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# GEODESICS ON ROTATIONAL SURFACES IN PSEUDO-GALILEAN SPACE

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**Abstract.** In this paper, we study rotational surfaces in the pseudo-Galilean three-space $\mathbb{G}_3^1$ with pseudo-Euclidean rotations and isotropic rotations. In particular, we investigate properties of geodesics on rotational surfaces in $\mathbb{G}_3^1$ and give some examples.

*MSC*: 53A35, 53B30
*Keywords*: Geodesic, pseudo-Galilean space, rotational surface

## 1. Introduction

Geodesics are curves in surfaces that plays a role analogous to straight lines in the plane. Geometrically, a geodesic in a surface is an embedded simple curve such that the portion of the curve between any two points is the shortest curve on the surface. A geodesic can be obtained as the solutions of the non-linear system of the second order ordinary differential equations (the Euler-Lagrange equations) with the given points and its tangent direction for the initial conditions. It is well-known that great circles are geodesics on a sphere. Also, meridians (lines), parallels (circles) and helices are geodesics on a circular cylinder. For more details about geodesics and some relative topics in Euclidean space, Minkowski space or simple isotropic space we refer to [1–3] and [5]. In this paper, we study geodesics on rotational surfaces in the pseudo-Galilean three-space.

## 2. Preliminaries

In 1872, F. Klein in his Erlangen program proposed how to classify and characterize geometries on the basis of projective geometry and group theory. He showed that the Euclidean and non-Euclidean geometries could be considered as spaces that are invariant under a given group of transformations. The geometry motivated by this approach is called a Cayley-Klein geometry. Actually, the formal definition of Cayley-Klein geometry is pair $(G, H)$, where $G$ is a Lie group and $H$ is a closed

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Lie subgroup of $G$ such that the (left) coset $G/H$ is connected. $G/H$ is called the space of the geometry or simply Cayley-Klein geometry.

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries with projective signature $(0,0,+,-)$. The absolute of the pseudo-Galilean geometry is an ordered triple $\{\omega, f, I\}$, where $\omega$ is the ideal (absolute) plane, $f$ the line in $\omega$ and I the fixed hyperbolic involution of $f$.

Homogenous coordinates in $\mathbb{G}_3^1$ are introduced in such a way that the absolute plane $\omega$ is given by $x_0 = 0$, the absolute line $f$ by $x_0 = x_1 = 0$ and the hyperbolic involution by $(0:0:x_2:x_3) \to (0:0:x_3:x_2)$. Metric relations are introduced with respect to the absolute figure. In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the distance between the points $\mathrm{P_i} = (\mathrm{x_i}, \mathrm{y_i}, \mathrm{z_i})$ $(i = 1, 2)$ is defined by (cf. [4])

$$\mathrm{d_G}(\mathrm{P_1}, \mathrm{P_2}) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2 \\ \sqrt{(y_2-y_1)^2 + (z_2-z_1)^2}, & \text{if } x_1 = x_2. \end{cases}$$

The group motions of $\mathbb{G}_3^1$ is a six-parameter group given (in affine coordinates) by

$$\bar{x} = a + x, \quad \bar{y} = b + cx + y\cosh\varphi + z\sinh\varphi, \quad \bar{z} = d + ex + y\sinh\varphi + z\cosh\varphi.$$

A pseudo-Galilean scalar product of two vectors $\mathbf{x} = (x_1, y_1, z_1)$ and $\mathbf{y} = (x_2, y_2, z_2)$ in the pseudo-Galilean three-space $\mathbb{G}_3^1$ is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathrm{G}} = \begin{cases} x_1 x_2, & \text{if} \quad x_1 \neq 0 \quad \text{or} \quad x_2 \neq 0 \\ y_1 y_2 - z_1 z_2, & \text{if} \quad x_1 = 0 \quad \text{and} \quad x_2 = 0 \end{cases} \tag{1}$$

and a pseudo-Galilean norm of $\mathbf{x}$ is given by

$$||\mathbf{x}||_{\mathrm{G}} = \begin{cases} |x_1|, & \text{if} \quad x_1 \neq 0 \\ \sqrt{|y_1^2 - y_2^2|}, & \text{if} \quad x_1 = 0. \end{cases}$$

A vector $\mathbf{x}$ is called isotropic if $x_1 = 0$, otherwise it is called non-isotropic. All unit non-isotropic vectors are the form $(1, y_1, z_1)$. An isotropic vector $\mathbf{x} = (0, y_1, z_1)$ of $\mathbb{G}_3^1$ is said to be spacelike if $y_1^2 - z_1^2 > 0$, timelike if $y_1^2 - z_1^2 < 0$ and lightlike if $y_1^2 - z_1^2 = 0$. A non-lightlike isotropic vector is a unit vector if $y_1^2 - z_1^2 = \pm 1$.

A pseudo-Galilean cross product of $\mathbf{x}$ and $\mathbf{y}$ on $\mathbb{G}_3^1$ is defined by

$$\mathbf{x} \times_{\mathrm{G}} \mathbf{y} = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \tag{2}$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Consider a $C^r$-regular surface $\Sigma$, $r \geq 1$, in $\mathbb{G}_3^1$ parameterized by

$$\mathbf{x}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).$$

We denote by $x_{u_i}$, $y_{u_i}$ and $z_{u_i}$ the partial derivatives of the functions $x$, $y$ and $z$ with respect to $u_i$ ($i = 1, 2$), respectively. A surface in $\mathbb{G}_3^1$ is admissible if it does not have pseudo-Euclidean tangent planes, that is, $x_{u_i} \neq 0$ for some $i = 1, 2$.

On the other hand, the unit normal vector field N of a regular admissible surface $\Sigma$ is defined by

$$\mathrm{N} = \frac{1}{\omega}(0, x_{u_1} z_{u_2} - x_{u_2} z_{u_1}, x_{u_1} y_{u_2} - x_{u_2} y_{u_1})$$

where the positive function $\omega$ is given by

$$\omega = \sqrt{|(x_{u_1} z_{u_2} - x_{u_2} z_{u_1})^2 - (x_{u_1} y_{u_2} - x_{u_2} y_{u_1})^2|}.$$

We put

$$g_i = x_{u_i}, \qquad h_{ij} = \langle \tilde{\mathbf{x}}_{u_i}, \tilde{\mathbf{x}}_{u_j} \rangle_{\mathrm{G}}, \qquad i = 1, 2$$

where $\tilde{\mathbf{x}}_{u_k}$ is the projection of a vector $\mathbf{x}_{u_k}$ onto the $yz$-plane. Then, the first fundamental form of $\Sigma$ is given by

$$\mathrm{d}s^2 = (g_1 \mathrm{d}u_1 + g_2 \mathrm{d}u_2)^2 + \delta(h_{11} \mathrm{d}u_1^2 + 2h_{12} \mathrm{d}u_1 \mathrm{d}u_2 + h_{22} \mathrm{d}u_2^2)$$

where

$$\delta = \begin{cases} 0, & \text{if direction } \mathrm{d}u_1 : \mathrm{d}u_2 \text{ is nonisotropic} \\ 1, & \text{if direction } \mathrm{d}u_1 : \mathrm{d}u_2 \text{ is isotropic.} \end{cases}$$

On the other hand, the function $\omega$ can be represented in terms of $g_i$ and $h_{ij}$ as follows

$$\omega^2 = -\epsilon(g_1^2 h_{22} - 2g_1 g_2 h_{12} + g_2^2 h_{11}) > 0$$

where $\epsilon(\pm 1)$ is the sign of the unit normal vector N. A surface is spacelike if $g_1^2 h_{22} - 2g_1 g_2 h_{12} + g_2^2 h_{11} > 0$, timelike otherwise. We can notice that if a surface is spacelike, both parts of the first fundamental form, $\mathrm{d}s_1^2 = (g_1 \mathrm{d}u_1 + g_2 \mathrm{d}u_2)^2$ and

$$\mathrm{d}s_2^2 = h_{11} \mathrm{d}u_1^2 + 2h_{12} \mathrm{d}u_1 \mathrm{d}u_2 + h_{22} \mathrm{d}u_2^2 = \begin{cases} -\epsilon \frac{\omega^2}{g_1^2} \mathrm{d}u_2^2, & \text{if } g_1 \neq 0 \\ -\epsilon \frac{\omega^2}{g_2^2} \mathrm{d}u_1^2, & \text{if } g_2 \neq 0 \end{cases}$$

are positive definite, while for a timelike surface, the form $\mathrm{d}s_1^2$ is positive definite and $\mathrm{d}s_2^2$ is negative definite. Thus, the matrix of the first fundamental form $\mathrm{d}s^2$ of a surface $\Sigma$ in $\mathbb{G}_3^1$ is given by

$$\mathrm{d}s^2 = \begin{pmatrix} \mathrm{d}s_1^2 & 0 \\ 0 & \mathrm{d}s_2^2 \end{pmatrix}.$$

In such case, we denote the components of $\mathrm{d}s^2$ by $g_{ij}^*$ for $i, j = 1, 2$.

## 3. Geodesics on Rotational Surfaces

A rotational surface in the Euclidean space is generated by revolving of an arbitrary curve about an arbitrary axis. In the pseudo-Galilean space, however, there are different cases of curves (nonisotropic or isotropic) as well as different cases of rotations (pseudo-Euclidean or isotropic).

**Case 1.** The curve is nonisotropic and the rotation is pseudo-Euclidean.

Suppose that a nonisotropic curve $C$ lies in the $xy$-plane or $xz$-plane. Then the curve $C$ is given by

$$C(u) = (f(u), g(u), 0) \qquad \text{or} \qquad C(u) = (f(u), 0, g(u))$$

where $g$ is a positive function and $f$ is a smooth function on an open interval $I$.

By a pseudo-Euclidean rotation given by the normal form

$$\bar{x} = x, \qquad \bar{y} = y\cosh t + z\sinh t, \qquad \bar{z} = y\sinh t + z\cosh t$$

the rotational surface is parametrized as

$$\mathbf{x}(u,v) = (f(u), g(u)\cosh v, g(u)\sinh v) \tag{3}$$

or

$$\mathbf{x}(u,v) = (f(u), g(u)\sinh v, g(u)\cosh v) \tag{4}$$

for any $v \in \mathbb{R}$ (see [4], [6]).

**Case 2.** The curve is isotropic and the rotation is isotropic. Without loss of generality, we may assume that an isotropic curve $C$ lies in the $yz$-plane and so it is given by

$$C(u) = (0, f(u), g(u))$$

for some smooth functions $f$ and $g$.

On the other hand, an isotropic rotation in $\mathbb{G}_3^1$ is given by the normal form

$$\bar{x} = x + bt, \qquad \bar{y} = y + xt + b\frac{t^2}{2}, \qquad \bar{z} = z$$

where $t \in \mathbb{R}$ and $b$ is a positive constant.

Thus the rotational surface generated by revolving the $z$-axis can be parameterized by

$$\mathbf{x}(u,v) = (v, f(u) + \frac{v^2}{2b}, g(u)) \tag{5}$$

where $f$ and $g$ are smooth functions and $b \neq 0$ (see [4], [6]).

First of all, we study geodesics on rotational surfaces generated by a nonisotropic curve.

Let $\Sigma$ be a rotational surface generated by a unit speed nonisotropic curve $C(u) = (u, g(u), 0)$ in $\mathbb{G}_3^1$. Then a parametrization of the surface is given by

$$\mathbf{x}(u,v) = (u, g(u)\cosh v, g(u)\sinh v) \tag{6}$$

where $g$ is a positive function. We have

$$g_1 = 1, \qquad g_2 = 0, \qquad h_{11} = h_{12} = 0, \qquad h_{22} = -g^2(u)$$

which imply the components of the first fundamental form $\mathrm{d}s^2$ on $M$ are given by

$$g_{11}^* = 1, \qquad g_{12}^* = 0, \qquad g_{22}^* = -g^2(u).$$

Let $\gamma(t) = \mathbf{x}(u(t), v(t))$ be a geodesic on $\Sigma$. Then the Euler-Lagrange equations of the geodesic become

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}(g^2(u))\dot{v}^2, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(g^2(u)\dot{v}) = 0 \tag{7}$$

where the "dot" denotes the derivative with respective to the parameter $t$.

Consider a meridian $v(t) = v_0$. From the first equation of (7) $\ddot{u}(t) = 0$, that is, $u(t)$ is a linear function. Also the second equation of (7) holds obviously. Thus, if $u(t) = at + b$ for some constants $a, b \in \mathbb{R}$, then the meridian is a geodesic.

Consider a parallel $u(t) = u_0$. Since $g(u_0) > 0$ the first equation of (7) implies $\dfrac{\mathrm{d}g(u)}{\mathrm{d}u} = 0$ when $u = u_0$. From this, the second equation of (7) gives $\ddot{v} = 0$, that is, $v(t) = at + b$ for some constants $a, b \in \mathbb{R}$.

**Theorem 1.** *Let $\gamma(t) = (u(t), g(u(t))\cosh v(t), g(u(t))\sinh v(t))$ be a curve on the rotational surface given by* (6) *in the pseudo-Galilean three-space. Then, the following statements are true*

1) *A meridian $v(t) = v_0$ is a geodesic if and only if $u(t) = at + b$ for some constants $a, b \in \mathbb{R}$.*

2) *A parallel $u(t) = u_0$ is a geodesic if and only if $u_0$ is a stationary point of $g$ and $v(t) = at + b$ for some constants $a, b \in \mathbb{R}$.*

We consider hyperbolic cylinders $z^2 - y^2 = r^2$ ($y^2 - z^2 = r^2$) which are everywhere a spacelike (timelike) surface. They are spheres of the space $\mathbb{G}_3^1$, called hyperbolic spheres. Planes $y^2 - z^2 = 0$ are everywhere lightlike surfaces.

If we consider the hyperbolic sphere parametrized by

$$\mathbf{x}(u,v)=(u, r\cosh v, r\sinh v), \qquad r\in\mathbb{R}-\{0\} \tag{8}$$

then the equations of the geodesics are given by

$$\ddot{u}=0, \qquad r^2\ddot{v}=0.$$

Thus we have

$$u(t)=at+b, \qquad v(t)=\frac{c}{r^2}t+d, \qquad a,b,c,d\in\mathbb{R}.$$

Consequently, we have

**Proposition 2.** *The geodesics of the hyperbolic sphere given by* (8) *in the pseudo-Galilean space* $\mathbb{G}_3^1$ *are the curves of a equation*

$$\gamma(t)=(at+b, r\cosh(ct+d), r\sinh(ct+d)), \qquad a,b,c,d\in\mathbb{R}$$

*that includes*: 1) *the meridians,* 2) *the parallels,* 3) *the helices.*

**Example 3.** *To find a partial solution of* (7) *we take a positive function* $g(u)=\mathrm{e}^u$ *on a real number* $\mathbb{R}$. *Then the equations* (7) *can be rewritten in the form*

$$\ddot{u}=-\mathrm{e}^{2u}\dot{v}^2, \qquad \mathrm{e}^{2u}\dot{v}=c_1$$

*for some constant* $c_1$*, and it follows that*

$$\ddot{u}+c_1^2\mathrm{e}^{-2u}=0.$$

*From this, a solution is given by*

$$u(t)=\ln(c_1t+c_2), \qquad c_2\in\mathbb{R}$$

*and it leads to*

$$v(t)=-(c_1t+c_2)^{-1}+c_3, \qquad c_3\in\mathbb{R}.$$

*Thus, a curve*

$$\gamma(t)=(\ln(c_1t+c_2), (c_1t+c_2)\cosh(c_3-\frac{1}{c_1t+c_2}), (c_1t+c_2)\sinh(c_3-\frac{1}{c_1t+c_2}))$$

*is a geodesic on the rotational surface* $\mathbf{x}(u,v)=(u,\mathrm{e}^u\cosh v,\mathrm{e}^u\sinh v)$.

Now, we consider a rotational surface $\Sigma$ defined by (4). Then, similarly as above, we have the following result

**Theorem 4.** *Let* $\gamma(t) = (u(t), g(u(t)) \sinh v(t), g(u(t)) \cosh v(t))$ *be a curve on the rotational surface given by* (4) *in the pseudo-Galilean three-space. Then, the following statements are true*

1) *A meridian* $v(t) = v_0$ *is a geodesic if and only if* $u(t) = at + b$ *for some constants* $a, b \in \mathbb{R}$.

2) *A parallel* $u(t) = u_0$ *is a geodesic if and only if* $u_0$ *is a stationary point of* $g$ *and* $v(t) = at + b$ *for some constants* $a, b \in \mathbb{R}$.

Last of all, let $\Sigma$ be a rotational surface in $\mathbb{G}_3^1$ generated by an isotropic curve $\alpha(u) = (0, f(u), g(u))$. Assume that the curve $\alpha$ is parametrized by arc-length, that is,

$$f'(u)^2 - g'(u)^2 = -\eta(= \pm 1).$$

Then the parametrization of $\Sigma$ is given by

$$\mathbf{x}(u, v) = (v, f(u) + \frac{v^2}{2b}, g(u)) \tag{9}$$

where $f$ and $g$ are smooth functions and $b \neq 0$. In this case, we have

$$g_1 = 0, \qquad g_2 = 1, \qquad h_{11} = -\eta, \qquad h_{12} = h_{22} = 0$$

which imply the components of the first fundamental form $\mathrm{d}s^2$ on $\Sigma$ are given by

$$g_{11}^* = 1, \qquad g_{12}^* = 0, \qquad g_{22}^* = -\eta.$$

From this, the Euler-Lagrange equations of the geodesics on $\Sigma$ become

$$\ddot{u} = 0, \qquad -\eta\ddot{v} = 0$$

and the solutions of the equations are given by

$$u(t) = a_1 t + a_2, \qquad v(t) = a_3 t + a_4, \qquad a_i \in \mathbb{R}.$$

Consequently, we have

**Theorem 5.** *Let* $\Sigma$ *be a rotational surface given by*

$$\mathbf{x}(u, v) = (v, f(u) + \frac{v^2}{2b}, g(u)) \tag{10}$$

*in the pseudo-Galilean three-space* $\mathbb{G}_3^1$. *A curve* $\gamma(t) = (v(t), f(u(t)) + \dfrac{v^2(t)}{2b},$ $g(u(t)))$ *is a geodesic on* $\Sigma$ *if and only if* $u(t)$ *and* $v(t)$ *are linear.*

**Example 6.** Take $f(u) = a_1 u + a_2$ and $g(u) = a_3 u + a_4$ in (10) with $a_1^2 - a_3^2 = -\eta$. Then the curve

$$\gamma(t) = (c_1 t + c_2, c_3 t + c_4 + \frac{b}{2}(c_1 t + c_2)^2, c_5 t + c_6), \qquad a_i \in \mathbb{R}$$

is a geodesic on the parabolic sphere in $\mathbb{G}_3^1$.

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