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CENTRALIZER OF REEB VECTOR FIELD IN CONTACT LIE GROUPS

BABAK HASSANZADEH

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Abstract. We consider the centralizer of Reeb vector field of a contact Lie group with a left invariant Riemannian metric while contact structure is left invariant. Then we decompose the Lie algebra of this Lie groups to centralizer of Reeb vector field and its orthogonal complement and using this decomposition the contact Lie group is investigated. Furthermore, in last section a special automorphism is defined and studied which it keeps the contact form.

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1. Introduction and Preliminaries

Lie groups play an important role in many areas of mathematics and physics, special in quantum mechanics. The Lie groups contain two main aspects of mathematics, Algebra and Geometry, therefore these two topics can be linked in the Lie groups. Based on this possibility we want to study differential geometry of a contact metric Lie group's properties. We commence with basic definitions and concepts of Lie group and its contact structure.

Let G be a simply connected Lie group equipped by Riemannian left invariant metric $\langle\, ,\rangle$ and $T_eG = \mathfrak{g}$ is Lie algebra of G, e is identity element of G. One would expect to find some properties that are similar to those in flat Euclidean space, which in this paper one can regard as a simply connected, abelian Lie group of translations with a canonical left invariant metric. Such features do exist, but other geometric features of $\{G, \langle\, ,\rangle\}$ are foreign to Euclidean geometry. A Lie group H of a Lie group G is a subgroup which is also a submanifold and $\mathfrak{h}$ is Lie subalgebra of $\mathfrak{g}$. For all $X, Y \in \mathfrak{g}$, we have

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

where ∇ is covariant derivative. Assume G is an odd dimensional Lie group with a left invariant metric $\langle\, ,\rangle$, then G is said to be an almost contact metric Lie group

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if there exists a tensor ϕ of type $(1,1)$, a vector field ξ called structure vector field and η, the left invariant dual one-form of ξ satisfying the following

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \qquad \langle X, \xi\rangle = \eta(X) \\ \eta(\xi) &= 1, \qquad \phi(\xi) = 0, \qquad \eta\circ\phi = 0 \\ \langle \phi X, \phi Y\rangle &= \langle X, Y\rangle - \eta(X)\eta(Y) \end{aligned} \tag{1}$$

for any $X, Y \in \mathfrak{g}$. In this case

$$\langle \phi X, Y\rangle = -\langle X, \phi Y\rangle$$

and the fundamental two-form Φ on G is given by

$$\Phi(X, Y) = \langle X, \phi Y\rangle$$

and the Lie group is said to be contact metric Lie group if $\Phi = \mathrm{d}\eta$. A contact metric Lie group is said to be an η-Einstein Lie group if

$$S(X, Y) = a\langle X, Y\rangle + b\eta(X)\eta(Y)$$

where a, b are smooth functions on G and S is the Ricci tensor. The Nijenhuis torsion of ϕ is defined by the identity

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

and

$$\begin{aligned} N^{(1)}(X, Y) &= [\phi, \phi](X, Y) + 2\mathrm{d}\eta(X, Y)\xi \\ N^{(2)}(X, Y) &= (\mathcal{L}_{\phi X}\eta)(Y) - (\mathcal{L}_{\phi Y}\eta)(X) \\ N^{(3)}(X) &= (\mathcal{L}_\xi \phi)X, \qquad N^{(4)}(X) = (\mathcal{L}_\xi \eta)X. \end{aligned} \tag{2}$$

An almost contact structure (ϕ, ξ, η) is normal if and only if these four tensors vanish. A normal contact metric manifold is called a Sasakian manifold. By using only these identities and combining a few permutations of variables obtain the formula

$$\begin{aligned} \langle \nabla_X Y, Z\rangle = \frac{1}{2}\{&X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &+ \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle\} \end{aligned} \tag{3}$$

for each nonzero left invariant vector fields $X, Y, Z \in \mathfrak{g}$. However, the first three terms are identically zero.

Proposition 1 ([1]). *Let* (g, η) *be a contact Lie algebra with* ξ *its Reeb vector and let* $\mathfrak{z}(\mathfrak{g})$ *be the center of* $\mathfrak{g}$. *Then*

1) $\dim \mathfrak{z}(\mathfrak{g}) \leq 1$

2) *If* $\dim \mathfrak{z}(\mathfrak{g}) = 1$, *then* $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\xi$.

In this paper center is trivial. If $\mathfrak{a}$ is a nonempty subset of $\mathfrak{g}$, then the centralizer of $\mathfrak{a}$ is the set

$$\mathfrak{c}(\mathfrak{a}) = \{X \in \mathfrak{g};\ [X, Y] = 0,\ Y \in \mathfrak{a}\}. \tag{4}$$

The centralizers is the most important concept in this paper and we will study it.

2. Centralizer of Reeb Vector Field

Let G be a Lie group with a left invariant contact structure. From (9) we will have

$$\mathfrak{c}_{\mathfrak{g}}(\xi) = \{X \in \mathfrak{g};\ [X, \xi] = 0\}$$

and the centralizer of ξ will be denoted further by $\mathfrak{c}$. Also from definition we have $\xi \in \mathfrak{c}$ and $\mathfrak{c}$ is an invariant subalgebra and $\mathfrak{n}$ is its orthogonal complement such that $\mathfrak{g} = \mathfrak{c} + \mathfrak{n}$. Assume $X \in \mathfrak{g}$ be an arbitrary vector field, then $X = X_{\mathfrak{c}} + X_{\mathfrak{n}}$ and applying ϕ we obtain

$$\phi X = \phi X_{\mathfrak{c}} + \phi X_{\mathfrak{n}}.$$

In rest of this paper we will investigate the contact metric Lie groups using this decomposition.

Theorem 2 ([3]). *If* η *is a contact form in a Lie algebra* $\mathfrak{g}$, *with Reeb vector* ξ, *then its kernel* (*nullspace*) $\mathrm{Ker}(\eta)$ *is not a Lie subalgebra of* $\mathfrak{g}$, *whereas the radical* (*nullspace*) $\mathrm{rad}(\partial\eta) = R\xi$ *of is a reductive subalgebra of G. Here* $\partial\eta(X, Y) = -\eta([X, Y])$ *and R is real numbers. Any Lie algebra is reductive if the redical of* $\mathfrak{g}$ *equals the center.*

Therefore $\mathfrak{n}$ is a subset of $\mathfrak{g}$. Throughout this paper each vector field $X \in \mathfrak{c}$ will show by $X_{\mathfrak{c}}$. From (2) and some straight calculations we get

$$N^{(2)}(X_{\mathfrak{c}}, Y_{\mathfrak{c}}) = N^{(4)}(X_{\mathfrak{c}}) = 0 \tag{5}$$

for all $X, Y \in \mathfrak{g}$. Also from definition of contact structure we have

$$\eta([X_{\mathfrak{c}}, Y_{\mathfrak{c}}]) = -2\langle X_{\mathfrak{c}}, \phi Y_{\mathfrak{c}}\rangle. \tag{6}$$

Easily we find out, if $Y_{\mathfrak{c}}$ is an arbitrary element of centeralizer of $\mathfrak{c}$, then we cannot claim always $\phi Y_{\mathfrak{c}} \in \mathfrak{c}$. Furthermore, $\mathrm{ad}\mathfrak{c}$ is a Lie subalgebra of $\mathfrak{c}$, then (6) is nonzero if $\xi \in \mathrm{ad}\mathfrak{c}$. Throughout this article the center of $\mathfrak{c}$ is empty set and it is ad surjective. Then $\mathfrak{c}$ is an invariant subalgebra and also $N^{(3)} = 0$. Suppose $X_{\mathfrak{c}}, Y_{\mathfrak{c}}$ are arbitrary elements, then

$$\eta([X_{\mathfrak{n}}, Y_{\mathfrak{c}}]) = -2\langle X_{\mathfrak{n}}, \phi Y_{\mathfrak{c}}\rangle = 0.$$

In the event that $[X_{\mathfrak{n}}, Y_{\mathfrak{c}}] \neq 0$, for any $Y_{\mathfrak{c}} \in \mathfrak{c}$, it results that $[\mathfrak{n}, \mathfrak{c}] \subset \mathfrak{n}$, also, easily it can concluded that $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{c}$. Furthermore, using (8) we have

$$2\langle \nabla_{X_{\mathfrak{c}}} Y_{\mathfrak{c}}, \xi\rangle = \langle [X_{\mathfrak{c}}, Y_{\mathfrak{c}}], \xi\rangle.$$

Then

$$\nabla_{X_{\mathfrak{c}}} Y_{\mathfrak{c}} = \frac{1}{2}[X_{\mathfrak{c}}, Y_{\mathfrak{c}}] \tag{7}$$

for all $X_{\mathfrak{c}}, Y_{\mathfrak{c}} \in \mathfrak{c}$. From (3) using same reasoning we get

$$2\langle \nabla_{X_{\mathfrak{c}}} Y_{\mathfrak{c}}, \xi\rangle = \langle [X_{\mathfrak{c}}, Y_{\mathfrak{c}}], \xi\rangle = \eta([X_{\mathfrak{c}}, Y_{\mathfrak{c}}]) = -2\langle X_{\mathfrak{c}}, \phi Y_{\mathfrak{c}}\rangle. \tag{8}$$

It is trivial that (8) is nonzero, because $\mathfrak{c}$ is an invariant subalgebra and from definition of totally geodesic Lie algebra in [4] we find out $\mathfrak{c}$ is totally geodesic Lie subalgebra. Next we will establish a formula for the covariant derivative of ϕ in the case of a general contact metric structure $(\phi, \xi, \eta, \langle\, ,\rangle)$.

Theorem 3 ([2]). *For a contact structure $(\phi, \xi, \eta, \langle\, ,\rangle)$, the covariant derivative of ϕ is given by*

$$2\langle (\nabla_X \phi)Y, Z\rangle = \langle N^{(1)}(X, Y), \phi X\rangle + 2\mathrm{d}\eta(\phi Y, X)\eta(Z) - 2\mathrm{d}\eta(\phi Z, X)\eta(Y).$$

We bring up a tensor field h on a contact metric manifold by

$$h = \frac{1}{2}\mathfrak{L}_{\xi}\phi = \frac{1}{2}N^{(3)}.$$

Lemma 4 ([2]). *On a contact metric manifold h is a symmetric operator*

$$\nabla_X \xi = -\phi X - \phi h X$$

which anticommutes with ϕ and besides $\mathrm{tr} h = 0$.

Theorem 5. *Let (G, η, ϕ, ξ) be a contact Lie group with a left invariant Riemannian metric. Then*

$$(\nabla_{X_{\mathfrak{n}}} \phi) Y_{\mathfrak{n}} = 2\langle X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle \xi$$

for all $X_{\mathfrak{n}}, Y_{\mathfrak{n}} \in \mathfrak{n}$.

Proof: From the definition of contact metric Lie groups we have

$$\eta([X_{\mathfrak{n}}, Y_{\mathfrak{n}}]) = -2\langle X_{\mathfrak{n}}, \phi Y_{\mathfrak{n}}\rangle \tag{9}$$

and

$$\eta([\phi X_{\mathfrak{n}}, \phi Y_{\mathfrak{n}}]) = -2\langle X_{\mathfrak{n}}, \phi Y_{\mathfrak{n}}\rangle \tag{10}$$

comparing (9) and (10) we conclude that

$$[X_{\mathfrak{n}}, Y_{\mathfrak{n}}] = [\phi X_{\mathfrak{n}}, \phi Y_{\mathfrak{n}}]. \tag{11}$$

Furthermore

$$\eta([\phi X_{\mathfrak{n}}, Y_{\mathfrak{n}}]) = -2\langle X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle \tag{12}$$

and

$$\eta([X_{\mathfrak{n}}, \phi Y_{\mathfrak{n}}]) = 2\langle X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle. \tag{13}$$

From (12) and (13) by similar calculations we obtain

$$[\phi X_{\mathfrak{n}}, Y_{\mathfrak{n}}] = -[X_{\mathfrak{n}}, \phi Y_{\mathfrak{n}}]. \tag{14}$$

Therefore $N^{(1)}(X_{\mathfrak{n}}, Y_{\mathfrak{n}}) = 0$. From Theorem 6.1 of [2] we have

$$N^{(2)} = N^{(3)} = N^{(4)} = 0.$$

Using Theorem 3 we have

$$\begin{aligned}\langle(\nabla_{X_{\mathfrak{n}}}\phi)Y_{\mathfrak{n}}, Z_{\mathfrak{c}}\rangle &= 2\mathrm{d}\eta(\phi Y_{\mathfrak{n}}, X_{\mathfrak{n}})\eta(Z_{\mathfrak{c}}) = -\eta([\phi Y_{\mathfrak{n}}, X_{\mathfrak{n}}])\eta(Z_{\mathfrak{c}}) \\ &= 2\langle Y_{\mathfrak{n}}, X_{\mathfrak{n}}\rangle\eta(Z_{\mathfrak{c}})\end{aligned}$$

for any $Z_{\mathfrak{c}} \in \mathfrak{c}$, and the proof is complete. Since $N^{(3)} = 0$ we have $[\xi, \phi X_{\mathfrak{n}}] = \phi[\xi, X_{\mathfrak{n}}]$, and using Theorem 5 we have the proof. ■

Theorem 6. *Let (G, η, ϕ, ξ) be a contact Lie group with a left invariant Riemannian metric. Then*

$$\langle[X_{\mathfrak{n}}, \xi], X_{\mathfrak{n}}\rangle = 0$$

for any $X_{\mathfrak{n}} \in \mathfrak{n}$.

Proof: From Theorem 5 we have $h(X_{\mathfrak{n}}) = 0$, then $\nabla_{X_{\mathfrak{n}}}\xi = -\phi X_{\mathfrak{n}}$. By straightforward calculations we get

$$\langle \nabla_{X_{\mathfrak{n}}}\xi, Y_{\mathfrak{n}}\rangle = -\langle \phi X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle \tag{15}$$

now from (3) we obtain

$$2\langle \nabla_{X_{\mathfrak{n}}}\xi, X_{\mathfrak{n}}\rangle = \langle [X_{\mathfrak{n}}, \xi], Y_{\mathfrak{n}}\rangle + \langle [\xi, Y_{\mathfrak{n}}], X_{\mathfrak{n}}\rangle - 2\langle Y_{\mathfrak{n}}, \phi X_{\mathfrak{n}}\rangle. \tag{16}$$

Using the lemma (15) and equation (16) we get

$$\langle [\xi, X_{\mathfrak{n}}], Y_{\mathfrak{n}}\rangle = -\langle [\xi, Y_{\mathfrak{n}}], X_{\mathfrak{n}}\rangle.$$

Assume $X_{\mathfrak{n}} = Y_{\mathfrak{n}}$ and the proof is complete. ■

Curvature tensor. Let X, Y, Z are arbitrary elements of $\mathfrak{g}$, then recall that the curvature tensor is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Then

$$R(X_{\mathfrak{c}}, \xi)\xi = 0, \qquad R(X_{\mathfrak{c}}, Y_{\mathfrak{c}})\xi = (\nabla_{Y_{\mathfrak{c}}}\phi)X_{\mathfrak{c}} - (\nabla_{X_{\mathfrak{c}}}\phi)Y_{\mathfrak{c}}$$
$$R(X_{\mathfrak{n}}, \xi)\xi = \phi[X_{\mathfrak{n}}, \xi]$$

and from Theorem 5 we have

$$R(X_{\mathfrak{n}}, Y_{\mathfrak{n}})\xi = (\nabla_{Y_{\mathfrak{n}}}\phi)X_{\mathfrak{n}} - (\nabla_{X_{\mathfrak{n}}}\phi)Y_{\mathfrak{n}} = 0.$$

Also, from (7) we obtain

$$R(\xi, X_{\mathfrak{c}})X_{\mathfrak{c}} = \frac{1}{4}[X_{\mathfrak{c}}, \phi X_{\mathfrak{c}}]$$

for any $X_{\mathfrak{c}} \in \mathfrak{c}$ and the sectional curvature is given by

$$K(\xi, X_{\mathfrak{c}}) = 1$$

where $X_{\mathfrak{c}}$ is unit vector, orthogonal to ξ. In the other case we have

$$K(X_{\mathfrak{c}}, Y_{\mathfrak{c}}) = \frac{1}{4} \mid [X_{\mathfrak{c}}, Y_{\mathfrak{c}}] \mid^2$$

where $X_{\mathfrak{c}}, Y_{\mathfrak{c}}$ are orthonormal vectors. Assume G is a η-Einstein Lie group, then

$$S(X_{\mathfrak{n}}, Y_{\mathfrak{n}}) = \frac{a}{2}\eta([X_{\mathfrak{n}}, \phi Y_{\mathfrak{n}}]), \qquad S(X_{\mathfrak{n}}, Y_{\mathfrak{n}}) = S(\phi X_{\mathfrak{n}}, \phi Y_{\mathfrak{n}}).$$

Theorem 7. *Let G be a contact Lie group with a left invariant Riemannian metric. Then*

$$[\xi, \phi X_{\mathfrak{n}}] = -2X_{\mathfrak{n}} - \phi[\xi, X_{\mathfrak{n}}]$$

for any $X_{\mathfrak{n}} \in \mathfrak{n}$.

Proof: Using Lemma 4 we obtain

$$[\xi, \phi X_{\mathfrak{n}}] = -\nabla_{\phi X_{\mathfrak{n}}}\xi = \phi^2 X_{\mathfrak{n}} + \frac{1}{2}\phi(\mathfrak{L}_\xi \phi)\phi X_{\mathfrak{n}} = -X_{\mathfrak{n}} - \frac{1}{2}\phi[\xi, X_{\mathfrak{n}}] + \frac{1}{2}[\xi, \phi X_{\mathfrak{n}}].$$

The rest of the proof is trivial. ■

Proposition 8. *Let G be a contact Lie group. Then*

$$\langle X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle = -\frac{1}{2}\left(\langle [\xi, \phi X_{\mathfrak{n}}], Y_{\mathfrak{n}}\rangle + \langle [Y_{\mathfrak{n}}, \xi], \phi X_{\mathfrak{n}}\rangle\right)$$

for any $X_n, Y_n \in \mathfrak{n}$.

Proof: Since G is contact Lie group then $\nabla_\xi \phi = 0$, from (3) we obtain

$$\langle \nabla_\xi \phi X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle = \frac{1}{2}\langle [\xi, \phi X_{\mathfrak{n}}], Y_{\mathfrak{n}}\rangle + \frac{1}{2}\langle [Y_{\mathfrak{n}}, \xi], \phi X_{\mathfrak{n}}\rangle + \langle X_{\mathfrak{n}}, Y_{\mathfrak{n}}\rangle.$$

Now the proof is trivial. ■

Proposition 9. *Let G be a contact Lie group with a left invariant Riemannian metric. Then* $\mid X_{\mathfrak{n}} \mid = \frac{\sqrt{2}}{2}$ *if and only if* $[X_{\mathfrak{n}}, \phi X_{\mathfrak{n}}] = \xi$.

Proof: ($\Rightarrow$) Let $\mid X_{\mathfrak{n}} \mid = \frac{\sqrt{2}}{2}$, then $\eta([X_{\mathfrak{n}}, \phi X_{\mathfrak{n}}]) = 2\langle X_{\mathfrak{n}}, X_{\mathfrak{n}}\rangle = 1$. Thus $\eta(\xi) = \eta([X_{\mathfrak{n}}, \phi X_{\mathfrak{n}}]) \neq 0$ and we conclude $[X_{\mathfrak{n}}, \phi X_{\mathfrak{n}}] = \xi$. ($\Leftarrow$) For this part of proof we have $\eta([X_{\mathfrak{n}}, \phi X_{\mathfrak{n}}]) = 2\langle X_{\mathfrak{n}}, X_{\mathfrak{n}}\rangle = 1$. Thus $\mid X_{\mathfrak{n}} \mid^2 = \frac{1}{2}$ and the proof is complete. ■

3. Automorphisms

Suppose $\psi \in \mathrm{aut}\mathfrak{g}$ and that the contact form η is invariant with respect to ψ. This means that we have

$$\eta(X) = \eta(\psi X) \tag{17}$$

for any $X \in \mathfrak{g}$. Easily we find also that $\psi(\xi) = \xi$. From definition of the contact metric Lie groups we have as well

$$\eta([X,Y]) = \eta(\psi[X,Y]) = \eta([\psi X, \psi Y]) = -2\langle \psi X, \phi\psi Y\rangle$$

and

$$\langle \psi X, \phi\psi Y\rangle = \langle X, \phi Y\rangle.$$

Theorem 10. *Let G be a contact Lie goup with a left invariant Riemannian metric and $\psi \in \mathfrak{g}$ is an automorphism which keep the contact form. Then*

$$\langle \phi\psi X_n, \psi\phi Y_n\rangle = \langle \psi\phi X_n, \phi\psi Y_n\rangle$$

for all $X_n, Y_n \in \mathfrak{n}$.

Proof: Using the definition of contact metric Lie groups and equations (12) and (13) we obtain

$$2\langle X_n, Y_n\rangle = \eta([X_n, \phi Y_n]) = \eta([\psi X_n, \psi\phi Y_n]) = -2\langle \psi X_n, \phi\psi\phi Y_n\rangle \qquad (18)$$

and

$$2\langle X_n, Y_n\rangle = -\eta([\phi X_n, Y_n]) = -\eta([\psi\phi X_n, \psi Y_n]) = 2\langle \psi\phi X_n, \phi\psi Y_n\rangle. \qquad (19)$$

Comparing (18) and (19) completes the proof. ■

Theorem 11. *Let G be a contact Lie group with a left invariant Riemannian metric. Then* $\ker\psi$ *is empty set, where ψ is an automorphism which preserve the contact form.*

Proof: Assume $X \in \mathfrak{g}$ is an arbitrary element such that $X \in \ker\psi$, then $\eta(X) = 0$ and

$$0 = \eta([X, \phi X]) = -2\langle \phi X, \phi X\rangle = -2\langle X, X\rangle$$

therefore $X = 0$ and the proof is complete. ■

Let us assume that $\psi(\mathfrak{c}) \subseteq \mathfrak{n}$, then

$$\psi[X_\mathfrak{c}, Y_\mathfrak{c}] = [\psi X_\mathfrak{c}, \psi Y_\mathfrak{c}]$$

for all $X_\mathfrak{c}, Y_\mathfrak{c} \in \mathfrak{c}$. The left hand is belong to $\mathfrak{n}$ and right hand is a member of $\mathfrak{c}$ and therefore $\psi(\mathfrak{c}) \subseteq \mathfrak{c}$. Using the well known properties of contact structure and from definition of ψ we have

$$\psi\phi^2 X = -\psi X + \eta(X)\psi\xi = -\psi X + \eta(\psi X)\xi = \phi^2\psi X$$

and therefore $\phi^2\psi X = \psi\phi^2 X$.

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Babak Hassanzadeh
Department of Mathematics
Azarbaijan Shahid Madani University
Tabriz, IRAN
E-mail address: babakmath777@gmail.com