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KÄHLER DYNAMICS FOR THE UNIVERSAL MULTI-ROBOT FLEET

VLADIMIR G. IVANCEVIC

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Abstract. A general model is formulated for a universal fleet of all unmanned vehicles, including Aerial Vehicles (UAVs), Ground Vehicles (UGVs), Sea Vehicles (USVs) and Underwater Vehicles (UUVs), as a geometric Kähler dynamics and control system. Based on the Newton-Euler dynamics of each vehicle, a control system for the universal autonomous fleet is designed as a combined Lagrangian and Hamiltonian form. The associated continuous system representing a very large universal fleet is given in Appendix in the form of the Kähler-Ricci flow.

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Contents

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1. Introduction

As a motivation for the present work, consider a hypothetical city coastline that has been attacked either by some natural disaster or by a terrorist group, leaving many victims both on the ground and in the sea. Fast recovery action is necessary, but the situation is still dangerous for humans. So, the only option for the quick response is to send swarms of many heterogenous robots to perform an *autonomous search-and-rescue operation.* The purpose of this paper is to develop a general formal model for such a large-scale multi-robot operation, including swarms of small UGVs (cars) and UAVs (quadcopters), as well as fleets of USVs (boats) and UUVs (submarines). This paper is a natural extension of [14], where a symplectic Hamiltonian dynamics [6, 9] and control [7, 11] model for swarms of UGVs and UAVs only was developed. The reason for this extension will be articulated below. Although there are not many realistic scenarios where such a universal autonomous fleet would be needed, the existence of such a general model would provide a new capability that can be easily specialized to suit a wide range of specific requirements.

To start this ambitious modeling attempt that would cover all possible unmanned vehicles, we first recall a well-known fact from physics that all conservative systems can be described by either Lagrangian or Hamiltonian formalisms, so that each degree-of-freedom (DOF) is governed either by a single second-order Lagrangian equation of motion or by two first-order Hamiltonian equations of motion. More generally, most non-conservative engineering systems (of mechanical, electrical, thermal or hydraulic nature, or their combination) can be described by dissipative and forced Lagrangian or Hamiltonian equations of motion. Even more generally, in modern geometric control theory (see [11] and the references therein), any nonlinear control system can be formulated as a general Lagrangian or Hamiltonian control system, where appropriate controllers (including, e.g., a linear PID controller, a quadratic Kalman regulator/filter, or higher-order nonlinear Lie-derivative controllers) are added to dissipative and forced Lagrangian or Hamiltonian dynamics, to give the nonlinear generalization of Kalman's state-space control theory. Still, it is possible to generalize this modeling approach and describe any nonlinear high-dimensional system as a union of Lagrangian and Hamiltonian control systems. This is the objective of this paper, called the *Kähler dynamics*, introduced in [12] and developed in [13].

In terms of modern mechanics (see, e.g. [7] and the references therein), based on the concept of the configuration n-manifold M (that includes all DOFs coordinated by generalized coordinates $x^i(t)$, $(i = 1, ..., n)$ in the swarm/fleet under consideration; see Figure 1 in [14]), the behavior of all vehicles is governed by the velocity vector-field $v^i(t)$, which is formally defined as a cross-section of the tangent bundle TM of the configuration manifold M. Thus, the $2n$-manifold TM, coordinated by $[x^i(t), v^i(t)]$, is called the *velocity phase space* and it is the stage for (dissipative, forced and controlled) Lagrangian dynamics, naturally endowed with Riemannian geometry. Alternatively, the behavior of all vehicles is also governed by the momentum covector-field $p_i(t)$, which is formally defined as a cross-section of the cotangent bundle T^*M of the configuration manifold M. Thus, the $2n$-manifold T^*M, coordinated by $[x^i(t), p_i(t)]$, is called the *momentum phase space* which is the stage for (dissipative, forced and controlled) Hamiltonian dynamics, naturally endowed with symplectic geometry.

For some engineering/robotics systems, like e.g., vehicles moving in the air, the Hamiltonian approach (presented in [14]) is stronger, as it allows both force and velocity controllers, while the Lagrangian approach allows only force controllers. However, for other systems like vehicles moving in the water, the Lagrangian (or, more precisely, Kirchhoff-Lagrangian) approach is the only possibility. In general, if we have a large-scale complex system, including four kinds of robots (namely, ground and air vehicles, boats and submarines), in which some components (ground and air vehicles) can be more naturally modeled via the Hamiltonian formalism, while other components (boats and submarines) can be more naturally modeled via the Lagrangian formalism, the union of both approaches, representing "the best of both worlds", would clearly be preferable, naturally leading to the universal autonomous *ground-air-sea-underwater operation model.*

Formally, both the standard formalisms, Lagrangian boats + submarines and Hamiltonian ground + air vehicles, can be unified in the basic definition of the multi-geometric *Kähler manifold* $\mathcal{K}$

$$\mathcal{K} \;=\; TM \;+\; \mathrm{i}\, T^*M \tag{1}$$

which states that the joint Kähler $4n$-manifold is defined as the complexified sum (i.e., the sum with the imaginary unit: $\mathrm{i} = \sqrt{-1}$) of the Lagrangian $2n$-manifold TM (with Riemannian geometry) and the Hamiltonian $2n$-manifold T^*M (with symplectic geometry). The Kähler dynamics are comprised of three mutually compatible geometrical and dynamical structures: (i) Lagrangian dynamics on the Riemannian tangent bundle TM, (ii) Hamiltonian dynamics on the symplectic cotangent bundle T^*M, and (iii) general complex-valued dynamics on their complexified sum manifold $\mathcal{K}$. Put simply, this universal approach can be described as follows: the output from the Lagrangian dynamics/control is a set of real numbers A and the output from the Hamiltonian dynamics/control is another set of real numbers B; their complexified sum: $C = A + \mathrm{i}B$ is the set of complex numbers that represents the Kähler dynamics.

In this paper, using the Kähler dynamics formalism, we will develop a general model for the universal unmanned ground-air-sea-underwater operation. Being an extension of the previous purely-Hamiltonian model, in the present paper we will mainly focus on the Lagrangian side of this universal unmanned vehicles problem.

2. Lagrangian and Hamiltonian Fleets

2.1. Individual Unmanned Vehicles: Basic Newton-Euler Mechanics

All unmanned/autonomous vehicles are Newtonian rigid bodies moving in 3D space. Formally, each vehicle is represented by a 6-parameter Euclidean Lie group $\mathrm{SE}(3)$ of rigid motions in 3D space $\mathbb{R}^3$, which consists of isometries of $\mathbb{R}^3$ and is defined as a semidirect (noncommutative) product of 3D rotations $\mathrm{SO}(3)$ and 3D translations $\mathbb{R}^3$: $\mathrm{SE}(3) := \mathrm{SO}(3) \rhd \mathbb{R}^3$ (see [9, 11, 17, 18] and the references therein).

Basic Newton-Euler mechanics for each unmanned $\mathrm{SE}(3)$-vehicle are given in vector form (with the overdot representing time derivative) as

$$\begin{aligned} \text{Newton}: \ \dot{\boldsymbol{p}} &\equiv \boldsymbol{M}\dot{\boldsymbol{v}} = \boldsymbol{F} + \boldsymbol{p} \times \boldsymbol{\omega} \\ \text{Euler}: \ \dot{\boldsymbol{\pi}} &\equiv \boldsymbol{I}\dot{\boldsymbol{\omega}} = \boldsymbol{T} + \boldsymbol{\pi} \times \boldsymbol{\omega} + \boldsymbol{p} \times \boldsymbol{v} \end{aligned} \tag{2}$$

and in tensor form, using *Einstein's summation convention* over repeated indices and the Levi-Civita permutation symbol ε^j_{ik}, as the following system of ordinary differential equations (ODEs)

$$\begin{aligned} \dot{p}_i &\equiv M_{ij}\dot{v}^j = F_i + \varepsilon^j_{ik} p_j \omega^k, \qquad i, j, k = 1, 2, 3 \\ \dot{\pi}_i &\equiv I_{ij}\dot{\omega}^j = T_i + \varepsilon^j_{ik} \pi_j \omega^k + \varepsilon^j_{ik} p_j v^k. \end{aligned} \tag{3}$$

In equations (2) and (3) the diagonal mass and inertia matrices

$$\boldsymbol{M} \equiv M_{ij} = \mathrm{diag}\{m_1, m_2, m_3\} \qquad \text{and} \qquad \boldsymbol{I} \equiv I_{ij} = \mathrm{diag}\{I_1, I_2, I_3\}$$

define the vehicle's mass–inertia distribution. The vehicle's linear and angular velocity vector fields are

$$\begin{aligned} \boldsymbol{v} &= \dot{\boldsymbol{x}} \equiv v^i = \dot{x}^i \equiv [v_1, v_2, v_3]^T = [\dot{x}_1, \dot{x}_2, \dot{x}_3]^T \\ \boldsymbol{\omega} &= \dot{\boldsymbol{\theta}} \equiv \omega^i = \dot{\theta}^i \equiv [\omega_1, \omega_2, \omega_3]^T = [\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3]^T \end{aligned}$$

where x^i are Cartesian coordinates of the vehicles center-of mass (CoM) and θ^i are its Euler angles (roll, pitch and yaw). The co-vector fields (or one-forms) of driving, gravitational and other external forces and torques acting on the vehicle are

$$\boldsymbol{F} \equiv F_i = [F_1, F_2, F_3] \qquad \text{and} \qquad \boldsymbol{T} \equiv T_i = [T_1, T_2, T_3]$$

while the corresponding linear and angular momentum co-vector fields are

$$\begin{aligned} \boldsymbol{p} &= \boldsymbol{M}\boldsymbol{v} \equiv p_i = [p_1, p_2, p_3] = [m_1 v_1, m_2 v_2, m_2 v_2] \\ \boldsymbol{\pi} &= \boldsymbol{I}\boldsymbol{\omega} \equiv \pi_i = [\pi_1, \pi_2, \pi_3] = [I_1\omega_1, I_2\omega_2, I_3\omega_3]. \end{aligned}$$

Now we move to a more general, Lagrangian formalism – in this subsection applied to individual vehicles only, to be generalized to the whole fleet in the next subsection. equations (2) and (3) can be derived as Lagrangian equations of motion from the *Lagrangian function* $L = L(v{=}\dot{x}; \omega{=}\dot{\theta})$ representing (translational + rotational) kinetic energy of each individual vehicle

$$L = \frac{1}{2}\boldsymbol{v}^T\boldsymbol{M}\boldsymbol{v} + \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega} = \frac{1}{2}M_{ij}v^i v^j + \frac{1}{2}I_{ij}\omega^i\omega^j. \tag{4}$$

From this Lagrangian function, we can immediately derive the conservative (dissipation-free and force-free) Lagrangian equations of motion for translations and rotations, respectively, in both vector and tensor form (using index notation for partial derivatives: $L_z = \frac{\partial L}{\partial z}$)

$$\dot{L}_{\boldsymbol{v}} = L_{\boldsymbol{x}} \Leftrightarrow \dot{L}_{v^i} = L_{x^i} \quad \text{and} \quad \dot{L}_{\boldsymbol{\omega}} = L_{\boldsymbol{\theta}} \Leftrightarrow \dot{L}_{\omega^i} = L_{\theta^i} \tag{5}$$

(this conservative approach is formally derived and explained in a more general settings in the next subsection).

Standard engineering extensions of conservative Lagrangian dynamics (5) include friction forces derived from the Rayleigh dissipative function $R = R(x, v) = \frac{1}{2}\left(v^i x^i\right)^2$, and other external forces and torques, including gradient ones E_{v^i} and E_{ω^i} derived from the vehicle's total potential energy $E(x, \theta)$. In such a way, we obtain the *dissipative and forced Lagrangian dynamics* in vector and tensor form

$$\begin{aligned} &\text{Translations}: \dot{L}_{\boldsymbol{v}} + R_{\boldsymbol{v}} = L_{\boldsymbol{x}} + \boldsymbol{F} \Longleftrightarrow \dot{L}_{v^i} + R_{v^i} = L_{x^i} + F_i \\ &\text{Rotations}: \quad \dot{L}_{\boldsymbol{\omega}} + R_{\boldsymbol{\omega}} = L_{\boldsymbol{\theta}} + \boldsymbol{T} \Longleftrightarrow \dot{L}_{\omega^i} + R_{\omega^i} = L_{\theta^i} + T_i\,. \end{aligned} \tag{6}$$

However, for the case of a vehicle immersed in water, equations (6) still need to be extended into the so-called *Kirchhoff–Lagrangian equations,* due to the addition of strong water influences which result in several mixed cross-products (see e.g. [15, 16] or the original work of Kirchhoff in German), which can be written in vector form

$$\begin{aligned} \dot{L}_{\boldsymbol{v}} + R_{\boldsymbol{v}} &= L_{\boldsymbol{x}} + \boldsymbol{F} + L_{\boldsymbol{v}} \times \boldsymbol{\omega} \\ \dot{L}_{\boldsymbol{\omega}} + R_{\boldsymbol{\omega}} &= L_{\boldsymbol{\theta}} + \boldsymbol{T} + L_{\boldsymbol{\omega}} \times \boldsymbol{\omega} + L_{\boldsymbol{v}} \times \boldsymbol{v} \end{aligned} \tag{7}$$

and in tensor form

$$\begin{aligned} \dot{L}_{v^i} + R_{v^i} &= L_{x^i} + F_i + \varepsilon^j_{ik} L_{v^j} \omega^k \\ \dot{L}_{\omega^i} + R_{\omega^i} &= L_{\theta^i} + T_i + \varepsilon^j_{ik} L_{\omega^j} \omega^k + \varepsilon^j_{ik} L_{v^j} v^k \,. \end{aligned} \tag{8}$$

Using equations (4)–(8), each vehicle's linear and angular momentum co-vector fields (or, one-forms) are defined as

$$\boldsymbol{p} = L_{\boldsymbol{v}} \;\Leftrightarrow\; p_i = L_{v^i} \,, \qquad \boldsymbol{\pi} = L_{\boldsymbol{\omega}} \;\Leftrightarrow \pi_i = L_{\omega^i}$$

with their corresponding time derivatives defining conservative force $\boldsymbol{F}^{\rm con}$ and torque $\boldsymbol{T}^{\rm con}$ one-forms (different from external/dissipative ones, $\boldsymbol{F}$ and $\boldsymbol{T}$)

$$\begin{aligned} \boldsymbol{F}^{\rm con} \equiv \dot{\boldsymbol{p}} = \dot{L}_{\boldsymbol{v}} \;\Leftrightarrow\; F^{\rm con}_i \equiv \dot{p}_i = \dot{L}_{v^i} \\ \boldsymbol{T}^{\rm con} \equiv \dot{\boldsymbol{\pi}} = \dot{L}_{\boldsymbol{\omega}} \;\Leftrightarrow\; T^{\rm con}_i \equiv \dot{\pi}_i = \dot{L}_{\omega^i} \,. \end{aligned}$$

2.2. Lagrangian Control for the Water Fleet

General Lagrangian dynamics for a large unmanned water fleet consisting of m vehicles (boats-USVs and submarines-UUVs, each with 6 DOFs) are defined on the fleet's configuration n-manifold $M_{\rm wat} = \prod_{k=1}^m \mathrm{SE}(3)^k$ (similar to the Figure 1 in [14]) with local coordinates $x^i(t)$,for $i = 1, ..., n = 6m$ and velocity vector-fields defined on its Riemannian tangent bundle $TM_{\rm wat}$ with local coordinates $(x^i; \dot{x}^i{=}v^i)$. We give a rigorous variational derivation of the water fleet's dynamics and finite control based on its Lagrangian energy function $L(x, \dot{x}) : TM_{\rm wat} \to \mathbb{R}$,using the formalism of exterior differential systems on the $(2n{+}1)$-dimensional time-extended tangent bundle, called the jet manifold $JM_{\rm wat} = j^1(\mathbb{R}, M_{\rm wat}) \cong \mathbb{R}{\times}TM_{\rm wat}$, with local canonical variables $(t; x^i; \dot{x}^i)$ (for technical details, see [2,7] and the references therein).

Consider a general variational problem $(I, \omega; \varphi)$ for the water fleet, where (I, ω) represents the Pfaffian exterior differential system on $JM_{\rm wat}$, given in local coordinates $(t; x^i; \dot{x}^i)$ as

$$\theta^i = \mathrm{d}x^i - \dot{x}^i\omega = 0, \qquad \omega \equiv \mathrm{d}t \neq 0$$

with structure equations

$$\mathrm{d}\theta^i = -\mathrm{d}\dot{x}^i \wedge \omega$$

where the symbols $\wedge$ and d denote exterior product and derivative, respectively. Integral manifolds $N \in j^1(\mathbb{R}, M_{\rm wat})$ of the Pfaffian system (I, ω) are locally defined by 1-jets $j^1 : t \to [t, x(t), \dot{x}(t)]$ of curves $x(t) : \mathbb{R} \to M_{\rm wat}$.

Next, we introduce a one-form $\varphi = L\,\omega$, where $L = L(t, x, \dot{x})$ is the system's Lagrangian defined on JM_{wat}, having unique coordinate and velocity partial derivatives, denoted by L_{x^i} and $L_{\dot{x}^i}$, respectively. A variational problem $(I, \omega; \varphi)$ is said to be *strongly non-degenerate*, or *well-posed* [2], if and only if (iff) the determinant of the matrix of mixed velocity partials of the Lagrangian is positive definite: $\det \|L_{\dot{x}^i \dot{x}^j}\| > 0$.

The corresponding extended Pfaffian system

$$\theta^i = 0, \qquad \mathrm{d}L_{\dot{x}^i} - L_{x^i}\,\omega = 0, \qquad \omega \equiv \mathrm{d}t \neq 0$$

generates conservative (dissipation-free and force-free) Lagrangian equations for the fleet of water vehicles (a generalization of equations (5))

$$\dot{L}_{\dot{x}^i} = L_{x^i}\,. \tag{9}$$

If an integral manifold N satisfies the Lagrangian equations (9) of a well-posed variational problem, then $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{N_t} \varphi\right)_{t=0} = 0$ for any admissible variation $\delta \in N$ with fixed endpoint conditions: $\omega = \theta^i = 0$.

Under the above conditions, the *Griffiths theorem* [2] states that both the (conservative) Lagrangian dynamics with initial conditions

$$\dot{L}_{\dot{x}^i} = L_{x^i}, \qquad x(t_0) = x_0, \qquad \dot{x}(t_0) = \dot{x}_0$$

and the Lagrangian dynamics with endpoint conditions (also called the finite control system)

$$\dot{L}_{\dot{x}^i} = L_{x^i}\,, \qquad x(t_0) = x_0, \qquad x(t_1) = x_1$$

have unique solutions (see [2] for the proof of this theorem).

To generalize this conservative theorem to include both dissipative and driving forces (as we did in the previous section), we use the fact that the tangent bundle TM_{wat} naturally represents a Riemannian $2n$-manifold, with the Riemannian positive-definite metric form

$$g_R = g_{ij}(x)\,\mathrm{d}x^i \mathrm{d}x^j \tag{10}$$

which defines both the material metric tensor g_{ij} (given by the smooth symmetric matrix $\|g_{ij}(x)\|$ and representing the mass-inertia distribution of the whole water fleet Ξ) and the kinetic energy $E_{\mathrm{kin}}(x, \dot{x}) = \frac{1}{2} g_R$ of the fleet Ξ. In addition, if the potential energy $E_{\mathrm{pot}}(x)$ of the fleet Ξ is also a smooth function, then the autonomous Lagrangian is defined as $L(x, \dot{x}) = E_{\mathrm{kin}}(x, \dot{x}) - E_{\mathrm{pot}}(x)$ and automatically satisfies the condition of well-posedness: $\det \|L_{\dot{x}^i \dot{x}^j}\| = \det \|g_{ij}(x)\| > 0$.

In the Riemannian settings, the *covariant* Lagrangian equations (9) can be immediately generalized to include both dissipative and driving (gradient) force one-forms $F_i(x, \dot{x})$ according to equations (6), giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}[x(t)]\,\dot{x}^j(t)\right) = \frac{1}{2}\,\left(\partial_{x^i} g_{jk}[x(t)]\dot{x}^j(t)\,\dot{x}^k(t)\right) - F_i\left[x(t), \dot{x}(t)\right]. \qquad (11)$$

Next, letting $\left\|g^{ij}(x)\right\|$ to be the inverse matrix of $\|g_{ij}(x)\|$ and introducing classical Christoffel symbols

$$\Gamma^i_{jk} = g^{il}\Gamma_{jkl}, \qquad \Gamma_{jkl} = \frac{1}{2}\left(\partial_{x^j} g_{kl} + \partial_{x^k} g_{jl} - \partial_{x^l} g_{jk}\right) \qquad (12)$$

the equations (11) resolve to the classical *contravariant form* (see [7])

$$\ddot{x}^i(t) + \Gamma^i_{jk}\dot{x}^j(t)\,\dot{x}^k(t) = F^i\left[x(t), \dot{x}(t)\right]. \qquad (13)$$

Then the Riemann-generalized Griffiths theorem implies that both the forced and dissipative Lagrangian dynamics with initial conditions

$$\ddot{x}^i(t) + \Gamma^i_{jk}\,\dot{x}^j(t)\,\dot{x}^k(t) = F^i\left[x(t), \dot{x}(t)\right], \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0 \qquad (14)$$

and the Lagrangian dynamics with endpoint conditions (the finite control system)

$$\ddot{x}^i(t) + \Gamma^i_{jk}\,\dot{x}^j(t)\,\dot{x}^k(t) = F^i\left[x(t), \dot{x}(t)\right], \quad x(t_0) = x_0, \quad x(t_1) = x_1 \qquad (15)$$

have unique solutions.

However, to make Lagrangian equations of motion suitable for the sea/underwater fleet dynamics, we need to generalize the external forces from (13), to include Kirchhoff-type torques and forces coming from the water medium, an nD analog of equations (8). If we denote such generalized torques and forces as $\mathcal{F}^i\left[t, x(t), \dot{x}(t)\right]$, we obtain the general form of the contravariant Lagrangian fleet dynamics

$$\ddot{x}^i(t) + \Gamma^i_{jk}[x(t)]\,\dot{x}^j(t)\,\dot{x}^k(t) = \mathcal{F}^i\left[t, x(t), \dot{x}(t)\right] \qquad (16)$$

which can be rewritten in the classical *covariant* Lagrangian form as

$$\dot{L}_{\dot{x}^i} - L_{x^i} = \mathcal{F}_i \qquad (17)$$

with

$$\mathcal{F}_i = \mathcal{F}_i\left[t, x(t), \dot{x}(t)\right] = F_i\left[x(t), \dot{x}(t)\right] + \mathcal{R}$$

where the term $\mathcal{R}$ represents an nD analog of Kirchhoff-type mixed rotational terms ($\varepsilon^j_{ik} L_{\dot{x}^j}\dot{x}^k, \varepsilon^j_{ik} L_{v^j}\omega^k\,, \varepsilon^j_{ik} L_{\omega^j}\omega^k$) of (8).

Equation (17) is our final Lagrangian dynamics model for the water fleet. However, due to Kirchhoff-type mixed rotational terms $\mathcal{R}$, its existence and uniqueness cannot be demonstrated because it does not adhere to the conditions imposed by the Griffiths theorem. Therefore, at present, we are unable to formulate a rigorous finite control system for the general Lagrangian dynamics of the water fleet.

So, we move to a stronger alternative, the so-called *affine Lagrangian control system* (see [11] and the references therein). For this we introduce the *affine Lagrangian control function* $L^a(x, \dot{x}, u) : TM_{\mathrm{wat}} \to \mathbb{R}$, which in local canonical coordinates on TM_{wat} given as

$$L^a(x, \dot{x}, u) = L^0(x, \dot{x}) - L^j(x, \dot{x})\, u_j(t, x, \dot{x})$$

where $L^0(x, \dot{x})$ is the above physical Lagrangian, $L^j(x, \dot{x})$, $j = 1, \ldots, m < n$ are the *coupling Lagrangians* corresponding to the *active* nearest-neighboring vehicles in the fleet, and the control one-forms $u_j = u_j(t, x, \dot{x})$ are defined via the *Lie derivative* control formalism (see equation (11) in [14]) giving the *control law* for asymptotic tracking of the predefined reference outputs. Then, the affine Lagrangian control system is governed by the following equations on TM_{wat}

$$\dot{L}_{\dot{x}^i} - L_{x^i} = \mathcal{F}_i + L^j_{x^i}\, u_j\, . \tag{18}$$

2.3. Hamiltonian Control for the Air Swarm

As already mentioned, the Hamiltonian dynamics and control model for a joint (UGV + UAV) swarm with the configuration n-manifold $M_{\mathrm{air}} = \prod_{k=1}^{m} \mathrm{SE}(3)^k$ consisting of m unmanned air vehicles (each with 6 DOFs) have been presented in detail in [14], based on the swarm's Hamiltonian energy function which is defined as a map $H(x, p) : T^*M_{\mathrm{air}} \to \mathbb{R}$ on its symplectic cotangent bundle T^*M_{air} with the *symplectic form*

$$\omega_S = \mathrm{d}p_i \wedge \mathrm{d}x^i, \qquad \text{for } i = 1, ..., n. \tag{19}$$

This Hamiltonian dynamics/control system can be summarized by the following two equations

Dynamics: The forced and dissipative Hamiltonian dynamics for a joint swarm of land/air vehicles are defined as

$$\dot{x}^i = H_{p_i} - R_{p_i}\, , \qquad \dot{p}_i = F_i - H_{x^i} + R_{x^i} \tag{20}$$

where x^i are the generalized coordinates associated to all active DOFs within each swarm, p_i are their corresponding momenta (both linear and angular), $F_i = F_i(t, x, p)$ are the generalized driving forces, while $R = R(x, p) = \frac{1}{2} p_i^2 x_i^2$ denotes the Rayleigh dissipative function.

Control: The affine Hamiltonian control system for a swarm of land/air vehicles is derived from the affine Hamiltonian control function $H^a(x,p,u) : T^*M_{\text{air}} \to \mathbb{R}$, in local canonical coordinates on T^*M_{air} given as

$$H^a(x,p,u) = H^0(x,p) - H^j(x,p)\,u_j(t,x,p), \qquad i = 1,\ldots,n$$

where $H^0(x,p)$ is the physical Hamiltonian, $H^j(x,p)$, $(j = 1,\ldots,m < n)$ are the affine Hamiltonians corresponding to the *active* nearest-neighboring vehicles in the swarm, while the control one-forms u_j, defined by the *Lie derivative* control formalism (see equation (11) in [14]) represent the *control law* for asymptotic tracking of reference swarm outputs. The whole control system includes both the contravariant velocity controllers $V^i = V^i(t,x,p)$ and the covariant force controllers $F_i = F_i(t,x,p)$, and is governed by the following canonical equations on T^*M_{air}

$$\begin{aligned} \dot{x}^i &= V^i + H^0_{p_i} - H^j_{p_i}\,u_j + R_{p_i} \\ \dot{p}_i &= F_i - H^0_{x^i} + H^j_{x^i}\,u_j + R_{x^i}\,. \end{aligned} \tag{21}$$

3. Kähler Dynamics for the Universal Fleet

A Kähler manifold, $\mathcal{K} \equiv (\mathcal{K}, g) \equiv (\mathcal{K}, \omega)$, is a Hermitian manifold [8] with the real dimension $4n$ (or, complex dimension $2n$) that admits three mutually compatible dynamical structures: (i) Riemannian/Lagrangian, (ii) symplectic/Hamiltonian, and (iii) complex-valued, formally defined as follows.

We start with a *universal configuration* n-manifold M, including all autonomous unmanned vehicles (UGVs, UAVS, USVs and UUVs, each of them formally defined as an $\mathrm{SE}(3)$-group). On the configuration n-manifold M we define two partial bundles, each being a $2n$-manifold: a Riemannian/Lagrangian tangent bundle TM_{wat} which governs the dynamics of water vehicles (USVs and UUVs) and a symplectic/Hamiltonian cotangent bundle T^*M_{air} which governs the dynamics of air vehicles (UGVs and UAVs). The 'water bundle' TM_{wat} is formally defined as a disjoint union of tangent spaces T_xM_{wat} at all water vehicles $x_{\text{wat}} \in M : TM_{\text{wat}} = \sqcup_{x_{\text{wat}} \in M} T_xM_{\text{wat}}$. Similarly, the 'air bundle' T^*M_{air} is formally defined as a disjoint union of cotangent spaces $T^*_xM_{\text{air}}$ at all air vehicles $x_{\text{air}} \in M : T^*M_{\text{air}} = \sqcup_{x_{\text{air}} \in M} T^*_xM_{\text{air}}$.

Now, the global Kähler $4n$-manifold $\mathcal{K}$, the stage of our dynamics, can be constructed in a similar way as a complexified tangent bundle of the configuration manifold M, that is, a disjoint union of complexified tangent spaces $T_xM^{\mathbb{C}}$ at all vehicles $x = x_{\text{wat}} + x_{\text{air}} \in M$

$$\mathcal{K} = TM \otimes \mathbb{C} = \sqcup_{x \in M} T_xM^{\mathbb{C}}.$$

The manifold $\mathcal{K}$ admits a Hermitian metric form g, such that its real part is the Riemannian metric form $g_R \in TM_{\rm wat}$ given by equation (10) and its imaginary part is the symplectic form $\omega_S \in T^*M_{\rm air}$ given by equation (19). Thus, we have

$$\begin{aligned}(\mathcal{K}, g) &= TM_{\rm wat} + \mathrm{i}\, T^*M_{\rm air} \qquad \text{with} \\ g &= g_R + \mathrm{i}\omega_S = g_{ij}\, \mathrm{d}x^i \mathrm{d}x^j + \mathrm{i}\, \mathrm{d}p_i \wedge \mathrm{d}x^i.\end{aligned}$$

The global Kähler manifold $\mathcal{K}$ has a very rich geometric and dynamical structure, defined as follows.

Any local open chart $U \subset \mathcal{K}$ defines a set of $2n$ holomorphic coordinates, in which x-components come from the Lagrangian water fleet on $TM_{\rm wat}$ and y-components come from the Hamiltonian air fleet on $T^*M_{\rm air}$. They locally identify $\mathbb{C}^{2n}$ with $\mathbb{R}^{4n}$, as $\{z^j = x^j + \mathrm{i}y^j \;;\; j = 1, \cdots, 2n\}$, with the corresponding holomorphic differentials

$$\mathrm{d}z^j = \mathrm{d}x^j + \mathrm{i}\mathrm{d}y^j \qquad \text{and} \qquad \mathrm{d}\bar{z}^j = \mathrm{d}x^j - \mathrm{i}\mathrm{d}y^j$$

and holomorphic velocities

$$\dot{z}^j = \dot{x}^j + \mathrm{i}\dot{y}^j \qquad \text{and} \qquad \dot{\bar{z}}^j = \dot{x}^j - \mathrm{i}\dot{y}^j.$$

The Hermitian metric tensor $g_{i\bar{j}} = g_{i\bar{j}}(z^i, z^{\bar{j}})$ of the Kähler manifold $\mathcal{K}$ represents the whole mass/inertia-distribution of the universal $(TM_{\rm wat} + \mathrm{i}\, T^*M_{\rm air})$ $-$fleet. The metric $g_{i\bar{j}}(z^i, z^{\bar{j}})$ obeys the following *Kähler condition* (independent of the choice of local holomorphic coordinates $z^j \in U$)

$$\partial_j g_{i\bar{k}} = \partial_i g_{j\bar{k}} \qquad \text{and} \qquad \partial_{\bar{j}} g_{k\bar{i}} = \partial_{\bar{i}} g_{k\bar{j}}\,, \quad (\partial_j \equiv \partial/\partial z^j,\; \partial_{\bar{j}} \equiv \partial/\partial z^{\bar{j}}) \tag{22}$$

where $\partial_j \equiv \partial : \Omega^{p,q}(\mathcal{K}) \to \Omega^{p+1,q}(\mathcal{K})$ and $\partial_{\bar{j}} \equiv \bar{\partial} : \Omega^{p,q}(\mathcal{K}) \to \Omega^{p,q+1}(\mathcal{K})$ are *Dolbeault's differential operators* on the space $\Omega^{p,q}(\mathcal{K})$ of exterior forms on $\mathcal{K}$, which are the additive components of the standard exterior derivative on $\mathcal{K}$: $\mathrm{d} = \partial + \bar{\partial}$. In a local z^k-coordinate chart $U \subset \mathcal{K}$, for any holomorphic function $f \in U$, ∂ and $\bar{\partial}$ operators are given by

$$\partial f = \left(\partial_{x^k} f - \mathrm{i}\partial_{y^k} f\right) \mathrm{d}z^k, \qquad \bar{\partial} f = \left(\partial_{x^k} f - \mathrm{i}\partial_{y^k} f\right) \mathrm{d}\bar{z}^k.$$

The *Kähler metric form* $g = g_R + \mathrm{i}\omega_S$ (defined by the Hermitian metric tensor $g_{i\bar{j}}$) is a positive-definite, symmetric (1,1)-form on $\mathcal{K}$ defined in local holomorphic coordinates as

$$g = g_{i\bar{j}}\, \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j > 0, \qquad i, j = 1, \cdots, 2n$$

which defines the complex kinetic-energy Lagrangian $L^0(\dot{z},\dot{\bar{z}}) : \mathcal{K} \to \mathbb{C}$ of the universal $(TM_{\rm wat} + \mathrm{i}\,T^*M_{\rm air})$ $-$fleet, in holomorphic coordinates z^i on $\mathcal{K}$ given as

$$L^0(\dot{z},\dot{\bar{z}}) = \frac{1}{2}g = \frac{1}{2}g_{i\bar{j}}\,\dot{z}^i \otimes \dot{\bar{z}}^j$$

so that the complex affine Lagrangian control function $L^a(\dot{z},\dot{\bar{z}},u) : \mathcal{K} \to \mathbb{C}$ is given in local coordinates on $\mathcal{K}$ as

$$L^a(\dot{z},\dot{\bar{z}},u) = L^0(\dot{z},\dot{\bar{z}}) - L^j(\dot{z},\dot{\bar{z}})\,u_j(t,z,\dot{z}), \qquad j = 1,\ldots,m < n$$

with coupling Lagrangians $L^j(\dot{z},\dot{\bar{z}})$ and complex control inputs $u_j(t,z,\bar{z})$. The contravariant Lagrangian equations are now derived from $L^a(\dot{z},\dot{\bar{z}},u)$ as

$$\ddot{z}^i(t) + \Gamma^{\bar{i}}_{jk}\,\dot{z}^j(t)\,\dot{\bar{z}}^k(t) = \mathcal{F}^i\,[t,z(t),\dot{z}(t)]$$

using the complex Christoffel symbols on $\mathcal{K}$ defined (via the Hermitian metric tensor $g_{i\bar{j}}$) as

$$\Gamma^k_{ij} = g^{k\bar{l}}\partial_j g_{i\bar{l}} \qquad \text{and} \qquad \Gamma^{\overline{k}}_{\bar{i}\,\bar{j}} = g^{\overline{k}l}\partial_{\bar{j}} g_{l\bar{i}}, \quad i,j,k = 1,\cdots,2n.$$

The covariant Lagrangian equations for the universal $(TM_{\rm wat} + \mathrm{i}\,T^*M_{\rm air})$ fleet (complexified and generalized from equation (18)) are

$$\dot{L}_{\dot{z}^i} - L_{z^i} = \mathcal{F}_i + L^j_{z^i}\,u_j \tag{23}$$

where $\mathcal{F}_i = \mathcal{F}_i\,[t,z(t),\dot{z}(t)]$ represent the general force one-forms (including dissipation and driving forces as well as Kirchhoff-type mixed rotational terms).

The associated *Kähler symplectic form* ω is a positive-definite exterior (1,1)-form on $\mathcal{K}$, which is also harmonic ($\delta\omega \equiv *\mathrm{d}*\omega = 0$), a result from the *Kähler-Hodge theory* (see, e.g. [20] and the references therein), given in holomorphic coordinates z^i on $\mathcal{K}$ as

$$\omega = \mathrm{i}g_{i\bar{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j > 0, \qquad i,j = 1,\cdots,2n \tag{24}$$

which is *closed*: $\mathrm{d}\omega = 0$, that follows directly from the Kähler condition (22). From the Kähler form (24) with $\bar{z}^i$ as canonical momenta, the Hamiltonian formalism can be derived from the complex kinetic-energy Hamiltonian $H^0(z,\bar{z}) : \mathcal{K} \to \mathbb{C}$ of the universal $(TM_{\rm wat} + \mathrm{i}\,T^*M_{\rm air})$ $-$fleet, in holomorphic coordinates z^i on $\mathcal{K}$ given as

$$H^0(z,\bar{z}) = \frac{1}{2}g_{i\bar{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j.$$

From the complexified affine Hamiltonian control function $H^a(z,\bar{z},u) : \mathcal{K} \to \mathbb{C}$, in local canonical coordinates on $\mathcal{K}$ given as

$$H^a(z,\bar{z},u) = H^0(z,\bar{z}) - H^j(z,\bar{z})\,u_j(t,z,\bar{z}), \qquad j = 1,\ldots,m < n$$

(with coupling Hamiltonians $H^j(z,\bar{z})$ and complex control inputs $u_j(t,z,\bar{z})$), the *complex affine Hamiltonian control system* (complexified and generalized from equation (21)) can be defined for the whole $(TM_{\mathrm{wat}} + \mathrm{i}\, T^*M_{\mathrm{air}})$ −fleet as

$$\begin{aligned} \dot{z}^i &= V^i + H^0_{\bar{z}^i} - H^j_{\bar{z}^i}\, u_j + R_{\bar{z}^i} \\ \dot{\bar{z}}^i &= F_i + \varepsilon^j_{ik} H^k_{z^j}\,\bar{z} - H^0_{z^i} + H^j_{z^i}\, u_j + R_{z^i} \end{aligned} \tag{25}$$

where $V^i = V^i(t,z,\bar{z})$ and $F_i = F_i(t,z,\bar{z})$ are the contravariant velocity controllers and the covariant force controllers, while the term $\varepsilon^j_{ik} H^k_{z^j}\,\bar{z}$ denotes (somewhat reduced) Kirchhoff-type mixed rotational terms in Hamiltonian form.

Equations (23) and (25) represent two alternative versions, Lagrangian and Hamiltonian, respectively, of the *discrete dynamics* for the universal $(\mathrm{UGV}, \mathrm{UAV}, \mathrm{USV}, \mathrm{UUV})$ fleet. They are both complexified extensions of our previously defined Lagrangian and Hamiltonian dynamics/control systems for the $(\mathrm{USV} + \mathrm{UUV})$-fleet and $(\mathrm{UGV} + \mathrm{UAV})$-swarm, respectively, each of them covering now the whole universal fleet. These two temporal systems are generalized in Appendix 5 to the spatiotemporal system called the complex Monge-Ampère equation.

4. Conclusion

The Kähler dynamics control model for a universal large-scale autonomous fleet (or, swarm of UGVs, UAVS, USVs and UUVs) is presented. Both Lagrangian and Hamiltonian formalisms, each capable of controlling the whole universal fleet, are developed on the Kähler manifold. For the case of a very large universal fleet, this Lagrangian/Hamiltonian model is extended in the Appendix to the Kähler–Ricci flow (or, complex Monge–Ampère equation) with the closed-form solution. We remark that the general dynamics formalism developed in this paper can be specialized to suit any kind of many-robot scenarios, be it on the ground, in the air, on the water surface and under the water, or any combination of these.

5. Appendix: Continuous Kähler Dynamics for a Very Large Fleet

Topology of the Kähler manifold $\mathcal{K}$ is based on the Kähler condition (22). Namely, in a local open chart $U \subset \mathcal{K}$, the Kähler symplectic form ω (24) can be alternatively defined as

$$\omega = \mathrm{i}\partial_i \partial_{\bar{j}} \varphi = \mathrm{i}\partial\overline{\partial}\varphi > 0\,, \qquad i,j = 1,\cdots,4n$$

where $\varphi \in U$ is a smooth real-valued function called the *Kähler potential*. This means that Kähler geometry (and therefore, dynamics) can be alternatively developed independently of any reference to the prior Lagrangian/Riemannian or Hamiltonian/symplectic geometry. So, although counter-intuitive, we could develop the universal fleet dynamics and control, in either Lagrangian or Hamiltonian form, using Kähler potentials φ instead of coordinates, velocities and momenta. Furthermore, as we can see below, using Kähler potentials φ we can develop a general model of *continuous dynamics* for a *very large-scale* universal $(\mathrm{UGV}, \mathrm{UAV}, \mathrm{USV}, \mathrm{UUV})$-fleet.

In general, any p-form α defined on the Kähler manifold $\mathcal{K}$ is called $\overline{\partial}$-*closed* iff $\overline{\partial}\alpha = 0$ and $\overline{\partial}$-*exact* iff $\alpha = \overline{\partial}\eta$ for some (p-1)-form η on $\mathcal{K}$. The associated *Dolbeault cohomology* group $H^{1,1}_{\overline{\partial}}(\mathcal{K}, \mathbb{R})$ is a complexification of the standard *de Rham cohomology* group $H^2_{\mathrm{d}}(\mathcal{K}, \mathbb{R})$, defined on $\mathcal{K}$ as the quotient

$$H^{1,1}_{\overline{\partial}}(\mathcal{K}, \mathbb{R}) = \frac{\{\overline{\partial}\text{-closed real (1,1)-forms}\}}{\{\overline{\partial}\text{-exact real (1,1)-forms}\}}.$$

The space $\mathcal{K}_{[\omega]}$ of Kähler forms ω with the same Kähler class $[\omega]$ is given by

$$\mathcal{K}_{[\omega]} = \{[\omega] \in H^2(\mathcal{K}, \mathbb{R})\ ;\ V = 0, \quad \omega + \mathrm{i}\partial\overline{\partial}\varphi > 0\}$$

i.e., the functional space $\mathcal{P}(\omega)$ of Kähler potentials φ on $(\mathcal{K}, \omega)$ is given by

$$\mathcal{P}(\omega) = \{\varphi \in C^\infty(\mathcal{K}, \mathbb{R})\ ;\ \omega_\varphi = \omega + \mathrm{i}\partial\overline{\partial}\varphi > 0\}.$$

More exactly, a Kähler form ω on $\mathcal{K}$ defines a nonzero element $[\omega] \in H^{1,1}_{\overline{\partial}}(\mathcal{K}, \mathbb{R})$. If a cohomology class $\alpha \in H^{1,1}_{\overline{\partial}}(\mathcal{K}, \mathbb{R})$ can be written as $\alpha = [\omega]$ for some Kähler form ω on $\mathcal{K}$ then we say that α is a *Kähler class* (and write $\alpha > 0$). Therefore, the Kähler class of ω is its cohomology class $[\omega] \in H^{1,1}_{\overline{\partial}}(\mathcal{K}, \mathbb{R})$. Alternatively, in terms of $H^2_{\mathrm{d}}(\mathcal{K}, \mathbb{R})$, the Kähler class of ω is its cohomology class $[\omega] \in H^2_{\mathrm{d}}(\mathcal{K}, \mathbb{R})$. Usually, all this is simply written: the Kähler class of ω is the cohomology class $[\omega] \in H^2(\mathcal{K}, \mathbb{R})$ represented by ω.

The $\partial\overline{\partial}$-*Lemma* (the holomorphic version of the *Poincaré Lemma,* also follows from the Hodge theory) states: "Let $\mathcal{K}$ be a compact Kähler manifold and suppose that $0 = [\alpha] \in H^{1,1}_{\overline{\partial}}(\mathcal{K}, \mathbb{R})$ for a real smooth $\overline{\partial}$-closed $(1,1)$-form α. Then there exists a real smooth Kähler potential φ (uniquely determined up to the addition of a constant) with $\alpha = \mathrm{i}\partial\overline{\partial}\varphi$." In other words, a real $(1,1)$-form α is $\overline{\partial}$-exact iff it is $\partial\overline{\partial}$-exact. It is an immediate consequence of the $\partial\overline{\partial}$-Lemma that if ω and ω_φ are Kähler forms in the same Kähler class on $\mathcal{K}$, then $\omega_\varphi = \omega + \mathrm{i}\partial\overline{\partial}\varphi$ for some smooth Kähler potential function φ.

Next we need to define curvatures on the Kähler manifold $\mathcal{K}$. The mixed and covariant *Riemannian curvature tensors* on $\mathcal{K}$ are

$$R^m_{ik\bar{l}} = -\partial_{\bar{l}}\Gamma^m_{ik} \qquad \text{and} \qquad R_{i\bar{j}k\bar{l}} = g_{m\bar{j}}R^m_{ik\bar{l}}\,.$$

Locally, in an open chart $U \subset \mathcal{K}$, the covariant Riemann tensor reads

$$R_{i\bar{j}k\bar{l}} = -\partial_i\partial_{\bar{j}}g_{k\bar{l}} + g^{\bar{q}p}(\partial_i g_{k\bar{q}})(\partial_{\bar{j}}g_{p\bar{l}}).$$

The curvature tensor $R_{i\bar{j}k\bar{l}}$ has the following three symmetries:

$\overline{R_{i\bar{j}k\bar{l}}} = R_{j\bar{i}l\bar{k}}$ (complex-conjugate)
$R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{i\bar{l}k\bar{j}}$ (I Bianchi identity), and
$\nabla_m R_{i\bar{j}k\bar{l}} = \nabla_i R_{m\bar{j}k\bar{l}}$ (II Bianchi identity)

where $\nabla_i \equiv \partial_i + g_{k\bar{q}}g^{\bar{q}j}\Gamma^k_{i\,j}$ is the complex covariant derivative on $(\mathcal{K}, g)$.
Then, the *Ricci curvature* tensor is defined as its contraction

$$R_{i\bar{j}} = g^{\bar{l}k}R_{i\bar{j}k\bar{l}} = g^{\bar{l}k}R_{k\bar{l}i\bar{j}} = R^k_{k\,i\bar{j}}$$

while its trace is the scalar curvature: $R = g^{\bar{j}i}R_{i\bar{j}}$. Locally, in an open chart $U \subset \mathcal{K}$, Ricci tensor is given by

$$R_{i\bar{j}}(g) = -\partial\bar{\partial}\log[\det(g_{i\bar{j}})], \qquad i,j = 1,\cdots,4n.$$

The associated *Ricci form* $\mathrm{Ric}(g)$ is the closed (1,1)-form on $\mathcal{K}$ given by

$$\mathrm{Ric}(g) \equiv \mathrm{Ric}(\omega) = \mathrm{i}R_{i\bar{j}}(g)\,\mathrm{d}z^i \wedge \mathrm{d}z^{\bar{j}} = -\mathrm{i}\partial\bar{\partial}\log[\det(g_{i\bar{j}})]. \tag{26}$$

Now we have all the necessary ingredients to derive the continuous dynamics model for a *very large-scale* universal autonomous $(\mathrm{UGV}, \mathrm{UAV}, \mathrm{USV}, \mathrm{UUV})$ fleet. For this, we recall that the *Ricci flow* on a Riemannian n-manifold M (introduced by R. Hamilton [3–5] and subsequently used by G. Perelman to prove the 100-year old Poincaré Conjecture), is governed by the nonlinear evolution equation of the Riemannian metric

$$\partial_t g_{ij}(t) = -2R_{ij}(t), \qquad i,j = 1,...,n \tag{27}$$

which in local harmonic coordinates on M can be rewritten as

$$\partial_t g_{ij}(t) = \Delta_M g_{ij} + Q_{ij}(g_{ij}, \partial g_{ij}) \tag{28}$$

where Δ_M is the Laplace-Beltrami operator defined locally on M as

$$\Delta_M \equiv \frac{1}{\sqrt{\det(g_{ij})}}\partial_{x^i}\left(\sqrt{\det(g)}g^{ij}\partial_{x^i}\right)$$

while the tensor function $Q_{ij}(g_{ij}, \partial g_{ij})$ is quadratic in g_{ij} and its first order partial derivatives ∂g_{ij}. Later, in [10], we proposed equations (27)-(28) as a general model for all real-valued nonlinear reaction-diffusion systems [with symmetric, positive-definite diffusion matrix $\mathbf{D}$, concentration state vector $\mathbf{u}(\mathbf{x}, t)$ and local reactions $\mathbf{R}(\mathbf{u})$] of the form

$$\begin{array}{ccccc} \partial_t \mathbf{u} & = & \mathbf{D}\Delta\mathbf{u} & + & \mathbf{R}(\mathbf{u}) \\ \updownarrow & & \updownarrow & & \updownarrow \\ \partial_t g_{ij} & = & \Delta_M g_{ij} & + & Q_{ij}(g_{ij}, \partial g_{ij}). \end{array}$$

To make the Kähler dynamics model, we need to generalize the Ricci flow (27)-(28) from the Riemannian n-manifold M to the Kähler $4n$-manifold $\mathcal{K}$, as shown in the previous subsection. To perform this generalization, we first remark that Kähler manifolds are usually classified into the following three subcategories, based on their first Chern class in Dolbeault cohomology. The first Chern class, denoted by $c_1(\mathcal{K})$, of a Kähler manifold $(\mathcal{K}, g)$, is defined as the cohomology class $[\mathrm{Ric}(g)] \in H^{1,1}_{\bar{\partial}}(\mathcal{K}, \mathbb{R})$. A compact (i.e., closed and bounded) Kähler manifold $(\mathcal{K}, g)$ with positive first Chern class, $c_1(\mathcal{K}) > 0$, is called the *Fano manifold* (the stage for our dynamics) in which case, $[\omega] = \pi c_1(\mathcal{K})$. A compact Kähler manifold with vanishing first Chern class, $c_1(\mathcal{K}) = 0$, is called the *Calabi–Yau manifold* (the stage for super-string theory). A compact Kähler manifold with negative first Chern class, $c_1(\mathcal{K}) < 0$, is called the *Kähler–Einstein manifold* (the stage for complex gravity theory), it admits the metric g defined by: $g = -\mathrm{Ric}(g)$.

We remark that the metric g on $\mathcal{K}$ is called the Kähler–Einstein metric iff

$$\mathrm{Ric}(\omega) = \lambda\omega, \qquad \text{for a real constant } \lambda = \frac{2\pi}{V}\int_{\mathcal{K}} c_1(\mathcal{K}) \wedge \omega^{n-1}$$

and if

$$\mathrm{Ric}(g) = 0 \quad \text{then } \ g \ \text{ is a Ricci–flat metric}$$

where $c_1(\mathcal{K})$ is the first Chern class of $\mathcal{K}$. If the manifold $\mathcal{K}$ admits a Ricci–flat metric [$\mathrm{Ric}(g) = 0$], then its first Chern class must vanish [$c_1(\mathcal{K}) = 0$]. This is the *Calabi conjecture*, proven by S.-T. Yau [21].

A Fano n-manifold $(\mathcal{K}, g)$ admits the (normalized) Kähler–Ricci flow [with the time-dependent Ricci form (26)]

$$\partial_t g_{ij}(t) = g_{ij}(t) - \mathrm{Ric}\,[g(t)] \tag{29}$$

which is locally, in an open chart $U \subset \mathcal{K}$, starting from some smooth initial Kähler metric tensor $g_0 = g_{i\bar{j}}(0)$, given by

$$\partial_t g_{i\bar{j}}(t) = g_{i\bar{j}}(t) - R_{i\bar{j}}(t), \qquad i, j = 1, \cdots, 4n.$$

The Kähler–Ricci flow (29) preserves the Kähler class $[\omega]$. It has a global solution $g(t) \equiv \omega(t)$ when $g_0 = g_{i\bar{j}}(0)$ has $[\omega] = 2\pi c_1(\mathcal{K})$ as its Kähler class, which is written as $g_0 \in 2\pi c_1(\mathcal{K})$. In particular, by the $\partial\bar{\partial}$-Lemma, there exists a family of real-valued functions $u(t)$, called Ricci potentials of the metric $g(t)$, which are special Kähler potentials. They are determined by

$$g_{i\bar{j}} - R_{i\bar{j}} \;=\; \partial_i \partial_{\bar{j}} u, \qquad \frac{1}{V_R}\int_{\mathcal{K}} \mathrm{e}^{-u(t)} \mathrm{d}v_g \;=\; 1$$

where $V_R = \int \mathrm{d}v_g$ is the volume of the Kähler–Ricci flow (29).

In terms of time-dependent Kähler potentials $\varphi = \varphi(t)$, the Kähler–Ricci flow (29) can be expressed as

$$\partial_t \varphi(t) = \varphi(t) + \log \frac{\omega_\varphi^n}{\omega^n} - g(t) \tag{30}$$

where the time-dependent Kähler metric form $g = g(t)$ is defined by

$$\mathrm{i}\partial\overline{\partial} g(t) = \mathrm{Ric}\,[\omega(t)] - \omega(t) \quad \text{and} \quad \int_{\mathcal{K}} (\mathrm{e}^{g(t)} - 1)\omega^n = 0.$$

The nth power ω^n of the Kähler symplectic form ω is the volume form on $\mathcal{K}$, given by

$$\omega^n = \frac{1}{n!}\mathrm{i}^n \det\left(g_{i\bar{j}}\right) \mathrm{d}z^1 \wedge \mathrm{d}z^{\bar{1}} \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}z^{\overline{n}}$$

so that $V = \int_{\mathcal{K}} \omega^n$ is the standard volume on $\mathcal{K}$.

The corresponding evolutions of the Ricci curvature $R_{i\bar{j}} = R_{i\bar{j}}(t)$ and the scalar curvature $R = R(t)$ on $(\mathcal{K}, g)$ are respectively governed by

$$\partial_t R_{i\bar{j}} = \triangle R_{i\bar{j}} + R_{i\bar{j}p\bar{q}} R_{q\overline{p}} - R_{i\overline{p}} R_{p\bar{j}}\,, \qquad \partial_t R = \triangle R + R_{i\bar{j}} R_{j\bar{i}} - R$$

starting from some smooth initial Ricci and scalar curvatures, $R_{i\bar{j}}(0)$ and $R(0)$.

The existence of the Kähler–Ricci flow in a time interval $t \in [0, t_1)$ can be established as follows: If $\omega(t)$ is a solution of the Kähler–Ricci flow

$$\partial_t \omega(t) = -\mathrm{Ric}\,[\omega(t)]\,, \qquad \omega(0) = \omega_0 \tag{31}$$

then the corresponding cohomology class $[\omega(t)]$ with $[\omega(0)] = [\omega_0]$ evolves on $\mathcal{K}$ as the following ODE

$$\begin{aligned} &\partial_t[\omega(t)] \;=\; -c_1(\mathcal{K}), \qquad \text{with the solution} \\ &[\omega(t)] \;=\; [\omega_0] - t\,c_1(\mathcal{K}) = [\mathrm{i}\partial\overline{\partial}\varphi(0)] - t\,c_1(\mathcal{K}) = [\mathrm{i}g_0\,\mathrm{d}z^i \wedge \mathrm{d}z^{\bar{j}}] - t\,c_1(\mathcal{K}). \end{aligned} \tag{32}$$

So, the Kähler–Ricci flow (31) exists for $t \in [0, t_1)$ iff $[\omega_0] - t\,c_1(\mathcal{K}) > 0$ (see [1]).

Equation (31) can be rewritten as a *complex parabolic Monge–Ampère equation* (see [19, 21])

$$\partial_t \varphi = \log \frac{(\omega_\varphi + \mathrm{i}\partial\overline{\partial}\varphi)^n}{\omega^n}, \qquad \text{with} \;\; \omega_\varphi + \mathrm{i}\partial\overline{\partial}\varphi > 0$$

while, the normalized Kähler–Ricci flow (29) can be rewritten as a normalized complex Monge–Ampère equation

$$\partial_t \varphi = \log \frac{(\omega_0 + \mathrm{i}\partial\overline{\partial}\varphi)^n}{\omega^n} - \varphi, \qquad \text{with} \;\; \omega_\varphi + \mathrm{i}\partial\overline{\partial}\varphi > 0.$$

The Kähler–Ricci flow (31) with the solution (32) is our continuous dynamics model for a very large universal $(\mathrm{UGV} + \mathrm{UAV} + \mathrm{USV} + \mathrm{UUV})-$ fleet (swarm). For further technical details on Kähler geometry and the Kähler–Ricci flow, see (e.g. [13]) and the references therein.

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Vladimir G. Ivancevic
Senior Research Scientist
Joint and Operations Analysis Division
Defence Science & Technology Group
Edinburgh, SA 5111, AUSTRALIA
E-mail address: `Vladimir.Ivancevic@dst.defence.gov.au`