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TWO TYPES OF LORENTZ TRANSFORMATIONS FOR MASSLESS FIELDS

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Abstract. In the present paper we demonstrate that the massless fields can be described by two types of potentials with different space-time properties and different Lorentz transformations. In particular, we discuss the possible applications of such approach to the description of electromagnetic field and weak gravity.

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Keywords: Lorentz transformations, sedeonic equations for massless fields, space-time sedeons

1. Introduction

The group of Lorentz transformations is widely discussed especially in application for electromagnetic field [18], [19]. In particular, there is an asymmetry between Lorentz transformations for potentials and field strengths in electrodynamics. The potentials are transformed as the components of four-vector, while the field strengths as the components of four-tensor [8]. However, it can be shown that there is an alternative possibility of constructing equations for massless field with different transformational properties.

In recent years, there have been a few publications devoted to the reformulation of linear equations for electromagnetic field and weak gravity (gravitoelectromagnetism [10]) in terms of hypercomplex field potentials. The first approach is based on four-component quaternions, which consist of scalar and vector parts that adequately describes the four-vector concept of special relativity [7], [9], [2]. However since the system of Maxwell equations consist of four equations for scalar, pseudoscalar, vector and pseudovector values, the application of multi-component algebras is more appropriate. Taking into account this spatial symmetry several approaches have been proposed to describe massless fields on the basis of eight-component octonions [6], [16], [1] and octons [11], [4], [3]. However, a consistent relativistic consideration implies equally the space and time symmetries that require using the extended sixteen-component space-time algebras.

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Recently we proposed the space-time algebra of sixteen-component sedeons, which takes into account the symmetry of physical values with respect to the space-time inversion and realizes the scalar-vector representation of Poincare group [12], [13]. In particular we considered the equations for massive and massless fields based on sedeonic potentials and space-time operators [14], [15]. In the present paper we consider the generalized approach to the description of massless field on the basis of equations obtained as the limiting transition from the sedeonic equations for massive field and discuss the transformational properties of these equations with respect to the Lorentz transformations.

2. Algebra of Space-Time Sedeons

To begin with we briefly review the basic properties of sedeons [13]. The sedeonic algebra encloses four groups of values, which are differed with respect to spatial and time inversion.

1. Absolute scalars (A) and absolute vectors ($\vec{A}$) are not transformed under spatial and time inversion.
2. Time scalars ($B_{\mathbf{t}}$) and time vectors ($\vec{B}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
3. Space scalars ($C_{\mathbf{r}}$) and space vectors ($\vec{C}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
4. Space-time scalars ($D_{\mathbf{tr}}$) and space-time vectors ($\vec{D}_{\mathbf{tr}}$) are changed under spatial and time inversion.

The indexes **t** and **r** indicate the transformations (**t** for time inversion and **r** for spatial inversion), which change the corresponding values. The space-time sedeon $\tilde{\mathbf{S}}$ is defined by the following expression

$$\tilde{\mathbf{S}} = A + \vec{A} + B_{\mathbf{t}} + \vec{B}_{\mathbf{t}} + C_{\mathbf{r}} + \vec{C}_{\mathbf{r}} + D_{\mathbf{tr}} + \vec{D}_{\mathbf{tr}}. \tag{1}$$

Here and further we indicate the sedeon by bold symbol with wave. The components of sedeon (1) can be written in the sedeonic space-time basis as

$$\begin{aligned}
A &= \mathbf{e_0} A \mathbf{a_0}, & \vec{A} &= \mathbf{e_0}\left(A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3}\right)\\
B_{\mathbf{t}} &= \mathbf{e_t} B \mathbf{a_0}, & \vec{B}_{\mathbf{t}} &= \mathbf{e_t}\left(B_1\mathbf{a_1} + B_2\mathbf{a_2} + B_3\mathbf{a_3}\right)\\
C_{\mathbf{r}} &= \mathbf{e_r} C \mathbf{a_0}, & \vec{C}_{\mathbf{r}} &= \mathbf{e_r}\left(C_1\mathbf{a_1} + C_2\mathbf{a_2} + C_3\mathbf{a_3}\right)\\
D_{\mathbf{tr}} &= \mathbf{e_{tr}} D \mathbf{a_0}, & \vec{D}_{\mathbf{tr}} &= \mathbf{e_{tr}}\left(D_1\mathbf{a_1} + D_2\mathbf{a_2} + D_3\mathbf{a_3}\right).
\end{aligned} \tag{2}$$

The values $\mathbf{a_0}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are scalar-vector basis ($\mathbf{a_0} \equiv 1$ is absolute scalar unit and the values $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are absolute unit vectors generating the right Cartesian basis) and values $\mathbf{e_0}$, $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ are space-time basis ($\mathbf{e_0} \equiv 1$ is a absolute scalar unit; $\mathbf{e_t}$ is a time unit; $\mathbf{e_r}$ is a space unit; $\mathbf{e_{tr}}$ is a space-time unit). Further we will omit the units $\mathbf{a_0}$ and $\mathbf{e_0}$ for simplicity.

The multiplication and commutation rules for the sedeonic absolute unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and space-time units $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ are presented in the Tables 1 and 2 respectively (in the tables and further the value "i" is the imaginary unit ($\mathrm{i}^2 = -1$)). Note that sedeonic units $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ commute with $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$.

Table 1

	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{a_1}$	1	$\mathrm{i}\mathbf{a_3}$	$-\mathrm{i}\mathbf{a_2}$
$\mathbf{a_2}$	$-\mathrm{i}\mathbf{a_3}$	1	$\mathrm{i}\mathbf{a_1}$
$\mathbf{a_3}$	$\mathrm{i}\mathbf{a_2}$	$-\mathrm{i}\mathbf{a_1}$	1

Table 2

	$\mathbf{e_t}$	$\mathbf{e_r}$	$\mathbf{e_{tr}}$
$\mathbf{e_t}$	1	$\mathrm{i}\mathbf{e_{tr}}$	$-\mathrm{i}\mathbf{e_r}$
$\mathbf{e_r}$	$-\mathrm{i}\mathbf{e_{tr}}$	1	$\mathrm{i}\mathbf{e_t}$
$\mathbf{e_{tr}}$	$\mathrm{i}\mathbf{e_r}$	$-\mathrm{i}\mathbf{e_t}$	1

Thus the sedeon $\tilde{\mathbf{S}}$ is the complicated space-time object consisting of absolute scalar, time scalar, space scalar, space-time scalar, absolute vector, time vector, space vector and space-time vector.

In sedeonic algebra we assume the Clifford multiplication of vectors. For example, the sedeonic product of two absolute vectors $\vec{A}$ and $\vec{B}$ can be presented in the following form

$$\vec{A}\vec{B} = \left(\vec{A} \cdot \vec{B}\right) + \left[\vec{A} \times \vec{B}\right]. \tag{3}$$

Here we denote the sedeonic scalar multiplication of two vectors (internal product) by symbol "·" and round brackets

$$\left(\vec{A} \cdot \vec{B}\right) = A_1B_1 + A_2B_2 + A_3B_3 \tag{4}$$

and sedeonic vector multiplication (external product) by symbol "×" and square brackets

$$\left[\vec{A} \times \vec{B}\right] = \mathrm{i}\left(A_2B_3 - A_3B_2\right)\mathbf{a_1} + \mathrm{i}\left(A_3B_1 - A_1B_3\right)\mathbf{a_2} + \mathrm{i}\left(A_1B_2 - A_2B_1\right)\mathbf{a_3}. \tag{5}$$

Note that in sedeonic algebra the definition of the vector product differs from analogous expression in Gibbs-Heaviside vector algebra. For the transition from sedeons to the common used vector algebra the replacement

$$\mathrm{i}\left[\vec{A} \times \vec{B}\right] \Rightarrow -\mathbf{A} \times \mathbf{B} \tag{6}$$

should be made in all sedeonic expressions. Here $\mathbf{A}$ and $\mathbf{B}$ are the vectors in Gibbs-Heaviside algebra.

3. Sedeonic Equations for Massive Field

Here we shortly recall the sedeonic equations for massive field [21]. Let us consider the massive field with mass of quantum m_0. We introduce the following operators

$$\partial = \frac{1}{c}\frac{\partial}{\partial t}, \qquad \vec{\nabla} = \frac{\partial}{\partial x}\,\mathbf{a_1} + \frac{\partial}{\partial y}\,\mathbf{a_2} + \frac{\partial}{\partial z}\,\mathbf{a_3}, \qquad m = \frac{m_0 c}{\hbar}\cdot \tag{7}$$

Here c is the speed of light, $\hbar$ is the Plank constant. Then the sedeonic second-order wave equation for massive field can be presented as [14]

$$\left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - \mathrm{i}\mathbf{e_{tr}}m\right)\left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - \mathrm{i}\mathbf{e_{tr}}m\right)\tilde{\mathbf{W}} = \tilde{\mathbf{J}} \tag{8}$$

where $\tilde{\mathbf{W}}$ is a sedeonic potential, $\tilde{\mathbf{J}}$ is a phenomenological sedeonic source of massive field. Let us choose the potential as

$$\tilde{\mathbf{W}} = \mathrm{i}a_1\mathbf{e_t} - \mathrm{i}a_2\mathbf{e_r} + a_3 - \mathrm{i}a_4\mathbf{e_{tr}} + \vec{A}_1\mathbf{e_r} + \vec{A}_2\mathbf{e_t} - \vec{A}_3\mathbf{e_{tr}} + \mathrm{i}\vec{A}_4 \tag{9}$$

where components a_s and $\vec{A}_\mathrm{s}$ are real functions of coordinates and time. Here and further the index s = 1, 2, 3, 4. Also we take the source in the following form

$$\tilde{\mathbf{J}} = -\mathrm{i}\rho_1\mathbf{e_t} + \mathrm{i}\rho_2\mathbf{e_r} - \rho_3 + \mathrm{i}\rho_4\mathbf{e_{tr}} - \vec{j}_1\mathbf{e_r} - \vec{j}_2\mathbf{e_t} + \vec{j}_3\mathbf{e_{tr}} - \mathrm{i}\vec{j}_4 \tag{10}$$

where $\rho_s = 4\pi{\rho_s}'$ (${\rho_s}'$ is the volume density of charges) and $\vec{j}_s = 4\pi\vec{j}_s'$ ($\vec{j}_s'$ is the volume density of currents). Let us introduce also the sedeon of field strength $\tilde{\mathbf{E}}$ as

$$\tilde{\mathbf{E}} = -\varepsilon_1 + \mathrm{i}\varepsilon_2\mathbf{e_{tr}} + \mathrm{i}\varepsilon_3\mathbf{e_t} - \mathrm{i}\varepsilon_4\mathbf{e_r} + \vec{E}_1\mathbf{e_{tr}} - \mathrm{i}\vec{E}_2 + \vec{E}_3\mathbf{e_r} + \vec{E}_4\mathbf{e_t} \tag{11}$$

where the scalar ε_s and vector $\vec{E}_\mathrm{s}$ components are defined as

$$\begin{aligned}
\varepsilon_1 &= \partial a_1 + \left(\vec{\nabla}\cdot\vec{A}_1\right) + m a_4, & \vec{E}_1 &= -\,\partial\vec{A}_1 - \vec{\nabla}a_1 + \mathrm{i}\left[\vec{\nabla}\times\vec{A}_2\right] + m\vec{A}_4\\
\varepsilon_2 &= \partial a_2 + \left(\vec{\nabla}\cdot\vec{A}_2\right) - m a_3, & \vec{E}_2 &= -\,\partial\vec{A}_2 - \vec{\nabla}a_2 - \mathrm{i}\left[\vec{\nabla}\times\vec{A}_1\right] - m\vec{A}_3\\
\varepsilon_3 &= \partial a_3 + \left(\vec{\nabla}\cdot\vec{A}_3\right) + m a_2, & \vec{E}_3 &= -\,\partial\vec{A}_3 - \vec{\nabla}a_3 - \mathrm{i}\left[\vec{\nabla}\times\vec{A}_4\right] + m\vec{A}_2\\
\varepsilon_4 &= \partial a_4 + \left(\vec{\nabla}\cdot\vec{A}_4\right) - m a_1, & \vec{E}_4 &= -\,\partial\vec{A}_4 - \vec{\nabla}a_4 + \mathrm{i}\left[\vec{\nabla}\times\vec{A}_3\right] - m\vec{A}_1.
\end{aligned} \tag{12}$$

Then

$$\left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - \mathrm{i}\mathbf{e_{tr}}m\right)\mathbf{W} = \tilde{\mathbf{E}} \tag{13}$$

and the sedeonic wave equation (8) takes the form

$$\left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - \mathrm{i}\mathbf{e_{tr}}m\right)\tilde{\mathbf{E}} = \tilde{\mathbf{J}}. \tag{14}$$

Producing the action of the operator on the left side of equation (14) and separating the values with different space-time properties, we obtain a system of equations for the field strengths, similar to the system of Maxwell equations in electrodynamics

$$\begin{aligned}
&\partial\varepsilon_1 + \left(\vec{\nabla}\cdot\vec{E}_1\right) - m\varepsilon_4 = \rho_1, \qquad \partial\vec{E}_1 + \vec{\nabla}\varepsilon_1 + \mathrm{i}\left[\vec{\nabla}\times\vec{E}_2\right] + m\vec{E}_4 = -\vec{j}_1\\
&\partial\varepsilon_2 + \left(\vec{\nabla}\cdot\vec{E}_2\right) + m\varepsilon_3 = \rho_2, \qquad \partial\vec{E}_2 + \vec{\nabla}\varepsilon_2 - \mathrm{i}\left[\vec{\nabla}\times\vec{E}_1\right] - m\vec{E}_3 = -\vec{j}_2\\
&\partial\varepsilon_3 + \left(\vec{\nabla}\cdot\vec{E}_3\right) - m\varepsilon_2 = \rho_3, \qquad \partial\vec{E}_3 + \vec{\nabla}\varepsilon_3 - \mathrm{i}\left[\vec{\nabla}\times\vec{E}_4\right] + m\vec{E}_2 = -\vec{j}_3\\
&\partial\varepsilon_4 + \left(\vec{\nabla}\cdot\vec{E}_4\right) + m\varepsilon_1 = \rho_4, \qquad \partial\vec{E}_4 + \vec{\nabla}\varepsilon_4 + \mathrm{i}\left[\vec{\nabla}\times\vec{E}_3\right] - m\vec{E}_1 = -\vec{j}_4.
\end{aligned} \tag{15}$$

All these equations are coupled by the mass terms. From the system of equations (15) we can get some relations for the energy and momentum of the massive field. Multiplying each of the equations (15) to the corresponding field strength and adding these equations to each other, we obtain

$$\begin{aligned}
&\frac{1}{2}\,\partial\left({\varepsilon_1}^2 + {\varepsilon_2}^2 + {\varepsilon_3}^2 + {\varepsilon_4}^2 + {\vec{E}_1}^2 + {\vec{E}_2}^2 + {\vec{E}_3}^2 + {\vec{E}_4}^2\right)\\
&\quad\left(\vec{\nabla}\cdot\left(\varepsilon_1\vec{E}_1 + \varepsilon_2\vec{E}_2 + \varepsilon_3\vec{E}_3 + \varepsilon_4\vec{E}_4 - \mathrm{i}[\vec{E}_1\times\vec{E}_2] + \mathrm{i}[\vec{E}_3\times\vec{E}_4]\right)\right)\\
&= \varepsilon_1\rho_1 + \varepsilon_2\rho_2 + \varepsilon_3\rho_3 + \varepsilon_4\rho_4\\
&\quad - \left(\vec{E}_1\cdot\vec{j}_1\right) - \left(\vec{E}_2\cdot\vec{j}_2\right) - \left(\vec{E}_3\cdot\vec{j}_3\right) - \left(\vec{E}_4\cdot\vec{j}_4\right).
\end{aligned} \tag{16}$$

Let us introduce the volume density of energy as

$$w = \frac{1}{2}\left({\varepsilon_1}^2 + {\varepsilon_2}^2 + {\varepsilon_3}^2 + {\varepsilon_4}^2 + {\vec{E}_1}^2 + {\vec{E}_2}^2 + {\vec{E}_3}^2 + {\vec{E}_4}^2\right) \tag{17}$$

and volume density of energy flow as

$$\vec{P} = \varepsilon_1\vec{E}_1 + \varepsilon_2\vec{E}_2 + \varepsilon_3\vec{E}_3 + \varepsilon_4\vec{E}_4 - \mathrm{i}\left[\vec{E}_1\times\vec{E}_2\right] + \mathrm{i}\left[\vec{E}_3\times\vec{E}_4\right]. \tag{18}$$

Then equation (16) have the sense of Pointing theorem for massive field

$$\begin{aligned}
\partial w + (\vec{\nabla}\cdot\vec{P}) &= \varepsilon_1\rho_1 + \varepsilon_2\rho_2 + \varepsilon_3\rho_3 + \varepsilon_4\rho_4\\
&\quad - \left(\vec{E}_1\cdot\vec{j}_1\right) - \left(\vec{E}_2\cdot\vec{j}_2\right) - \left(\vec{E}_3\cdot\vec{j}_3\right) - \left(\vec{E}_4\cdot\vec{j}_4\right).
\end{aligned} \tag{19}$$

Corresponding expression for the energy gradient is

$$\begin{aligned}
\vec{\nabla} w & +2\mathrm{i}m\left[\vec{E}_1\times\vec{E}_3\right]+2\mathrm{i}m\left[\vec{E}_2\times\vec{E}_4\right] \\
& -\mathrm{i}\partial\left[\vec{E}_1\times\vec{E}_2\right]+\varepsilon_1\partial\vec{E}_1+\varepsilon_2\partial\vec{E}_2-\vec{E}_1\partial\varepsilon_1-\vec{E}_2\partial\varepsilon_2 \\
& -\left(\vec{E}_1\cdot\vec{\nabla}\right)\vec{E}_1-\left(\vec{E}_2\cdot\vec{\nabla}\right)\vec{E}_2-\vec{E}_1\left(\vec{\nabla}\cdot\vec{E}_1\right)-\vec{E}_2\left(\vec{\nabla}\cdot\vec{E}_2\right) \\
& +\mathrm{i}\varepsilon_1\left[\vec{\nabla}\times\vec{E}_2\right]-\mathrm{i}\varepsilon_2\left[\vec{\nabla}\times\vec{E}_1\right]+\mathrm{i}\left[\vec{E}_2\times\vec{\nabla}\varepsilon_1\right]-\mathrm{i}\left[\vec{E}_1\times\vec{\nabla}\varepsilon_2\right] \\
& -\mathrm{i}\partial\left[\vec{E}_3\times\vec{E}_4\right]+\varepsilon_3\partial\vec{E}_3+\varepsilon_4\partial\vec{E}_4-\vec{E}_3\partial\varepsilon_3-\vec{E}_4\partial\varepsilon_4 \\
& -\left(\vec{E}_3\cdot\vec{\nabla}\right)\vec{E}_3-\left(\vec{E}_4\cdot\vec{\nabla}\right)\vec{E}_4-\vec{E}_3\left(\vec{\nabla}\cdot\vec{E}_3\right)-\vec{E}_4\left(\vec{\nabla}\cdot\vec{E}_4\right) \\
& -\mathrm{i}\varepsilon_3\left[\vec{\nabla}\times\vec{E}_4\right]+\mathrm{i}\varepsilon_4\left[\vec{\nabla}\times\vec{E}_3\right]-\mathrm{i}\left[\vec{E}_4\times\vec{\nabla}\varepsilon_3\right]+\mathrm{i}\left[\vec{E}_3\times\vec{\nabla}\varepsilon_4\right] \\
= & -\vec{E}_1\rho_1-\vec{E}_2\rho_2-\varepsilon_1\vec{j}_1-\varepsilon_2\vec{j}_2-\mathrm{i}\left[\vec{E}_2\times\vec{j}_1\right]+\mathrm{i}\left[\vec{E}_1\times\vec{j}_2\right] \\
& -\vec{E}_3\rho_3-\vec{E}_4\rho_4-\varepsilon_3\vec{j}_3-\varepsilon_4\vec{j}_4-\mathrm{i}\left[\vec{E}_4\times\vec{j}_3\right]+\mathrm{i}\left[\vec{E}_3\times\vec{j}_4\right].
\end{aligned} \tag{20}$$

Note that this expression contains two terms with mass.

4. Two Types of Lorentz Transformations

In the frames of sedeonic algebra the transformation of values from one inertial coordinate system to another are carried out with the following sedeons [13]

$$\tilde{\mathbf{L}}=\cosh\vartheta-\mathbf{e_{tr}}\vec{n}\sinh\vartheta, \qquad \tilde{\mathbf{L}}^*=\cosh\vartheta+\mathbf{e_{tr}}\vec{n}\sinh\vartheta \tag{21}$$

where $\tanh(2\vartheta)=v/c$ and v is velocity of motion along the unit vector $\vec{n}$. The transformed sedeonic potential can be presented as

$$\tilde{\mathbf{W}}'=\tilde{\mathbf{L}}^*\tilde{\mathbf{W}}\,\tilde{\mathbf{L}}. \tag{22}$$

In the transition from one inertial system to another the components of potential are transformed in different ways. The components of the first group (Group I), which comprises $a_1, a_2, \vec{A}_1, \vec{A}_2$, are transformed as follows

$$\begin{aligned}
a_1' &= a_1\cosh(2\vartheta)-\left(\vec{n}\cdot\vec{A}_1\right)\sinh(2\vartheta) \\
a_2' &= a_2\cosh(2\vartheta)-\left(\vec{n}\cdot\vec{A}_2\right)\sinh(2\vartheta) \\
\vec{A}_1' &= \vec{A}_1+\left(\vec{n}\cdot\vec{A}_1\right)\vec{n}\left(\cosh(2\vartheta)-1\right)-a_1\vec{n}\sinh(2\vartheta) \\
\vec{A}_2' &= \vec{A}_2+\left(\vec{n}\cdot\vec{A}_2\right)\vec{n}(\cosh(2\vartheta)-1)-a_2\vec{n}\sinh(2\vartheta).
\end{aligned} \tag{23}$$

If we take the x axis directed along the vector $\vec{n}$, then we get

$$\begin{aligned}
A'_{1y} &= A_{1y}, & a'_1 &= a_1\frac{1}{\sqrt{1-(v/c)^2}} - A_{1x}\frac{v/c}{\sqrt{1-(v/c)^2}}\\
A'_{1z} &= A_{1z}, & a'_2 &= a_2\frac{1}{\sqrt{1-(v/c)^2}} - A_{2x}\frac{v/c}{\sqrt{1-(v/c)^2}}\\
A'_{2y} &= A_{2y}, & A'_{1x} &= A_{1x}\frac{1}{\sqrt{1-(v/c)^2}} - a_1\frac{v/c}{\sqrt{1-(v/c)^2}}\\
A'_{2z} &= A_{2z}, & A'_{2x} &= A_{2x}\frac{1}{\sqrt{1-(v/c)^2}} - a_2\frac{v/c}{\sqrt{1-(v/c)^2}}.
\end{aligned} \tag{24}$$

The components of the second group (Group II), which comprises $a_3, a_4, \vec{A}_3, \vec{A}_4$ transformed as follows

$$\begin{aligned}
a'_3 &= a_3\\
a'_4 &= a_4\\
\vec{A}'_3 &= \vec{A}_3\cosh(2\vartheta) - \left(\vec{n}\cdot\vec{A}_3\right)\vec{n}(\cosh(2\vartheta)-1) - \mathrm{i}\left[\vec{n}\times\vec{A}_4\right]\sinh(2\vartheta)\\
\vec{A}'_4 &= \vec{A}_4\cosh(2\vartheta) - \left(\vec{n}\cdot\vec{A}_4\right)\vec{n}(\cosh(2\vartheta)-1) + \mathrm{i}\left[\vec{n}\times\vec{A}_3\right]\sinh(2\vartheta).
\end{aligned} \tag{25}$$

For the x axis directed along the vector $\vec{n}$ we get

$$\begin{aligned}
a'_3 &= a_3, & A'_{3y} &= A_{3y}\frac{1}{\sqrt{1-(v/c)^2}} - A_{4z}\frac{v/c}{\sqrt{1-(v/c)^2}}\\
a'_4 &= a_4, & A'_{3z} &= A_{3z}\frac{1}{\sqrt{1-(v/c)^2}} + A_{4y}\frac{v/c}{\sqrt{1-(v/c)^2}}\\
A'_{3x} &= A_{3x}, & A'_{4y} &= A_{4y}\frac{1}{\sqrt{1-(v/c)^2}} + A_{3z}\frac{v/c}{\sqrt{1-(v/c)^2}}\\
A'_{4x} &= A_{4x}, & A'_{4z} &= A_{4z}\frac{1}{\sqrt{1-(v/c)^2}} - A_{3y}\frac{v/c}{\sqrt{1-(v/c)^2}}.
\end{aligned} \tag{26}$$

Thus, these two groups of potentials are differed by their space-time properties and by Lorentz transformations. Similarly, the field sources are also divided into two groups differing by Lorentz transformations

$$\begin{aligned}
\rho'_1 &= \rho_1\cosh(2\vartheta) - \left(\vec{n}\cdot\vec{j}_1\right)\sinh(2\vartheta)\\
\rho'_2 &= \rho_2\cosh(2\vartheta) - \left(\vec{n}\cdot\vec{j}_2\right)\sinh(2\vartheta)\\
\vec{j}'_1 &= \vec{j}_1 + \left(\vec{n}\cdot\vec{j}_1\right)\vec{n}(\cosh(2\vartheta)-1) - \rho_1\vec{n}\sinh(2\vartheta)\\
\vec{j}'_2 &= \vec{j}_2 + \left(\vec{n}\cdot\vec{j}_2\right)\vec{n}(\cosh(2\vartheta)-1) - \rho_2\vec{n}\sinh(2\vartheta)
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
\rho_3' &= \rho_3 \\
\rho_4' &= \rho_4 \\
\vec{j}_3' &= \vec{j}_3 \cosh(2\vartheta) - \left(\vec{n} \cdot \vec{j}_3\right) \vec{n}(\cosh(2\vartheta) - 1) - \mathrm{i} \left[\vec{n} \times \vec{j}_4\right] \sinh(2\vartheta) \\
\vec{j}_4' &= \vec{j}_4 \cosh(2\vartheta) - \left(\vec{n} \cdot \vec{j}_4\right) \vec{n}(\cosh(2\vartheta) - 1) + \mathrm{i} \left[\vec{n} \times \vec{j}_3\right] \sinh(2\vartheta).
\end{aligned} \tag{28}$$

Also we have the following Lorentz transformations for the field strengths

$$\begin{aligned}
\varepsilon_1' &= \varepsilon_1 \\
\varepsilon_2' &= \varepsilon_4 \\
\vec{E}_3' &= \vec{E}_3 \cosh(2\vartheta) - \left(\vec{n} \cdot \vec{E}_3\right) \vec{n}(\cosh(2\vartheta) - 1) - \mathrm{i} \left[\vec{n} \times \vec{E}_4\right] \sinh(2\vartheta) \\
\vec{E}_4' &= \vec{E}_4 \cosh(2\vartheta) - \left(\vec{n} \cdot \vec{E}_4\right) \vec{n}(\cosh(2\vartheta) - 1) + \mathrm{i} \left[\vec{n} \times \vec{E}_3\right] \sinh(2\vartheta)
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
\varepsilon_3' &= \varepsilon_3 \cosh(2\vartheta) - \left(\vec{n} \cdot \vec{E}_3\right) \sinh(2\vartheta) \\
\varepsilon_4' &= \varepsilon_4 \cosh(2\vartheta) - \left(\vec{n} \cdot \vec{E}_4\right) \sinh(2\vartheta) \\
\vec{E}_3' &= \vec{E}_3 + \left(\vec{n} \cdot \vec{E}_3\right) \vec{n}(\cosh(2\vartheta) - 1) - \varepsilon_3 \vec{n} \sinh(2\vartheta) \\
\vec{E}_4' &= \vec{E}_4 + \left(\vec{n} \cdot \vec{E}_4\right) \vec{n}(\cosh(2\vartheta) - 1) - \varepsilon_2 \vec{n} \sinh(2\vartheta).
\end{aligned} \tag{30}$$

5. Sedeonic Equations for Massless Electromagnetic Fields

If the mass of field quantum m_0 is zero, then the wave equation (8) describes the massless field [15]. In this case we have

$$\left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right) \left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right) \tilde{\mathbf{W}} = \tilde{\mathbf{J}}. \tag{31}$$

The sedeonic potential $\tilde{\mathbf{W}}$ and field source $\tilde{\mathbf{J}}$ have the same space-time structure (9), (10) and the same Lorentz transformations. In massless case we can define using (12) two groups of field strengths

$$\begin{aligned}
\varepsilon_1 &= \partial a_1 + \left(\vec{\nabla} \cdot \vec{A}_1\right), \qquad & \vec{E}_1 &= -\,\partial \vec{A}_1 - \vec{\nabla} a_1 + \mathrm{i} \left[\vec{\nabla} \times \vec{A}_2\right] \\
\varepsilon_2 &= \partial a_2 + \left(\vec{\nabla} \cdot \vec{A}_2\right), \qquad & \vec{E}_2 &= -\,\partial \vec{A}_2 - \vec{\nabla} a_2 - \mathrm{i} \left[\vec{\nabla} \times \vec{A}_1\right]
\end{aligned} \tag{32}$$

and

$$\begin{aligned} \varepsilon_3 &= \partial a_3 + \left(\vec{\nabla} \cdot \vec{A}_3\right), & \vec{E}_3 &= -\,\partial \vec{A}_3 - \vec{\nabla} a_3 - \mathrm{i}\left[\vec{\nabla} \times \vec{A}_4\right] \\ \varepsilon_4 &= \partial a_4 + \left(\vec{\nabla} \cdot \vec{A}_4\right), & \vec{E}_4 &= -\,\partial \vec{A}_4 - \vec{\nabla} a_4 + \mathrm{i}\left[\vec{\nabla} \times \vec{A}_3\right]. \end{aligned} \tag{33}$$

These field strengths satisfy two independent systems of Maxwell equations

$$\begin{aligned} \partial\varepsilon_1 + \left(\vec{\nabla} \cdot \vec{E}_1\right) &= \rho_1, & \partial \vec{E}_1 + \vec{\nabla}\varepsilon_1 + \mathrm{i}\left[\vec{\nabla} \times \vec{E}_2\right] &= -\,\vec{j}_1 \\ \partial\varepsilon_2 + \left(\vec{\nabla} \cdot \vec{E}_2\right) &= \rho_2, & \partial \vec{E}_2 + \vec{\nabla}\varepsilon_2 - \mathrm{i}\left[\vec{\nabla} \times \vec{E}_1\right] &= -\,\vec{j}_2 \end{aligned} \tag{34}$$

and

$$\begin{aligned} \partial\varepsilon_3 + \left(\vec{\nabla} \cdot \vec{E}_3\right) &= \rho_3, & \partial \vec{E}_3 + \vec{\nabla}\varepsilon_3 - \mathrm{i}\left[\vec{\nabla} \times \vec{E}_4\right] &= -\,\vec{j}_3 \\ \partial\varepsilon_4 + \left(\vec{\nabla} \cdot \vec{E}_4\right) &= \rho_4, & \partial \vec{E}_4 + \vec{\nabla}\varepsilon_4 + \mathrm{i}\left[\vec{\nabla} \times \vec{E}_3\right] &= -\,\vec{j}_4. \end{aligned} \tag{35}$$

For simplicity let us consider the equations without magnetic charges ($\rho_2 = \rho_3 = 0$) and magnetic currents ($\vec{j}_2 = \vec{j}_3 = 0$). Taking into account the Lorentz gauge

$$\begin{aligned} \partial a_1 + \left(\vec{\nabla} \cdot \vec{A}_1\right) &= 0, & \partial a_2 + \left(\vec{\nabla} \cdot \vec{A}_2\right) &= 0 \\ \partial a_3 + \left(\vec{\nabla} \cdot \vec{A}_3\right) &= 0, & \partial a_4 + \left(\vec{\nabla} \cdot \vec{A}_4\right) &= 0 \end{aligned} \tag{36}$$

the Maxwell equations (34) and (35) can be rewritten as

$$\begin{aligned} \left(\vec{\nabla} \cdot \vec{E}_1\right) &= \rho_1, & \partial \vec{E}_1 + \mathrm{i}\left[\vec{\nabla} \times \vec{E}_2\right] &= -\,\vec{j}_1 \\ \left(\vec{\nabla} \cdot \vec{E}_2\right) &= 0, & \partial \vec{E}_2 - \mathrm{i}\left[\vec{\nabla} \times \vec{E}_1\right] &= 0 \end{aligned} \tag{37}$$

and

$$\begin{aligned} \left(\vec{\nabla} \cdot \vec{E}_3\right) &= 0, & \partial \vec{E}_3 - \mathrm{i}\left[\vec{\nabla} \times \vec{E}_4\right] &= 0 \\ \left(\vec{\nabla} \cdot \vec{E}_4\right) &= \rho_4, & \partial \vec{E}_4 + \mathrm{i}\left[\vec{\nabla} \times \vec{E}_3\right] &= -\,\vec{j}_4. \end{aligned} \tag{38}$$

Then the expression for the gradient of volume density of energy (20) in this case takes the following form

$$\begin{aligned} &\vec{\nabla}\left(\vec{E}_1^{\,2} + \vec{E}_2^{\,2} + \vec{E}_3^{\,2} + \vec{E}_4^{\,2}\right) - \mathrm{i}\partial\left[\vec{E}_1 \times \vec{E}_2\right] - \mathrm{i}\partial\left[\vec{E}_3 \times \vec{E}_4\right] \\ &- \left(\vec{E}_1 \cdot \vec{\nabla}\right)\vec{E}_1 - \left(\vec{E}_2 \cdot \vec{\nabla}\right)\vec{E}_2 - \vec{E}_1\left(\vec{\nabla} \cdot \vec{E}_1\right) - \vec{E}_2\left(\vec{\nabla} \cdot \vec{E}_2\right) \\ &- \left(\vec{E}_3 \cdot \vec{\nabla}\right)\vec{E}_3 - \left(\vec{E}_4 \cdot \vec{\nabla}\right)\vec{E}_4 - \vec{E}_3\left(\vec{\nabla} \cdot \vec{E}_3\right) - \vec{E}_4\left(\vec{\nabla} \cdot \vec{E}_4\right) \\ &\qquad = -\vec{E}_1\rho_1 - \mathrm{i}\left[\vec{E}_2 \times \vec{j}_1\right] - \vec{E}_4\rho_4 - \mathrm{i}\left[\vec{E}_3 \times \vec{j}_4\right]. \end{aligned} \tag{39}$$

It can be clearly seen that in this expression the field strengths and charges of first group are not mixed with the field strengths and charges of second group. So the expressions for the electromagnetic forces have the following form

$$\vec{F}_{eI} = \vec{E}_1\rho_1 + \mathrm{i}\left[\vec{E}_2 \times \vec{j}_1\right], \qquad \vec{F}_{eII} = \vec{E}_4\rho_4 + \mathrm{i}\left[\vec{E}_3 \times \vec{j}_4\right]. \tag{40}$$

Thus we have the same Maxwell equations for the field strengths $\vec{E}_1, \vec{E}_2$ (37) and $\vec{E}_3, \vec{E}_4$ (38), but different Lorentz transformations (23) and (25).

6. Sedeonic Equations for Weak Gravitational Fields

In the frames of gravitoelectromagnetism the weak gravitational field can be described by the following sedeonic equation [15]

$$\left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\tilde{\mathbf{V}} = \tilde{\mathbf{I}}. \tag{41}$$

where $\tilde{\mathbf{V}}$ is a sedeonic gravitational potential, $\tilde{\mathbf{I}}$ is a phenomenological sedeonic source of gravitational field. Let us choose the potential as

$$\tilde{\mathbf{V}} = \mathrm{i}b_1\mathbf{e_t} - \mathrm{i}b_2\mathbf{e_r} + b_3 - \mathrm{i}b_4\mathbf{e_{tr}} + \vec{B}_1\mathbf{e_r} + \vec{B}_2\mathbf{e_t} - \vec{B}_3\mathbf{e_{tr}} + \mathrm{i}\vec{B}_4. \tag{42}$$

Also we take the source in the following form

$$\tilde{\mathbf{I}} = -\mathrm{i}\beta_1\mathbf{e_t} + \mathrm{i}\beta_2\mathbf{e_r} - \beta_3 + \mathrm{i}\beta_4\mathbf{e_{tr}} - \vec{l}_1\mathbf{e_r} - \vec{l}_2\mathbf{e_t} + \vec{l}_3\mathbf{e_{tr}} - \mathrm{i}\vec{l}_4 \tag{43}$$

where $\beta_s = 4\pi{\beta_s}'$ (${\beta_s}'$ is the volume density of gravitational charges) and $\vec{l}_s = 4\pi\vec{l}'_s$ ($\vec{l}'_s$ is the volume density of gravitational currents). The sedeonic potential $\tilde{\mathbf{V}}$ and field source $\tilde{\mathbf{I}}$ have the same space-time structure (9), (10) and the same Lorentz transformations as (23), (25). Let us introduce two groups of scalar g_s and vector $\vec{G}_s$ field strengths according to the following definitions

$$\begin{aligned} g_1 &= \partial b_1 + \left(\vec{\nabla}\cdot\vec{B}_1\right), &\qquad \vec{G}_1 &= -\,\partial\vec{B}_1 - \vec{\nabla}b_1 + \mathrm{i}\left[\vec{\nabla}\times\vec{B}_2\right] \\ g_2 &= \partial b_2 + \left(\vec{\nabla}\cdot\vec{B}_2\right), &\qquad \vec{G}_2 &= -\,\partial\vec{B}_2 - \vec{\nabla}b_2 - \mathrm{i}\left[\vec{\nabla}\times\vec{B}_1\right] \end{aligned} \tag{44}$$

and

$$\begin{aligned} g_3 &= \partial b_3 + \left(\vec{\nabla}\cdot\vec{B}_3\right), &\qquad \vec{G}_3 &= -\,\partial\vec{B}_3 - \vec{\nabla}b_3 - \mathrm{i}\left[\vec{\nabla}\times\vec{B}_4\right] \\ g_4 &= \partial b_4 + \left(\vec{\nabla}\cdot\vec{B}_4\right), &\qquad \vec{G}_4 &= -\,\partial\vec{B}_4 - \vec{\nabla}b_4 + \mathrm{i}\left[\vec{\nabla}\times\vec{B}_3\right]. \end{aligned} \tag{45}$$

These field strengths satisfy two independent systems of Maxwell equations

$$\begin{aligned} &\partial g_1 + \left(\vec{\nabla}\cdot\vec{G}_1\right) = \beta_1, \qquad && \partial\vec{G}_1 + \vec{\nabla} g_1 + \mathrm{i}\left[\vec{\nabla}\times\vec{G}_2\right] = -\,\vec{l}_1 \\ &\partial g_2 + \left(\vec{\nabla}\cdot\vec{G}_2\right) = \beta_2, \qquad && \partial\vec{G}_2 + \vec{\nabla} g_2 - \mathrm{i}\left[\vec{\nabla}\times\vec{G}_1\right] = -\,\vec{l}_2 \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\partial g_3 + \left(\vec{\nabla}\cdot\vec{G}_3\right) = \beta_3, \qquad && \partial\vec{G}_3 + \vec{\nabla} g_3 - \mathrm{i}\left[\vec{\nabla}\times\vec{G}_4\right] = -\,\vec{l}_3 \\ &\partial g_4 + \left(\vec{\nabla}\cdot\vec{G}_4\right) = \beta_4, \qquad && \partial\vec{G}_4 + \vec{\nabla} g_4 + \mathrm{i}\left[\vec{\nabla}\times\vec{G}_3\right] = -\,\vec{l}_4. \end{aligned} \tag{47}$$

Let us consider these equations without magnetic charges ($\beta_2 = \beta_3 = 0$) and magnetic currents ($\vec{l}_2 = \vec{l}_3 = 0$). Taking into account the Lorentz gauge

$$\begin{aligned} &\partial b_1 + \left(\vec{\nabla}\cdot\vec{B}_1\right) = 0, \qquad && \partial b_2 + \left(\vec{\nabla}\cdot\vec{B}_2\right) = 0 \\ &\partial b_3 + \left(\vec{\nabla}\cdot\vec{B}_3\right) = 0, \qquad && \partial b_4 + \left(\vec{\nabla}\cdot\vec{B}_4\right) = 0 \end{aligned} \tag{48}$$

we get Maxwell equations for gravitational field in the following form

$$\begin{aligned} &\left(\vec{\nabla}\cdot\vec{G}_1\right) = \beta_1, \qquad && \partial\vec{G}_1 + \mathrm{i}\left[\vec{\nabla}\times\vec{G}_2\right] = -\,\vec{l}_1 \\ &\left(\vec{\nabla}\cdot\vec{G}_2\right) = 0, \qquad && \partial\vec{G}_2 - \mathrm{i}\left[\vec{\nabla}\times\vec{G}_1\right] = 0 \end{aligned} \tag{49}$$

and

$$\begin{aligned} &\left(\vec{\nabla}\cdot\vec{G}_3\right) = 0, \qquad && \partial\vec{G}_3 - \mathrm{i}\left[\vec{\nabla}\times\vec{G}_4\right] = 0 \\ &\left(\vec{\nabla}\cdot\vec{G}_4\right) = \beta_4, \qquad && \partial\vec{G}_4 + \mathrm{i}\left[\vec{\nabla}\times\vec{G}_3\right] = -\,\vec{l}_4. \end{aligned} \tag{50}$$

Then the gradient of gravitational energy is

$$\begin{aligned} &\vec{\nabla}\left(\vec{G}_1^{\,2} + \vec{G}_2^{\,2} + \vec{G}_3^{\,2} + \vec{G}_4^{\,2}\right) - \mathrm{i}\partial\left[\vec{G}_1\times\vec{G}_2\right] - \mathrm{i}\partial\left[\vec{G}_3\times\vec{G}_4\right] \\ &- \left(\vec{G}_1\cdot\vec{\nabla}\right)\vec{G}_1 - \left(\vec{G}_2\cdot\vec{\nabla}\right)\vec{G}_2 - \vec{G}_1\left(\vec{\nabla}\cdot\vec{G}_1\right) - \vec{G}_2\left(\vec{\nabla}\cdot\vec{G}_2\right) \\ &- \left(\vec{G}_3\cdot\vec{\nabla}\right)\vec{G}_3 - \left(\vec{G}_4\cdot\vec{\nabla}\right)\vec{G}_4 - \vec{G}_3\left(\vec{\nabla}\cdot\vec{G}_3\right) - \vec{G}_4\left(\vec{\nabla}\cdot\vec{G}_4\right) \\ &\qquad = -\vec{G}_1\beta_1 - \mathrm{i}\left[\vec{G}_2\times\vec{l}_1\right] - \vec{G}_4\beta_4 - \mathrm{i}\left[\vec{G}_3\times\vec{l}_4\right]. \end{aligned} \tag{51}$$

It can be clearly seen that in this expression the field strengths, charges and currents of first group are not mixed with the field strengths, charges and currents of second group. So the expressions for the gravitational forces have the following form

$$\vec{F}_{gI} = \vec{G}_1\beta_1 + \mathrm{i}\left[\vec{G}_2\times\vec{l}_1\right], \qquad \vec{F}_{gII} = \vec{G}_4\beta_4 + \mathrm{i}\left[\vec{G}_3\times\vec{l}_4\right]. \tag{52}$$

Thus we have the same Maxwell equations for the gravitational field strengths $\vec{G}_1, \vec{G}_2$ (49) and $\vec{G}_3, \vec{G}_4$ (50), but different Lorentz transformations (23) and (25).

7. Massless Fields Described by Sedeonic Firs-Order Wave Equations

There is the special class of electromagnetic and weak gravitational fields described by first-order wave equations [15]

$$\left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\tilde{\mathbf{W}} = 0 \tag{53}$$

and

$$\left(\mathrm{i}\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla}\right)\tilde{\mathbf{V}} = 0. \tag{54}$$

Taking into account the expressions for potentials $\tilde{\mathbf{W}}$ (9) and $\tilde{\mathbf{V}}$ (42) one can see that these two sedeonic wave equations are equivalent to the following systems

$$\begin{aligned}
&\partial a_1 + \left(\vec{\nabla}\cdot\vec{A}_1\right) = 0, && -\partial\vec{A}_1 - \vec{\nabla}a_1 + \mathrm{i}\left[\vec{\nabla}\times\vec{A}_2\right] = 0\\
&\partial a_2 + \left(\vec{\nabla}\cdot\vec{A}_2\right) = 0, && -\partial\vec{A}_2 - \vec{\nabla}a_2 - \mathrm{i}\left[\vec{\nabla}\times\vec{A}_1\right] = 0\\
&\partial a_3 + \left(\vec{\nabla}\cdot\vec{A}_3\right) = 0, && -\partial\vec{A}_3 - \vec{\nabla}a_3 - \mathrm{i}\left[\vec{\nabla}\times\vec{A}_4\right] = 0\\
&\partial a_4 + \left(\vec{\nabla}\cdot\vec{A}_4\right) = 0, && -\partial\vec{A}_4 - \vec{\nabla}a_4 + \mathrm{i}\left[\vec{\nabla}\times\vec{A}_3\right] = 0
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
&\partial b_1 + \left(\vec{\nabla}\cdot\vec{B}_1\right) = 0, && -\partial\vec{B}_1 - \vec{\nabla}b_1 + \mathrm{i}\left[\vec{\nabla}\times\vec{B}_2\right] = 0\\
&\partial b_2 + \left(\vec{\nabla}\cdot\vec{B}_2\right) = 0, && -\partial\vec{B}_2 - \vec{\nabla}b_2 - \mathrm{i}\left[\vec{\nabla}\times\vec{B}_1\right] = 0\\
&\partial b_3 + \left(\vec{\nabla}\cdot\vec{B}_3\right) = 0 && -\partial\vec{B}_3 - \vec{\nabla}b_3 - \mathrm{i}\left[\vec{\nabla}\times\vec{B}_4\right] = 0\\
&\partial b_4 + \left(\vec{\nabla}\cdot\vec{B}_4\right) = 0, && -\partial\vec{B}_4 - \vec{\nabla}b_4 + \mathrm{i}\left[\vec{\nabla}\times\vec{B}_3\right] = 0.
\end{aligned} \tag{56}$$

As seen in this case there are also two types of electromagnetic fields and two types of gravitational fields, which are differed in Lorentz transformations.

8. Summary

Thus, we have shown that in the frames of sedeonic approach there are two types of massless fields, which have the same equations but are described by potentials with different space-time properties and different Lorentz transformations. The sources of these fields do not interact with each other. This model of matter can be investigated for the possible explanation of the dark matter and dark energy properties [17], [5].

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