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DEFORMATIONS OF SYMPLECTIC STRUCTURES BY MOMENT MAPS

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Abstract. We study deformations of symplectic structures on a smooth manifold M via the quasi-Poisson theory. We can deform a given symplectic structure ω with a Hamiltonian G-action to a new symplectic structure ω^t parametrized by some element t in $\Lambda^2\mathfrak{g}$. We can obtain concrete examples for the deformations of symplectic structures on the complex projective space and the complex Grassmannian. Moreover applying the deformation method to any symplectic toric manifold, we show that manifolds before and after deformations are isomorphic as a symplectic toric manifold.

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Contents

1. Introduction

In the context of symplectic geometry, deformation-equivalence assumptions and conditions are often appeared, for example, in the statement of Moser's theorem [9], Donaldson's four-six conjecture [10] and so on. However, it seems that a method of constructing deformation-equivalent symplectic structures specifically is not well known. In this paper, we construct a method of producing new symplectic structures deformation-equivalent to a given symplectic structure with a Hamiltonian action. Our approach to deformations of symplectic structures is to use quasi-Poisson theory which was introduced by Alekseev and Kosmann-Schwarzbach [1], and this approach is carried out by using the fact that a moment map for a symplectic-Hamiltonian action σ is also a moment map for a quasi-Poisson action σ. The former moment map satisfies conditions for only one symplectic structure, whereas the latter does conditions for a family of quasi-Poisson structures parametrized by elements in $\Lambda^2\mathfrak{g}$. From here we call these elements *twists*. Regarding the quasi-Poisson structure induced by a symplectic structure as that with twist 0, which is denoted by π_0, we can find different quasi-Poisson structures π_t which induce symplectic structures ω^t by the choice of "good" twists t. The quasi-Poisson structure inducing a symplectic structure must be a non-degenerate Poisson structure. We describe the conditions for the quasi-Poisson structure with a twist t to be a non-degenerate Poisson structure. Our method of using the family of quasi-Poisson structures is one of interesting geometry frameworks [1].

From here, we explain briefly the difference among moment maps for symplectic, Poisson and quasi-Poisson actions on a smooth manifold.

I) Symplectic-Hamiltonian actions

In symplectic geometry, a moment map $\mu : M \to \mathfrak{g}^*$ for a symplectic action σ of a Lie group G on a symplectic manifold (M, ω) is defined with two conditions: one is for the symplectic structure ω

$$\mathrm{d}\mu^X = \iota_{X_M}\omega, \qquad X \in \mathfrak{g}. \tag{1}$$

Here $\mu^X(p) := \langle \mu(p), X\rangle$ and X_M is a vector field on M defined by

$$X_{M,p} := \left.\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{\exp tX}(p)\right|_{t=0} \tag{2}$$

for p in M. The other is the G-equivariance condition with respect to the action σ on M and the coadjoint action Ad^* on $\mathfrak{g}^*$

$$\mu \circ \sigma_g = \mathrm{Ad}^*_g \circ \mu \tag{3}$$

for all g in G. In this paper, we call symplectic actions with moment maps *symplectic-Hamiltonian actions* to distinguish it from other actions with moment maps.

II) Poisson-Hamiltonian actions

A Poisson Lie group, which was introduced by Drinfel'd [4], is a Lie group with a Poisson structure π compatible with the group structure. Namely, the structure π satisfies

$$\pi_{gh} = L_{g*}\pi_h + R_{h*}\pi_g \tag{4}$$

for any g and h in G, where L_g and R_h are the left and right translations in G by g and h, respectively. Such a structure is called *multiplicative*. Then the simply connected Lie group G^* called the dual Poisson Lie group is obtained uniquely from a Poisson Lie group (G, π) and a local action λ of G on G^* is defined naturally. We call a multiplicative Poisson structure π on G *complete* if the action λ is global. Then (G, π) is called a *complete Poisson Lie group*. A moment map $\mu : M \to G^*$ for a Poisson action σ of a Poisson Lie group (G, π) on a Poisson manifold (M, π_M) is defined with a condition

$$X_M = \pi_M^\sharp(\mu^*(X^L)) \tag{5}$$

for any X in $\mathfrak{g}$, where X^L is the left invariant one-form on G^* with value X at e. In this paper, we call Poisson actions with moment maps *Poisson-Hamiltonian actions*. If (G, π) is complete, we can also consider the G-equivariance of a moment map with respect to σ and λ. An equivariant moment map for a Poisson action of a Poisson Lie group on a complete Poisson manifold is a generalization of a moment map for a symplectic action on a symplectic manifold, which was given by Lu in [5].

III) Quasi-Poisson-Hamiltonian actions (See Section 2 for details.)

Quasi-Poisson theory, which was originated with [1] by Alekseev and Kosmann-Schwarzbach, is a generalization of Poisson theory with Poison actions. More specifically, the theory gives an unified view for various moment map theories [9], [5], [7], [2]. In quasi-Poisson geometry, quasi-triples $(D, G, \mathfrak{h})$ and its infinitesimal version, Manin quasi-triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, play important roles. A quasi-triple $(D, G, \mathfrak{h})$ defines a quasi-Poisson Lie group $G_D^{\mathfrak{h}}$ and we can obtain the notion of a quasi-Poisson action of such a quasi-Poisson Lie group $G_D^{\mathfrak{h}}$. A moment map μ for the action is a map from M into the quotient D/G and satisfies a condition not for one quasi-Poisson structure but for a family of quasi-Poisson structures parametrized by elements in $\Lambda^2\mathfrak{g}$. In this paper, we call quasi-Poisson actions with moment maps *quasi-Poisson-Hamiltonian actions*. An equivariant moment map for a Poisson action in (II) is an example of a moment map for a quasi-Poisson action. We use the moment map theory for quasi-Poisson actions to deform symplectic structures on a smooth manifold.

This paper is constructed as follows. It is the contents of Section 2 to review the moment map theory for quasi-Poisson actions. In Section 3, we describe a deformation method of symplectic structures on a smooth manifold via the quasi-Poisson theory. This method is the subject of this paper. Theorem 14 gives a sufficient condition for a twist to deform a symplectic structure to a new one. In addition, Theorem 16 gives a simple condition for a twist to satisfy the assumption of Theorem 14. In Section 4, we introduce concrete examples and applications for deformations of symplectic structures. We give deformations of the Fubini-Study and the Kirillov-Kostant forms on $\mathbb{CP}^n$ and the complex Grassmannians, respectively. Moreover, as an application of our deformation, we show that for any symplectic toric manifold, manifolds before and after deformations are isomorphic as a symplectic toric manifold.

2. Moment Maps for Quasi-Poisson Actions on Quasi-Poisson Manifolds

In this section, we shall recall the quasi-Poisson theory [1]. We start with the definition of quasi-Poisson Lie groups, which is a generalization of Poisson Lie groups.

Definition 1. *Let G be a Lie group with the Lie algebra $\mathfrak{g}$. Then a pair (π, φ) is a quasi-Poisson structure on G if a multiplicative two-vector field π on G and an element φ of $\Lambda^3\mathfrak{g}$ satisfy*

$$\frac{1}{2}[\pi,\pi] = \varphi^R - \varphi^L, \qquad [\pi, \varphi^L] = [\pi, \varphi^R] = 0 \tag{6}$$

where the bracket $[\cdot,\cdot]$ is the Schouten bracket on the multi-vector fields on G, and φ^L and φ^R denote the left and right invariant three-vector fields on G with value φ at e respectively. A triple (G, π, φ) is called a quasi-Poisson Lie group.

Remark 2. *In a quasi-Poisson structure (π, φ) on G, the two-vector field π is a multiplicative Poisson structure if $\varphi = 0$. Namely, (G, π) is a Poisson Lie group.*

We use a "quasi-triple" to obtain a quasi-Poisson Lie group. To define a quasi-triple, we describe its infinitesimal version, a Manin quasi-triple.

Definition 3. *Let $\mathfrak{d}$ be a $2n$-dimensional Lie algebra with an invariant non-degenerate symmetric bilinear form of signature (n,n), which is denoted by $(\cdot|\cdot)$. Let $\mathfrak{g}$ be an n-dimensional Lie subalgebra of $\mathfrak{d}$ and $\mathfrak{h}$ be an n-dimensional vector subspace of $\mathfrak{d}$. Then a triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin quasi-triple if $\mathfrak{g}$ is a maximal isotropic subspace with respect to $(\cdot|\cdot)$ and $\mathfrak{h}$ is an isotropic complement subspace of $\mathfrak{g}$ in $\mathfrak{d}$.*

Remark 4. *For a given Lie algebra $\mathfrak{d}$ and a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{d}$, a choice of an isotropic complement subspace $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$ is not unique.*

A Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ defines the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$. Then the linear isomorphism

$$j : \mathfrak{g}^* \to \mathfrak{h}, \qquad (j(\xi)|x) := \langle \xi, x \rangle, \qquad \xi \in \mathfrak{g}^*, x \in \mathfrak{g} \tag{7}$$

is determined by the decomposition. We denote the projections from $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ to $\mathfrak{g}$ and $\mathfrak{h}$ by $p_{\mathfrak{g}}$ and $p_{\mathfrak{h}}$ respectively. We introduce an element $\varphi_{\mathfrak{h}}$ in $\Lambda^3\mathfrak{g}$ which is defined by the map from $\Lambda^2\mathfrak{g}^*$ to $\mathfrak{g}$, denoted by the same letter

$$\varphi_{\mathfrak{h}}(\xi, \eta) = p_{\mathfrak{g}}([j(\xi), j(\eta)]) \tag{8}$$

for any ξ, η in $\mathfrak{g}^*$. We define the linear map $F_{\mathfrak{h}} : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ by setting

$$F_{\mathfrak{h}}^*(\xi, \eta) = j^{-1}(p_{\mathfrak{h}}([j(\xi), j(\eta)])) \tag{9}$$

for any ξ, η in $\mathfrak{g}^*$, where $F_{\mathfrak{h}}^* : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ is the dual map of $F_{\mathfrak{h}}$. These elements will be used later to define a quasi-Poisson structure and a quasi-Poisson action respectively.

Next we define a quasi-triple $(D, G, \mathfrak{h})$ and construct a quasi-Poisson structure on G using $(D, G, \mathfrak{h})$.

Definition 5. *Let D be a connected Lie group with a bi-invariant scalar product with the Lie algebra $\mathfrak{d}$ and G be a connected closed Lie subgroup of D with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a vector subspace of $\mathfrak{d}$. Then a triple $(D, G, \mathfrak{h})$ is a quasi-triple if $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin quasi-triple.*

A method of constructing a quasi-Poisson structure by a quasi-triple is as follows. Let $(D, G, \mathfrak{h})$ be a quasi-triple with a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. Using the inverse $j^{-1} : \mathfrak{h} \to \mathfrak{g}^*$ of the linear isomorphism (7), we identify $\mathfrak{d}$ with $\mathfrak{g} \oplus \mathfrak{g}^*$. Consider the map

$$r_{\mathfrak{h}} : \mathfrak{d}^* \to \mathfrak{d}, \qquad \xi + X \mapsto \xi$$

for any ξ in $\mathfrak{g}^*$ and X in $\mathfrak{g}$. This map defines an element $r_{\mathfrak{h}} \in \mathfrak{d} \otimes \mathfrak{d}$ which we denote by the same letter. We set

$$\pi_D^{\mathfrak{h}} := r_{\mathfrak{h}}^L - r_{\mathfrak{h}}^R$$

where $r_{\mathfrak{h}}^L$ and $r_{\mathfrak{h}}^R$ is denoted as the left and right invariant two-tensors on D with value $r_{\mathfrak{h}}$ at the identity element e in D respectively, and we can see that it is a multiplicative two-vector field on D. Furthermore, the two-vector field $\pi_D^{\mathfrak{h}}$ and the element $\varphi_{\mathfrak{h}}$ defined by (8) satisfy (6). We set

$$\pi_{G,g}^{\mathfrak{h}} := \pi_{D,g}^{\mathfrak{h}} \tag{10}$$

for any g in G. Then we can see that $\pi_G^{\mathfrak{h}}$ is well-defined and that $\pi_G^{\mathfrak{h}}$ is a multiplicative two-vector field on G. Moreover, $\pi_G^{\mathfrak{h}}$ and $\varphi_{\mathfrak{h}}$ satisfy (6). Therefore $(G, \pi_G^{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ is a quasi-Poisson Lie group. We sometimes denote a Lie group with such a structure by $G_D^{\mathfrak{h}}$.

From here, we consider only connected quasi-Poisson Lie group $G_D^{\mathfrak{h}}$ defined as above by a quasi-triple $(D, G, \mathfrak{h})$. For a smooth manifold M with a two-vector field π_M, a quasi-Poisson action is defined as follows. It is a generalization of Poisson actions of connected Poisson Lie groups [7].

Definition 6. *Let $G_D^{\mathfrak{h}}$ be a connected quasi-Poisson Lie group acting on a smooth manifold M with a two-vector field π_M. The action σ of G on M is a quasi-Poisson action if for each X in $\mathfrak{g}$*

$$\frac{1}{2}[\pi_M, \pi_M] = (\varphi_{\mathfrak{h}})_M, \qquad \mathfrak{L}_{X_M}\pi_M = F_{\mathfrak{h}}(X)_M \tag{11}$$

where x_M is a fundamental multi-vector field for any x in $\wedge^*\mathfrak{g}$. Here $F_{\mathfrak{h}}$ is the dual of the map (9)*. Then a two-vector field π_M is called a quasi-Poisson $G_D^{\mathfrak{h}}$-structure on M and (M, π_M) is called a quasi-Poisson $G_D^{\mathfrak{h}}$-manifold.*

Remark 7. *A quasi-Poisson Lie group $G_D^{\mathfrak{h}}$ with the natural left action is not a quasi-Poisson $G_D^{\mathfrak{h}}$-manifold. In fact, $(\varphi_{\mathfrak{h}})_G = \varphi_{\mathfrak{h}}^R$.*

Finally we define a moment map for a quasi-Poisson action to carry out the deformation of symplectic structures using the moment map theory for quasi-Poisson actions in Section 3. We need some preliminaries to define a moment map. For any quasi-triple $(D, G, \mathfrak{h})$, since G is a closed subgroup of D, the quotient space D/G is a smooth manifold, which is the range of moment maps. The action of D on itself by left multiplication induces an action of D on D/G. We call it *dressing action* of D on D/G and denote the corresponding infinitesimal action by $X \mapsto X_{D/G}$ for X in $\mathfrak{d}$. Let $p_{D/G} : D \to D/G$ be the natural projection. Then

$$\pi_{D/G}^{\mathfrak{h}} := p_{D/G*}\pi_D^{\mathfrak{h}}$$

is a two-vector field on D/G. We consider the dressing action on D/G restricted to G, and can see that $\pi_{D/G}^{\mathfrak{h}}$ satisfies (11). Therefore $(D/G, \pi_{D/G}^{\mathfrak{h}})$ is a quasi-Poisson

$G_D^{\mathfrak{h}}$-manifold. The following definition is one of the important notions to define moment maps.

Definition 8. *An isotropic complement* $\mathfrak{h}$ *of* $\mathfrak{g}$ *in* $\mathfrak{d}$ *is called admissible at a point* s *in* D/G *if the infinitesimal dressing action restricted to* $\mathfrak{h}$ *defines an isomorphism from* $\mathfrak{h}$ *onto* $T_s(D/G)$*, that is, the map* $\mathfrak{h} \to T_s(D/G)$*,* $\xi \mapsto \xi_{D/G,s}$ *is an isomorphism. A quasi-triple* $(D, G, \mathfrak{h})$ *is complete if* $\mathfrak{h}$ *is admissible everywhere on* D/G*.*

It is clear that any isotropic complement $\mathfrak{h}$ of $\mathfrak{g}$ is admissible at eG in D/G. If the complement $\mathfrak{h}$ is admissible at a point s in D/G, then it is also admissible on some open neighborhood U of s. For a quasi-triple $(D, G, \mathfrak{h})$, we assume that $\mathfrak{h}$ is admissible on an open subset U of D/G. Then for any X in $\mathfrak{g}$, we define the one-form $\hat{X}_{\mathfrak{h}}$ on U by

$$\langle \hat{X}_{\mathfrak{h}}, \xi_{D/G} \rangle = (X|\, \xi) \tag{12}$$

for any ξ in $\mathfrak{h}$. If a quasi-triple $(D, G, \mathfrak{h})$ is complete, then $\hat{X}_{\mathfrak{h}}$ is a global one-form on D/G. Next we define a twist between isotropic complement subspaces $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$ in $\mathfrak{d}$. Twists also play an important role in the moment map theory for quasi-Poisson actions. Let j and j' be the linear isomorphism (7) defined by Manin quasi-triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}')$ respectively. Consider the map

$$t := j' - j : \mathfrak{g}^* \to \mathfrak{d}.$$

It is easy to show that t takes values in $\mathfrak{g}$ and that it is anti-symmetric, so that the map t defines an element t in $\Lambda^2\mathfrak{g}$ which we denote by the same letter. The element t is called the *twist* from $\mathfrak{h}$ to $\mathfrak{h}'$. Fix a quasi-triple $(D, G, \mathfrak{h})$. Let $\mathfrak{h}_t$ be an isotropic complement of $\mathfrak{g}$ with a twist t from $\mathfrak{h}$. Then we can represent the elements $\varphi_{\mathfrak{h}_t}, F_{\mathfrak{h}_t}$ and $\pi_G^{\mathfrak{h}_t}$ defined by a quasi-triple $(D, G, \mathfrak{h}_t)$ as follows

$$\varphi_{\mathfrak{h}_t} = \varphi_{\mathfrak{h}} + \frac{1}{2}[t, t] + \varphi_t, \qquad F_{\mathfrak{h}_t} = F_{\mathfrak{h}} + F_t, \qquad \pi_G^{\mathfrak{h}_t} = \pi_G^{\mathfrak{h}} + t^L - t^R \tag{13}$$

where $[t, t] := [t^L, t^L]_e$, $\varphi_t(\xi) := \overline{\mathrm{ad}_\xi^* t}$ and $F_t(X) := \mathrm{ad}_X t$. Here ad denotes the adjoint action of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$ and $\overline{\mathrm{ad}_\xi^* t}$ denotes the projection of $\mathrm{ad}_\xi^* t$ onto $\Lambda^2\mathfrak{g} \subset \Lambda^2\mathfrak{d}$, where $\mathfrak{d}^*$ including $\mathfrak{g}^*$ acts on $\Lambda^2\mathfrak{d}$ by the coadjoint action. Let $\{e_i\}$ be a basis on $\mathfrak{g}$ and $\{\varepsilon^i\}$ be the basis on $\mathfrak{h}$ identified with the dual basis of $\{e_i\}$ on $\mathfrak{g}^*$ by j^{-1}. Then the basis $\{\varepsilon_t^i\}$ on $\mathfrak{h}_t$ identified with the dual basis of $\{e_i\}$ on $\mathfrak{g}^*$ by j'^{-1} can be written by

$$\varepsilon_t^i = \varepsilon^i + t^{ij} e_j \tag{14}$$

where $t = \frac{1}{2}t^{ij}e_i \wedge e_j$. Moreover components of φ_t with respect to the basis $\{\varepsilon^i\}$ are written as

$$\varphi_t^{ijk} = (F_{\mathfrak{h}})_l^{jk} t^{il} - (F_{\mathfrak{h}})_l^{ik} t^{jl}. \tag{15}$$

This indication is useful later. Let $(M, \pi_M^{\mathfrak{h}})$ be a quasi-Poisson $G_D^{\mathfrak{h}}$-manifold. We set that $\pi_M^{\mathfrak{h}_t} := \pi_M^{\mathfrak{h}} - t_M$. Then it follows that $(M, \pi_M^{\mathfrak{h}_t})$ is a quasi-Poisson $G_D^{\mathfrak{h}_t}$-manifold. Now we define moment maps for quasi-Poisson actions.

Definition 9. *Let $G_D^{\mathfrak{h}}$ be a connected quasi-Poisson Lie group defined by a quasi-triple $(D, G, \mathfrak{h})$ and $(M, \pi_M^{\mathfrak{h}})$ be a quasi-Poisson $G_D^{\mathfrak{h}}$-manifold. Then a map $\mu : M \to D/G$ which is equivariant with respect to the G-action on M and the dressing action on D/G is a moment map for the quasi-Poisson action of $G_D^{\mathfrak{h}}$ on $(M, \pi_M^{\mathfrak{h}})$ if for any open subset $\Omega \subset M$ and any isotropic complement $\mathfrak{h}'$ admissible on $\mu(\Omega)$*

$$(\pi_M^{\mathfrak{h}'})^\sharp(\mu^*(\hat{X}_{\mathfrak{h}'})) = X_M \tag{16}$$

on Ω for any X in $\mathfrak{g}$. Here $\langle (\pi_M^{\mathfrak{h}'})^\sharp(\alpha), \beta \rangle := \pi_M^{\mathfrak{h}'}(\alpha, \beta)$. We call a quasi-Poisson action with a moment map a quasi-Poisson-Hamiltonian action.

Actually we need not impose the equation (16) on all admissible complements because we have the following proposition.

Proposition 10 ([1]). *Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two complements admissible at a point s in D/G, and p in M be such that $\mu(p) = s$. Then, at the point p, conditions* (16) *for $\mathfrak{h}$ and $\mathfrak{h}'$ are equivalent, namely*

$$(\pi_M^{\mathfrak{h}})^\sharp(\mu^*(\hat{X}_{\mathfrak{h}}))_p = (\pi_M^{\mathfrak{h}'})^\sharp(\mu^*(\hat{X}_{\mathfrak{h}'}))_p.$$

For a quasi-Poisson manifold with a quasi-Poisson-Hamiltonian action, the following theorem holds.

Theorem 11 ([1]). *Let $(M, \pi_M^{\mathfrak{h}})$ be a quasi-Poisson manifold on which a quasi-Poisson Lie group $G_D^{\mathfrak{h}}$ defined by a quasi-triple $(D, G, \mathfrak{h})$ acts by a quasi-Poisson-Hamiltonian action σ . For any p in M, if both $\mathfrak{h}'$ and $\mathfrak{h}''$ are admissible at $\mu(p)$ in D/G, then*

$$\mathrm{im}(\pi_M^{\mathfrak{h}'})_p^\sharp = \mathrm{im}(\pi_M^{\mathfrak{h}''})_p^\sharp$$

where μ is a moment map for σ.

Now we show important examples for quasi-Poisson-Hamiltonian actions.

Example 12 (Poisson manifolds [1], [3], [7]). *Let (M,π) be a Poisson manifold on which a connected Poisson Lie group (G,π_G) acts by a Poisson action σ. Then (M,π) is a quasi-Poisson $(G,\pi_G,0)$-manifold and σ is a quasi-Poisson action on (M,π). In fact, the Manin triple $(\mathfrak{g}\oplus\mathfrak{g}^*,\mathfrak{g},\mathfrak{g}^*)$ corresponding to (G,π_G) is a Manin quasi-triple and the multiplicative two-vector field π_G on G coincides with the two-vector field $\pi_G^{\mathfrak{g}^*}$ defined by the corresponding quasi-triple $(D,G,\mathfrak{g}^*)$. Since $[\pi,\pi]=0$ and the Poisson action σ satisfies*

$$\mathfrak{L}_{X_M}\pi = F_{\mathfrak{g}^*}(X)_M \tag{17}$$

for any X in $\mathfrak{g}$, the action σ is a quasi-Poisson action by Definition 6. *Here the dual of $F_{\mathfrak{g}^*}$ coincides with the bracket on $\mathfrak{g}^*$ defined by (G,π_G).*

We assume that π_G is complete and that there exists a G-equivariant moment map $\mu : M \to G^*$ for the Poisson action σ, where G^* is the dual Poisson Lie group of (G,π_G) and G acts on G^* by the dressing action (see Lu and Weinstein [7]). *Then σ is a quasi-Poisson-Hamiltonian action. Actually, by the definition, the map μ satisfies*

$$\pi^\sharp(\mu^*(X^L)) = X_M \tag{18}$$

for any X in $\mathfrak{g}$, where X^L is a left-invariant one-form on G^* with value X at e in G^*. The quotient manifold D/G is diffeomorphic to G^* as a manifold. The quasi-triple $(D,G,\mathfrak{g}^*)$ is complete since π_G is complete. Then one-form $\hat{X}_{\mathfrak{g}^*}$ defined by (12) *is global for any X in $\mathfrak{g}$. Furthermore the one-form $\hat{X}_{\mathfrak{g}^*}$ on $D/G \cong G^*$ coincides with X^L. The complement $\mathfrak{g}^*$ is admissible at any point in D/G, so that the map $\mu : M \to G^* \cong D/G$ is a moment map for the quasi-Poisson action σ because of* (18) *and* Proposition 10.

Example 13 (Symplectic manifolds [1]). *Let (M,ω) be a symplectic manifold on which a connected Lie group G acts by a symplectic-Hamiltonian action σ. Since the symplectic structure ω induces a Poisson structure π, the pair (M,π) is a Poisson manifold. Then the action σ is a Poisson action of a trivial Poisson Lie group $(G,0)$ on (M,π). The trivial Poisson structure 0 on G is complete and the quasi-triple corresponding to $(G,0)$ is $(T^*G,G,\mathfrak{g}^*)$, where $T^*G \cong G\times\mathfrak{g}^*$ is the cotangent bundle of G equipped with the group structure of a semi-direct product with respect to coadjoint action of G on $\mathfrak{g}^*$ (see* [1]). *The dual group G^* of $(G,0)$ is the additive group $\mathfrak{g}^*$ and the moment map μ for symplectic action σ is G-equivariant with respect to σ on M and* Ad^* *on $\mathfrak{g}^*$ by the definition. Furthermore the dressing action of G on $G^* = \mathfrak{g}^*$ coincides with the coadjoint action* Ad^*. *Thus the map $\mu : M \to \mathfrak{g}^* = G^*$ is a moment map for the Poisson action σ. Therefore, by* Example 12*, the map $\mu : M \to \mathfrak{g}^* \cong T^*G/G$ is a moment map for the quasi-Poisson action σ on the quasi-Poisson $(G,0,0)$-manifold (M,π).*

3. Main Result

Here, we carry out deformations of symplectic structures on a smooth manifold. We use the moment map theory for quasi-Poisson actions for it. A moment map for the quasi-Poisson action on a quasi-Poisson $G_D^{\mathfrak{h}}$-manifold $(M, \pi_M^{\mathfrak{h}})$ are defined with the conditions for the family of quasi-Poisson $G_D^{\mathfrak{h}'}$-structures $\left\{\pi_M^{\mathfrak{h}'}\right\}_{\mathfrak{h}'}$ on M. For each complement $\mathfrak{h}'$, there exists a twist t in $\Lambda^2\mathfrak{g}$ such that $\mathfrak{h}' = \mathfrak{h}_t$, so that the family $\left\{\pi_M^{\mathfrak{h}'}\right\}_{\mathfrak{h}'}$ is regarded as the family parametrized by twist, $\left\{\pi_M^{\mathfrak{h}_t}\right\}_{t\in\Lambda^2\mathfrak{g}}$. When the quasi-Poisson $G_D^{\mathfrak{h}_t}$-structure with twist $t = 0$ is induced by a given symplectic structure, we will give the method of finding a quasi-Poisson $G_D^{\mathfrak{h}_t}$-structure which induced a symplectic structure in $\left\{\pi_M^{\mathfrak{h}_t}\right\}_t$. That is, we can deform a given symplectic structure to a new one by a twist t. This deformation can be carried out due to using the family $\left\{\pi_M^{\mathfrak{h}_t}\right\}_t$ as moment map conditions for quasi-Poisson actions. In this regard, it is described as follows in [1]: It would be interesting to find a geometric framework for considering the family $\left\{\pi_M^{\mathfrak{h}_t}\right\}_t$. Our deformation is one of the answers for this proposal.

Let (M, ω) be a symplectic manifold on which an n-dimensional connected Lie group G acts by symplectic-Hamiltonian action σ with a moment map $\mu : M \to \mathfrak{g}^*$. Let π be the non-degenerate Poisson structure on M induced by ω. Then μ is a moment map for the quasi-Poisson-Hamiltonian action σ of $(G, 0, 0)$ on (M, π) by Example 13 in Section 2.

Let $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ be the Manin triple corresponding to the trivial Poisson Lie group $(G, 0)$, where $\mathfrak{g} \oplus \mathfrak{g}^*$ has the Lie bracket

$$[X, Y] = [X, Y]_{\mathfrak{g}}, \qquad [X, \xi] = \mathrm{ad}^*_X \xi, \qquad [\xi, \eta] = [\xi, \eta]_{\mathfrak{g}^*} = 0 \tag{19}$$

for any X, Y in $\mathfrak{g}$ and ξ, η in $\mathfrak{g}^*$. Here the bracket $[\cdot,\cdot]_{\mathfrak{g}}$ and $[\cdot,\cdot]_{\mathfrak{g}^*}$ are the brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Then the Manin (quasi-)triple $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ defines $F := F_{\mathfrak{g}^*} = 0$ and $\varphi := \varphi_{\mathfrak{g}^*} = 0$ (see (8) and (9)). Since the corresponding quasi-triple $(T^*G, G, \mathfrak{g}^*)$ is complete by Example 12 and Example 13, an isotropic complement $\mathfrak{g}^*$ is admissible at any ξ in $\mathfrak{g}^*$ by Definition 8, and hence it is admissible at any ξ in $\mu(M)$.

Let $\mathfrak{g}_t^*$ be an isotropic complement of $\mathfrak{g}$ in $\mathfrak{g} \oplus \mathfrak{g}^*$ with a twist t in $\Lambda^2\mathfrak{g}$ from $\mathfrak{g}^*$. When we deform π to $\pi_M^t := \pi - t_M$ by a twist t, the quasi-Poisson Lie group $(G, 0, 0)$ is deformed to $(G, \pi_G^t, \varphi_{\mathfrak{g}_t^*})$, where $\pi_G^t = t^L - t^R$ and $\varphi_{\mathfrak{g}_t^*} = \frac{1}{2}[t, t] + \varphi_t$ by (13). Moreover it follows from $F = 0$ and (15) that $\varphi_t = 0$. So $\varphi_{\mathfrak{g}_t^*} = \frac{1}{2}[t, t]$.

On the other hand, it follows from Definition 6 that the quasi-Poisson $(G, \pi_G^t, \varphi_{\mathfrak{g}_t^*})$-manifold (M, π_M^t) satisfies

$$\frac{1}{2}\left[\pi_M^t, \pi_M^t\right] = (\varphi_{\mathfrak{g}_t^*})_M, \qquad \mathfrak{L}_{X_M}\pi_M^t = F_{\mathfrak{g}_t^*}(X)_M. \tag{20}$$

If $(\varphi_{\mathfrak{g}_t^*})_M = 0$, i.e., $[t,t]_M = 0$, then the two-vector field π_M^t is a Poisson structure on M by (20).

Assume that a twist t in $\Lambda^2\mathfrak{g}$ is an *r-matrix*, namely that $[t,t]$ is ad-invariant. Then $\pi_G^t = t^L - t^R$ is a multiplicative Poisson structure (see [7]). Therefore (G, π_G^t) is a Poisson Lie group. Then it follows that $F_{\mathfrak{g}_t^*}$ coincides with the dual of the bracket map $[\cdot,\cdot]_{\pi_G^t} : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ defined by the Poisson Lie group (G, π_G^t). In fact, by the relation (18), we have

$$F_{\mathfrak{g}_t^*}^*(\xi,\eta) = \mathrm{ad}_{t^\sharp(\xi)}\eta - \mathrm{ad}_{t^\sharp(\eta)}\xi \tag{21}$$

where $\langle t^\sharp(\xi), \eta\rangle := t(\xi,\eta)$. In addition, the bracket on $\mathfrak{g}^*$ induced by a multiplicative Poisson structure defined by an r-matrix is represented by the right-hand side of (21) (see [6], Ex.2.19). Therefore, since G is connected, the condition (20) means that the action σ is a Poisson action of (G, π_G^t) on (M, π_M^t) under the assumption that t is an r-matrix and that $[t,t]_M = 0$.

Let $\{e_i\}$ be a basis on $\mathfrak{g}$, a set $\{\varepsilon^i\}$ the dual basis of $\{e_i\}$ on $\mathfrak{g}^*$. Then we can write by (14)

$$\mathfrak{g}_t^* = \mathrm{span}\{\varepsilon^i + t^{ij}e_j\,;\, i = 1,\ldots,n\} \tag{22}$$

where $t = \frac{1}{2}t^{ij}e_i \wedge e_j$ in $\Lambda^2\mathfrak{g}$. If $\mathfrak{g}_t^*$ is admissible at any point in $\mu(M)$, then it satisfies $\mathrm{im}\pi_p^\sharp = \mathrm{im}(\pi_M^t)_p^\sharp$ for any p in M by Theorem 11. The non-degeneracy of π means that $\mathrm{im}\pi_p^\sharp = T_pM$ for any p in M. Therefore, by the fact that $\mathrm{im}(\pi_M^t)_p^\sharp = T_pM$ for any p in M, a quasi-Poisson structure π_M^t is also non-degenerate.

Here we shall examine the condition for a isotropic complement to be admissible at a point in $\mathfrak{g}^*$ in more detail. Let (ξ_i) be the linear coordinates for $\{\varepsilon^i\}$. Then it follows that for $i = 1,\ldots,n$

$$\begin{aligned}(\varepsilon^i + t^{ij}e_j)_{\mathfrak{g}^*} &= -\frac{\partial}{\partial\xi_i} + t^{ij}c_{jl}^k\xi_k\frac{\partial}{\partial\xi_l} \\ &= -t^{ij}\sum_{l\neq i}c_{lj}^k\xi_k\frac{\partial}{\partial\xi_l} - (1 + t^{ij}c_{ij}^k\xi_k)\frac{\partial}{\partial\xi_i}\end{aligned} \tag{23}$$

where $X \mapsto X_{\mathfrak{g}^*}$, for X in $\mathfrak{g}\oplus\mathfrak{g}^*$, is the infinitesimal action of the dressing action on $\mathfrak{g}^* \cong T^*G/G$. The isotropic constant $\mathfrak{g}_t^*$ is admissible at $\xi = (\xi_1,\ldots,\xi_n)$ in

$\mathfrak{g}^*$ if and only if the elements (23) form a basis on $T_\xi(\mathfrak{g}^*) \cong \mathfrak{g}^*$. Hence this means that the matrix

$$A_t(\xi) := \begin{pmatrix} -1 - t^{1j} c_{1j}^k \xi_k & -t^{1j} c_{2j}^k \xi_k & \cdots & -t^{1j} c_{nj}^k \xi_k \\ -t^{2j} c_{1j}^k \xi_k & -1 - t^{2j} c_{2j}^k \xi_k & \cdots & -t^{2j} c_{nj}^k \xi_k \\ \vdots & \vdots & \ddots & \vdots \\ -t^{nj} c_{1j}^k \xi_k & -t^{nj} c_{2j}^k \xi_k & \cdots & -1 - t^{nj} c_{nj}^k \xi_k \end{pmatrix} \tag{24}$$

is regular.

Since any non-degenerate Poisson structure on M defines a symplectic structure on M, the following theorem holds.

Theorem 14. *Let (M, ω) be a symplectic manifold on which a connected Lie group G with the Lie algebra $\mathfrak{g}^*$ acts by a symplectic-Hamiltonian action σ, and μ a moment map for σ. Then the following holds*

1. *If a twist t in $\Lambda^2\mathfrak{g}$ satisfies that $[t,t]_M = 0$, then t deforms the Poisson structure π induced by ω to a Poisson structure $\pi_M^t := \pi - t_M$. Moreover, if t is an r-matrix, then σ is a Poisson action of (G, π_G^t) on (M, π_M^t), where $\pi_G^t = t^L - t^R$.*
2. *For a twist t in $\Lambda^2\mathfrak{g}$, if the isotropic complement $\mathfrak{g}_t^*$ is admissible on $\mu(M)$, then t deforms the non-degenerate two-vector field π induced by ω to a non-degenerate two-vector field π_M^t. This condition is equivalent to that the matrix $A_t(\xi)$ defined by* (24) *is regular for any ξ in $\mu(M)$.*

Therefore, if a twist t satisfies both assumptions in Theorem 14*, then t deforms ω to a symplectic structure ω^t induced by the non-degenerate Poisson structure π_M^t.*

Remarks 15. i) *In* Section 4*, we will show that the condition in* Theorem 14 *is not a necessary condition for π_M^t to be a non-degenerate Poisson structure.*

ii) *If a twist t satisfies both assumptions in* Theorem 14 *and is an r-matrix, then the Poisson action σ of (G, π_G^t) on a symplectic manifold (M, ω^t) has a moment map* (*although not necessarily G-equivariant*) *due to* Theorem 3.16 in [6].

The following theorem gives a sufficient condition for a twist to deform a symplectic structure in the sense of Theorem 14.

Theorem 16. *Let (M, ω) be a symplectic manifold on which an n-dimensional connected Lie group G acts by a symplectic-Hamiltonian action σ. Assume that X, Y in $\mathfrak{g}$ satisfy $[X, Y] = 0$. Then the twist $t = \frac{1}{2} X \wedge Y$ in $\Lambda^2\mathfrak{g}$ deforms the* *symplectic structure ω to a symplectic structure ω_t. For example, a twist t in $\Lambda^2\mathfrak{h}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, satisfies the assumption of the theorem.*

Proof: For X and Y in $\mathfrak{g}$, we set

$$X = X^i e_i, \qquad Y = Y^j e_j$$

where $\{e_i\}_{i=1}^n$ is a basis on the Lie algebra $\mathfrak{g}$. Then since $[X,Y] = X^i Y^j c_{ij}^k e_k = 0$, we obtain the following conditions:

$$X^i Y^j c_{ij}^k = 0 \qquad \text{for any } k$$

where c_{ij}^k are the structure constants of $\mathfrak{g}$ with respect to the basis $\{e_i\}$. Moreover, since we have

$$[t,t] = \left[\frac{1}{2}X\wedge Y, \frac{1}{2}X\wedge Y\right] = \frac{1}{2}X\wedge[X,Y]\wedge Y = 0$$

the twist t is an r-matrix such that $[t,t]_M = 0$ obviously. Hence π_M^t is a Poisson structure, and if π_M^t is non-degenerate, then the twist t induces the symplectic structure ω_t.

We shall show the non-degeneracy of π_M^t. Let μ be the moment map for a given symplectic-Hamiltonian action ψ. We must show that $\mathfrak{g}_t^*$ is admissible at any point in $\mu(M)$. We prove a stronger condition that the quasi-triple $(T^*G, G, \mathfrak{g}_t^*)$ is complete.

Let $\{\varepsilon^i\}$ be the dual basis of $\{e_i\}$ on $\mathfrak{g}^*$ and (ξ_i) be the linear coordinates for $\{\varepsilon^i\}$. Since $t = \frac{1}{2}X^i Y^j e_i \wedge e_j$

$$\mathfrak{g}_t^* = \text{span}\{\varepsilon^i + X^i Y^j e_j\,;\, i = 1, \cdots n\}.$$

Then it follows that for $i = 1, \ldots, n$

$$(\varepsilon^i + X^i Y^j e_j)_{\mathfrak{g}^*} = -X^i Y^j \sum_{l\neq i} c_{lj}^k \xi_k \frac{\partial}{\partial \xi_l} - (1 + X^i Y^j c_{ij}^k \xi_k)\frac{\partial}{\partial \xi_i}. \tag{25}$$

The quasi-triple $(T^*G, G, \mathfrak{g}_t^*)$ is complete if and only if the elements (25) form a basis on $T_\xi(\mathfrak{g}^*) \cong \mathfrak{g}^*$ for any $\xi = (\xi_1, \ldots, \xi_n)$. Therefore we shall prove that the matrix

$$\begin{pmatrix} -1 - X^1 Y^j c_{1j}^k \xi_k & -X^1 Y^j c_{2j}^k \xi_k & \cdots & -X^1 Y^j c_{nj}^k \xi_k \\ -X^2 Y^j c_{1j}^k \xi_k & -1 - X^2 Y^j c_{2j}^k \xi_k & \cdots & -X^2 Y^j c_{nj}^k \xi_k \\ \vdots & \vdots & \ddots & \vdots \\ -X^n Y^j c_{1j}^k \xi_k & -X^n Y^j c_{2j}^k \xi_k & \cdots & -1 - X^n Y^j c_{nj}^k \xi_k \end{pmatrix} \tag{26}$$

is regular. In the case of $X=0$, this matrix is equal to the opposite of the identity matrix, so that it is regular. In the case of $X\neq 0$, using $X^iY^jc_{ij}^k=0$, we can transform the matrix to the opposite of the identity matrix. Thus the matrix (26) is regular. Therefore $\mathfrak{g}_t^*$ is admissible at any point in $\mathfrak{g}^*$. That is, $(T^*G, G, \mathfrak{g}_t^*)$ is complete. ■

Remark 17. *We try to generalize the assumption of* Theorem 16 *and consider X, Y in $\mathfrak{g}$ such that $[X,Y]=aX+bY$ $(a,b\in\mathbb{R})$, that is, the subspace spanned by X, Y is also a Lie subalgebra. We set $t=\frac{1}{2}X\wedge Y$ in $\Lambda^2\mathfrak{g}$. Since $[t,t]=0$, the twist t is an r-matrix such that $[t,t]_M=0$. Therefore the symplectic action ψ is a Poisson action of (G,π_G^t) on (M,π_M^t). Then we research whether $\mathfrak{g}_t^*$ is admissible at each point in $\mathfrak{g}^*$. Similarly to the proof of* Theorem 16, *a matrix to check the regularity can be deformed to*

$$\begin{pmatrix} -1-(aX^k+bY^k)\xi_k & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}.$$

Therefore this matrix is regular if and only if

$$-1-(aX^k+bY^k)\xi_k\neq 0.$$

In the case of $[X,Y]=0$, by Theorem 16, the space $\mathfrak{g}_t^*$ is admissible at all points in $\mathfrak{g}^*$. In the case of $[X,Y]\neq 0$, the above condition means

$$\langle [X,Y],\xi\rangle\neq -1.$$

Let ξ' be an element satisfying that $\langle [X,Y],\xi'\rangle\neq 0$. By setting

$$\xi := -\frac{\xi'}{\langle [X,Y],\xi'\rangle}$$

we obtain $\langle [X,Y],\xi\rangle=-1$, so that $\mathfrak{g}_t^*$ is not admissible at ξ. Eventually, to make sure of the admissibility of $\mathfrak{g}_t^*$, we need check whether such a point ξ is included in $\mu(M)$.

4. Examples on $\mathbb{CP}^n$ and $\mathrm{Gr}(r,\mathbb{C}^n)$

In this section, we compute specifically which element t in $\Lambda^2\mathfrak{g}$ defines a different symplectic structure ω_t from given one ω on a smooth manifold. One example is the complex projective line $(\mathbb{CP}^1,\omega_{\mathrm{FS}})$, where ω_{FS} is the Fubini-Study form, with an action of $\mathrm{SU}(2)$. The other is the complex Grassmannian $(\mathrm{Gr}(r,\mathbb{C}^n),\omega_{\mathrm{KK}})$, where ω_{KK} is the Kirillov-Kostant form, with an action of $\mathrm{SU}(n+1)$.

First we review the relation between $\mathrm{SU}(n+1)$ and $\mathbb{CP}^n$. For any $[z_1:\cdots:z_{n+1}]$ in $\mathbb{CP}^n$ and $g=(a_{ij})$ in $\mathrm{SU}(n+1)$, the action is given by

$$g\cdot[z_1:\cdots:z_{n+1}] := \left[\sum_{j=1}^{n+1} a_{1j}z_j:\cdots:\sum_{j=1}^{n+1} a_{n+1,j}z_j\right].$$

The isotropic subgroup of $[1:0:\cdots:0]$ is

$$\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n)) = \left\{\begin{pmatrix}\mathrm{e}^{\mathrm{i}\theta} & O\\ O & B\end{pmatrix}\in\mathrm{SU}(n+1)\,;\,\theta\in\mathbb{R}, B\in U(n)\right\}.$$

Therefore it follows

$$\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))\cong\mathbb{CP}^n.$$

The complex projective space $\mathbb{CP}^n$ has the coordinate neighborhood system $\{(U_i,\varphi_i)\}_i$ consisting of $n+1$ open sets U_i given by

$$U_i := \{[z_1:\cdots:z_{n+1}]\in\mathbb{CP}^n\,;\,z_i\neq 0\},\qquad \varphi_i: U_i\to\mathbb{C}^n\cong\mathbb{R}^{2n}$$

$$\begin{aligned}[z_1:\cdots:z_{n+1}] &\mapsto \left(\frac{z_1}{z_i},\ldots,\frac{z_{i-1}}{z_i},\frac{z_{i+1}}{z_i},\ldots,\frac{z_{n+1}}{z_i}\right)\\ &\mapsto \left(\Re\frac{z_1}{z_i},\Im\frac{z_1}{z_i},\ldots,\Re\frac{z_{n+1}}{z_i},\Im\frac{z_{n+1}}{z_i}\right)\end{aligned}$$

for $i=1,\ldots,n+1$. By using this coordinate system, the Fubini-Study form ω_{FS} on $\mathbb{CP}^n$ is defined by setting

$$\varphi_j^*\left(\frac{\mathrm{i}}{2}\partial\bar\partial\log\left(\sum_k|z_k|^2+1\right)\right)$$

on each U_j.

The action of $\mathrm{SU}(n+1)$ on $(\mathbb{CP}^n,\omega_{\mathrm{FS}})$ is a symplectic-Hamiltonian action and its moment map μ satisfies

$$\langle\mu([z_1:\cdots:z_{n+1}]),X\rangle = \frac{1}{2}\mathrm{im}\frac{\langle{}^t(z_1,\ldots,z_{n+1}),X\,{}^t(z_1,\ldots,z_{n+1})\rangle}{\langle{}^t(z_1,\ldots,z_{n+1}),{}^t(z_1,\ldots,z_{n+1})\rangle}$$

for any $[z_1 : \cdots : z_{n+1}]$ in $\mathbb{CP}^n$ and X in $\mathfrak{su}(n+1)$. We use

X_{jk} : the (j,k)-element is 1, the (k,j)-element is -1, and the rest are 0
Y_{jk} : the (j,k)- and (k,j)-elements are i, and the rest are 0
Z_l : the (l,l)–element is i, the $(n+1,n+1)$-element is $-$i
and the rest are 0

for $1 \leq j < k \leq n+1$ and $l = 1, \ldots, n$, as a basis of $\mathfrak{su}(n+1)$ which is defined by a Chevalley basis of the complexified Lie algebra $\mathfrak{sl}(n+1, \mathbb{C})$ of $\mathfrak{su}(n+1)$. The subspace spanned by Z_l's is a Cartan subalgebra of $\mathfrak{su}(n+1)$.

We consider the case of $n = 1$. The complex projective line $\mathbb{CP}^1$ has the coordinate neighborhood system $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$. The Fubini-Study form ω_{FS} on $\mathbb{CP}^1$ is

$$\omega_{\mathrm{FS}} = \frac{\mathrm{d}x_1 \wedge \mathrm{d}y_1}{(x_1^2 + y_1^2 + 1)^2}$$

on U_1, where $(x_1, y_1) := \left(\Re \frac{z_2}{z_1}, \Im \frac{z_2}{z_1}\right)$. Then $e_1 := X_{12}$, $e_2 := Y_{12}$ and $e_3 := Z_1$ form a basis of $\mathfrak{su}(2)$. Let $\{\varepsilon^i\}$ be the dual basis of $\mathfrak{su}(2)^*$. We obtain

$$\mu(x_1, y_1) = \frac{y_1}{1 + x_1^2 + y_1^2}\varepsilon^1 + \frac{x_1}{1 + x_1^2 + y_1^2}\varepsilon^2 + \frac{1 - x_1^2 - y_1^2}{2(1 + x_1^2 + y_1^2)}\varepsilon^3.$$

Hence $\mu(\mathbb{CP}^1) \subset \mathfrak{su}(2)^*$ is the two-sphere with center at the origin and with radius $\frac{1}{2}$.

Let (ξ_i) be the linear coordinates for $\{\varepsilon^i\}$. We set $\mathfrak{g} := \mathfrak{su}(2)$. Any twist t is an r-matrix on $\mathfrak{g}$ because $e_1 \wedge e_2 \wedge e_3$ is ad-invariant. Since $\mathbb{CP}^1$ is two-dimensional, it follows that $[t,t]_{\mathbb{CP}^1} = 0$. Therefore we can deform the Poisson structure π_{FS} induced by ω_{FS} to a Poisson structure π_{FS}^t on $\mathbb{CP}^1$ by t and the natural action is a Poisson action of $(\mathrm{SU}(2), t^L - t^R)$.

Let $\mathfrak{g}_t^*$ be the space twisted $\mathfrak{g}^*$ by t in $\Lambda^2\mathfrak{g}$. We consider what is the condition for t under which $\mathfrak{g}_t^*$ is admissible on $\mu(\mathbb{CP}^1)$. For any twist

$$t = \sum_{i<j} \frac{1}{2}\lambda_{ij} e_i \wedge e_j \in \Lambda^2\mathfrak{g}, \qquad \lambda_{ij} \in \mathbb{R}$$

we obtain

$$\mathfrak{g}_t^* = \mathrm{span}\{\varepsilon^1 + \lambda_{12}e_2 + \lambda_{13}e_3,\ \varepsilon^2 - \lambda_{12}e_1 + \lambda_{13}e_3,\ \varepsilon^3 - \lambda_{13}e_1 - \lambda_{23}e_2\}.$$

Then $\mathfrak{g}_t^*$ is admissible at $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathfrak{g}^*$ if and only if the matrix

$$A_t(\xi) = \begin{pmatrix} 1 + 2\lambda_{12}\xi_3 - 2\lambda_{13}\xi_2 & 2\lambda_{13}\xi_1 & -2\lambda_{12}\xi_1 \\ -2\lambda_{23}\xi_2 & 1 + 2\lambda_{12}\xi_3 + 2\lambda_{23}\xi_1 & -2\lambda_{12}\xi_2 \\ -2\lambda_{23}\xi_3 & 2\lambda_{13}\xi_3 & 1 - 2\lambda_{13}\xi_2 + 2\lambda_{23}\xi_1 \end{pmatrix}$$

is regular. By computing the determinant of the matrix, we have

$$\det A_t(\xi) = (1 + 2\lambda_{23}\xi_1 - 2\lambda_{13}\xi_2 + 2\lambda_{12}\xi_3)^2.$$

So the complement $\mathfrak{g}_t^*$ is admissible at $\xi = (\xi_1, \xi_2, \xi_3)$ if and only if $1 + 2\lambda_{23}\xi_1 - 2\lambda_{13}\xi_2 + 2\lambda_{12}\xi_3 \neq 0$.

Therefore $\mathfrak{g}_t^*$ is admissible on $\mu(\mathbb{CP}^1)$ if and only if the "non-admissible surface" $\{\xi = (\xi_1, \xi_2, \xi_3) \in \mathfrak{g}^*;\ 1 + 2\lambda_{23}\xi_1 - 2\lambda_{13}\xi_2 + 2\lambda_{12}\xi_3 \neq 0\}$ for $\mathfrak{g}_t^*$ and the image $\mu(\mathbb{CP}^1)$ have no common point. Since $\mu(\mathbb{CP}^1)$ is the two-sphere with center at the origin and with radius $\frac{1}{2}$, we can see that this condition is equivalent to the condition

$$\lambda_{12}^2 + \lambda_{13}^2 + \lambda_{23}^2 < 1.$$

From the above discussion, we obtain the following theorem.

Theorem 18. *If a twist $t := \sum_{i<j} \frac{1}{2}\lambda_{ij} e_i \wedge e_j$ satisfies $\lambda_{12}^2 + \lambda_{13}^2 + \lambda_{23}^2 < 1$, then the Fubini-Study form ω_{FS} on $\mathbb{CP}^1$ can be deformed by t in the sense of* Section 3.

We shall see an example of a concrete twists on $\mathbb{CP}^1$.

Example 19. *We use a twist $t = \frac{1}{2}X_{12} \wedge Y_{12}$ in $\Lambda^2\mathfrak{su}(2)$ and a real number λ, where $-1 < \lambda < 1$. The symplectic structure $\omega_{\mathrm{FS}}^{\lambda t}$ deformed ω_{FS} by λt is written by*

$$\omega_{\mathrm{FS}}^{\lambda t} = \left\{ \left(1 + \frac{1}{2}\lambda\right)(x_1^2 + y_1^2)^2 + 2(x_1^2 + y_1^2) + \left(1 - \frac{1}{2}\lambda\right) \right\}^{-1} \mathrm{d}x_1 \wedge \mathrm{d}y_1$$

on U_1. Then it follows from an elementary calculation that the symplectic volume $\mathrm{vol}(\mathbb{CP}^1, \omega_{\mathrm{FS}}^{\lambda t})$ of $(\mathbb{CP}^1, \omega_{\mathrm{FS}}^{\lambda t})$ is

$$\mathrm{vol}(\mathbb{CP}^1, \omega_{\mathrm{FS}}^{\lambda t}) = \begin{cases} \pi, & \lambda = 0 \\ \frac{\pi}{\lambda}\log\left|\frac{2+\lambda}{2-\lambda}\right|, & \lambda \neq 0. \end{cases} \tag{27}$$

Next, we consider a cohomology class of each $\omega_{\mathrm{FS}}^{\lambda t}$. Since $H_{\mathrm{DR}}^2(\mathbb{CP}^1) = \mathbb{R}$, there exists a real number k_λ in $\mathbb{R}$ such that $\left[\omega_{\mathrm{FS}}^{\lambda t}\right] = k_\lambda\,[\omega_{\mathrm{FS}}]$. By integrating, we obtain

$$k_\lambda = \frac{1}{\lambda}\log\left|\frac{2+\lambda}{2-\lambda}\right|$$

where $\lambda \neq 0$. Since the function k_λ of λ is smooth, even and strictly monotone increasing when λ is positive, $\omega_{\mathrm{FS}}^{\lambda t}$ and $\omega_{\mathrm{FS}}^{-\lambda t}$ are cohomologous. This means that we obtain a lot of non-trivial symplectic structures different from original ω_{FS} and non-trivial symplectomorphisms $(M, \omega_{\mathrm{FS}}^{\lambda t}) \longrightarrow (M, \omega_{\mathrm{FS}}^{-\lambda t})$.

In the above example, the condition $-1 < \lambda < 1$ is *not* a necessary condition for $\omega_{\mathrm{FS}}^{\lambda t}$ to be a symplectic structure. In fact, it follows that $\omega_{\mathrm{FS}}^{\lambda t}$ is a symplectic structure for $-2 < \lambda < 2$. Therefore in general, the non-degeneracy for π^t is not equivalent to that the isotropic complement $\mathfrak{g}_t^*$ is admissible on $\mu(M)$.

The next example is the complex Grassmannian $\mathrm{Gr}(r, \mathbb{C}^n) := \mathrm{SU}(n)/(\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(n-r)))$ with the Kirillov-Kostant form ω_{KK}. With respect to ω_{KK}, the natural $\mathrm{SU}(n)$-action is symplectic-Hamiltonian.

Then we consider the following r-matrix of $\mathfrak{su}(n)$

$$t = \frac{1}{4n} \sum_{1 \leq i < j \leq n} X_{ij} \wedge Y_{ij}$$

where the r-matrix t is the canonical one defined on any compact semi-simple Lie algebra over $\mathbb{R}$ (for example, see [4]). This is an r-matrix such that $[t,t] \neq 0$. We show that it satisfies $[t,t]_M = 0$, where $M := \mathrm{Gr}(r, \mathbb{C}^n)$. Since t is an r-matrix, the element $[t,t]$ is ad-invariant by the definition. Therefore $[t,t]$ is Ad-invariant because $\mathrm{SU}(n)$ is connected. By the definition of the $\mathrm{SU}(n)$-action on $\mathrm{Gr}(r, \mathbb{C}^n)$, it follows that

$$[t,t]_M = p_*[t,t]^R$$

where $p : \mathrm{SU}(n) \to \mathrm{Gr}(r, \mathbb{C}^n) = \mathrm{SU}(n)/(\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(n-r)))$ is the natural projection. Since any point m in $\mathrm{Gr}(r, \mathbb{C}^n)$ is represented by gH, where g is in $\mathrm{SU}(n)$ and $H := \mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(n-r))$, we compute

$$[t,t]_{M,m} = p_*[t,t]^R_g = p_* R_{g*}[t,t].$$

Because of the Ad-invariance of $[t,t]$, we obtain

$$p_* R_{g*}[t,t] = p_* L_{g*} L_{g^{-1}*} R_{g*}[t,t] = p_* L_{g*} \mathrm{Ad}_{g^{-1}}[t,t] = p_* L_{g*}[t,t].$$

Let $\mathfrak{h}$ be the Lie algebra of H. For any X in $\mathfrak{h}$ and g in $\mathrm{SU}(n)$, we compute

$$p_* L_{g*} X = p_* L_{g*} \frac{\mathrm{d}}{\mathrm{d}s} \exp sX \bigg|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} (g \exp sX) H \bigg|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} gH \bigg|_{s=0} = 0$$

where we have used that $\exp sX$ is in H in the third equality. Therefore it holds that $[t,t]_M = 0$ if each term of $[t,t]$ includes elements in $\mathfrak{h}$. We notice that

$$\mathfrak{h} = \mathrm{span}_{\mathbb{R}}\{X_{ij}, Y_{ij}, Z_k \,;\, 1 \leq i < j \leq r \text{ or } r{+}1 \leq i < j \leq n, \quad k = 1, \dots, n{-}1\}.$$

If $X_{ij}, Y_{ij} \in \mathfrak{h}$, then

$$[\,\cdot\,, X_{ij} \wedge Y_{ij}] = [\,\cdot\,, X_{ij}] \wedge Y_{ij} - X_{ij} \wedge [\,\cdot\,, Y_{ij}]\,.$$

So these terms include an element in $\mathfrak{h}$. Hence we investigate terms of the form

$$\begin{aligned}[X_{ij} \wedge Y_{ij}, X_{kl} \wedge Y_{kl}] = &- [X_{ij}, X_{kl}] \wedge Y_{ij} \wedge Y_{kl} - X_{ij} \wedge [Y_{ij}, X_{kl}] \wedge Y_{kl} \\ &- Y_{ij} \wedge [X_{ij}, Y_{kl}] \wedge X_{kl} - X_{ij} \wedge X_{kl} \wedge [Y_{ij}, Y_{kl}]\end{aligned}$$

where X_{ij}, Y_{ij}, X_{kl} and Y_{kl} are not in $\mathfrak{h}$. In the case of $i = k$ and $j = l$, we get

$$[X_{ij}, X_{ij}] = [Y_{ij}, Y_{ij}] = 0, \qquad [X_{ij}, Y_{ij}] = 2(Z_i - Z_j) \in \mathfrak{h}$$

where we set $Z_n := 0$. In the case of $i = k$ and $j < l$ (resp. $l < j$), since it follows that $r + 1 \leq j,\ l \leq n$, we obtain

$$\begin{aligned}[X_{ij}, X_{kl}] &= [Y_{ij}, Y_{kl}] = -X_{jl}, &&\text{respectively } X_{lj} \in \mathfrak{h} \\ [Y_{ij}, X_{kl}] &= [Y_{kl}, X_{ij}] = -Y_{jl}, &&\text{respectively } Y_{lj} \in \mathfrak{h}.\end{aligned}$$

We can also show the case of $i < k$ respectively $k < i$ and $j = l$ in the similar way. Therefore all terms of $[t, t]$ include elements in $\mathfrak{h}$, so that $[t, t]_M = 0$. Therefore π^t_{KK} is Poisson by Theorem 14, where π_{KK} is the Poisson structure induced by ω_{KK}. Since $\mathrm{Gr}(r, \mathbb{C}^n)$ is compact, for sufficiently small $|\lambda|$, the Poisson structure $\pi^{\lambda t}_{\mathrm{KK}}$ is non-degenerate. Example 19 is the special case of this example.

5. Symplectic Toric Manifolds

In this section, we consider deformations of symplectic toric manifolds, i.e., $2n$-dimensional symplectic manifolds with effective Hamiltonian n-dimensional torus actions. First, we consider the case of $\mathbb{CP}^n$.

Example 20. *A symplectic toric manifold $\mathbb{CP}^n$ has the torus action σ*

$$(e^{\mathrm{i}\theta_2}, e^{\mathrm{i}\theta_3}, \dots, e^{\mathrm{i}\theta_{n+1}}) \cdot [z_1 : \cdots : z_{n+1}] := [z_1 : e^{\mathrm{i}\theta_2} z_2 : \cdots : e^{\mathrm{i}\theta_{n+1}} z_{n+1}]$$

for any θ_i in $\mathbb{R}$. The moment map $\mu : \mathbb{CP}^n \to \mathbb{R}^n$ for this action on $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ is

$$\mu([z_1 : \cdots : z_{n+1}]) := -\frac{1}{2}\left(\frac{|z_2|^2}{|z|^2}, \dots, \frac{|z_{n+1}|^2}{|z|^2}\right)$$

where $z = (z_1, \dots, z_{n+1})$ in $\mathbb{C}^n$. We set $X_1 := (1, 0, \dots, 0), \dots, X_n := (0, \dots, 0, 1)$. Since $\mathbb{T}^n$ is commutative, the brackets $[X_i, X_j]$ vanish for all i and j. Hence for any λ_{12} in $\mathbb{R}$, the twist $t_{12} := \lambda_{12} X_1 \wedge X_2$ deforms ω_{FS} to a symplectic structure $\omega^{t_{12}}_{\mathrm{FS}}$ induced by a Poisson structure $\pi^{t_{12}}_{\mathrm{FS}} := \pi_{\mathrm{FS}} - (t_{12})_{\mathbb{CP}^n}$ by Theorem 16. *On the other hand it follows $\pi^t_{\mathbb{T}^n} := t^L - t^R = 0$ for any twist t by the commutativity of $\mathbb{T}^n$. Therefore, after deformation, the multiplicative Poisson structure*

0 *on* $\mathbb{T}^n$ *is invariant and the action* σ *is a symplectic action. Then moreover* σ *is a symplectic-Hamiltonian action on* $(\mathbb{CP}^n, \omega_{\mathrm{FS}})^{t_{12}}$. *In fact, since the action* σ *is symplectic-Hamiltonian on* $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$, *the actions of each of the factor circles is Hamiltonian. Since* $H^1_{DR}(\mathbb{CP}^n;\mathbb{R}) = H^{2n-1}_{DR}(\mathbb{CP}^n;\mathbb{R}) = \{0\}$, *the condition means that the actions of each of the factor circles have fixed point* (*see* [9, Theorem 5.5]). *This condition is independent on a symplectic structure on* $\mathbb{CP}^n$, *so that the action* σ *is symplectic-Hamiltonian on* $(\mathbb{CP}^n, \omega_{\mathrm{FS}}^{t_{12}})$. *Therefore, by* Theorem 16 *again, the twist* $t_{13} := \lambda_{13} X_1 \wedge X_3$ *deforms* $\omega_{\mathrm{FS}}^{t_{12}}$ *to* $(\omega_{\mathrm{FS}}^{t_{12}})^{t_{13}} = \omega_{\mathrm{FS}}^{t_{12}+t_{13}}$ *induced by* $(\pi_{\mathrm{FS}}^{t_{12}})^{t_{13}} = \pi_{\mathrm{FS}}^{t_{12}+t_{13}}$. *Then we see that the trivial Poisson structure on* $\mathbb{T}^n$ *is invariant and that the action* σ *is symplectic-Hamiltonian. By repeating this operation, it follows that we can deform* ω_{FS} *to* ω_{FS}^t *for any twist* $t = \sum_{i<j} \lambda_{ij} X_i \wedge X_j$ *and that* σ *is symplectic-Hamiltonian. Hence* $(\mathbb{CP}^n, \omega_{\mathrm{FS}}^t)$ *is a symplectic toric manifold. On* U_1, *since we obtain*

$$\begin{aligned}(X_i \wedge X_j)_{\mathbb{CP}^n} &= y_i y_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} - y_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} \\ &\quad - x_i y_j \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_j} + x_i x_j \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}, \qquad 1 \le i < j \le n\end{aligned}$$

where $x_i := \Re \frac{z_{i+1}}{z_1}$ *and* $y_i := \Im \frac{z_{i+1}}{z_1}$, *for example on* $\mathbb{CP}^2$, *it follows that*

$$\begin{aligned}\omega_{\mathrm{FS}}^{t_{12}} &= \omega_{\mathrm{FS}} + \frac{\lambda_{12}\left((x_1^2+y_1^2)(x_2^2+y_2^2)-1\right)}{(x_1^2+y_1^2+x_2^2+y_2^2+1)^4}(x_1x_2\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\ &\quad + x_1y_2\mathrm{d}x_1 \wedge \mathrm{d}y_2 + y_1x_2\mathrm{d}y_1 \wedge \mathrm{d}x_2 + y_1y_2\mathrm{d}y_1 \wedge \mathrm{d}y_2).\end{aligned}$$

Obviously the above example can be generalized to any $2n$-dimensional compact connected symplectic toric manifold (M, ω) satisfying that the map

$$\wedge\omega^{n-1} : H^1_{DR}(M;\mathbb{R}) \longrightarrow H^{2n-1}_{DR}(M;\mathbb{R}) \tag{28}$$

is an isomorphism, which is the assumption of Theorem 5.5 in [9]. Moreover the following theorem holds.

Theorem 21. *For any* $2n$*-dimensional compact connected symplectic toric manifold* (M, ω) *such that the map* (28) *is an isomorphism, and any twist* t *in* $\Lambda^2\mathbb{R}^n$, *the manifold* (M, ω^t) *deformed by* t *in the sense of* Section 3 *is a symplectic toric manifold with the same action as on* (M, ω). *Moreover* (M, ω^t) *is isomorphic to* (M, ω) *as a symplectic toric manifold. Therefore each element in* $\Lambda^2\mathbb{R}^n$ *gives a canonical transformation on* (M, ω).

Proof: We shall prove the latter claim. It is sufficient to prove the claim for $t = X_1 \wedge X_2$ $(X_i \in \mathbb{R}^n)$ due to the same argument as in Example 20. Let σ be the symplectic-Hamiltonian action on the symplectic toric manifold (M, ω) with a moment map μ. Then, by Delzant theorem (for example, see [9]), it is sufficient to show that the action σ on (M, ω^t) has the same moment map μ. Since the map μ satisfies (3), the map μ is a moment map on (M, ω^t) if and only if

$$\mathrm{d}\mu^X = \iota_{X_M}\omega^t \tag{29}$$

for any X in $\mathbb{R}^n$. This condition is equivalent to

$$t^\sharp_M \mathrm{d}\mu^X = 0 \tag{30}$$

for any X in $\mathbb{R}^n$ since μ satisfies (1) with respect to ω. Then we calculate

$$t^\sharp_M \mathrm{d}\mu^X = \omega(X_{1,M}, X_M)X_{2,M} - \omega(X_{2,M}, X_M)X_{1,M}.$$

Using the facts that for any Hamiltonian G-space (M, ω, G, μ),

$$\omega(Y_M, Z_M) = \mu^{[Y,Z]}$$

for any Y and Z in $\mathfrak{g}$, and that the Lie algebra $\mathbb{R}^n$ is commutative, we obtain the condition (30). ■

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