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A COMPLEX STRUCTURE ON THE MODULI SPACE OF RIGGED RIEMANN SURFACES

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Abstract. The study of Riemann surfaces with parametrized boundary components was initiated in conformal field theory (CFT). Motivated by general principles from Teichmüller theory, and applications to the construction of CFT from vertex operator algebras, we generalize the parametrizations to quasisymmetric maps. For a precise mathematical definition of CFT (in the sense of G. Segal), it is necessary that the moduli space of these Riemann surfaces be a complex manifold, and the sewing operation is holomorphic. We report on the recent proofs of these results by the authors.

1. Introduction

The results described in this paper are proved in detail in [19], which also contains an extensive introduction. As well as giving an overview of some of those results, we expand on certain conceptually important points. In particular, we explain why the use of quasisymmetric boundary parametrizations, in the geometric framework for conformal field theory, is both natural and necessary from the point of view of Teichmüller theory.

1.1. Description of Problem and Results

Let Σ^B be a bordered Riemann surface of type (g, n) with $n \geq 1$, i.e., a Riemann surface of genus g bounded by n closed curves $\partial_i \Sigma^B$, $i = 1, \ldots, n$, which are homeomorphic to S^1. An important object in conformal field theory is the **rigged Riemann surface**, which is a Riemann surface Σ^B together with a set of homeomorphisms $\psi_i : \partial_i \Sigma^B \to S^1$. As well as the ordering, each boundary component is labeled as either **incoming** or **outgoing**. Keeping track of this data is not important for our results, so it is subsequently neglected. We denote the rigged Riemann surface by an ordered pair (Σ^B, ψ), where $\psi = (\psi_1, \ldots, \psi_n)$, and the ordered n-tuple of homeomorphisms ψ is referred to as a **rigging**.

Generally speaking, one specifies that the riggings are in some subclass of homeomorphisms. In the conformal field theory literature, riggings are taken to be either analytic homeomorphisms or diffeomorphisms. We will take the riggings to be quasisymmetric homeomorphisms, which will be defined in Section 2.2.

The **rigged Riemann moduli space** $\widetilde{\mathcal{M}}^B(g,n)$ is defined as the space of rigged Riemann surfaces up to biholomorphic equivalence. More precisely

Definition 1. *The rigged Riemann moduli space is defined by*

$$\widetilde{\mathcal{M}}^B(g,n) = \{(\Sigma^B, \psi)\}/\sim$$

where $(\Sigma_1^B, \psi^1) \sim (\Sigma_2^B, \psi^2)$ *if and only if there exists a biholomorphism* $\sigma : \Sigma_1^B \to \Sigma_2^B$ *such that* $\psi^2 \circ \sigma = \psi^1$.

(The superscript "B" stands for "border", and serves to distinguish $\widetilde{\mathcal{M}}^B$ from the equivalent puncture model in Section 3.1.)

An important operation is the **sewing operation**, in which two rigged Riemann surfaces (Σ_1, ψ^1) and (Σ_2, ψ^2) are joined along a boundary curve by using the riggings. That is, for boundary curves $\partial_i \Sigma_1^B$ and $\partial_j \Sigma_2^B$, for fixed i and j, define $\Sigma_1^B \#_{ij} \Sigma_2^B = (\Sigma_1^B \sqcup \Sigma_2^B)/\sim$, where two boundary points $p_1 \in \partial_i \Sigma_1^B$ and $p_2 \in \partial_j \Sigma_2^B$ are equivalent if and only if $p_2 = (\psi_j^2)^{-1}(\psi_i^1(p_1))$. The role of the reciprocal is to produce an orientation reversing map.

Our main results are that the (infinite-dimensional) quasisymmetrically rigged Riemann moduli space $\widetilde{\mathcal{M}}^B(g,n)$ is a complex manifold, and that the sewing operation is holomorphic. We also give, for the first time, the precise relation of $\widetilde{\mathcal{M}}^B(g,n)$ to the Teichmüller space of bordered surfaces of genus g bounded by n closed curves, which is not possible without the use of quasisymmetries. These results are summarized in Theorem 10, Section 3.4.

The first proof of the holomorphicity of the sewing operation, in the case of analytic riggings, was given by the first author in his thesis [18], but this has not yet been published. In genus-zero with analytic riggings, the holomorphicity of the sewing was proved by Huang [7].

1.2. Why Use Quasisymmetric Parametrizations?

Naturally, in tackling the problem of constructing the complex structure on rigged moduli space, one is led to apply the Teichmüller theory of bordered Riemann surfaces. This theory is constructed using quasisymmetries [10, 14], which arise as boundary values of quasiconformal maps, as will be outlined in Section 2. Thus one is led to use quasisymmetric boundary parametrizations.

Although it may be possible to construct the complex structure on the moduli space of Riemann surfaces rigged with other classes of mappings, such as diffeomorphisms (see e.g. [13]), it remains an interesting question to establish the connection of such a moduli space to the standard Teichmüller space. For this, the results presented in this paper seem to be necessary.

1.3. Motivation of the Problem from Conformal Field Theory

Conformal field theory (CFT) is a special class of two-dimensional quantum field theories that first arose in statistical mechanics. Mathematically the algebraic structure is encoded in the notion of a vertex operator algebra. In string theory, the study of the geometry of CFT was initiated in [4]. The rigged Riemann surfaces, described above, appear as the worldsheets of interacting strings.

Around 1986, Segal [20] and Kontsevich independently extracted the mathematical properties the non-rigorous path integrals in CFT should have, and gave a purely mathematical definition of CFT. Substantial work was done recently by Fiore, Hu and Kriz in [3, 6] to make the categorical structures in this definition rigorous. Problems in the complex analytic aspects of the definition have been solved by the authors in [18, 19].

Although this definition has existed since 1986, no general construction for arbitrary genus has been given. Significant progress has been made by developing and using the theory of vertex operator algebras and their representations. With a series of papers, culminating in [8, 9], Huang has completed a general construction of genus-zero CFT. The genus-one theory is also essentially complete.

To construct higher-genus CFT completely however, many holomorphicity issues must be addressed. In the notion of a *weakly conformal field theory*, as defined by Segal [20], the operators in the CFT are required to depend holomorphically on the associated rigged Riemann surface. For this definition to make sense, the rigged moduli spaces must be complex manifolds and the sewing operation is required to be holomorphic. Our results (Theorem 10) solve this particular problem. Moreover, in constructing CFT from vertex operator algebras it will be necessary to sew using parametrizations that are more general than analytic. This was the original motivation for our generalization to quasisymmetries in [19].

The Teichmüller space of the disk $T(\mathbb{D})$, which contains the Teichmüller spaces of all Riemann surfaces covered by the disk, is closely related to the homogeneous space Diff$(S^1)/$Möb(S^1). Convincing evidence has been advanced that it might serve as a basis for a non-perturbative formulation of closed bosonic string theory (see [5] and [16] for an overview and references). Thus, it is of interest to establish the relation of the rigged moduli spaces to the Teichmüller spaces of bordered Riemann surfaces. See Section 4 for further comments.

2. Teichmüller Theory of Bordered Riemann Surfaces

2.1. Quasiconformal Maps

Let Σ be a Riemann surface. A **Beltrami differential** on Σ is a $(-1,1)$ differential ω, i.e., a differential given in a local biholomorphic coordinate z by $\mu(z)\mathrm{d}\bar{z}/\mathrm{d}z$, such that μ is Lebesgue-measurable in every choice of coordinate and $||\mu||_\infty < 1$. The expression $||\omega||_\infty$ is well-defined, since μ transforms under a local biholomorphic change of parameter $w = g(z)$ according to the rule $\tilde{\mu}(g(z))\overline{g'(z)}g'(z)^{-1} = \mu(z)$ and thus $|\tilde{\mu}(g(z))| = |\mu(z)|$. Denote the space of Beltrami differentials on Σ by $L^\infty_{(-1,1)}(\Sigma)_1$.

The **Beltrami equation** is the differential equation given in local coordinates by $\overline{\partial} f = \omega \partial f$ where ω is a Beltrami differential. We have the important theorem

Theorem 2. *Given any Beltrami differential on a Riemann surface Σ, there exists a homeomorphism $f : \Sigma \to \Sigma_1$, onto a Riemann surface Σ_1, which is differentiable almost everywhere and is a solution of the Beltrami equation almost everywhere. This solution is unique in the sense that given any other solution $\tilde{f} : \Sigma \to \widetilde{\Sigma}_1$, there exists a biholomorphism $g : \Sigma_1 \to \widetilde{\Sigma}_1$ such that $g \circ f = \tilde{f}$.*

In fact, such a solution must have derivatives in L^2. If $||\omega||_\infty = 0$, then f must be a biholomorphism. The solutions of the Beltrami equation are called **quasiconformal mappings**. We will take this as the definition for the purposes of this paper, although there are various equivalent definitions of quasiconformal mappings.

Although the Teichmüller space of a compact or punctured Riemann surface without boundary can be constructed with the use of diffeomorphisms, it is well known that quasiconformal maps are necessary in defining the Teichmüller space of bordered Riemann surfaces. Note that a quasiconformal map on a bordered Riemann surface need not be even differentiable. Conversely, a diffeomorphism need not be quasiconformal if no restrictions are placed on its boundary behavior.

Given a Beltrami differential and the corresponding quasiconformal solution to the Beltrami equation $f : \Sigma \to \Sigma_1$, one can pull back the complex structure on

Σ_1 to obtain a new complex structure on Σ. Thus, one can regard a Beltrami differential as a change of the complex structure on Σ.

2.2. Quasisymmetric Maps

A **quasisymmetric mapping** of the extended real line $h : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ is an increasing homeomorphisms such that $h(\infty) = \infty$ and satisfying, for some $k > 0$, the inequality

$$\frac{1}{k} \leq \left| \frac{h(x+y) - h(x)}{h(x) - h(x-y)} \right| \leq k$$

for all $x \in \mathbb{R}$ and $y > 0$. Every quasiconformal self-map of the upper half-plane f satisfying $f(\infty) = \infty$ has quasisymmetric boundary values. Beurling and Ahlfors [2] demonstrated that the converse is true, namely that every quasisymmetric map extends to a quasiconformal map of the upper half-plane. This accounts for the importance of quasisymmetries in the Teichmüller theory of bordered surfaces.

Quasisymmetric mappings of S^1 are defined by mapping the real line to the unit circle S^1 with a Möbius transformation and applying the above definition. For a Riemann surface, one can similarly define a quasisymmetry of a component of the border that is homeomorphic to S^1, by mapping an annular neighborhood of the boundary curve to an annular neighborhood of the disc. By standard extension results for quasiconformal maps [12], we have the following characterization of quasisymmetries of boundary curves [19].

Theorem 3. *Let Σ^B be a bordered Riemann surface, with boundary curve $\partial_i \Sigma^B$ homeomorphic to S^1. A map $h : \partial_i \Sigma^B \to S^1$ is a quasisymmetry if and only if h has a quasiconformal extension to an annular neighbourhood of $\partial_i \Sigma^B$.*

Note that not every quasisymmetry is a diffeomorphism. Denote homeomorphisms, diffeomorphisms and quasisymmetries of S^1 by $\mathrm{Homeo}(S^1)$, $\mathrm{Diff}(S^1)$ and $\mathrm{QS}(S^1)$ respectively. Then

$$\mathrm{Diff}(S^1) \subsetneq \mathrm{QS}(S^1) \subsetneq \mathrm{Homeo}(S^1).$$

2.3. Conformal Welding and the Sewing Operation

Let $h \in \mathrm{QS}(S^1)$ be normalized so that it fixes three points, say 1, -1 and i. Let $\mathbb{D}$ be the open unit disk and let $\mathbb{D}^* = \hat{\mathbb{C}} \setminus \bar{\mathbb{D}}$. It is a result of Pfluger [17] and Lehto and Virtanen [11] that there exists a domain $\Omega \subset \mathbb{C}$ and a pair of conformal mappings $f : \mathbb{D} \to \Omega$, and $g : \mathbb{D}^* \to \hat{\mathbb{C}} \setminus \bar{\Omega}$, that have quasiconformal extensions to the plane and satisfy $h = g^{-1} \circ f$ when restricted to S^1. If it is specified that f and g must be normalized so that their extensions preserve $1, 0$, and -1, then these maps are uniquely determined. This pair of maps is referred to as a solution to the "sewing problem". The sewing problem plays a prominent role in the construction of the Teichmüller spaces of Riemann surfaces which are covered by the disc.

Joining the two halves of the disc together using the quasisymmetry h, we obtain a topological space homeomorphic to the sphere. That is, let

$$\bar{\mathbb{D}} \#_h \bar{\mathbb{D}}^* = \bar{\mathbb{D}} \sqcup \bar{\mathbb{D}}^* / \sim$$

where $\sqcup$ denotes the disjoint union and two points $z_1 \in \partial\bar{\mathbb{D}}$ and $z_2 \in \partial\bar{\mathbb{D}}^*$ are equivalent if and only if $h(z_1) = z_2$; i.e., $f(z_1) = g(z_2)$. Since f and g are conformal on $\mathbb{D}$ and $\mathbb{D}^*$ respectively, they can be used to pull back the complex structure on $\hat{\mathbb{C}}$ in such a way that with the resulting complex structure, $\bar{\mathbb{D}} \#_h \bar{\mathbb{D}}^*$ is biholomorphic to $\hat{\mathbb{C}}$ via the continuous extension of the map

$$F(z) = \begin{cases} f(z) & \text{when} \quad z \in \mathbb{D} \\ g(z) & \text{when} \quad z \in \mathbb{D}^*. \end{cases}$$

The procedure outlined in the last two paragraphs is referred to as **conformal welding**.

Given a surface obtained by applying the sewing operation to two rigged Riemann surfaces as described in Section 1.1, it is not difficult to extend the conformal welding procedure to define a complex structure on this new surface. For details and references see [19].

If the two Riemann surfaces are sewn together with analytic riggings, it is a trivial matter to construct a complex structure on the new surface. However, if the riggings are quasisymmetric, then some kind of extension theorem for quasiconformal maps is necessary.

2.4. Teichmüller Space and Its Complex Structure

We now define the Teichmüller space of a bordered Riemann surface and give a brief description of its complex structure. For a comprehensive treatment see [10] or [14]. Note that this Teichmüller space is infinite-dimensional.

Given a bordered Riemann surface Σ^B, two quasiconformal mappings g_1 and g_2 are said to be **homotopic rel boundary** if they are equal on $\partial\Sigma^B$ and there is a homotopy $F : \Sigma^B \times [0,1] \to \Sigma^B$ such that $F(p,t) = g_1(p) = g_2(p)$ for all $t \in [0,1]$ and $p \in \partial\Sigma^B$.

Definition 4. *Let Σ^B be a bordered Riemann surface of type (g,n). The Teichmüller space of Σ^B, denoted by $\mathrm{T}^B(\Sigma^B)$, is defined by*

$$\mathrm{T}^B(\Sigma^B) = \{(\Sigma^B, f, \Sigma_1)\}/\sim$$

where $f : \Sigma^B \to \Sigma_1$ is a quasiconformal map onto a Riemann surface Σ_1. Two triples are equivalent $(\Sigma^B, f_1, \Sigma_1) \sim (\Sigma^B, f_2, \Sigma_2)$ if and only if there exists a biholomorphism $\sigma : \Sigma_1 \to \Sigma_2$ such that $f_2^{-1} \circ \sigma \circ f_1$ is homotopic to the identity rel $\partial\Sigma^B$.

The Teichmüller space of the disc $\mathrm{T}^B(\mathbb{D})$ can be identified with $\mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$. It is called the **universal Teichmüller space** since it canonically contains the Teichmüller spaces of every Riemann surface covered by $\mathbb{D}$.

By Theorem 2 there is a map $\Phi_{\Sigma^B} : L^\infty_{(-1,1)}(\Sigma^B)_1 \to \mathrm{T}^B(\Sigma^B)$ from the space of Beltrami differentials to the Teichmüller space, given by mapping a Beltrami differential $\mu \mathrm{d}\bar{z}/\mathrm{d}z$ to the corresponding quasiconformal solution of the Beltrami equation. The map Φ_{Σ^B} is called the **fundamental projection**.

It was shown by Bers [1] that this Teichmüller space is an infinite-dimensional manifold with complex structure modeled on a complex Banach space. The following two facts regarding this complex structure are essential for our purposes.

Theorem 5. *The fundamental projection $\Phi_{\Sigma^B} : L^\infty_{(-1,1)}(\Sigma^B)_1 \to \mathrm{T}^B(\Sigma^B)$ is holomorphic. It possesses local holomorphic sections; that is, for any point $p \in \mathrm{T}^B(\Sigma^B)$ there is an open neighbourhood U of p and a holomorphic mapping $\eta : U \to L^\infty_{(-1,1)}(\Sigma^B)$ such that $\Phi_{\Sigma^B} \circ \eta$ is the identity.*

3. Construction of the Complex Structure on Rigged Riemann Moduli Space and Holomorphicity of Sewing

3.1. The Puncture Model of Rigged Riemann Moduli Space

In the conformal field theory literature, the rigged Riemann moduli space is often represented in an equivalent form in terms of punctured, rather than bordered, Riemann surfaces with local biholomorphisms at the punctures for riggings. This puncture picture is obtained from the border picture by sewing caps onto the boundary curves.

Our use of quasisymmetries for the border model requires that we make a corresponding adjustment to the class of maps used for riggings in the puncture model. Let Σ^P be a Riemann surface with an ordered set of punctures $\mathbf{p} = (p_1, \ldots, p_n)$, such that filling in the punctures results in a compact Riemann surface of genus g. We call this a punctured Riemann surface of type (g, n). Note that for bordered surfaces, n refers to the number of boundary curves, whereas here it refers to the number of punctures.

Definition 6. *A rigging at one of the punctures p_i is given by a quasiconformal map ϕ_i, from a neighbourhood of p_i into an open neighbourhood of the unit disc $\bar{\mathbb{D}}$, which is conformal on $\phi_i^{-1}(\mathbb{D})$. Let $\mathcal{O}_{qc}(\mathbf{p})$ denote the set of ordered n-tuples of riggings $(\phi_1, \ldots, \phi_n)$ whose domains of definition are non-overlapping.*

Note that the restriction of a rigging ϕ to $\phi^{-1}(S^1)$ is quasisymmetric, but need not be analytic or even a diffeomorphism.

This definition of rigging is an imitation of Bers' model of the universal Teichmüller space.

Definition 7. *The puncture model of rigged Riemann moduli space is given by*

$$\widetilde{\mathcal{M}}^P(g, n) = \{(\Sigma^P, \phi)\}/\sim$$

where Σ^P is a Riemann surface with punctures $\mathbf{p}$ and $\phi \in \mathcal{O}_{qc}(\mathbf{p})$. Two pairs are equivalent $(\Sigma_1^P, \phi^1) \sim (\Sigma_2^P, \phi^2)$ if and only if there exists a biholomorphism $\sigma : \Sigma_1^P \to \Sigma_2^P$ such that $\phi^2 \circ \sigma = \phi^1$ on $(\phi^1)^{-1}(\mathbb{D})$.

Note that for two pairs to be equivalent it suffices that $\phi^2 \circ \sigma = \phi^1$ on $(\phi^1)^{-1}(S^1)$.

By sewing punctured disks onto the boundary components, a bijection between $\widetilde{\mathcal{M}}^B(g, n)$ and $\widetilde{\mathcal{M}}^P(g, n)$ can easily be established.

3.2. Rigged Teichmüller Spaces

We now define the rigged Teichmüller spaces in both the border and puncture models. These are notions motivated by conformal field theory. It is by using quasisymmetric riggings that we will be able to establish the relation of the rigged Teichmüller spaces to the standard Teichmüller space (described in Section 2.4) and hence endow them with complex structures.

Definition 8. *Let Σ^B be a bordered Riemann surface of type (g, n). Consider the set of quadruples $(\Sigma^B, f, \Sigma_1, \psi)$ where $f : \Sigma^B \to \Sigma_1$ is a quasiconformal map onto the Riemann surface Σ_1, and ψ is a quasisymmetric rigging on Σ_1. The border model of rigged Teichmüller space is*

$$\widetilde{\mathrm{T}}_{\#}^B(\Sigma^B) = \{(\Sigma^B, f, \Sigma_1, \psi)\}/\sim$$

where $(\Sigma^B, f_1, \Sigma_1, \psi^1) \sim (\Sigma^B, f_2, \Sigma_2, \psi^2)$ *if and only if there exists a biholomorphism* $\sigma : \Sigma_1 \to \Sigma_2$ *such that* $\psi^2 \circ \sigma = \psi^1$ *and* $f_2^{-1} \circ \sigma \circ f_1$ *is homotopic to the identity.*

In this definition we do not require that the homotopy be rel boundary, so this is a kind of 'reduced' Teichmüller space (the $\#$ is used to denote this).

Next, we define the puncture model of the rigged Teichmüller space.

Definition 9. *Let* Σ^P *be a punctured Riemann surface of type* (g,n) *with punctures* $\mathbf{p}$. *Consider the set of quadruples* $(\Sigma^P, f, \Sigma_1, \phi)$ *where* Σ_1 *is a punctured Riemann surface with punctures* $\mathbf{p}^1$, $f : \Sigma^P \to \Sigma_1$ *is a quasiconformal map such that* $f(\mathbf{p}) = \mathbf{p}^1$ *(preserving the order of the individual points), and* $\phi \in \mathcal{O}_{qc}(\mathbf{p}^1)$. *The puncture model of rigged Teichmüller space is*

$$\widetilde{\mathrm{T}}^P(\Sigma^P) = \{(\Sigma^P, f, \Sigma_1, \phi)\}/\sim$$

where $(\Sigma^P, f_1, \Sigma_1, \phi^1) \sim (\Sigma^P, f_2, \Sigma_2, \phi^2)$ *if and only if there exists a biholomorphism* $\sigma : \Sigma_1 \to \Sigma_2$ *such that* $\phi^2 \circ \sigma = \phi^1$ *on* $(\phi^1)^{-1}(\mathbb{D})$ *and* $f_2^{-1} \circ \sigma \circ f_1$ *is homotopic to the identity in such a way that* $\mathbf{p}$ *remains fixed throughout the homotopy.*

3.3. Modular Groups

Let Σ^B be a bordered surface of type (g,n). Although our main results hold for all such surfaces, we now assume that Σ^B is not the disk or an annulus. These cases can easily be treated separately. Let $\mathrm{PQCI}^B(\Sigma^B)$ be the space of quasiconformal self-mappings of Σ^B that are the identity on $\partial\Sigma^B$. Let $\mathrm{PQCI}_0^B(\Sigma^B)$ be the subspace whose elements are isotopic to the identity rel $\partial\Sigma^B$. Let $\mathrm{PModI}^B(\Sigma^B) = \mathrm{PQCI}^B(\Sigma^B)/\mathrm{PQCI}_0^B(\Sigma^B)$. For a punctured Riemann surface Σ^P of type (g,n), we define, in an analogous way, $\mathrm{PMod}^P(\Sigma^P)$.

The space $\mathrm{PModI}^B(\Sigma^B)$ is a subgroup of the pure (quasiconformal) mapping class group and is finitely generated by Dehn twists. Let $\mathrm{DB}(\Sigma^B)$ be the subgroup generated by Dehn twists about curves that are isotopic to boundary curves, and let $\mathrm{DI}(\Sigma^B)$ be the subgroup generated by Dehn twists around curves that are not isotopic to boundary curves. From standard theory we know that $\mathrm{DB}(\Sigma^B)$ is isomorphic to $\mathbb{Z}^n$, and $\mathrm{PModI}(\Sigma^B)/\mathrm{DB}(\Sigma^B) \simeq \mathrm{DI}(\Sigma^B)$.

The usual action of the mapping class group on Teichmüller space is given by $[\rho] \cdot [\Sigma^B, f, \Sigma_1] = [\Sigma^B, f \circ \rho, \Sigma_1]$. Actions on the rigged Teichmüller spaces can be defined in an identical way. If G is a mapping class group or subgroup, then we denote the projection map, defined by the above action, by P_G.

3.4. Covering of the Rigged Moduli Spaces by the Teichmüller Space $\mathrm{T}^{B}(\Sigma^{B})$

The following commutative diagram captures the relation between the Teichmüller space, the rigged Teichmüller spaces and the rigged moduli spaces.

$$\begin{array}{ccc} & T^B(\Sigma^B) & \\ \swarrow{\scriptstyle P^{\#}_{\mathrm{DB}}} & & \searrow{\scriptstyle P_{\mathrm{DB}}} \\ \widetilde{T}^B(\Sigma^B) & \xrightarrow{\cong} & \widetilde{T}^P(\Sigma^P) \\ {\scriptstyle P_{\mathrm{DI}}}\downarrow & & \downarrow{\scriptstyle P_{\mathrm{mod}}} \\ \widetilde{\mathcal{M}}^B(g,n) & \xrightarrow{\cong} & \widetilde{\mathcal{M}}^P(g,n) \end{array} \tag{1}$$

Recall that the "n" in $\widetilde{\mathcal{M}}^B(g,n)$ stands for the number of boundary curves, whereas in $\widetilde{\mathcal{M}}^P(g,n)$ it stands for the number of punctures. The surface Σ^P is obtained from Σ^B by sewing on copies of the punctured disc $\bar{\mathbb{D}} \setminus \{0\}$.

Theorem 10 (Summary of results) 1) *All the spaces in* Diagram 1 *are obtained from* $T^B(\Sigma^B)$ *by quotienting by the action of the mapping class group and certain subgroups. These actions are biholomorphic, properly discontinuous and fixed-point free.*

2) *With the complex structures inherited from* $T^B(\Sigma^B)$*, all the spaces in Diagram 1 become complex Banach manifolds. These complex structures are the unique ones that make all the maps holomorphic. All the projections possess local holomorphic sections. The horizontal bijections are biholomorphisms.*

3) *The sewing operation is holomorphic.*

4. Concluding Remarks

As remarked in Section 1.3, the connection between the Teichmüller space of the unit disc $\mathrm{T}(\mathbb{D})$ and string theory has been observed by several authors, and the "sum over paths" might be formalized using $\mathrm{T}(\mathbb{D})$. Since $\mathrm{T}(\mathbb{D})$ contains all the Teichmüller spaces $\mathrm{T}^B(\Sigma^B)$, Diagram 1 and Theorem 10 give further evidence for this. Recently certain obstacles to that program have been overcome by Takhtajan and Teo [22], who constructed the Weil-Petersson metric on $\mathrm{T}(\mathbb{D})$ and gave its relation to the Kähler structures on $\mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1)$ and $\mathrm{Diff}(S^1)/S^1$.

Secondly, $\mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1) \cong \mathrm{T}(\mathbb{D})$ contains $\mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1)$ as one leaf of a holomorphic foliation [15, 21, 22]. Thus, the problem of relating the quasisymmetrically rigged moduli space to the diffeomorphic one, though difficult, may be tractable.

Finally, the recognition that the rigged moduli spaces are intermediate between the Teichmüller space and the (un-rigged) Riemann moduli space may have applications to defining a local fiber-like structure of Teichmüller space. For some preliminary work in this direction see [19].

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