,x) = (v,xy) \quad \text{and} \quad D(v,xy) = 0 \\ &D(t,x) = -(v,xz), \quad D(v,u) = (v,xz) \quad \text{and} \quad D(v,xz) = 0. \end{aligned}$$

Hence, we deduce that

$$H_{2r}\left(\operatorname{Der}(\Lambda V)\right)=H_{2r-1}\left(\operatorname{Der}(\Lambda V)\right)=H_{2r-2}\left(\operatorname{Der}(\Lambda V)\right)=0.$$

Next, since

$$\begin{array}{ll} D(x,1)=(u,z)+(t,y) & D(u,z)=-(v,yz) \\ D(t,y)=(v,yz), & D(v,yz)=0. \end{array}$$

Furthermore, we have

$$D(u,y)=-\left(v,y^{2}\right),\quad D(t,z)=\left(v,z^{2}\right)\text{ and } D\left(v,y^{2}\right)=D\left(v,z^{2}\right)=0$$

$$D(y,x)=(v,xu),\quad D(z,x)=-(v,xt)\ \text{ and } D(v,xu)=D(v,xt)=0.$$

So, we deduce that

$$H_{i}\left(\operatorname{Der}(\Lambda V)\right)=0 \text{ for } i=2s+1,\ 2s,\ 2s-1,\ 2r-2s-1,\ 2r-2s-2.$$

Finally, we have shown that

$$\pi_{i}\left(\operatorname{Baut}_{1}(X_{r,s})\right)\cong\mathbb{Q}\ \text{ for }\ i=4r-1, i=4r+2s \text{ and zero otherwise.}$$

Second case: $\mathbf{r}\leq$ s. Here, we have $(\Lambda V,d)=(\Lambda(y,z,x,t,u,v),d)$ with the same degrees and differential as in the previous case. Recall we need to prove that $\pi_{i}\left(\operatorname{Baut}_{1}(X_{r,s})\right)$ is concentrated in degrees $4r-1$ and $4r+2s$. Instead we prove that the rank of $H_{i}\left(\operatorname{Der}(\Lambda V)\right)$ is equal to 1 for $i=4r-2$, $4r+2s-1$ and is equal to zero for other i's. As in the previous case, although here we have

$$\begin{array}{lll} D(y,1)=-(t,x)-(v,u), & D(t,x)=-(v,xz)\ \text{ and } & D(v,u)=(v,xz) \\ D(z,1)=-(u,x)+(v,t), & D(u,x)=(v,xy)\ \text{ and } & D(v,t)=(v,xy). \end{array}$$

Further for $\beta_{1},\beta_{2}\geq 1$ and $\beta_{1}+\beta_{2}\leq\dfrac{2s-1}{2r}+2$ we have

$$D\left(x,y^{\beta_{1}-1}z^{\beta_{2}-1}\right)=\left(u,y^{\beta_{1}-1}z^{\beta_{2}}\right)+\left(t,y^{\beta_{1}}z^{\beta_{2}-1}\right)$$
$$D\left(u,y^{\beta_{1}-1}z^{\beta_{2}}\right)=-\left(v,y^{\beta_{1}}z^{\beta_{2}}\right)\ \text{ and }\ D\left(t,y^{\beta_{1}}z^{\beta_{2}-1}\right)=\left(v,y^{\beta_{1}}z^{\beta_{2}}\right).$$

Next, from the above computation and by claim 30 we deduce that the elements which could be cycles in $\operatorname{Der}_{*}(\Lambda V)$ are on the form: $(v,1)$, (v,x), (v,yx), (v,zx), (v,y^{α}), (v,z^{α}), $\left(v,y^{\beta_{1}}z^{\beta_{2}}\right)$, $(t,x)+(v,u)$, $(u,x)-(v,t)$, $\left(u,y^{\beta_{1}-1}z^{\beta_{2}}\right)+\left(t,y^{\beta_{1}}z^{\beta_{2}-1}\right)$ for $1\leq\alpha\leq\frac{2s-1}{2r}+2$, $\beta_{1},\beta_{2}\geq 1$ and $\beta_{1}+\beta_{2}\leq\dfrac{2s-1}{2r}+2.$

On the other hand, it is clear that $(v,1)$ and (v,x) are non-bounding D-cycles. Now again from the computation above and

$$D\left(u,y^{\alpha-1}\right)=(v,y^{\alpha})\qquad\text{and}\qquad D\left(t,z^{\alpha-1}\right)=(v,z^{\alpha})$$

We obtain

$$H_*\left(\mathrm{Der}\left(\Lambda V\right)\right)=\mathbb{Q}\left\{[(v,x)]\right\}\oplus\mathbb{Q}\left\{[(v,1)]\right\}.$$

Using Theorem 6 completes this proof. ■

There is another situation in which a similar approach gives the following

Theorem 32. *For $r\geq 1$ and $s\geq 2r-1$, suppose $X_{r,s}$ is the space with minimal Sullivan model $\left(\Lambda\left(y,z,x,t,u,v\right),d\right)$ where degrees of the generators given by $|y|=|z|=2r$, $|x|=2s$, $|t|=|u|=2r+2s-1$ and $|v|=4r+2s-2$. The differential is defined as follows: $dy=dz=0$, $dx=0$, $dt=xy$, $du=xz$, $dv=yu-zt$. Then*

$$\mathrm{Baut}_1(X_{r,s})\simeq_{\mathbb{Q}}K\left(\mathbb{Q},4r-1\right)\times K\left(\mathbb{Q},4r+2s-1\right).$$

Proof: We will prove that $\pi_i(\mathrm{Baut}_1(X_{r,s}))$ is concentrated in degrees $4r-1$ and $4r+2s-1$. Also, since

$$\begin{array}{ll} D\left(y,1\right)=-\left(t,x\right)-\left(v,u\right), & D\left(t,x\right)=-\left(v,xz\right)\ \text{ and }\ D\left(v,u\right)=\left(v,xz\right)\\ D\left(z,1\right)=-\left(u,x\right)+\left(v,t\right), & D\left(u,x\right)=\left(v,yx\right)\ \text{ and }\ D\left(v,u\right)=\left(v,yx\right). \end{array}$$

Further for $\beta_1,\beta_2\geq 1$ and $\beta_1+\beta_2\leq\dfrac{s-1}{r}+2$ we have

$$\begin{aligned} D\left(x,y^{\beta_1-1}z^{\beta_2-1}\right)&=-\left(u,y^{\beta_1-1}z^{\beta_2}\right)-\left(t,y^{\beta_1}z^{\beta_2-1}\right)\\ D\left(u,y^{\beta_1-1}z^{\beta_2}\right)&=\left(v,y^{\beta_1}z^{\beta_2}\right)\text{ and }D\left(t,y^{\beta_1}z^{\beta_2-1}\right)=-\left(v,y^{\beta_1}z^{\beta_2}\right). \end{aligned}$$

We deduce that the elements which could be cycles in $\mathrm{Der}_*\left(\Lambda V\right)$ are on the form: $(v,1)$, (v,x), (v,yx), (v,zx), (v,y^{α}), (v,z^{α}), $\left(v,y^{\beta_1}z^{\beta_2}\right)$, $(t,x)+(v,u)$, $(u,x)-(v,t)$, $\left(u,y^{\beta_1-1}z^{\beta_2}\right)+\left(t,y^{\beta_1}z^{\beta_2-1}\right)$ for $1\leq\alpha\leq\dfrac{s-1}{r}+2$, $\beta_1,\beta_2\geq 1$ and $\beta_1+\beta_2\leq\dfrac{s-1}{r}+2$.

Finally, from the computation above and the fact

$$D\left(u,y^{\alpha-1}\right)=(v,y^{\alpha})\ \text{ and }D\left(t,z^{\alpha-1}\right)=(v,z^{\alpha})\ \text{ for }1\leq\alpha\leq\frac{s-1}{r}+2.$$

We conclude that

$$H_*\left(\mathrm{Der}\left(\Lambda V\right)\right) \cong \mathbb{Q}\left\{\left[(v,x)\right]\right\} \oplus \mathbb{Q}\left\{\left[(v,1)\right]\right\}.$$

■

We finish this section with some comments on the constructions above:

i) Note that the restriction $s \geq 2r-1$ in Theorem 32 is necessary. Indeed, if $s < 2r-1$, then the derivations on the forme $\left(v, x^{\lambda}\right)$ for $\lambda \geq 2$ are non-exact D-cycles.

ii) If we take $|z|$ odd, then the derivation (t,z) is a non-exact D-cycle and if $|y|$ is odd, (u,y) is a non-exact D-cycle.

iii) If $|y|$ and $|z|$ are even and $|y| \neq |z|$, we will have $|u| \neq |t|$ and one of the derivations $(u,t)+(z,y)$ or $(t,u)-(y,z)$ is a non-exact D-cycle.

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