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COMPOSITION ALGEBRAS, EXCEPTIONAL JORDAN ALGEBRA AND RELATED GROUPS*

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Abstract. Normed division rings are reviewed in the more general framework of composition algebras that include the split (indefinite metric) case. The Jordan - von Neumann - Wigner classification of finite dimensional Jordan algebras is outlined with special attention to the 27 dimensional exceptional Jordan algebra $\mathfrak{J}$. The automorphism group F_4 of $\mathfrak{J}$ and its maximal Borel-de Siebenthal subgroups $\frac{\mathrm{SU}(3)\times\mathrm{SU}(3)}{\mathbb{Z}_3}$ and $\mathrm{Spin}(9)$ are studied in some detail with an eye to possible applications to the fundamental fermions in the Standard Model of particle physics.

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*Dedicated to the memory of Professor Vasil V. Tsanov 1948-2017.

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1. Introduction

Division and Clifford algebras were introduced in 19th century with an eye for applications in geometry and physics (for a historical survey see the last chapter of [24]). Pascual Jordan introduced and studied his algebras in the 1930's in order to describe observables in quantum mechanics (for a taste of Jordan algebras see [25] along with the original paper [23]). Yet, the first serious applications of these somewhat exotic structures appeared (in mid-twentieth century) in pure mathematics: in the theory of exceptional Lie groups and symmetric spaces (cf. [17, 19]) as well as the later surveys [1, 5, 26] in topology [2]. Possible applications to particle physics were first advocated by Feza Gürsey and his students in the 1970's - see his lecture and his posthumous book (with C.-H.Tze) [21] and references therein). They continue in various guises to attract attention until these days, never becoming a mainstream activity. The present lectures are meant as a background for the ongoing work [14, 30]. Although this proposal of an "exceptional quantum geometry" is still tentative, we feel that it is worth pursuing.[1] In any case, the mathematical background which is the main subject of these notes is sound and beautiful - and deserves to be known by particle theorists.

2. Composition and Clifford Algebras

2.1. Normed Alternative Algebras

A composition (or Hurwitz) algebra $\mathcal{A}$ is a vector space over a field $\mathbb{K} = (\mathbb{R}, \mathbb{C}, \dots)$ equipped with a bilinear (not necessarily associative) product xy with a unit 1 ($1x = x1 = x$) and a nondegenerate quadratic form $N(x)$, the norm satisfying

$$N(xy) = N(x)N(y), \qquad N(\lambda x) = \lambda^2 N(x) \quad \text{for} \quad x \in \mathcal{A}, \quad \lambda \in \mathbb{K}. \qquad (1)$$

[1] For related attempts to provide an algebraic counterpart of the Standard Model of particle physics see [8, 11–13, 18, 28] and references to earlier work cited therein.

The norm allows to define by polarization a *symmetric bilinear form* $\langle x, y\rangle$ setting

$$2\langle x, y\rangle = N(x+y) - N(x) - N(y) = 2\langle y, x\rangle. \tag{2}$$

Nondegeneracy of N means that if $\langle x, y\rangle = 0$ for all $y \in \mathcal{A}$ then $x = 0$. By repeated polarization of the identity $\langle xy, xy\rangle = \langle x, x\rangle\langle y, y\rangle$ one obtains

$$\langle ab, ac\rangle = N(a)\langle b, c\rangle = \langle ba, ca\rangle \tag{3}$$

$$\langle ac, bd\rangle + \langle ad, bc\rangle = 2\langle a, b\rangle\langle c, d\rangle. \tag{4}$$

Setting in (4) $a = c = x,\ b = 1,\ d = y$ and using (3) we find

$$\langle x^2 + N(x)1 - t(x)x, y\rangle = 0$$

where $t(x) := 2\langle x, 1\rangle$ is by definition the *trace*, or, using the non-degeneracy of the form $\langle\cdot\,,\cdot\rangle$

$$x^2 - t(x)x + N(x)1 = 0, \qquad t(x) = 2\langle x, 1\rangle. \tag{5}$$

Thus every $x \in \mathcal{A}$ satisfies a quadratic relation with coefficients the trace $t(x)$ and the norm $N(x)$ (a linear and a quadratic scalar functions) taking values in $\mathbb{K}$.

The trace functional (5) allows to introduce *Cayley conjugation*

$$x \to x^* = t(x) - x, \qquad t(x) = t(x)1 \in \mathcal{A} \tag{6}$$

an important tool in the study of composition algebras. It is an (orthogonal) reflection ($\langle x^*, y^*\rangle = \langle x, y\rangle$) that leaves the scalars $\mathbb{K}1$ invariant (in fact, $t(\lambda 1) = 2\lambda$ implying $(\lambda 1)^* = \lambda 1$ for $\lambda \in \mathbb{K}$). It is also an involution and an antihomomorphysm

$$(x^*)^* = x, \qquad (xy)^* = y^* x^*. \tag{7}$$

Furthermore equations (5) and (6) allow to express the trace and the norm as a sum and a product of x and x^*

$$t(x) = x + x^*, \qquad N(x) = xx^* = x^*x = N(x^*).$$

The relation (4) allows to deduce

$$\langle ax, y\rangle = \langle x, a^*y\rangle, \qquad \langle xa, y\rangle = \langle x, ya^*\rangle. \tag{8}$$

From these identities it follows $\langle ab, 1\rangle = \langle a, b^*\rangle = \langle ba, 1\rangle$, hence, the trace is commutative

$$t(ab) = \langle b, a^*\rangle = \langle a, b^*\rangle = t(ba). \tag{9}$$

Similarly, one proves that t is associative and symmetric under cyclic permutations

$$t\big((ab)c\big) = t(a(bc)) =: t(abc) = t(cab) = t(bca). \tag{10}$$

Moreover, using the quadratic relation (5) and the above properties of the trace one proves the identities that define an *alternative algebra*

$$x^2y = x(xy), \qquad yx^2 = (yx)x \tag{11}$$

(see [26, Section 1] for details). The conditions (11) guarantee that the *associator*

$$\big[x, y, z\big] = \big(xy\big)z - x\big(yz\big) \tag{12}$$

changes sign under odd permutations (and is hence preserved by even, cyclic, permutations). This implies, in particular, the *flexibility conditions*

$$\big(xy\big)x = x\big(yx\big). \tag{13}$$

An unitar alternative algebra with an involution $x \to x^*$ satisfying (7) is a composition algebra if the norm N and the trace t defined by (9) are scalars (i.e., belong to $\mathbb{K}(= \mathbb{K}1)$) and the norm is non-degenerate.

Given a finite dimensional composition algebra $\mathcal{A}$ Cayley and Dickson have proposed a procedure to construct another composition algebra $\mathcal{A}'$ with twice the dimension of $\mathcal{A}$. Each element x of $\mathcal{A}'$ is written in the form

$$x = a + eb, \qquad a,\ b \in \mathcal{A} \tag{14}$$

where e is a new "imaginary unit" such that

$$e^2 = -\mu, \qquad \mu \in \{1, -1\}. \tag{15}$$

Thus $\mathcal{A}$ appears as a subalgebra of $\mathcal{A}'$. The product of two elements $x = a + eb$, $y = c + ed$ of $\mathcal{A}'$ is defined as

$$xy = ac - \mu d\bar{b} + e(\bar{a}d + cb) \tag{16}$$

where $a \to \bar{a}$ is the Cayley conjugation in $\mathcal{A}$. (The order of the factors becomes important, when the product in $\mathcal{A}$ is noncommutative.) The Cayley conjugation $x \to x^*$ and the norm $N(x)$ in $\mathcal{A}'$ are defined by

$$\begin{aligned} x^* &= (a + eb)^* = \bar{a} + \bar{b}e^* = \bar{a} - \bar{b}e = \bar{a} - eb \\ N(x) &= xx^* = a\bar{a} + \mu b\bar{b} = x^*x. \end{aligned} \tag{17}$$

Let us illustrate the meaning of (16) and (17) in the first simplest cases.

For $\mathcal{A} = \mathbb{R}$, $\overline{a} = a$, (16) coincides with the definition of complex numbers for $\mu = 1$ ($e = \mathrm{i}$) and defines the split complex numbers for $\mu = -1$. Taking next $\mathcal{A} = \mathbb{C}$ and $\mu = 1$ we can identify $\mathcal{A}'$ with a 2×2 matrix representations setting

$$\begin{aligned} \mathbf{a} &= a_0 + e_1 a_1 = a_0 + \mathrm{i}\sigma_3 a_1 = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}, \qquad a = a_0 + \mathrm{i}a_1 \\ x &= \mathbf{a} + e\mathbf{b}, \qquad e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}, \qquad \mathbf{b} = b_0 + e_1 b_1. \end{aligned} \tag{18}$$

Anticipating Baez *Fano plane* [3] notation for the octonion imaginary units (see Application A) we shall set $e = e_4$, $e_4 e_1 = e_2$ $(= \mathrm{i}\sigma_1)$.

It is then easily checked that the multiplication law (16) reproduces the standard matrix multiplication, the Cayley conjugation $x \to x^*$ coincides with the hermitian conjugation of matrices, while the norm $N(x)$ in $\mathcal{A}'$ is given by the determinant

$$\mathbb{H} = \{x \in \mathbb{C}[2]\,;\, xx^* = \mathrm{det}x \ (\geq 0)\}. \tag{19}$$

Similarly, starting with the split complex numbers, we can write

$$\mathbf{a_s} = \mathbf{a_0} + \tilde{\mathbf{e}}_\mathbf{1}\mathbf{a_1}, \quad \tilde{\mathbf{e}}_\mathbf{1} = \boldsymbol{\sigma_3} \leftrightarrows \mathbf{a_s} = \begin{pmatrix} a_s & 0 \\ 0 & \overline{a}_s \end{pmatrix}, \quad \mathbf{a_s} = \mathbf{a_0} + \mathbf{a_1}, \quad \overline{\mathbf{a}}_\mathbf{s} = \mathbf{a_0} - \mathbf{a_1}$$

and choosing the same e as above we can identify the *split quaternions* $\mathbb{H}_s$ with real 2×2 matrices

$$\mathbb{H}_s = \{x = \begin{pmatrix} a_s & -\overline{b}_s \\ b_s & \overline{a}_s \end{pmatrix} \in \mathbb{R}[2]\,;\, x^* = \begin{pmatrix} \overline{a}_s & \overline{b}_s \\ -b_s & a_s \end{pmatrix},\ xx^* = \mathrm{det}x\} \tag{20}$$

its norm having signature $(2, 2)$.

The next step in Cayley-Dickson construction gives the octonions, which have a nonassociative (but alternative) multiplication and thus do not have matrix realization.

2.2. Relation to Clifford Algebras and Classification

Given a composition algebra $\mathcal{A}$ we define subspace $\mathcal{A}_0 \subset \mathcal{A}$ of pure imaginary elements with respect to the Cayley conjugation (6)

$$\mathcal{A}_0 = \{y \in \mathcal{A};\, y^* = -y\}. \tag{21}$$

It is a subspace of co-dimension one, orthogonal to the unit 1 of $\mathcal{A}$. For any $x \in \mathcal{A}$ we define its imaginary part as

$$x_0 = \frac{1}{2}(x - x^*) = x - \langle x, 1\rangle \qquad \Rrightarrow \qquad \langle x_0, 1\rangle = 0. \tag{22}$$

Table 1. The types of algebra $Cl(p, q)$ depend on $p - q$ mod 8.

n	$\mathrm{Cliff}_{(1-\mathrm{n})}$	Irreducible spinor	n	$\mathrm{Cliff}_{(1-\mathrm{n})}$	Irreducible spinor
1	$\mathbb{R}$	$S_1 = \mathbb{R}$	5	$\mathbb{H}[2]$	$S_5 = \mathbb{H}^2$
2	$\mathbb{C}$	$S_2 = \mathbb{C}$	6	$\mathbb{C}[4]$	$S_6 = \mathbb{C}^4$
3	$\mathbb{H}$	$S_3 = \mathbb{H}$	7	$\mathbb{R}[8]$	$S_7 = \mathbb{R}^8$
4	$\mathbb{H} \oplus \mathbb{H}$	$S_4^+ = \mathbb{H}$, $S_4^- = \mathbb{H}$	8	$\mathbb{R}[8] \oplus \mathbb{R}[8]$	$S_8^+ = \mathbb{R}^8$, $S_8^- = \mathbb{R}^8$

From the expression $N(x) = xx^*$ (8) and from the defining property (21) of imaginary elements it follows that

$$x_0 \in \mathcal{A}_0 \qquad \Rightarrow \qquad x_0^2 = -N(x_0). \tag{23}$$

In other words, if the composition algebra $\mathcal{A}$ is n-dimensional then its $(n-1)$-dimensional subalgebra $\mathcal{A}_0$ gives rise to a Clifford algebra. If the norm N is positive definite then[2] $\mathcal{A}_0 = \mathrm{Cliff}(0, \mathrm{n}-1) = \mathrm{Cliff}_{(1-\mathrm{n})}$. In the case of *split* complex numbers, quaternions and octonions one encounters instead the algebras $\mathrm{Cliff}_1 \equiv \mathrm{Cliff}(1,0)$, $\mathrm{Cliff}(2,1)$ and $\mathrm{Cliff}(4,3)$, respectively.

It turns out that the classification of the Clifford algebras $\mathrm{Cliff}_{(1-\mathrm{n})}$ implies the classification of normed division rings of dimension n. So we recall it in the following table: Here we use the notation $\mathbb{A}[n]$ for the algebra $n \times n$ matrices with entries in the (associative) algebra $\mathbb{A}$. As discovered by Elie Cartan in 1908 $\mathrm{Cliff}_{(-\nu-8)} = \mathrm{Cliff}_{-\nu} \otimes \mathbb{R}[16]$ so that the above table suffices to reconstruct all Clifford algebras of type $\mathrm{Cliff}_{-\nu}$. We see that the (real) dimension of the irreducible representation of $\mathrm{Cliff}_{(1-\mathrm{n})}$ coincides with n for $n = 1, 2, 4, 8$ only thus implying Hurwitz theorem (see [3, Theorem 1] and the subsequent discussion).

Proceeding to the split alternative composition algebras we note that the type of $\mathrm{Cliff}(\mathrm{p}, \mathrm{q})$ only depends on the signature $p - q$ which is 1 (similar to -7) for all above cases $\mathrm{Cliff}(1,0) = \mathbb{R} \oplus \mathbb{R}$, $\mathrm{Cliff}(2,1) = \mathbb{R}[2] \oplus \mathbb{R}[2]$, $\mathrm{Cliff}(4,3) = \mathbb{R}[8] \oplus \mathbb{R}[8]$ and

$$\mathrm{Cliff}(\mathrm{p}, \mathrm{p}-1) \cong \mathbb{R}[2^{(\mathrm{p}-1)}] \oplus \mathbb{R}[2^{(\mathrm{p}-1)}]. \tag{24}$$

All these cases are summarized in Table 1. We note here the difference in the treatment of the representations of $\mathrm{Cliff}(\mathrm{p}, \mathrm{p}-1)$ in the cases $p = 1, 2$, in which we are dealing with real associative composition algebras $\mathbb{C}_s$ and $\mathbb{H}_s$, and $p = 4$ of the split octonions. In the associative case we deal with the action of $\mathrm{Cliff}(\mathrm{p}, \mathrm{p}-1)$ on the direct sum $\mathbb{R}^n \oplus \mathbb{R}^n$, $n = 2^{(p-1)}$ (for $p = 1, 2$) while in the non-associative case it acts on the irreducible subspace $\mathbb{R}^n$ ($n = 8$), thus again fitting the dimension of the corresponding alternative algebra.

[2] We adopt the sign convention of [24], [21], [29]. The opposite sign convention, $\mathrm{Cliff}_{(\mathrm{n}-1)}$ for the positive definite $N(x)$, is used e.g. in [3].

Remark 1. *The spinors S_n are here understood as quantities transforming under the lowest order faithful irreducible representation of the (compact) group* $\mathrm{Spin}(n)$ *which consists of the norm one even elements of* $\mathrm{Cliff}_{-\mathrm{n}}$. *In fact, the even part* $\mathrm{Cliff}_0(\mathrm{p},\mathrm{q})$ *of* $\mathrm{Cliff}(\mathrm{p},\mathrm{q})$ *is isomorphic, for $q > 0$ to* $\mathrm{Cliff}(\mathrm{p},\mathrm{q}-1)$. $\mathrm{Spin}(n)$ *is the double cover of the rotation group* $\mathrm{SO}(\mathrm{n})$. *The group of all norm one elements* $\mathrm{Cliff}_{(1-\mathrm{n})}$ *is the double cover* $\mathrm{Pin}(\mathrm{n}-1)$ *of the full orthogonal group* $\mathrm{O}(\mathrm{n}-1)$ *and its irreducible representations are called "pinors" - see* [3, Section 2.3].

In summary, the alternative algebras are classified as follows ([26, Proposition 1.6])

Theorem 2. *Let $(\mathcal{A}, N)$ be a composition algebra. For $\mu = \pm 1$, denote by $\mathcal{A}(\mu)$ the algebra $\mathcal{A}(\mu) = \mathcal{A} \oplus e\mathcal{A}$ with $e^2 = -\mu$ and product* (16). *Then*

- *$\mathcal{A}(\mu)$ is commutative iff $\mathcal{A} = \mathbb{K}$*
- *$\mathcal{A}(\mu)$ is associative iff $\mathcal{A}$ is associative and commutative*
- *$\mathcal{A}(\mu)$ is alternative iff $\mathcal{A}$ is associative.*

Theorem 3 ([26], see also the relations (7)-(10)). *A composition algebra is, as a vector space, $1, 2, 4$ or 8 dimensional. There are four composition algebras $\mathcal{A}_j$ over $\mathbb{C}$ of dimension 2^j, $j = 0, 1, 2, 3$. There are seven composition algebras over $\mathbb{R}$: the division algebras $\mathcal{A}_j^+$, $(j = 0, 1, 2, 3)$ with $N(x) \geq 0$ and $x^{-1} = \dfrac{x^*}{N(x)}$ for $x \neq 0$, and the split algebras $\mathcal{A}_j^s$, $j = 1, 2, 3$ of signature $(2^{j-1}, 2^{j-1})$.*

All above algebras are unique up to isomorphism. The multiplication rule (16) varies in different expositions. Different conventions are related by algebra automorphisms. (Our notation differs from Roos [26] only by the sign of μ, as we set $e^2 = -\mu$.) The only nontrivial automorphism of the algebra of complex numbers is the complex conjugation. The automorphism group of the (real) quaterions is $\mathrm{SO}(3)$ realized by

$$x \to uxu^*, \qquad u \in \mathrm{SU}(2), \qquad \mathrm{u}^* = \mathrm{u}^{-1}. \tag{25}$$

Similarly, the automorphism group of the split quaternions is $\mathrm{SO}(2,1)$

$$\mathbb{H} \ni x \to gxg^{-1}, \qquad g \in \mathrm{SL}(2,\mathbb{R}). \tag{26}$$

We shall survey the octonions and their automorphisms in the next section.

2.3. Historical Note

The simplest relation of type (1), the one applicable to the absolute value square of a product of complex numbers

$$(xu - yv)^2 + (xv + yu)^2 = (x^2 + y^2)(u^2 + v^2)$$

$(x, y, u, v \in \mathbb{R})$, was found by Diophantus of Alexandria around 250 BC. A more general relation of this type

$$(xu + Dyv)^2 - D(xv + yu)^2 = (x^2 - Dy^2)(u^2 - Dv^2)$$

occurs for special values of D in Indian mathematics (cf. Brahmegupta 598 AD) - see [6, Section 2]. For D positive it applies to the *split complex numbers*. The geometric interpretation by Gauss comes much later. (The fact that complex numbers are useful and should be taken seriously is sometimes attributed to Gerolamo Cardano (1501-1576), whose book Arts Magna (The Great Art) contains the solution of the cubic equation. In fact, it was his contemporary, Bologna's mathematician Rafael Bombelli (1526-1572) who first thoroughly understood the complex numbers and described them in his L'Algebra, published in 1572.)

The multiplicativity of the norm of the quaternions was noted by Euler in 1748, a century before Hamilton discovered the algebra of quaternions in 1843 (when "in a famous act of a mathematical vandalism, he carved the equations $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$ into the stone of Brougham Bridge" [3, p. 145]). The corresponding relation for the octonions was discovered by the Danish mathematician Degen in 1818 - again before the discovery of the octonions (in late 1843 - in a letter to Hamilton by his college friend J. Graves). The first publication about octonions appears as an appendix to an otherwise erroneous paper of the English mathematician (at the time, lawyer) Arthur Cayley (1821-1895) in 1845 (see Introduction of [3] and references 17 and 18 therein)

The American algebraist and author of the three volumes *History of the Theory of Numbers*, Leonard E. Dickson (1874-1954) contributed to the construction of composition algebras in 1919 [10]. The theorem that the only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ was proven by A. Hurwitz (1859 -1919) in 1898. The extension of this result to alternative (including split) algebras belongs to M. Zorn (1906-1993) in 1930 and 1933. The fact that the only division algebras (without extra structure) have dimensions $1, 2, 4, 8$ was established as late as in 1958 (independently by R. Bott and J. Milnor and by M. Kervaire).

3. Octonions. Isometries and Automorphisms

3.1. Eight Dimensional Alternative Algebras

The multiplication table of the imaginary octonions (see Application A) can be introduced by first selecting a quaternion subalgebra

$$e_j e_k = \epsilon_{jkl} e_l - \delta_{jk}, \quad j,k,l = 1,2,4, \quad \epsilon_{124} = 1 = \epsilon_{412} = \epsilon_{241} = 1 = -\epsilon_{214}. \tag{27}$$

The somewhat exotic labeling of the units (jumping over 3) is justified by the following memorable multiplication rules $\mathrm{mod} 7$

$$\begin{gathered} e_i e_j = e_k \Rightarrow e_{i+1} e_{j+1} = e_{k+1}, \qquad e_{2i} e_{2j} = e_{2k} \equiv e_{2k(\mathrm{mod} 7)} \\ e_7 e_j = e_{3j(\mathrm{mod} 7)} \quad \text{for} \quad j = 1,2,4, \quad e_7 e_4 = e_5. \end{gathered} \tag{28}$$

A convenient complex isotropic basis for the vector representation of the isometry Lie algebra $\mathfrak{so}(8)$ of $\mathbb{O}$ (which contains the automorphism algebra $\mathfrak{g}_2$ of the octonions) is given by

$$\rho^\epsilon = \frac{1}{2}(1 + \mathrm{i}\epsilon e_7), \quad \zeta_j^\epsilon = \rho^\epsilon e_j = \frac{1}{2}(e_j + \mathrm{i}\epsilon e_{3j}), \quad j = 1,2,4, \quad \epsilon = \pm \tag{29}$$

(the imaginary unit i commutes with octonion units e_a). The multiplication table of the octonion units is summarized by the following relations

$$\begin{gathered} (\zeta_j^\epsilon)^2 = 0 = \rho^+ \rho^-, \qquad (\rho^\epsilon)^2 = \rho^\epsilon, \qquad \rho^+ + \rho^- = 1, \qquad \zeta_j^\epsilon \zeta_k^\epsilon = \epsilon_{jkl} \zeta_l^{-\epsilon} \\ \zeta_j^\epsilon \zeta_k^{-\epsilon} = -\rho^{-\epsilon} \delta_{jk} \quad \Rightarrow \quad [\zeta_j^+, \zeta_k^-]_+ = \delta_{jk}, \qquad j,k,l = 1,2,4. \end{gathered} \tag{30}$$

The idempotents $\rho^\pm$ (which go back to Gürsey) are also exploited in [12]. The last equation (30) coincides with the canonical anticommutation relations for fermionic creation and annihilation operators (cf. [8]).

The *split octonions* x_s with units $\tilde{e}_a$ can be embedded in the algebra $\mathbb{CO}$ of complexified octonions by setting $\tilde{e}_\mu = e_\mu$, $\mu = 0,1,2,4$, $\tilde{e}_7 = \mathrm{i}e_7$, $\tilde{e}_{3j} = \mathrm{i}e_{3j(\mathrm{mod} 7)}$, so that

$$\begin{aligned} x_s &= \sum_{a=0}^{7} x_s^a \tilde{e}_a \Rightarrow N(x_s) = x_s x_s^* \\ &= \sum_{\mu=0,1,2,4} (x_s^\mu)^2 - (x_s^7)^2 - (x_s^3)^2 - (x_s^6)^2 - (x_s^5)^2. \end{aligned} \tag{31}$$

The quark-lepton correspondence suggests the splitting of octonions into a direct sum,

$$\begin{gathered} \mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3, \qquad x = a + \mathbf{z}\,\mathbf{e} = a + z^1 e_1 + z^2 e_2 + z^4 e_4, \qquad e_1 e_2 = e_4 \\ a = x^0 + x^7 e_7, \qquad z^j = x^j + x^{3j(\mathrm{mod} 7)} e_7, \qquad x^{12} \equiv x^5 \end{gathered} \tag{32}$$

thus e_7 playing the role of *imaginary unit within the real octonions*. The Cayley-Dickson construction corresponds to the splitting of $\mathbb{O}$ into two quaternions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}, \qquad x = u + e_7 v, \qquad u = x^0 + x^j e_j, \qquad v = x^7 + x^{3j} e_j. \quad (33)$$

One may speculate that upon complexification u and v could be identified with the "up-" and "down-", leptons and quarks: $u = (\nu, u^j,\ j = 1, 2, 4)$. $v = (e^-, d_j)$ j playing the role of a colour index. For $x_r = u_r + e_7 v_r,\ r = 1, 2$ the Cayley-Dickson formula (16) and its expression in terms of the complex variable a_r, z_r^j read

$$\begin{aligned} x_1 x_2 &= u_1 u_2 - v_2 v_1^* + e_7(u_1^* v_2 + u_2 v_1) \\ &= a_1 a_2 - \mathbf{z_1} \overline{\mathbf{z}} + (a_1 \mathbf{z_2} + \overline{a}_2 \mathbf{z_1} + \overline{\mathbf{z}}_\mathbf{1} \times \overline{\mathbf{z}}_\mathbf{2})\mathbf{e} \end{aligned} \quad (34)$$

where the star indicates quaternionic conjugation while the bar stands for a change of the sign of e_7 ($\bar{z}^j = x^j - e_7 x^{3j}$). The two representations (34) display the covariance of the product under the action of two subgroups of maximal rank of the automorphism group of the octonions. If p and q are two unit quaternions

$$\begin{aligned} &p = p^0 + p^j e_j, \qquad q = q^0 + q^j e_j \\ &pp^* = N(p) = (p^0)^2 + \mathbf{p^2} = 1 = qq^* \ \Leftrightarrow\ (p, q) \in \mathrm{SU}(2) \times \mathrm{SU}(2). \end{aligned} \quad (35)$$

It is easy to verify, using the first equation (34), that the transformation

$$\begin{aligned} x = u + e_7 v \qquad &\to \qquad pup^* + e_7 pvq^* \\ (p, q) \in \frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathbb{Z}_2^{\mathrm{diag}}} & \end{aligned} \quad (36)$$

where $\mathbb{Z}_2^{\mathrm{diag}} = \{(p, q) = (1, 1), (-1, -1)\}$, is an automorphism of the octonion algebra. Similarly, if $U \in \mathrm{SU}(3)$ acts on x (32) as

$$x = a + z^j e_j \to a + U_j^i z^j e_i, \qquad U(x_1)U(x_2) = U(x_1 x_2). \quad (37)$$

The subgroups (36), (37) are the two closed connected subgroups of maximal rank of the compact group G_2 corresponding to the *Borel-de Siebenthal theory* [7] that plays a central role in [30].

3.2. Isometry Group of the (Split) Octonions. Triality

The norm $N(x) = \sum_0^7 (x^a)^2$ and the associated scalar product are preserved by the orthogonal group $\mathrm{O}(8)$ in 8-dimensions. Similarly, the isometry group of the norm of the split octonions $\tilde{x} \in \mathbb{O}_s$

$$N_s(\tilde{x}) = (\tilde{x}^0)^2 + (\tilde{x}^1)^2 + (\tilde{x}^2)^2 + (\tilde{x}^4)^2 - (\tilde{x}^7)^2 - (\tilde{x}^3)^2 - (\tilde{x}^6)^2 - (\tilde{x}^5)^2 \quad (38)$$

is the noncompact pseudoorthogonal group $\mathrm{O}(4,4)$. What is remarkable is that the trilinear form $t(xyz)$ (10) is invariant under the 2-fold cover $\mathrm{Spin}(8)$ (respectively, $\mathrm{Spin}(4,4)$) of the connected group $\mathrm{SO}(8)$ (respectively, $\mathrm{SO}(4,4)$). More precisely, there exist two involutive outer automorphisms[3] κ and π of $\mathrm{Spin}(8)$ such that for each element $g \in \mathrm{Spin}(8)$ the trilinear form

$$t_8(x,y,z) = \frac{1}{2}t(xyz) = \langle xyz, 1\rangle \tag{39}$$

is invariant under the combined transformation $x \to gx,\ y \to \kappa(g)y,\ z \to \pi(g)z$

$$t_8(gx, \kappa(g)y, \pi(g)z) = t_8(x,y,z). \tag{40}$$

The factor $\frac{1}{2}$ in (39) follows from the definition of *normed triality* of [3] which demands $|t_8(x,y,z)|^2 \leqq N(x)N(y)N(z)$. We shall illustrate the meaning of κ and π in the special case when g acts on x by left multiplication with a norm-one octonion a: $L_a x = ax$. Note that not all $\mathrm{Spin}(8)$ elements can be written in this form. For instance, due to non-associativity, the product $L_{a_1}L_{a_2}x = a_1(a_2x)$ cannot, in general be written in the form $L_a x$. As emphasized in the thesis [18], the composition of maps $L_{a_1}L_{a_2}$ is associative while the product of octonions is not. We define, in general, the map κ: $\mathrm{Spin}(8) \to \mathrm{Spin}(8)$ by $\kappa(g)x := (gx^*)^*$ $(x \in \mathbb{O},\ g = \mathrm{Spin}(8))$. It ia easy to verify that

$$\kappa L_a = R_a, \qquad \text{where} \qquad R_a y = ya^*, \qquad a, y \in \mathbb{O}; \quad aa^* = 1. \tag{41}$$

(Note that if the algebra $\mathcal{A}$ of elements a were associative then the group law $R_{a_1}R_{a_2} = R_{a_1a_2}$ would be only satisfied if one uses the conjugation for right action as in (41).) Finally, setting

$$\pi L_a = T_a, \qquad \text{where} \qquad T_a z = aza^*, \qquad \pi R_a = R_a, \quad \pi T_a = L_a \tag{42}$$

we verify the invariance condition (40). The case of a general g can be deduced from here by first proving that any element of g can be represented as a product of a finite number of left multiplications (cf. [11, 32]).

3.3. Automorphism Group and Derivations of Octonions

As proven by Elie Cartan in 1914, the automorphism group of the octonions is the exceptional Lie group

$$\mathrm{G}_2 = \{\mathrm{g} \in \mathrm{L}(\mathbb{O}, \mathbb{R});\ (\mathrm{gx})(\mathrm{gy}) = \mathrm{g}(\mathrm{xy}), \quad \mathrm{x}, \mathrm{y} \in \mathbb{O}\} \tag{43}$$

[3] We adopt the notation of [32] where κ, π and $\nu = \kappa\pi$ are defined as Lie algebra automorphisms. Baez ([3] Section 2.4), who works with the group action, denotes them $\alpha^{\pm}$.

where $\mathrm{L}(\mathbb{O}, \mathbb{R})$ is the group of non-singular linear transformations of the 8-dimensional real vector space $\mathbb{O}$. It follows from (38) that G_2 preserves the octonion unit and commutes with the Cayley conjugation, so that it preserves the norm $N(x)$

$$g1 = 1, \qquad (gx)^* = g(x^*), \qquad N(gx) = N(x). \tag{44}$$

Thus G_2 is a subgroup of the isometry (orthogonal) group $\mathrm{O}(\mathbb{O}) = \mathrm{O}(8)$ of the 8-dimensional euclidean space of the octonions. In fact, it is a subgroup of the connected orthogonal group $\mathrm{SO}(7)$ of the 7-dimensional space $\mathbb{O}_0$ of imaginary octonions, the Lie algebra $\mathfrak{so}(7)$ splitting as a vector space into a direct sum of the Lie algebra $\mathfrak{g}_2$ of G_2 and its lowest order faithful 7-dimensional representation

$$\mathfrak{so}(7) \approxeq \mathfrak{g}_2 \oplus \underline{7} \cong \mathfrak{g}_2 \oplus \mathbb{R}^7. \tag{45}$$

We see, in particular, that the dimension of G_2 is 14.

The maximal subgroups of G_2, whose action was defined by (36) and (37) (and which correspond to the Borel-de Siebenthal theory) can be characterized as follows. The *complex conjugation* γ (in the notation of [32]): $e_7 \to -e_7$ belongs to the automorphism group G_2 of $\mathbb{O}$ (corresponding, in fact, to the reflection of four imaginary units $e_7,\ e_7e_1 = e_3,\ e_7e_2 = e_6$ and $e_7e_4 = e_5$) and has square one

$$\gamma x = \gamma(u + e_7 v) = u - e_7 v, \qquad \gamma^2 = 1. \tag{46}$$

The rank two (semisimple) subgroup (35), (36) of G_2 can be characterized as the commutant of γ in G_2

$$\mathrm{G}_2^\gamma = \{\mathrm{g} \in \mathrm{G}_2; \gamma\mathrm{g} = \mathrm{g}\gamma\} \equiv \frac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathbb{Z}_2}. \tag{47}$$

Denote, on the other hand, by ω the generator of the center of $\mathrm{SU}(3)$ acting on z by (37)

$$\omega x = a + \omega_7 z^j e_j, \qquad \omega_7 = -\frac{1}{2} + \frac{\sqrt{3}}{2} e_7, \qquad \omega_7^3 = 1 = \omega^3. \tag{48}$$

Then the subgroup (37) of G_2 is characterized by

$$\mathrm{G}_2^\omega := \{\omega \in \mathrm{G}_2\ ;\ \omega\mathrm{g} = \mathrm{g}\omega\} \equiv \mathrm{SU}(3). \tag{49}$$

It is convenient to embed the Lie algebra $\mathfrak{so}(7)$ (and hence $\mathfrak{g}_2 \subset \mathfrak{so}(7)$) into the isometry algebra $\mathfrak{so}(8)$ of the quadratic form $N(x)$. Following Yokota [32] we introduce two bases[4] $G_{ab}(= G_{ba})$ and F_{ab} in the 28-dimensional vector space

[4]We note that the convention $e_1e_2 = e_3$ (rather then our $e_1e_2 = e_4$) is used in [32] so that the relations connecting G and F (Application B) are different.

$\mathfrak{so}(8)$

$$G_{ab}e_c = \delta_{bc}e_a - \delta_{ac}e_b, \qquad a,b,c = 0,1,\ldots,7, \qquad e_0 = 1 \tag{50}$$
$$F_{ab}x = \frac{1}{2}e_a(e_b^*x) = -F_{ba}x, \qquad F_{aa} \equiv 0, \qquad e_0^* = e_0$$

related by an involution π

$$(\pi G)_{ab} = F_{ab}, \qquad (\pi F)_{ab} = G_{ab}, \qquad \pi^2 = 1. \tag{51}$$

We shall demonstrate in Application B that the involution π splits into seven four-dimensional involutive transformations. Here, we display one of them which involves our choice of the Cartan subalgebra of $\mathfrak{so}(8)$

$$F_7 = X_7G_7,\ G_7 = \begin{pmatrix} G_{07} \\ G_{13} \\ G_{26} \\ G_{45} \end{pmatrix},\ F_7 = \begin{pmatrix} F_{07} \\ F_{13} \\ F_{26} \\ F_{45} \end{pmatrix},\ X_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{52}$$

a straightforward calculation gives $X_7^2 = \mathbb{1}$, $\det X_7 = -1$. The generators G_{ab} obey the following commutation relations

$$\big[G_{ab}, G_{cd}\big] = 2\big(\delta_{bc}G_{ad} - \delta_{ac}G_{bd} + \delta_{ad}G_{bc} - \delta_{bd}G_{ac}\big). \tag{53}$$

In order to work with real Cartan matrices we shall first display the isometry algebra $\mathfrak{so}(4,4)$ and the derivations of the split octonions defined by (31). The algebra $\mathfrak{so}(4,4)$ has 12 compact and 16 noncompact generators. The maximal compact subalgebra $\mathfrak{so}(4) \times \mathfrak{so}(4)$ of [5] $D_{4(4)} = \mathfrak{so}(4,4)$ is spanned by the generators $G_{\mu\nu}$, $\mu, \nu = 0,1,2,4$, $(\mu\langle\nu)$; $G_{3j,7}$, $G_{3j,3k}$, $j,k = 1,2,4$, $j\langle k$ while the 16 noncompact generators are given by

$$\tilde{G}_{\mu,3\rho} = \frac{1}{\mathrm{i}}G_{\mu,3\rho} = \tilde{G}_{3\rho,\mu}, \qquad \mu = 0,1,2,4, \qquad 3\rho = 3,6,5,7$$
$$3\rho = 3\rho\,(\mathrm{mod}7), \quad \text{so that} \quad 3\times 4 = 5,\ 3\times 7 = 7. \tag{54}$$

3.4. Roots and Weights of $\mathfrak{g}_{2(2)} \subset \mathfrak{so}(4,4)$

The Lie algebra $\mathfrak{so}(8)$ being simply laced, the roots α_j coincide with coroots $\alpha_j^\vee = 2\alpha_j/\alpha_j^2$. The four-dimensional root space is spanned by the orthogonal weight basis $\{\lambda_\mu\}$

$$\lambda_0 \leftrightarrow H_0 = \tilde{G}_{07},\ \lambda_1 \leftrightarrow H_1 = \tilde{G}_{13},\ \lambda_2 \leftrightarrow H_2 = \tilde{G}_{26},\ \lambda_4 \leftrightarrow H_4 = \tilde{G}_{45}. \tag{55}$$

[5]The notation $\mathrm{L_{r(s)}}$ for a rank r Lie algebra $\mathrm{L_r}$ means that its signature, the difference between the number of $+$ and $-$ signs of its Cartan Killing form, is s.

In the isotropic basis (29) H_μ are 8×8 diagonal matrices with two non-zero eigenvalues

$$H_\mu \zeta_\nu^\epsilon = \epsilon \delta_{\mu\nu} \zeta_\nu^\epsilon, \qquad \epsilon = \pm, \qquad \mu, \nu = 0, 1, 2, 4 \tag{56}$$

so that we have

$$\langle \lambda_\mu, \lambda_\nu \rangle = \frac{1}{2} \mathrm{tr} H_\mu H_\nu = \delta_{\mu\nu}, \qquad \zeta_0^\epsilon = \rho^\epsilon. \tag{57}$$

The simple roots α_ν and the fundamental weights λ_μ of $\mathfrak{so}(4,4)$ are

$$\begin{aligned} \alpha_0 &= \lambda_0 - \lambda_1 \leftrightarrow H_{01} = H_0 - H_1, \quad \alpha_1 = \lambda_1 - \lambda_2 \leftrightarrow H_{12} = H_1 - H_2 \\ \alpha_2^- &= \lambda_2 - \lambda_3 \leftrightarrow H_2 - H_3 \; \alpha_2^+ = \lambda_2 + \lambda_3 \leftrightarrow H_2 + H_3 \\ \Lambda_0 &= \lambda_0, \qquad \Lambda_1 = \lambda_0 + \lambda_1, \qquad \Lambda_2^\mp = \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 \mp \lambda_3) \end{aligned} \tag{58}$$

$$\langle \Lambda_\mu^\epsilon, \alpha_\nu^{\epsilon'} \rangle = \delta_{\mu\nu} \delta^{\epsilon\epsilon'}, \qquad \epsilon = \pm, \qquad \mu, \nu = 0, 1, 2. \tag{59}$$

The rank three- Lie subalgebra $\mathfrak{so}(7) \subset \mathfrak{so}(8)$ is spanned by the 21 generators G_{kl} with $k, l > 0$. The simple (co)roots and the fundamental weights of the noncompact real form $\mathfrak{so}(4,3)$ and the corresponding Cartan matrix of $\mathfrak{so}(7,\mathbb{C})$ are

$$\begin{aligned} \alpha_1 &= \lambda_1 - \lambda_2, \qquad \alpha_2^- = \lambda_2 - \lambda_3, \qquad \alpha_3 = \lambda_3 \Rightarrow \alpha_3^\vee = 2\alpha_3 \\ \Lambda_1^{\mathfrak{so}(7)} &= \lambda_1, \qquad \Lambda_2^{\mathfrak{so}(7)} = \lambda_1 + \lambda_2, \qquad \Lambda_3^{\mathfrak{so}(7)} = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) \end{aligned} \tag{60}$$

$$\langle \Lambda_j^{\mathfrak{so}(7)}, \alpha_k^\vee \rangle = \delta_{jk}, \qquad (c_{jk}) = (\langle \alpha_j^\vee, \alpha_k \rangle) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}. \tag{61}$$

Finally, the derivation algebra $\mathfrak{g}_2 (\subset \mathfrak{so}(7)$ of the octonions is spanned by

$$\begin{aligned} &\lambda G_{24} + \mu G_{37} + \nu G_{56}, && \lambda G_{14} - \mu G_{35} + \nu G_{76} \\ &\lambda G_{17} + \mu G_{25} - \nu G_{46}, && -\lambda G_{12} + \mu G_{36} + \nu G_{75} \\ &\lambda G_{16} - \mu G_{23} + \nu G_{47}, && -\lambda G_{15} + \mu G_{27} + \nu G_{43} \\ &\lambda G_{13} + \mu G_{26} + \nu G_{45}, && \lambda + \mu + \nu = 0. \end{aligned} \tag{62}$$

For instance, the last line of (63) is a symmetric way of saying that the Cartan subalgebra of $\mathfrak{g}_2$ is spanned by $G_{13} - G_{26}, G_{26} - G_{45}$. Thus $\mathfrak{g}_2$ is $7 \times 2 = 14$ dimensional. Its simple (co-)roots are expressed conveniently in terms of the barycentric weights

$$\overline{\lambda}_j := \lambda_j - \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_4), \qquad j = 1, 2, 4, \qquad \sum_j \overline{\lambda}_j = 0 \tag{63}$$

as follows

$$\begin{aligned} \alpha_1 &= \overline{\lambda}_1 - \overline{\lambda}_2 \qquad (= \lambda_1 - \lambda_2) \quad \longleftrightarrow \quad H_{12} = \tilde{G}_{13} - \tilde{G}_{26} \\ \alpha_2 &= \overline{\lambda}_2 \qquad \Rightarrow \qquad \alpha_2^{\vee} = 3\alpha_2 \end{aligned} \tag{64}$$

yielding the $\mathfrak{g}_2$ Cartan matrix

$$(\langle \alpha_i^{\vee}, \alpha_j \rangle) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The remaining four positive roots of $\mathfrak{g}_2$ (including the fundamental weights $\Lambda_i^{\mathfrak{g}_2}$) are given by

$$\begin{aligned} \alpha_1 + \alpha_2 &= \overline{\lambda}_1, \qquad \alpha_1 + 2\alpha_2 = \overline{\lambda}_1 + \overline{\lambda}_2 = -\overline{\lambda}_4 = \Lambda_2^{\mathfrak{g}_2} \\ \alpha_1 + 3\alpha_2 &= \overline{\lambda}_2\overline{\lambda}_4 = \lambda_2 - \lambda_4 \quad \longleftrightarrow \quad H_{24} = \tilde{G}_{26} - \tilde{G}_{45} \\ \theta : &= 2\alpha_1 + 3\alpha_2 = \overline{\lambda}_1 - \overline{\lambda}_4 \\ &= \lambda_1 - \lambda_4 = \Lambda_1^{\mathfrak{g}_2} \quad \longleftrightarrow \quad H_{14} = \tilde{G}_{13} - \tilde{G}_{45}. \end{aligned}$$

The $\mathfrak{su}(3)$ subalgebra of $\mathfrak{g}_2$ becomes $\mathfrak{sl}_3 = \mathfrak{s}l(3, \mathbb{R})$ in $\mathfrak{g}_{2(2)}$. Its Cartan matrices are H_{12}, H_{24} while the raising (and the lowering) generators E_α (and F_α) are labeled by the long roots $\alpha = \alpha_1, \alpha_1 + 3\alpha_2, \theta$. The maximal compact subalgebra $\mathfrak{so}(3)$ of $\mathfrak{sl}_3$ is spanned by

$$\begin{aligned} L_{12} &:= G_{12} + G_{36}, \qquad L_{24} = G_{24} + G_{56}, \qquad L_{14} = G_{14} + G_{35} \\ \left[L_{12}, L_{24}\right] &= L_{14}, \qquad \left[L_{12}, L_{14}\right] = -L_{24} = L_{42} \end{aligned} \tag{65}$$

the non-diagonal non-compact generators accompanying (64) are

$$\tilde{G}_{16} + \tilde{G}_{23}, \qquad \tilde{G}_{15} + \tilde{G}_{13}, \qquad \tilde{G}_{14} + \tilde{G}_{35}.$$

4. Jordan Algebras and Related Groups

4.1. Classification of Finite Dimensional Jordan Algebras

Pascual Jordan (1902-1980) the “unsung hero among the creators of quantum theory” (in the words of Schweber, 1994) asked himself in 1932 a question you would expect of an idle mathematician: Can one construct an algebra of (hermitian) observables without introducing an auxiliary associative product? He arrived, after some experimenting with the *special Jordan product*

$$A \circ B = \frac{1}{2}(AB + BA) = B \circ A \tag{66}$$

at two axioms (Jordan, 1933)

$$\text{i) } A \circ B = B \circ A, \qquad \text{ii) } A^2 \circ (B \circ A) = (A^2 \circ B) \circ A \tag{67}$$

where $A^2 := (A \circ A)$. They imply, in particular, power associativity and

$$A^m \circ A^n = A^{m+n}, \qquad m, n = 0, 1, 2, ..., \qquad A^0 = 1. \tag{68}$$

Being interested in extracting the properties of the algebra of hermitian matrices (or self-adjoint operators) for which $A^2 \geq 0$, Jordan adopted Artin's *formal reality* condition

$$A_1^2 + \cdots + A_n^2 = 0 \quad \Longrightarrow \quad A_1 = 0 = \cdots = A_n. \tag{69}$$

In a fundamental paper of 1934 Jordan, von Neumann and Wigner [23] classified all finite dimensional formally real *Jordan algebras* (i.e., algebras over the field of real numbers satisfying (67). They split into a direct sum of *simple algebras*, which belong to four infinite families

$$\mathcal{H}_n(\mathbb{R}),\ \mathcal{H}_n(\mathbb{C}),\ \mathcal{H}_n(\mathbb{H}),\ \mathrm{JSpin}(n) \tag{70}$$

and a single exceptional one

$$\mathfrak{J}(= \mathfrak{J}_3^8) = \mathcal{H}_3(\mathbb{O}). \tag{71}$$

Here $\mathcal{H}_n(\mathbb{A})$ stands for the set of $n \times n$ hermitian matrices with entry in the division ring $\mathbb{A}(= \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$, equipped with the commutative product (66). (One uses the same notation when $\mathbb{A}$ is replaced by one of the alternative split composition rings , $\mathbb{C}_s, \mathbb{H}_s$ or $\mathbb{O}_s$ albeit the resulting algebra is not formally real in that case.) $\mathrm{JSpin}(n)$ is an algebra of elements $(\xi, x;\, \xi \in \mathbb{R}, x \in \mathbb{R}^n)$ where $\mathbb{R}^n$ is equipped with the (real) euclidean scalar product $\langle x, y\rangle$ and the product in $\mathrm{JSpin}(n)$ is given by

$$(\xi, x)(\eta, y) = (\xi\eta + \langle x, y\rangle, \xi y + \eta x). \tag{72}$$

The first three algebras $\mathcal{H}_n(\mathbb{A})$ (70) are equipped with the special Jordan product (66) where AB stands for the (associative) matrix product. The algebra $\mathrm{JSpin}(n)$ is special as a Jordan subalgebra of the 2^n dimensional (associative) Clifford algebra Cliff(n).

Remark 4. *The Jordan algebras* $\mathcal{H}_2(\mathbb{A})$ *for* $\mathbb{A} = \mathbb{R},\ \mathbb{C},\ \mathbb{H},\ \mathbb{O}$ *are isomorphic to* $\mathrm{JSpin}(n)$ *for* $n = 2, 3, 5, 9$ *respectively. Indeed in all four cases the elements of* $\mathbb{A}$ *split into a multiple of the unit matrix and a traceless part of scalar square*

$$\begin{aligned} X &= \begin{pmatrix} \xi_1 & x \\ x^* & \xi_2 \end{pmatrix} = \xi_0 1\!\!1 + X_0 \xi_0 = \frac{\xi_1 + \xi_2}{2}, \qquad X_0 = \begin{pmatrix} \xi & x \\ x^* & -\xi \end{pmatrix} \\ \xi &= \frac{\xi_1 - \xi_2}{2}, \qquad X_0^2 = (\xi^2 + x x^*) 1\!\!1, \qquad x x^* = x^* x \in \mathbb{R}_+. \end{aligned} \tag{73}$$

We note that in each case the determinant of X has a Minkovski space signature

$$\det X = \xi_0^2 - \xi^2 - xx^* = \xi_1\xi_2 - xx^* \tag{74}$$

and is thus invariant under the Lorentz group in $3, 4, 6$ *and* 10 *dimensions, respectively.*

On the other hand, the algebras $\mathcal{H}_n(\mathbb{O})$ for $n > 3$ are not Jordan since they violate condition ii) of (67). The exceptional Jordan algebra $\mathfrak{J} = \mathcal{H}_3(\mathbb{O})$ did not seem to be special but the authors of [23] assigned the proof that the product $A \circ B$ of two elements of $\mathfrak{J}$ cannot be represented in the form (66) with an *associative* product. The PhD student of L. Dickson Abraham Albert (1905-1972) proved this.

We introduce the one-dimensional projectors E_i and the hermitian octonionic matrices $F_i(x_i)$ writing down a general element of $\mathcal{H}_3(\mathbb{O})$ as

$$X = \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix} = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(x_1) + F_2(x_2) + F_3(x_3). \tag{75}$$

We can then write the Jordan multiplication $X \circ Y$ setting

$$E_i \circ E_j = \delta_{ij} E_j, \qquad E_i \circ F_j(x) = \begin{cases} 0, & \text{if} \quad i = j \\ \frac{1}{2} F_j(x), & \text{if} \quad i \neq j \end{cases}$$

$$F_i(x) \circ F_i(y) = \langle x, y \rangle (E_{i+1} + E_{i+2}), \quad F_i(x) F_{i+1}(y) = \frac{1}{2} F_{i+2}(y^* x^*) \tag{76}$$

where the indices are counted $\mathrm{mod}3$: $E_4 \equiv E_1,\ F_5 \equiv F_2, \ldots$. We define the trace, a symmetric bilinear inner product and a trilinear scalar product in $\mathfrak{J}$ by

$$\mathrm{tr}X = \xi_1 + \xi_2 + \xi_3, \quad \langle X, Y \rangle = \mathrm{tr}(X \circ Y), \quad \mathrm{tr}(X, Y, Z) = \langle X, Y \circ Z \rangle. \tag{77}$$

The Jordan algebra $\mathfrak{J}$ also admits a (symmetric) *Freudenthal product*

$$X \times Y = \frac{1}{2}\left[2X \circ Y - X\mathrm{tr}Y - Y\mathrm{tr}X + (\mathrm{tr}X\mathrm{tr}Y - \langle X, Y \rangle E)\right] \tag{78}$$

where E is the 3×3 unit matrix, $E = E_1 + E_2 + E_3$. Finally, we define a three-linear form (X, Y, Z) and the determinant $\det X$ by

$$\begin{aligned} (X, Y, Z) = \langle X, Y \times Z \rangle = \langle X \times Y, Z \rangle, \qquad \det X &= \frac{1}{3}(X, X, X) \\ = \xi_1\xi_2\xi_3 + 2\mathrm{Re}(x_1x_2x_3) - \xi_1 x_1 x_1^* &- \xi_2 x_2 x_2^* - \xi_3 x_3 x_3^*. \end{aligned} \tag{79}$$

The following identities hold

$$X \times X \circ X = (\det X)E, \qquad (X \times X) \times (X \times X) = (\det X)X. \tag{80}$$

4.2. Automorphism Groups of the Exceptional Jordan Algebras $\mathcal{H}_3(\mathbb{O}_{(s)})$ and their Maximal Subgroups

Classical Lie groups appear as symmetries of classical symmetric spaces. For quite some time there was no such interpretation for the exceptional Lie groups. The situation only changed with the discovery of the exceptional Jordan algebra $\mathcal{H}_3(\mathbb{O})$ and its split octonions' cousin $\mathcal{H}_3(\mathbb{O}_s)$.

The automorphism group of the $\mathcal{H}_3(\mathbb{O})$ algebra is the rank four compact simple Lie group[6] F_4. It clearly leaves the unit element E invariant and is proven to preserve the trace (77) (see Lemma 2.2.1 in [32]). The stabilizer of E_1 in F_4 is the double covering $\mathrm{Spin}(9)$ of the rotation group in nine dimensions (which preserves X_0^2 (73)). Moreover, we have

$$\mathrm{F}_4/\mathrm{Spin}(9) \simeq \mathbb{OP}^2 \Longrightarrow \dim \mathrm{F}_4 = \dim \mathrm{Spin}(9) + \dim \mathbb{O}^2 = 36 + 16 = 52. \quad (81)$$

Building upon our treatment of $\mathfrak{g}_{2(2)}$ (Section 2.2) we shall first construct the Lie algebra $\mathfrak{f}_{4(4)}$ of *derivations* (infinitesimal automorphisms) of $\mathcal{H}_3(\mathbb{O}_s)$ which admits a real Cartan subalgebra spanned by the orthonormal basis $\lambda_0, \lambda_1, \lambda_2, \lambda_4$ (55) (normalized by (57)). The simple roots of $\mathfrak{f}_{4(4)}$ (and the associated Cartan matrices in the basis ζ_μ^ϵ (29)) are:

$$\begin{aligned}
\alpha_1 &= \lambda_1 - \lambda_2 \quad \longleftrightarrow \quad H_{12} = \epsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\alpha_2 &= \lambda_2 - \lambda_4 \quad \longleftrightarrow \quad H_{24} = H_2 - H_4 \\
\alpha_4 &= \lambda_4 \quad \longleftrightarrow \quad H_4 = \epsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\alpha_0 &= \frac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 - \lambda_4) \\
&\longleftrightarrow \quad \frac{1}{2}(H_{01} - H_2 - H_4) = \frac{\epsilon}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned} \quad (82)$$

[6] This was proven by Claude Chevalley and Richard Schafer in 1950. The result was prepared by Ruth Moufang's study in 1933 of the octonionic projective plane, then Jordan's construction in 1949 of $\mathbb{OP}^2$ in terms of one-dimensional projections in $\mathcal{H}_3(\mathbb{O})$ and Armand Borel's observation that F_4 is the isometry group of $\mathbb{OP}^2$. For a review and references - see [3, Section 4.2]. Octonionic quantum mechanics in the Moufang plane was considered in [20].

The corresponding coroots and Cartan matrix for $\mathfrak{f}_4$ are

$$\alpha_1^\vee = \alpha_1, \qquad \alpha_2^\vee = \alpha_2, \qquad \alpha_4^\vee = 2\alpha_4, \qquad \alpha_0^\vee = 2\alpha_0$$

$$(c_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{83}$$

The Lie algebra $\mathfrak{f}_{4(4)}$ has 24 positive roots: the 12 long roots coincide with the positive roots $\{\lambda_0 \pm \lambda_j,\ j = 1, 2, 4,\ \lambda_j \pm \lambda_k,\ 1 \leqq j \langle k \leqq 4\}$ of $\mathfrak{so}(4,4)$; the 4 short roots $\lambda_\mu,\ \mu = 0, 1, 2, 4$ coincide with the short positive roots of $\mathfrak{so}(5,4)$; finally, $\mathfrak{f}_{4(4)}$ has 8 additional short roots of the form $\frac{1}{2}(\lambda_0 \pm \lambda_1, \pm\lambda_2 \pm \lambda_4)$; the highest root θ of $\mathfrak{f}_{4(4)}$ coincide with that of its rank four simple subalgebras $\mathfrak{so}(5,4)$ and $\mathfrak{so}(4,4)$

$$\theta = \lambda_0 + \lambda_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_4 + 2\alpha_0. \tag{84}$$

The elements D of $\mathfrak{so}(8)$ act on X of $\mathfrak{J}$ (75) through their action on the octonions.

$$DX = F_1(Dx_1) + F_2(\kappa(D)x_2) + F_3(\pi(D)x_3) \tag{85}$$

where $D =: D_1$, $\kappa(D) =: D_2$, $\pi(D) =: D_3$, obey the principle of infinitesimal triality

$$(D_1 x)y + x(D_2 y) = (D_3((xy)^*))^*. \tag{86}$$

For $D \in \mathrm{G}_2$ we have $D_1 = D_2 = D_3 = D$.

The remaining 24 generators of $\mathfrak{f}_4$ (outside $\mathfrak{so}(8)$) can be identified with the skew-hermitian matrices $A_i(e_a)$, $i = 1, 2, 3$, $a = 0, 1, \ldots, 7$

$$A_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & -x^* & 0 \end{pmatrix}, \ A_2(x) = \begin{pmatrix} 0 & 0 & -x^* \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \ A_3(x) = \begin{pmatrix} 0 & x & 0 \\ -x^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{87}$$

They act on $\mathfrak{J}$ through the commutators

$$\tilde{A}_i(e_a)X = \frac{1}{2}\big[A_i(e_a), X\big], \qquad i = 1, 2.3, \quad a = 0, 1, ..., 7. \tag{88}$$

The derivation algebra $\mathfrak{f}_{4(4)}$ of $\mathcal{H}_3(\mathbb{O}_3)$ is obtained from here by substituting the action of $\mathfrak{g}_2$ by that of $\mathfrak{g}_{2(2)}$ and by replacing $A_i(e_a)$ by $A_i(\tilde{e}_a)$ for $a = 3, 6, 5, 7$. It is for this non-compact form $\mathfrak{f}_{4(4)}$, that the Cartan elements are represented by real diagonal matrices in the isotropic basis ξ_μ^ϵ (29)

$$\begin{aligned} \alpha_1^\vee &= H_{12}(= \alpha_1), \qquad \alpha_2^\vee = H_{23}(= \alpha_2), \qquad \alpha_3^\vee = 2H_3(= 2\alpha_3) \\ \alpha_0^\vee &= H_{01} - H_2 - H_3 = 2\alpha_0. \end{aligned} \tag{89}$$

The physical meaning of the F_4 covariance of the algebra $\mathfrak{J}$ is revealed by exhibiting the action of

$$\mathrm{F}_4^{\omega} = \frac{\mathrm{SU}(3) \times \mathrm{SU}(3)}{\mathbb{Z}_3} \subset \mathrm{F}_4 \tag{90}$$

(one of the maximal subgroup according to the Borel-de Siebentahal theory on $\mathcal{H}_3(\mathbb{O})$). To do that we shall first extend the splitting of the octonions $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ into a splitting of the exceptional Jordan algebra, $H_3(\mathbb{O}) = H_3(\mathbb{C}) \oplus \mathbb{C}[3]$

$$\mathcal{H}_3(\mathbb{O}) \ni X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix} = X(\xi, a) + X(0, \mathbf{ze}) \tag{91}$$

where

$$\begin{aligned} X(\xi, a) &= \begin{pmatrix} \xi_1 & a_3 & \overline{a}_2 \\ \overline{a}_3 & \xi_2 & a_1 \\ a_2 & \overline{a}_1 & a_3 \end{pmatrix} \\ a_r &= x_r^0 + x_r^7 e_7, \qquad \overline{a}_r = x_r^0 - x_r^7 e_7, \qquad r = 1,2,3 \\ X(0, \mathbf{ze}) &= \begin{pmatrix} 0 & \mathbf{z_3 e} & -\mathbf{z_2 e} \\ -\mathbf{z_3 e} & 0 & \mathbf{z_1 e} \\ \mathbf{z_2 e} & -\mathbf{z_1 e} & 0 \end{pmatrix} \\ \mathbf{z_r e} &= z_r^1 e_1 + z_r^2 e_2 + z_r^4 e_4, \qquad z_r^j = x_r^j + x_r^{3j(\mathrm{mod} 7)} e_7 \end{aligned} \tag{92}$$

in which we have used the conjugation property $(\mathbf{ze})^* = -\mathbf{ze}$ of imaginary octonions. Multiplications mixes the two terms in the right hand side of (91). The Freudenthal product $X(\xi, x) \times Y(\eta, b)$ can be expressed in a nice compact way if we substitute the skew symmetric octonionic matrices $X(0, \mathbf{ze}), X(0, \mathbf{we})$ by 3×3 complex matrices Z, W

$$X(0, \mathbf{ze}) \longleftrightarrow Z = (z_r^j,\ r = 1,2,3,\ \jmath = 1,2,4) \in \mathbb{C}[3] \tag{93}$$

which transform naturally under the subgroup (90).

Indeed, using the fact that the matrices $X(0, \mathbf{ze})$ and $X(0, \mathbf{we})$ are traceless their Fredenthal product (78) simplifies and we find

$$\begin{aligned} X(\xi, a) \ \times \ X(0, \mathbf{we}) &= X(\xi, a) \circ X(0, \mathbf{we}) - \frac{\xi_1 + \xi_2 + \xi_3}{2} X(0, \mathbf{we}) \\ \Longrightarrow X(\xi, a) \times W &= -\frac{1}{2} W X(\xi, a),\ \text{for } W = (w_r^j) \end{aligned} \tag{94}$$

$$\begin{aligned} X(0,\mathbf{ze}) \times X(0,\mathbf{we}) &= X(0,\mathbf{ze}) \circ X(0,\mathbf{we}) - \frac{1}{2}\mathrm{tr}\big(X(0,\mathbf{ze})X(0,\mathbf{we})E\big) \\ X(0,\mathbf{ze}) \times X(0,\mathbf{we}) &\leftrightarrow -\frac{1}{2}\big(W^*Z + Z^*W + \overline{Z} \times \overline{W}\big) \end{aligned} \tag{95}$$

where $Z \times W = (\epsilon_{rst}(\mathbf{z}_s \times \mathbf{w}_t)^j)$, so that

$$\begin{aligned} (X(\xi,a) + Z) \times (X(\eta,b) + W) &= X(\zeta,c) + V \\ X(\zeta,c) &= X(\xi,a) \times X(\eta,b) - \frac{1}{2}(Z^*W - W^*Z) \\ V &= -\frac{1}{2}\left(WX(\xi,a) + ZX(\eta,b) + \overline{Z} \times \overline{W}\right). \end{aligned} \tag{96}$$

Thus, if we set $V = (v_r^j)$ we would have

$$\begin{aligned} 2\mathbf{v_1} &= -\xi_1\mathbf{w_1} - \overline{a}_3\mathbf{w_2} - a_2\mathbf{w_3} - \overline{\mathbf{z}}_\mathbf{2} \times \overline{\mathbf{w}}_\mathbf{3} \\ 2\mathbf{v_2} &= -a_3\mathbf{w_1} - \xi_2\mathbf{w_2} - \overline{a}_1\mathbf{w_3} - \overline{\mathbf{z}}_\mathbf{3} \times \overline{\mathbf{w}}_\mathbf{1} \\ 2\mathbf{v_3} &= -\overline{a}_2\mathbf{w}_1 - a_1\mathbf{w}_2 - \xi_3\mathbf{w}_3 - \overline{\mathbf{z}}_\mathbf{1} \times \overline{\mathbf{w}}_\mathbf{2}. \end{aligned}$$

The inner product in $\mathfrak{J}$ is express in terms of the components $X(\xi,a)$ and Z as

$$\begin{aligned} (X,Y) &= \mathrm{tr}X \circ Y = \big(X(\xi,a), X(\eta,b)\big) + 2(Z,W) \\ 2(Z,W) &= \mathrm{Tr}\big(Z^*W + W^*Z\big) = 2\sum_{r=1}^{3}\sum_{j=1,2,4}(\overline{z}_r^j w_r^j + \overline{w}_r^j z_r^j). \end{aligned} \tag{97}$$

In the applications to the Standard Model of particle physics the (upper) index j of z ($j = 1,2,4$) labels quark's colour while $r \in \{1,2,3\}$ is a *family* (or flavour) index. The SU(3) subgroup of G_2, displayed in Section 2 acting on individual (imaginary) octonions is the colour group.

The subgroup F_4^ω (cf. (90)) is defined as the commutant of the automorphism ω of order three in F_4

$$\begin{aligned} \omega X(\xi,x) &= \begin{pmatrix} \xi_1 & \omega x_3 & (\omega x_2)^* \\ (\omega x_3)^* & \xi_2 & \omega x_1 \\ \omega x_2 & (\omega x_1)^* & \xi_3 \end{pmatrix}, \quad \omega(a + \mathbf{ze}) = a + \omega_7\mathbf{ze} \\ \omega(X(\xi,a) + Z) &= X(\xi,a) + \omega_7 Z, \quad \omega_7 = -\frac{1}{2} + \frac{\sqrt{3}}{2}e_7, \qquad \omega_7^3 = 1 = \omega^3. \end{aligned} \tag{98}$$

We leave to the reader to verify that the restriction of ω to $\mathbb{O}$, given by

$$\omega x = \omega(a + \mathbf{ze}) = a + \omega_7\mathbf{ze}$$

is an automorphism of $\mathbb{O}$ and that its commutant in G_2 is $\mathrm{SU}(3)$. (One uses, in particular, the relation $\omega_7 \mathbf{z e} \omega_7 = \omega_7 \overline{\omega}_7 \mathbf{z e} = \mathbf{z e}$). The automorphisms $\mathfrak{g} \in \mathrm{F}_4^{\omega}$ that commute with ω (98) are given by pairs $g = (A, U) \in \mathrm{SU}(3) \times \mathrm{SU}(3)$ acting on $\mathcal{H}_3(\mathbb{O})$ by

$$(A, U)\,(X(\xi, a) + Z) = AX(\xi, a)A^* + UZA^*. \tag{99}$$

The central subgroup

$$\mathbb{Z}_3 = \{(1,1), (\omega_7, \omega_7), (\omega_7^2, \omega_7^2)\} \in \mathrm{SU}(3) \times \mathrm{SU}(3) \tag{100}$$

acts trivially on $\mathcal{H}_3(\mathbb{O})$. We see that unitary matrix U acts (in (98)) on the colour index j and hence belongs to the (unbroken) *colour group* $\mathrm{SU}(3)_{\mathrm{c}}$, while A spans the (badly broken) family symmetry.

4.3. The Jordan Subalgebra JSpin_9 of $\mathcal{H}_3(\mathbb{O})$ and its Automorphism Group $\mathrm{Spin}(9) \subset \mathrm{F}_4$

The ten dimensional Jordan algebra JSpin_9 can be identified with the algebra of 2×2 hermitian octonionic matrices $\mathcal{H}_2(\mathbb{O})$ equipped with the Jordan matrix product. It is generated by the 9-dimensional vector subspace $s\mathcal{H}_2(\mathbb{O})$ of traceless matrices of $\mathcal{H}_2(\mathbb{O})$ whose square is, in fact, a positive real scalar

$$X = \begin{pmatrix} \xi & x \\ x^* & -\xi \end{pmatrix} \Rightarrow X^2 = (\xi^2 + x^* x)\mathbb{1}, \qquad x \in \mathbb{O}, \qquad \xi \in \mathbb{R}. \tag{101}$$

JSpin_9 is a (special) Jordan subalgebra of the (associative) matrix algebra $\mathbb{R}[2^4]$ that provides an IR of Cliff_9. Clearly, it is a subalgebra of $\mathcal{H}_3(\mathbb{O})$-consisting of 3×3 matrices with vanishing first row and first column. Its automorphism group is the subgroup $\mathrm{Spin}(9) \subset \mathrm{F}_4$ which stabilizes the projector E_1: $\mathrm{Spin}(9) = (\mathrm{F}_4)_{\mathrm{E}_1} \subset \mathrm{F}_4$.

With our interpretation of $\mathfrak{J} = \mathcal{H}_3(\mathbb{O})$ as (possibly part of) the finite quantum algebra of the Standard Model of particle physics JSpin_9 is the subalgebra corresponding to the first generation of (left chiral) quarks and leptons

$$\begin{pmatrix} \nu_L & u_L^j \\ e_L^- & d_L^j \end{pmatrix}, \qquad j \quad \text{is the colour index.}$$

They are physically distinguished as being much lighter than the particles in the second and third generation and therefore the relevant ones for the low energy physics. This justifies a more detailed study of the Jordan spin factor JSpin_9 and its symmetry group.

To begin with we shall interpret the 16-dimensional (real) spinor representation S_9 of $\mathrm{Spin}(9)$ as describing the two four-dimensional *complex* representations of the first generation of particles. They correspond to the splitting of S_9 into two 8-dimensional spinor representations $S_8^\uparrow$ and $S_8^\downarrow$ of $\mathrm{Spin}(8)$ that appear as eigenvectors of the Coxeter element ω_8 of Cl_8 (see Table 1)

$$S_9 = S_8^\uparrow \oplus S_8^\downarrow, \qquad (\omega_8 - 1)S_8^\uparrow = 0 = (\omega_8 + 1)S_8^\downarrow. \tag{102}$$

In fact, Cl_9 is isomorphic (as ungraded algebra) to the direct sum of two 16×16 matrix algebras

$$Cl_9 \approxeq \mathbb{R}[2^4] \oplus \mathbb{R}[2^4]. \tag{103}$$

Each of the irreducible components is spanned by Cliff_8. Here is a (real) basis of Cliff_8 Γ -matrices with diagonal ω_8

$$\begin{aligned}
&\Gamma_0 = \sigma_1 \otimes P_0, && \Gamma_a = c \otimes P_a, && P_0 = 1\!\!1_8 = 1\!\!1 \otimes 1\!\!1 \otimes 1\!\!1 \\
&P_1 = 1\!\!1 \otimes c \otimes \sigma_3, && P_2 = c^* \otimes \sigma_3 \otimes \sigma_3, && P_4 = \sigma_1 \otimes c^* \otimes \sigma_1 \\
&P_3 = \sigma_3 \otimes c^* \otimes \sigma_1, && P_6 = c \otimes 1\!\!1 \otimes \sigma_1, && P_5 = c^* \otimes \sigma_1 \otimes \sigma_3 \\
&P_7 = 1\!\!1 \otimes 1\!\!1 \otimes c^*, && a = 1, \dots, 7 &&
\end{aligned} \tag{104}$$

$$\Gamma_8 \equiv \omega_8 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7 = \sigma_3 \otimes 1\!\!1 \otimes 1\!\!1 \otimes 1\!\!1 = \sigma_3 \otimes \omega_{-7}$$

$$c^* = -c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{105}$$

(Each factor $1\!\!1$ in (104) is a 2×2 matrix.) The $\binom{9}{2} = 36$ generators of the Lie algebra $\mathrm{Spin}(9)$ can be chosen as the commutators of these matrices. The maximal Lie subalgebra $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$ of $\mathrm{Spin}(9)$ is spanned by

$$\begin{aligned}
&\mathfrak{su}(4) : \{\Gamma_{ab} = \frac{1}{2}\left[\Gamma_a, \Gamma_b\right] = \Gamma_a\Gamma_b \quad \text{for} \quad a < b, \quad a, b = 1, \dots, 6\} \\
&\mathrm{i}\mathfrak{su}(2) : \quad 2I_1 = \mathrm{i}\Gamma_7\omega_8 = \sigma_1 \otimes 1\!\!1 \otimes 1\!\!1 \otimes \sigma_2, \; 2I_2 = \mathrm{i}\Gamma_0\omega_8 = \sigma_2 \otimes 1\!\!1 \otimes 1\!\!1 \otimes 1\!\!1 \\
&\qquad 2I_3 = \mathrm{i}\Gamma_0\Gamma_7 = \sigma_3 \otimes 1\!\!1 \otimes 1\!\!1 \otimes \sigma_2, \quad \left[I_1, I_2\right] = \mathrm{i}I_3, \quad \left[I_j, \Gamma_{ab}\right] = 0.
\end{aligned} \tag{106}$$

The general element X (101) of the space $s\mathcal{H}_2(\mathbb{O})$ is written as

$$X = \xi\omega_8 + x^a\Gamma_a = (\xi\sigma_3 + x^0\sigma_1) \otimes P_0 + c \otimes \sum_{a=1}^{7} x^a P_a. \tag{107}$$

The $8 \otimes 8$ matrices P_a are an 8-dimensional counterpart of the $2 \otimes 2$ matrix realization of the quaternion units $1\!\!1$, $q_j = -\mathrm{i}\sigma_j$. They are characterized by similar anticommutation relations and product of imaginary units

$$\begin{aligned}
P_a P_b^* + P_b P_a^* &= 2\delta_{ab}, \qquad a, b = 0, 1, \dots, 7 \\
P_{\bar{a}}^* &= -P_{\bar{a}}, \qquad \bar{a} = 1, \dots, 7, \qquad P_1 \dots P_7 = \omega_{-7} = 1
\end{aligned} \tag{108}$$

while the 28 skew symmetric matrices $\frac{1}{2}(P_a P_b^* - P_b P_a^*)$ span the Lie algebra $\mathfrak{so}(8)$.

We stress that the map $e_a \to P_a$ (unlike the representation $q_j = -\mathrm{i}\sigma_j$) only respects the Jordan products (i.e. the anticcommutators) of the octonion units, not their commutators. It could not have been otherwise as the P_a belong to the 64-dimensional associative algebra of real 8×8 matrices which can be identified (due to the last relation (108)) with the Clifford algebra Cliff_{-6} .

The correspondence between octonions $x = x^a e_a$ and 8×8 matrices $\hat{x} = x^a P_a$ is norm preserving

$$x = x^a e_a \quad \longleftrightarrow \quad \hat{x} = x^a P_a \qquad \Longrightarrow \qquad xx^* = N(x) = \hat{x}\hat{x^*}. \tag{109}$$

The norm preserving action of the Lie algebra $\mathfrak{so}(8)$ on x given by the operators G_{ab} (51) corresponds to commutation with $\frac{1}{2}\hat{G}_{ab}$ where

$$\begin{aligned} \hat{G}_{ab} &= \frac{1}{2}(P_a P_b^* - P_b P_a^*), \qquad \text{i.e.,} \qquad \hat{G}_{a0} = P_a = -\hat{G}_{0a} \\ \hat{G}_{ab} &= -P_{ab} = P_{ba} \qquad (= -\hat{G}_{ba}) \qquad \text{for} \qquad 0 < a < b. \end{aligned} \tag{110}$$

We have

$$\frac{1}{2}[\hat{G}_{ab}, \hat{x}] = P_a x_b - x_a P_b \quad \longleftrightarrow \quad G_{ab} x = e_a x_b - x_a e_b. \tag{111}$$

From now on we shall omit the hat on $\hat{G}_{ab}$ and will identify it with G_{ab}. The Lie subalgebra $\mathfrak{su}(4) = \mathfrak{so}(6)$ is spanned by G_{ab}, $a, b = 1, \ldots, 6$. Its commutant in $\mathfrak{so}(8)$ consists of the multiples of the $\mathrm{U}(1)$ generator G_{07}, that corresponds to the third component of the weak isospin I_3 (106). In order to reveal the physical meaning of the $\mathfrak{su}(4)$ Lie algebra we shall identify its colour $\mathfrak{su}(3)$ subalgebra which belongs to $\mathfrak{g}_2 \subset \mathfrak{so}(7)$. According to the results of Section 2.3 it is spanned by

$$\begin{aligned} -\mathrm{i}H_{12} &= G_{13} - G_{26}, \qquad -\mathrm{i}H_{24} = G_{26} - G_{45} \\ L_{12} &= G_{12} + G_{36}, \qquad L_{24} = G_{24} + G_{65}, \qquad L_{14} = G_{14} + G_{35} \\ N_{12} &= G_{16} + G_{23}, \qquad N_{24} = G_{25} + G_{46}, \qquad N_{14} = G_{15} + G_{43}. \end{aligned} \tag{112}$$

The commutant $\mathfrak{u}(1)$ of $\mathfrak{su}(3)$ in $\mathfrak{su}(4)$ is spanned by the Cartan element

$$\mathrm{i}(h_1 + h_2 + h_3) = G_{13} + G_{26} + G_{45} \tag{113}$$

whose physical meaning will be made clear shortly. The operators G_{13}, G_{26} and G_{45} form a basis of the Cartan subalgebra of $\mathfrak{so}(6)$

$$\begin{aligned} G_{13} &= -P_1 P_3 = \sigma_3 \otimes 1\!\!1 \otimes c^*, \qquad G_{26} = 1\!\!1 \otimes \sigma_3 \otimes c^* \\ G_{45} &= \sigma_3 \otimes \sigma_3 \otimes c^* \implies G_{13} G_{26} G_{45} = \omega_6 = 1\!\!1 \otimes 1\!\!1 \otimes c = P_7^* \\ w_{-6} &= P_1 P_2 P_3 P_4 P_5 P_6 = P_1 P_3^* P_2 P_6^* P_4 P_5^*. \end{aligned} \tag{114}$$

The equation $\omega_{-6} = P_7^*$ and $\omega_{-7} = \omega_{-6} P_7$) imply the last equation (108). The resulting real 8-dimensional representation of $\mathfrak{su}(4)$ (given by the matrices $\hat{G}_{ab} = G_{ab}$ (110) for $1 \le a, b \le 6$) is equivalent to a complex four-dimensional representation which can be obtained from (110) by identifying the matrix $P_7 = 1\!\!1 \otimes 1\!\!1 \otimes c^*$ (which commutes with the above G_{ab}) with the imaginary unit i. For the Cartan generators (115) we obtain, in particular

$$\begin{aligned} P_7 \to \mathrm{i} \Longrightarrow G_{ab} \to \gamma_{ab}, \qquad \gamma_{13} = \mathrm{i}\sigma_3 \otimes 1\!\!1, \qquad \gamma_{26} = \mathrm{i}1\!\!1 \otimes \sigma_3 \\ \gamma_{45} \;=\; \mathrm{i}\sigma_3 \otimes \sigma_3 \qquad \longrightarrow \qquad \gamma_{13}\gamma_{26}\gamma_{45} = -\mathrm{i}. \end{aligned} \tag{115}$$

We shall go instead in the opposite direction: diagonalizing the 16-dimensional hermitian Cartan matrices $h_j = \mathrm{i}\Gamma_{j3j}$, $j = 1, 2, 4$, in a complexified basis in which h_j become real diagonal. To this end we use the unitary similarity transformation

$$\Gamma \to S\Gamma_a S^*, \qquad S = 1\!\!1 \otimes 1\!\!1 \otimes 1\!\!1 \otimes \frac{1 + \mathrm{i}\sigma_1}{\sqrt{2}} \quad \Rightarrow \quad SS^* = 1\!\!1_{16} \tag{116}$$

with the result

$$\begin{aligned} h_1 :&= S\mathrm{i}\Gamma_{13}S^* = -1\!\!1 \otimes \sigma_3 \otimes 1\!\!1 \otimes \sigma_3 \\ h_2 :&= S\mathrm{i}\Gamma_{26}S^* = -1\!\!1 \otimes 1\!\!1 \otimes \sigma_3 \otimes \sigma_3 \\ h_3 :&= S\mathrm{i}\Gamma_{45}S^* = -1\!\!1 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3. \end{aligned} \tag{117}$$

Inserting the matrices in the tensor products, so that

$$1\!\!1 \otimes \sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \sigma_3 \otimes 1\!\!1 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}, \quad \sigma_3 \otimes \sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}$$

we find for the sum (113) the product of the 2×2 unit matrix with a 4×4 diagonal matrix with 2×2 diagonal bloks

$$h_1 + h_2 + h_4 = 1\!\!1 \otimes \begin{pmatrix} -3\sigma_3 & 0 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_3 \end{pmatrix}. \tag{118}$$

4.4. The Structure Group $\mathrm{Spin}(9,1)$ of JSpin_9 and its 32-Dimensional Dirac Spinor Representation

The spectrum of the operator $h_1 + h_2 + h_4$ consisting of four eigenvalues ± 3 and ± 1 corresponds to the odd part of the operator $3Y$, where Y is the weak hypercharge acting on the 32-dimensional space of fundamental fermions and antifermions of the Standard Model (SM). This motivates us to go one step further, to the *structure group* $\mathrm{Spin}(9,1)$ of the Jordan spin factor JSpin_9 which, by definition, preserves the determinant

$$\det\begin{pmatrix}\xi_1 & x\\ x^* & \xi_2\end{pmatrix} = \xi_1\xi_2 - xx^*, \qquad \xi_{1,2}\in\mathbb{R}, x\in\mathbb{O}. \tag{119}$$

Its Lie algebra is spanned by the 32×32 matrices

$$\begin{aligned} T_{ab} &= \frac{1}{2}[T_a, T_b] \qquad \text{for} \qquad a,b = -1,0,1,\dots,8\\ T_a &= \sigma_1\otimes S\Gamma_a S^*, \quad a = 0,1,\dots,8, \quad \Gamma_8 = \omega_8, \quad T_{-1} = c\otimes \mathbb{1}_{16}\end{aligned} \tag{120}$$

with T_a being the generators of $\mathrm{Cliff}(9,1)$. It is another real form of the complexification of $\mathfrak{so}(10)$, the ultimate "grand unified" Lie algebra (see [4] for a pedagogical review and references). The (complexified) Cartan subalgebra of $\mathfrak{so}(9,1)$ is spanned by five diagonal matrices

$$\begin{aligned} H_j &= \mathbb{1}\otimes h_j = \mathrm{i}T_{j3j}, \qquad j = 1,2,4, \qquad H_0 = \mathrm{i}T_{07}\\ H_8 &= T_{-18} = \sigma_3\otimes\sigma_3\otimes\mathbb{1}_8\\ \gamma &= H_0H_1H_2H_4H_8 = -\sigma_3\otimes\mathbb{1}_{16} = -\omega_{9,1}.\end{aligned} \tag{121}$$

Here γ is the *parity operator* taking value 1 for the left chiral fermions and -1 for the right chiral fermions. The fundamental fermions are labeled by three quantum numbers: the parity γ, the third component I_3 of the weak isospin and the weak hypercharge Y defined as eigenvalue of the corresponding operators

$$\begin{aligned} 2I_3 &= \frac{1}{2}(H_8 - H_0) = \mathrm{diag}\{(\sigma_3^-)^{\times 4}, (-\sigma_3^-)^{\times 4}, (-\sigma_3^+)^{\times 4}, (\sigma_3^+)^{\times 4}\}\\ \sigma_3^- &= \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \qquad \sigma_3^+ = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\end{aligned} \tag{122}$$

where $\sigma^{\times 4}$ stands for a 4×4 block diagonal matrix with 2×2 matrix blocks σ on the diagonal, and

$$\begin{aligned} 3Y &= \frac{3}{2}(H_8 + H_0) + H_1 + H_2 + H_4 = \begin{pmatrix}A & 0\\ 0 & \tilde{A}\end{pmatrix}\\ A &= \mathrm{diag}\{\begin{pmatrix}0 & 0\\ 0 & 3\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & -1\end{pmatrix}^{\times 3}, \begin{pmatrix}-6 & 0\\ 0 & 3\end{pmatrix}, \begin{pmatrix}-2 & 0\\ 0 & -1\end{pmatrix}^{\times 3}\}\\ \tilde{A} &= \mathrm{diag}\{\begin{pmatrix}-3 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -4\end{pmatrix}^{\times 3}, \begin{pmatrix}-3 & 0\\ 0 & 6\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}^{\times 3}\}.\end{aligned} \tag{123}$$

The 16 fundamental particles of negative parity ($\gamma = -1$) corresponding to right chiral fields, are characterized by the pairs of eigenvalues $(2I_3, 3Y)$ as follows

$$\begin{aligned} &\nu_R \backsim (0,0), \quad e_R^+ \backsim (1,3), \quad u_R \backsim (0,4), \quad \overline{u}_R \backsim (-1,-1) \\ &\overline{d}_R \backsim (1,-1), \quad e_R^- \backsim (0,-6), \quad \overline{\nu}_R \backsim (-1,3), \quad d_R \backsim (0,-2). \end{aligned} \tag{124}$$

The corresponding left chiral fermions ($\gamma = 1$) are obtained from (124) by going to antiparticles and changing the sign of the quantum numbers

$$\begin{aligned} &\overline{\nu}_L \backsim (0,0), \quad e_L^- \backsim (-1,-3), \quad \overline{u}_L \backsim (0,-4), \quad u_L \backsim (1,1) \\ &d_L \backsim (-1,1), \quad e_L^+ \backsim (0,6), \quad \nu_L \backsim (1,-3), \quad \overline{d}_L \backsim (0,2). \end{aligned} \tag{125}$$

The projection operators to this states are polynomial functions of the Cartan elements. The electric charge Q is related to Y by

$$Q = I_3 + \frac{1}{2}Y \quad \Rightarrow \quad 2Q - Y \in \mathbb{Z}, \qquad Q + Y \in \mathbb{Z}. \tag{126}$$

We see that the 32-dimensional Dirac spinor splits into two 16-dimensional left and right Weyl spinors which transform under irreducible representations of $\mathrm{Spin}(9,1)$. The splitting between left and right (or, equivalently, between even and odd element of the Clifford algebra $\mathrm{Cliff}(5,\mathbb{C})$) thus appears to be more relevant also mathematically than the difference between particles and antiparticles.

5. The Symmetry Algebra of the Standard Model

It has been observed by Baez and Huerta [4] that the gauge group of the SM

$$S(\mathrm{U}(2) \times \mathrm{SU}(3)) = \frac{\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{U}(1)}{\mathbb{Z}_6} \tag{127}$$

can be obtained as the intersection of the Georgi-Glashow and Pati-Salam *grand unified theory groups* $\mathrm{SU}(5)$ and $(\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$ viewed as subgroup of $\mathrm{Spin}(10)$. It was suggested in [30] that one can deduce the symmetry of the SM by applying the Borel-de Siebenthal theory to the automorphism group F_4 of the exceptional Jordan algebra J.

Here we shall consider instead maximal rank subalgebras of the complexified Lie algebra $\mathbb{C} \otimes \mathfrak{so}(9,1)$ of the structure group $\mathrm{Spin}(9,1)$ of JSpin_9 which commute with the exact symmetry Lie algebra

$$\mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_Q, \qquad Q = I_3 + \frac{1}{2}Y. \tag{128}$$

The resulting *superselected observables* (in the terminology introduced in [31]) give rise to a three dimensional abelian algebra spanned by the operators $2I_3, 3Y$ and

$$H = H_1 + H_2 + H_4 = \mathbb{1} \otimes (h_1 + h_2 + h_4) = \mathbb{1}_4 \otimes \begin{pmatrix} -3\sigma_3 & 0 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_3 \end{pmatrix} \quad (129)$$

(cf. (118)), which are all traceless with integer eigenvalues. In order to reveal the physical meaning of H we shall recall the realization of the 32-dimensional fermionic Fock space $\mathcal{F}$ as the exterior algebra

$$\Lambda\mathbb{C}^5 = \bigoplus_{\nu=0}^{5} \Lambda^\nu, \qquad \Lambda^\nu = \Lambda^\nu\mathbb{C}^5 \quad (130)$$

where

$$\Lambda^0 = \overline{\nu}_L, \qquad \Lambda^1 = e_R^+, \overline{\nu}_R, \qquad d_R = (d_R^c) \quad (131)$$

see [4, Section 2].

More generally, the 2^n-dimensional vector space of the Clifford algebra $\mathrm{Cliff}(\mathrm{n}, \mathbb{C})$ is isomorphic to the exterior algebra $\Lambda\mathbb{C}^n$. If $(e_1, \ldots, e_n)$ is basis in $\Lambda^1\mathbb{C}^n$ we can turn $\Lambda\mathbb{C}^n$ into a Hilbert space by introducing the orthonormal basis

$$1, e_1, \ldots, \quad e_n, \quad e_i \wedge e_j, \quad 1 \le i < j \le n, \ldots, \quad e_1 \wedge \cdots \wedge e_n. \quad (132)$$

There is an equivalent fermionic Fock space realization of $\Lambda\mathbb{C}^n \simeq \mathcal{F}$ in terms of n creation and n annihilation operators a_i^* and a_j acting on $\mathcal{F}$ such that

$$a_i^*\Psi = e_i \wedge \Psi, \qquad \langle a_i\Phi, \Psi\rangle = \langle \Phi, a_i^*\Psi\rangle \qquad \Phi, \Psi \in \mathcal{F}. \quad (133)$$

It follows from (133) that (a_i, a_j^*) satisfy the *canonical anticommutation onrelations*

$$[a_i, a_j]_+ = 0 = [a_i^*, a_j^*]_+, \qquad [a_i, a_j^*]_+ = \delta_{ij}. \quad (134)$$

Furthermore, if we identify the "vector" $1 \in \Lambda^0\mathbb{C}$ with the Fock space vacuum $|0\rangle$ we shall have

$$a_i|0\rangle = 0, \qquad a_j^*|0\rangle = e_j, \qquad a_i^*a_j^*|0\rangle = e_i \wedge e_j, \quad \text{etc.} \quad (135)$$

Returning to the case $n = 5$ with the identification (131) we introduce the operators $r^* = (r_+^*, r_0^*), d^* = (d_c^*)$ which create the states in Λ^1

$$r_+^*|0\rangle = e_R^+, \qquad r_0^*|0\rangle = \overline{\nu}_R, \qquad d^*|0\rangle = d_R, \qquad |0\rangle = \overline{\nu}_L \quad (136)$$

with quantum numbers $(2I_3, 3Y, H) = (1,3,3), (-1,3,3), (0,-2,1)^{\times 3}$, respectively (and their annihilation counterparts r, d). The eigenvalues $(3,3,1)^{\times 3}$ of H are the same as those of the operator

$$H = [d, d^*] = 3 - 2d^*d \tag{137}$$

(where we skip the sum sign over the colour index in d^*d). We note that H is i-times a generator of $\mathfrak{so}(6)(\subset \mathfrak{so}(9))$ but not of $\mathfrak{su}(5)$, so the sum of its eigenvalues in Λ^1 does not vanish but its trace on the entire right and left sectors, separately, is zero

$$(\mathrm{tr}H)_{\mathrm{odd}} = \mathrm{tr}_{\Lambda^1} H + \mathrm{tr}_{\Lambda^3} H + \mathrm{tr}_{\Lambda^5} H = 9 - 6 - 3 = 0. \tag{138}$$

Furthermore, in the 16-dimensional subspace $\mathcal{F}_1$ of $\mathcal{F}$ spanned by 8 isotopic doublets

$$\begin{pmatrix} \nu_L \\ e_L^- \end{pmatrix}, \qquad \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad \begin{pmatrix} e_R^+ \\ \overline{\nu}_R \end{pmatrix}, \qquad \begin{pmatrix} \overline{d}_R \\ \overline{u}_R \end{pmatrix}$$

in which $3Y$ and $2I_3$ have odd eigenvalues and actually belong to $\mathfrak{so}(8)$

$$(-1)^{3Y} = (-1)^{2I_3} = -1, \quad 3Y = H, \quad 2I_3 = \mathrm{i}\Gamma_{07},\ Y = B - L \ \in \mathcal{F}_1 \tag{139}$$

where B and L are the baryon and lepton numbers. The space $\mathcal{F}_1$ in fact coincides with the spinor representation of $\mathrm{Spin}(9)$. The eight fundamental left chiral particles and their (right chiral) antiparticles that take part in the weak interaction span (inequivalent) 8-dimensional irreducible representations of $\mathrm{Spin}(8)$. All (anti)particles in the doublets are characterized by the pairs of eigenvalues $(2I_3, 3Y)$ (cf. (124) and (125))

$$\begin{array}{llll} \nu_L \backsim (1,-3), & e_L^- \backsim (-1,-3), & u_L \backsim (1,1), & d_L \backsim (-1,1) \\ e_R^+ \backsim (1,3), & \overline{\nu}_R \backsim (-1,3), & \overline{d}_R \backsim (1,-1), & \overline{u}_R \backsim (-1,-1). \end{array} \tag{140}$$

6. Outlook

The idea that exceptional structures in mathematics should characterize the fundamental constituents of matter has been with us since the ancient Greeks first contemplated the Platonic solids. The octonions, the elements of the ultimate division algebra, have been linked to the Standard Model of particle physics ever since

Günaydin and Gürsey related them to the coloured quarks around 1973. When the idea of a finite quantum geometry emerged [9, 15] it became natural to look for a role of the exceptional algebraic structures in such a context. A promising step in this direction was made by Dubois-Violette [14] who pointed out that the exceptional Jordan algebras $\mathfrak{J} = \mathcal{H}_3(\mathbb{O})$ with its three octonions and three real elements offers room to the three families of quarks and leptons along with three Majorana neutrinos.

The next step [30], continued in the present notes, puts more emphasis on the automorphism (and isometry) groups of the algebraic structures. We identify such basic observables as the weak hypercharge and isospin as elements of the subalgebra $\mathfrak{so}(9,1)$ of the Lie algebra of the structure group $\mathrm{E}_{6(-26)}$ that preserves the determinant of the elements of $\mathfrak{J}$.

In fact, the exceptional Jordan algebra is intimately related to all exceptional Lie groups [5]. It will be interesting to reveal further the role of the structure group $\mathrm{E}_{6(-26)}$ and the conformal group $\mathrm{E}_{7(-25)}$ of $\mathfrak{J}$ in the physics of the Standard Model. We intend to return to this problem in future work.

Intriguingly, the basic doublets that participate in the weak interaction fit in the 16-dimensional spinor representation of the subgroup $\mathrm{Spin}(9)$ of the (compact) automorphism group F_4 of the exceptional Jordan algebra which in turn splits into two 8-dimensional $\mathrm{Spin}(8)$ spinors corresponding to the eight fundamental particles and to their antiparticles, respectively.

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Application A. The Fano Plane of Imaginary Octonions

The multiplication table for the seven octonionic imaginary units can be recovered from the following properties

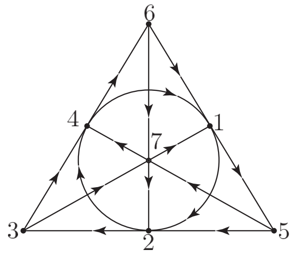
$$e_i^2 = -1\,, \quad i = 1, \ldots, 7, \quad e_i\, e_j = -e_j\, e_i \tag{A.1}$$

$$e_i\, e_j = e_k \Rightarrow e_{i+1}\, e_{j+1} = e_{k+1}\,, \quad e_{2i}\, e_{2j} = e_{2k} \tag{A.2}$$

where indices are counted modulo seven, and a single relation of the type

$$e_1\, e_2 = e_4 \tag{A.3}$$

producing a quaternionic line. We have displayed on Fig. 1 the points e_i as non-zero triples of homogeneous coordinates taking values 0 and 1 such that the product $e_i\, e_j$ (in clockwise order) is obtained by adding the coordinates (a,b,c), $a,b,c \in \{0,1\}$, modulo two.



$$e_1 = (0,0,1), e_2 = (0,1,0) \Longrightarrow e_1 e_2 = e_4 = (0,1,1)$$

$$e_3 = (1,0,0) \Longrightarrow e_2\, e_3 = e_5 = (1,1,0)$$

$$e_1\, e_5 = e_6 = (1,1,1), \quad e_4\, e_5 = e_7 = (1,0,1).$$

Figure 1. Projective plane in $\mathbb{Z}_2^3$ with seven points and seven lines.

Application B.

Two bases of $\mathfrak{so}(8)$ related by the outer automorphism π. The generators G_{ab} of $\mathfrak{so}(8)$ are given directly by their action on the octonion units (51)

$$G_{ab} e_b = e_a, \qquad G_{ab} e_a = -e_b, \qquad G_{ab} e_c = 0 \qquad \text{for} \qquad a \neq c \neq b. \quad \text{(B.1)}$$

The action of F_{ab} can also be deduced from definition (51) and multiplication rules:

$$\begin{gathered} F_{ab} e_b = \frac{1}{2} e_a, \qquad F_{ab} e_a = -\frac{1}{2} e_b, \qquad (a \neq b) F_{ab} = -F_{ba} \\ F_{j0} e_{2j} = \frac{1}{2} e_{4j\mathrm{mod}7} (= -F_{0j} e_{2j}), \qquad j = 1,2,4 \\ F_{j0} e_{3j} = \frac{1}{2} e_7, \qquad F_{70} e_7 = \frac{1}{2} e_{3j} \qquad F_{07} = \frac{1}{2} e_j \\ F_{0j} e_{6j} = \frac{1}{2} e_{5j}, \qquad F_{j0} e_{5j} = \frac{1}{2} e_{6j}, \qquad [F_{j0}, F_{0k}] = F_{jk}. \end{gathered} \qquad \text{(B.2)}$$

(All indices are counted $\mathrm{mod}7$.) From (B.1) and (B.2) we find

$$\begin{aligned} 2F_{0j} &= G_{0j} + G_{2j4j} + G_{3j7} + G_{5j6j} \\ 2F_{03j} &= G_{03j} - G_{J7} - G_{2j5j} + G_{4j6j}, \qquad j = 1,2,4 \\ 2F_{07} &= G_{07} + G_{13} + G_{26} + G_{45}. \end{aligned} \tag{B.3}$$

In particular, taking the show symmetry of G_{ab} and the counting $\mathrm{mod}7$ into account we can write

$$\begin{aligned} 2F_{02} &= G_{02} - G_{14} + G_{35} - G_{76} \\ 2F_{04} &= G_{04} + G_{12} - G_{36} - G_{75} \\ 2F_{03} &= G_{03} - G_{17} - G_{25} + G_{46} \\ 2F_{06} &= G_{06} + G_{15} - G_{27} - G_{43} \\ 2F_{05} &= G_{05} - G_{16} + G_{23} - G_{47}. \end{aligned} \tag{B.4}$$

Note that with the abc are ordering $(1,2,4,3,7,5,6)$ The first (positive) indices of G $(2,3,5;1,3,7;1,2,4)$ correspond to quaternionic triples:$e_2e_3 = e_5$, $e_1e_3 = e_7$, $e_1e_2 = e_4$. Setting

$$G_1 = \begin{pmatrix} G_{01} \\ G_{24} \\ G_{37} \\ G_{56} \end{pmatrix}, \quad G_2 = \begin{pmatrix} G_{02} \\ G_{14} \\ G_{35} \\ G_{76} \end{pmatrix}, \quad G_4 = \begin{pmatrix} G_{04} \\ G_{12} \\ G_{36} \\ G_{75} \end{pmatrix}, \quad G_3 = \begin{pmatrix} G_{03} \\ G_{17} \\ G_{25} \\ G_{46} \end{pmatrix}$$

$$G_7 = \begin{pmatrix} G_{07} \\ G_{13} \\ G_{26} \\ G_{45} \end{pmatrix}, \quad G_5 = \begin{pmatrix} G_{05} \\ G_{16} \\ G_{23} \\ G_{47} \end{pmatrix}, \quad G_6 = \begin{pmatrix} G_{06} \\ G_{15} \\ G_{27} \\ G_{43} \end{pmatrix}$$

and similarly for $F_1, \ldots F_6$ we find

$$F_a = X_a G_a, \qquad a = 1, \ldots, 7, \qquad \text{with}$$

$$X_1 = X_7 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad X_2 = X_5 = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$X_4 = X_6 = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \qquad X_3 = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

They all define involutive transformations

$$X_k^2 = \mathbb{1}, \qquad \det X_k = -1, \qquad k = 1, 2, 3, 4. \tag{B.5}$$

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