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LAPLACE-BELTRAMI OPERATOR OF A HELICOIDAL HYPERSURFACE IN FOUR-SPACE

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Abstract. We introduce helicoidal hypersurface in the four dimensional Euclidean space. We calculate the mean and the Gaussian curvature, and some relations of the helicoidal hypersurface. Then we give the Laplace-Beltrami operator of the helicoidal hypersurface.

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Contents

1. Introduction

The notion of finite type immersion of submanifolds of a Euclidean space has been used in classifying and characterizing well known Riemannian submanifolds [3]. Chen [3] posed the problem of classifying the finite type surfaces in the three-dimensional Euclidean space $\mathbb{E}^3$. A Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ.

Further, the notion of finite type can be extended to any smooth function on a submanifold of a Euclidean space or a pseudo-Euclidean space. Then the theory of submanifolds of finite type has been studied by many geometers.

Takahashi [18] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$. Ferrandez, Garay and Lucas [8] prove that the surfaces of $\mathbb{E}^3$ satisfying $\Delta H = AH$, $A \in \mathrm{Mat}(3,3)$ are either minimal, or an open piece of sphere or of a right circular cylinder. Choi and Kim [5] characterize the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind.

Dillen, Pas and Verstraelen [6] prove that the only surfaces in $\mathbb{E}^3$ satisfying $\Delta r = Ar + B$, $A \in \mathrm{Mat}(3,3)$, $B \in \mathrm{Mat}(3,1)$ are the minimal surfaces, the spheres and the circular cylinders.

Senoussi and Bekkar [17] study helicoidal surfaces M^2 in $\mathbb{E}^3$ which are of finite type in the sense of Chen with respect to the fundamental forms I, II and III, i.e., their position vector field $r(u,v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A = (a_{ij})$ is a constant 3×3 matrix and Δ^J denotes the Laplace operator with respect to the fundamental forms I, II and III.

In classical surface geometry in Euclidean space, it is well known that the right helicoid (respectively catenoid) is the only ruled (respectively rotational) surface which is minimal. If we focus on the ruled (helicoid) and rotational characters, we have Bour's theorem in [2].

About helicoidal surfaces in Euclidean three-space, do Carmo and Dajczer [8] prove that, by using a result of Bour [3], there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface. Some relations among the Laplace-Beltrami operator and curvatures of the helicoidal surfaces in Euclidean three-space are shown by Güler, Yaylı and Hacısalihoğlu [10].

Lawson [11] gives the general definition of the Laplace-Beltrami operator in his lecture notes. Magid, Scharlach and Vrancken [12] introduce the affine umbilical surfaces in four-space. Vlachos [20] consider hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. Scharlach [16] studies the affine geometry of surfaces and hypersurfaces in four-space. Cheng and Wan [4] consider complete hypersurfaces of four-space with constant mean curvature.

Arvanitoyeorgos, Kaimakamais and Magid [1] show that if the mean curvature vector field of M_1^3 satisfies the equation

$$\Delta H = \alpha H$$

(α a constant), then M_1^3 has constant mean curvature in Minkowski four-space $\mathbb{E}_1^4$. This equation is a natural generalization of the biharmonic submanifold equation $\Delta H = 0$.

General rotational surfaces as a source of examples of surfaces in the four dimensional Euclidean space were introduced by Moore [13, 14]. Ganchev and Milousheva [9] consider the analogue of these surfaces in the Minkowski four-space. They classify completely the minimal general rotational surfaces and the general rotational surfaces consisting of parabolic points. Moruz and Munteanu [15] have considered hypersurfaces in the Euclidean space $\mathbb{E}^4$ defined as the sum of a curve and a surface whose mean curvature vanishes. They call them minimal translation hypersurfaces in $\mathbb{E}^4$ and give a classification of these hypersurfaces. Verstraelen, Walrave and Yaprak [19] study the minimal translation surfaces in $\mathbb{E}^n$ for arbitrary dimension n.

In this paper, we introduce the helicoidal hypersurfaces in Euclidean four-space $\mathbb{E}^4$. We give some basic notions of the four dimensional Euclidean geometry in Section 2. In Section 3, we give the definition of a helicoidal hypersurface. We calculate some relations among the mean curvature and the Gaussian curvature of the helicoidal hypersurface in Section 4. We introduce the Laplace-Beltrami operator in Section 5. Moreover, we calculate the Laplace-Beltrami operator of the helicoidal hypersurface. Finally, we give the helicoidal hypersurface with $\Delta^I \mathbf{H} = A\mathbf{H}$ in $\mathbb{E}^4$ in the last section.

2. Preliminaries

In this section, we will introduce the first and second fundamental forms, matrix of the shape operator $\mathbf{S}$, Gaussian curvature K, and the mean curvature H of hypersurface $\mathbf{M} = \mathbf{M}(u, v, w)$ in Euclidean four-space $\mathbb{E}^4$. In the rest of this work, we shall identify a vector (a,b,c,d) with its transpose (a,b,c,d)^t.

Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in the $\mathbb{E}^4$. The vector product of $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4)$, $Z = (z_1, z_2, z_3, z_4)$ on $\mathbb{E}^4$ is defined as follows

$$X \times Y \times Z = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

For a hypersurface $\mathbf{M}(u,v,w)$ in four-space we have

$$\det I=\det\begin{pmatrix}E & F & A\\ F & G & B\\ A & B & C\end{pmatrix}=(EG-F^2)C-A^2G+2ABF-B^2E \tag{1}$$

and

$$\det II=\det\begin{pmatrix}L & M & P\\ M & N & T\\ P & T & V\end{pmatrix}=\left(LN-M^2\right)V-P^2N+2PTM-T^2L \tag{2}$$

where $A=\mathbf{M}_u\cdot\mathbf{M}_w$, $B=\mathbf{M}_v\cdot\mathbf{M}_w$, $C=\mathbf{M}_w\cdot\mathbf{M}_w$, $P=\mathbf{M}_{uw}\cdot e$, $T=\mathbf{M}_{vw}\cdot e$, $V=\mathbf{M}_{ww}\cdot e$, e is the Gauss map (i.e., the unit normal vector). We compute

$$\begin{pmatrix}E & F & A\\ F & G & B\\ A & B & C\end{pmatrix}^{-1}\begin{pmatrix}L & M & P\\ M & N & T\\ P & T & V\end{pmatrix}$$

and it gives the matrix of the shape operator $\mathbf{S}$ as follows

$$\mathbf{S}=\frac{1}{\det I}\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} \tag{3}$$

where

$$\begin{aligned}
a_{11} &= ABM-CFM-AGP+BFP+CGL-B^2L\\
a_{12} &= ABN-CFN-AGT+BFT+CGM-B^2M\\
a_{13} &= ABT-CFT-AGV+BFV+CGP-B^2P\\
a_{21} &= ABL-CFL+AFP-BPE+CME-A^2M\\
a_{22} &= ABM-CFM+AFT-BTE+CNE-A^2N\\
a_{23} &= ABP-CFP+AFV-BVE+CTE-A^2T\\
a_{31} &= -AGL+BFL+AFM-BME+GPE-F^2P\\
a_{32} &= -AGM+BFM+AFN-BNE+GTE-F^2T\\
a_{33} &= -AGP+BFP+AFT-BTE+GVE-F^2V.
\end{aligned}$$

So, we get the formulas of the Gaussian and the mean curvatures, respectively as follow

$$K=\det(\mathbf{S})=\frac{\det II}{\det I} \tag{4}$$

and

$$H=\frac{1}{3}\operatorname{tr}(\mathbf{S}) \tag{5}$$

where $\operatorname{tr}(\mathbf{S}) = \frac{1}{3\det I}((EN + GL - 2FM)C + (EG - F^2)V - A^2N - B^2L - 2(APG + BTE - ABM - ATF - BPF))$.

A hypersurface $\mathbf{M}$ is minimal if $H = 0$ is fulfilled identically on $\mathbf{M}$.

3. Helicoidal Hypersurface

We define the rotational hypersurface and helicoidal hypersurface in $\mathbb{E}^4$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, and let ℓ be a straight line in Π.

A rotational hypersurface in $\mathbb{E}^4$ is defined as a hypersurface rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Then the resulting hypersurface is called the *helicoidal hypersurface* with axis ℓ and pitches $a, b \in \mathbb{R}\backslash\{0\}$.

We may suppose that ℓ is the line spanned by the vector $(0,0,0,1)^t$. The orthogonal matrix which fixes the above vector is

$$Z(v,w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

where $v, w \in \mathbb{R}$. The matrix Z can be found by solving the following equations simultaneously

$$Z\ell = \ell, \qquad Z^tZ = ZZ^t = I_4, \qquad \det Z = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_4-axis of $\mathbb{E}^4$. Parametrization of the profile curve is given by

$$\gamma(u) = (u, 0, 0, \varphi(u))$$

where $\varphi(u) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function for all $u \in I$. So, the helicoidal hypersurface which is spanned by the vector $(0,0,0,1)$ with pitches $a, b \in \mathbb{R}\backslash\{0\}$, is as follows

$$\mathbf{H}(u,v,w) = Z(v,w)\gamma(u)^t + (av + bw)(0,0,0,1)^t \tag{7}$$

in $\mathbb{E}^4$, where $u \in I$, $v, w \in [0, 2\pi]$. When $w = 0$, we have helicoidal surface in the three dimensional Euclidean space $\mathbb{E}^3$. When $a = b = 0$, the surface is just a rotational hypersurface as follows

$$\mathbf{R}(u,v,w) = (u \cos v \cos w, u \sin v \cos w, u \sin w, \varphi(u)). \tag{8}$$

4. Curvatures

In this section, we obtain the mean curvature and the Gaussian curvature of a helicoidal hypersurface

$$\mathbf{H}(u,v,w)=\begin{pmatrix} u\cos v\cos w \\ u\sin v\cos w \\ u\sin w \\ \varphi(u)+av+bw \end{pmatrix} \tag{9}$$

where $u,a,b\in\mathbb{R}\setminus\{0\}$ and $0\leq v,w\leq 2\pi.$

Using the first differentials of (9) we get the first quantities as follow

$$I=\begin{pmatrix} 1+\varphi'^2 & a\varphi' & b\varphi' \\ a\varphi' & a^2+u^2\cos^2 w & ab \\ b\varphi' & ab & b^2+u^2 \end{pmatrix}.$$

We also have

$$\det I=u^2\left((b^2+u^2(1+\varphi'^2))\cos^2 w+a^2\right)$$

where $\varphi=\varphi(u)$, $\varphi'=\dfrac{\mathrm{d}\varphi}{\mathrm{d}u}$.

The line element of the helicoidal hypersurface is given by

$$\begin{aligned}\mathrm{d}s^2 =& (1+\varphi'^2)\mathrm{d}u^2+2a\varphi'\mathrm{d}u\mathrm{d}v+\left(a^2+u^2\cos^2 w\right)\mathrm{d}v^2\\ &+2b\varphi'\mathrm{d}u\mathrm{d}w+2ab\mathrm{d}v\mathrm{d}w+(b^2+u^2)\mathrm{d}w^2.\end{aligned}$$

Using the second differentials with respect to u,v,w, we have the second quantities as follow

$$II=\begin{pmatrix} -\frac{u^2\varphi''\cos w}{\sqrt{\det I}} & \frac{au\cos w}{\sqrt{\det I}} & \frac{ub\cos w}{\sqrt{\det I}} \\ \frac{au\cos w}{\sqrt{\det I}} & -\frac{u^2\cos^2 w(u\varphi'\cos w-b\sin w)}{\sqrt{\det I}} & -\frac{au^2\sin w}{\sqrt{\det I}} \\ \frac{ub\cos w}{\sqrt{\det I}} & -\frac{au^2\sin w}{\sqrt{\det I}} & -\frac{u^3\varphi'\cos w}{\sqrt{\det I}} \end{pmatrix}$$

and

$$\begin{aligned}\det II =& -\frac{u^4\cos w}{(\det I)^{3/2}}(u^4\varphi'^2\varphi''\cos^4 w-bu^3\varphi'\varphi''\sin w\cos^3 w\\ &-a^2u^2\varphi''\sin^2 w-u\cos^2 w(b^2\cos^2 w+a^2)\varphi'\\ &+b(b^2\cos^2 w+2a^2)\sin w\cos w).\end{aligned}$$

The Gauss map of the helicoidal hypersurface is

$$e_{\mathbf{H}} = \frac{1}{\sqrt{\det I}} \begin{pmatrix} u\left((u\varphi' \cos w - b \sin w) \cos v \cos w - a \sin v\right) \\ u\left((u\varphi' \cos w - b \sin w) \sin v \cos w + a \cos v\right) \\ u\left((u\varphi' \sin w + b \cos w)\right) \cos w \\ -u^2 \cos w \end{pmatrix} \tag{10}$$

where

$$\det I = u^2 \left((b^2 + u^2(1 + \varphi'^2)) \cos^2 w + a^2\right).$$

The shape operator of the helicoidal hypersurface is

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}$$

where

$$s_{11} = -\frac{u^3 \cos w}{(\det I)^{3/2}}((a^2 + (b^2 + u^2) \cos^2 w)u\varphi'' + (a^2 + b^2 \cos^2 w)\varphi')$$

$$s_{12} = \frac{au \cos w}{\sqrt{\det I}}$$

$$s_{13} = \frac{u^3}{(\det I)^{3/2}}(bu^2\varphi'^2 \cos^3 w + a^2 u\varphi' \sin w + (a^2 + (b^2 + u^2) \cos^2 w)b \cos w)$$

$$s_{21} = \frac{au^3 \cos w(u\varphi'\varphi'' + \varphi'^2 + 1)}{(\det I)^{3/2}}, \qquad s_{22} = \frac{-u\varphi' \cos w + b \sin w}{\sqrt{\det I}}$$

$$s_{23} = -\frac{au^2 \sin w(u^2\varphi'^2 + b^2 + u^2)}{(\det I)^{3/2}}$$

$$s_{31} = \frac{bu^3 \cos^3 w(u\varphi'\varphi'' + \varphi'^2 + 1)}{(\det I)^{3/2}}, \qquad s_{32} = -\frac{a \sin w}{\sqrt{\det I}}$$

$$s_{33} = \frac{u^2}{(\det I)^{3/2}}(-u^3\varphi'^3 \cos^3 w - u \cos w((b^2 + u^2) \cos^2 w + a^2)\varphi' + a^2 b \sin w).$$

Finally, we calculate the Gaussian curvature and the mean curvature of the helicoidal hypersurface as follow

$$K = \det(\mathbf{S}) = \frac{\det II}{\det I} = \frac{\lambda_1\varphi'^2\varphi'' + \lambda_2\varphi'\varphi'' + \lambda_3\varphi'' + \lambda_4\varphi' + \lambda_5}{(\det I)^{5/2}}$$

and

$$H = \frac{1}{3}\mathrm{tr}(\mathbf{S}) = \frac{\zeta_1\varphi'' + \zeta_2\varphi'^3 + \zeta_3\varphi'^2 + \zeta_4\varphi' + \zeta_5}{3\,(\det I)^{3/2}}$$

where

$$\begin{aligned}
\lambda_1 &= -u^8\cos^5 w, \qquad \lambda_2 = bu^7 \sin w \cos^4 w, \qquad \lambda_3 = a^2u^6 \cos w \sin^2 w \\
\lambda_4 &= u^5\cos^3 w(b^2\cos^2 w + a^2), \qquad \lambda_5 = -bu^4(b^2\cos^2 w + 2a^2)\sin w \cos^2 w \\
\zeta_1 &= -u^4 \cos w \left(\left(u^2+b^2\right)\cos^2 w + a^2\right), \qquad \zeta_2 = -2u^5\cos^3 w \\
\zeta_3 &= bu^4\cos^2 w \sin w, \qquad \zeta_4 = -u^3\cos w\left(3a^2 + \left(3b^2+2u^2\right)\cos^2 w\right) \\
\zeta_5 &= bu^2 \sin w\left(\left(u^2+b^2\right)\cos^2 w + 2a^2\right).
\end{aligned}$$

Corollary 1. *When* $\varphi = c = \text{const}$, *then we get*

$$K = \frac{-b(b^2\cos^2 w + 2a^2)\sin w \cos^2 w}{u\left(((b^2+u^2)\cos^2 w + a^2)\right)^{5/2}}$$

$$H = \frac{b\left(\left(u^2+b^2\right)\cos^2 w + 2a^2\right)\sin w}{3u\left((b^2+u^2)\cos^2 w + a^2\right)^{3/2}}.$$

Corollary 2. *When* $\varphi = c = \text{const}$, $b = 0$, *we get*

$$K = 0, \qquad H = 0.$$

Thus, the hypersurface is flat and minimal.

Corollary 3. *When* $\varphi = c = \text{const}$, $w = 0$ *or* π, *then we have*

$$K = 0, \qquad H = 0.$$

Hence, the hypersurface is again flat and minimal.

Corollary 4. *When* $\varphi = c = \text{const}$, $a = b = 0$, *i.e., the hypersurface is a rotational hypersurface, then we have the results of* Corollary 2 (*and also* Corollary 3).

5. Laplace-Beltrami Operator

The inverse of the matrix

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

is as follows

$$\frac{1}{g}\begin{pmatrix} g_{22}g_{33} - g_{23}g_{32} & g_{13}g_{32} - g_{12}g_{33} & g_{12}g_{23} - g_{13}g_{22} \\ g_{31}g_{23} - g_{21}g_{33} & g_{11}g_{33} - g_{13}g_{31} & g_{21}g_{13} - g_{11}g_{23} \\ g_{21}g_{32} - g_{22}g_{31} & g_{12}g_{31} - g_{11}g_{32} & g_{11}g_{22} - g_{12}g_{21} \end{pmatrix}$$

where

$$g=\det\left(g_{ij}\right)=g_{11}g_{22}g_{33}-g_{11}g_{23}g_{32}+g_{12}g_{31}g_{23}$$
$$-g_{12}g_{21}g_{33}+g_{21}g_{13}g_{32}-g_{13}g_{22}g_{31}.$$

So, we get

$$(I)^{-1}=\frac{1}{\det I}\begin{pmatrix} CG-B^2 & AB-CF & BF-AG \\ AB-CF & CE-A^2 & AF-BE \\ BF-AG & AF-BE & EG-F^2 \end{pmatrix}$$

where

$$\det I=(EG-F^2)C+2ABF-A^2G-B^2E.$$

Therefore, the Laplace-Beltrami operator of a smooth function $\phi=\phi(u,v,w)\mid_{\mathbf{D}}$ ($\mathbf{D}\subset\mathbb{R}^3$) of class C^3 with respect to the first fundamental form of hypersurface $\mathbf{M}$ is the operator Δ which is defined by as follows

$$\Delta^I\phi=\frac{1}{\sqrt{g}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial\phi}{\partial x^j}\right) \tag{11}$$

where $\left(g^{ij}\right)=\left(g_{kl}\right)^{-1}$ and $g=\det\left(g_{ij}\right)$.

Clearly, we write $\Delta^I\phi$ as follows

$$\frac{1}{\sqrt{g}}\begin{bmatrix} \frac{\partial}{\partial x^1}\left(\sqrt{g}g^{11}\frac{\partial\phi}{\partial x^1}\right)-\frac{\partial}{\partial x^1}\left(\sqrt{g}g^{12}\frac{\partial\phi}{\partial x^2}\right)+\frac{\partial}{\partial x^1}\left(\sqrt{g}g^{13}\frac{\partial\phi}{\partial x^3}\right) \\ -\frac{\partial}{\partial x^2}\left(\sqrt{g}g^{21}\frac{\partial\phi}{\partial x^1}\right)+\frac{\partial}{\partial x^2}\left(\sqrt{g}g^{22}\frac{\partial\phi}{\partial x^2}\right)-\frac{\partial}{\partial x^2}\left(\sqrt{g}g^{23}\frac{\partial\phi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^3}\left(\sqrt{g}g^{31}\frac{\partial\phi}{\partial x^1}\right)-\frac{\partial}{\partial x^3}\left(\sqrt{g}g^{32}\frac{\partial\phi}{\partial x^2}\right)+\frac{\partial}{\partial x^3}\left(\sqrt{g}g^{33}\frac{\partial\phi}{\partial x^3}\right) \end{bmatrix}. \tag{12}$$

Using more transparent notation we get

$$\Delta^I\phi=\frac{1}{\sqrt{\det I}}\begin{bmatrix} \frac{\partial}{\partial u}\left(\frac{\left(CG-B^2\right)\phi_u-(AB-CF)\phi_v+(BF-AG)\phi_w}{\sqrt{\det I}}\right) \\ -\frac{\partial}{\partial v}\left(\frac{(AB-CF)\phi_u-\left(CE-A^2\right)\phi_v+(AF-BE)\phi_w}{\sqrt{\det I}}\right) \\ +\frac{\partial}{\partial w}\left(\frac{(BF-AG)\phi_u-(AF-BE)\phi_v+\left(EG-F^2\right)\phi_w}{\sqrt{\det I}}\right) \end{bmatrix} \tag{13}$$

where $g=\det\left(g_{ij}\right)=\det I$.

Now, we study on the Laplace Beltrami operator of a helicoidal hypersurface (9). The inverse matrix of I is as follows

$$\left(I^{-1}\right)=\frac{1}{\det I}\left(\Omega\right)_{3\times3}$$

where

$$\begin{aligned} \Omega_{11} =& u^2\left[a^2+\left(b^2+u^2\right)\cos^2 w\right], & \Omega_{12} =& -au^2\varphi' = \Omega_{21} \\ \Omega_{13} =& -bu^2\varphi'\cos^2 w = \Omega_{31}, & \Omega_{22} =& b^2+\left(1+\varphi'^2\right)u^2 \\ \Omega_{23} =& -ab = \Omega_{32}, & \Omega_{33} =& a^2+u^2\left(1+\varphi^{2\prime}\right)\cos^2 w. \end{aligned}$$

Now we continue our calculations to find the Laplace Beltrami operator $\Delta^I \mathbf{H}$ of the helicoidal hypersurface $\mathbf{H}$.

The Laplace Beltrami operator of a hypersurface $\mathbf{H}$ is given by

$$\Delta^I \mathbf{H} = \frac{1}{\sqrt{\det I}}\left(\frac{\partial}{\partial u}\mathbf{U} - \frac{\partial}{\partial v}\mathbf{V} + \frac{\partial}{\partial w}\mathbf{W}\right)$$

where

$$\mathbf{U} = \frac{\left(CG-B^2\right)\mathbf{H}_u - \left(AB-CF\right)\mathbf{H}_v + \left(BF-AG\right)\mathbf{H}_w}{\sqrt{\det I}}$$

$$\mathbf{V} = \frac{\left(AB-CF\right)\mathbf{H}_u - \left(CE-A^2\right)\mathbf{H}_v + \left(AF-BE\right)\mathbf{H}_w}{\sqrt{\det I}}$$

$$\mathbf{W} = \frac{\left(BF-AG\right)\mathbf{H}_u - \left(AF-BE\right)\mathbf{H}_v + \left(EG-F^2\right)\mathbf{H}_w}{\sqrt{\det I}}$$

$$\det I = u^2\left(\left(b^2+u^2\left(1+\varphi'^2\right)\right)\cos^2 w + a^2\right).$$

We obtain the matrices

$$\mathbf{U} = \begin{pmatrix} \frac{u^2\left(a^2+\left(b^2+u^2\right)\cos^2 w\right)\cos v\cos w - au^3\varphi'\sin v\cos w + bu^3\varphi'\cos v\sin w\cos^2 w}{\sqrt{\det I}} \\ \frac{u^2\left(a^2+\left(b^2+u^2\right)\cos^2 w\right)\sin v\cos w + au^3\varphi'\cos v\cos w + bu^3\varphi'\sin v\sin w\cos^2 w}{\sqrt{\det I}} \\ \frac{u^2\left(a^2+\left(b^2+u^2\right)\cos^2 w\right)\sin w - bu^3\varphi'\cos^3 w}{\sqrt{\det I}} \\ \frac{u^4\varphi\cos^2 w + 2a^2u^2\varphi'}{\sqrt{\det I}} \end{pmatrix}$$

and

$$\mathbf{V} = \begin{pmatrix} \frac{-au^2\varphi'\cos v\cos w + u\left(b^2+\left(1+\varphi'^2\right)u^2\right)\sin v\cos w + abu\cos v\sin w}{\sqrt{\det I}} \\ \frac{-au^2\varphi'\sin v\cos w - u\left(b^2+\left(1+\varphi'^2\right)u^2\right)\cos v\cos w + abu\sin v\sin w}{\sqrt{\det I}} \\ \frac{-au^2\varphi'\sin w - abu\cos w}{\sqrt{\det I}} \\ \frac{au^2}{\sqrt{\det I}} \end{pmatrix}$$

and

$$\mathbf{W}=\begin{pmatrix} \frac{-bu^2\varphi'\cos v\cos^3 w-abu\sin v\cos w-\left(a^2+u^2\left(1+\varphi^{2\prime}\right)\cos^2 w\right)u\cos v\sin w}{\sqrt{\det I}} \\ \frac{-bu^2\varphi'\sin v\cos^3 w+abu\cos v\cos w-\left(a^2+u^2\left(1+\varphi^{2\prime}\right)\cos^2 w\right)u\sin v\sin w}{\sqrt{\det I}} \\ \frac{-bu^2\varphi'\sin w\cos^2 w+\left(a^2+u^2\left(1+\varphi^{2\prime}\right)\cos^2 w\right)u\cos w}{\sqrt{\det I}} \\ \frac{-bu^2\varphi'^2\cos^2 w+a^2b+b\left(a^2+u^2\left(1+\varphi^{2\prime}\right)\cos^2 w\right)}{\sqrt{\det I}} \end{pmatrix}.$$

Using differentials of u, v, w on $\mathbf{U}, \mathbf{V}, \mathbf{W}$, respectively, we get

$$\frac{\partial}{\partial u}\mathbf{U}=\frac{1}{2\left(\det I\right)^{3/2}}\mathbf{A}=\frac{1}{2\left(\det I\right)^{3/2}}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

where

$$\begin{aligned} A_1 = {} & (2ua^2\cos v\cos w-(3u^2\varphi'+u^3\varphi'')a\sin v\cos w+2ub^2\cos v\cos^3 w \\ & +(3u^2\varphi'+u^3\varphi'')b\cos v\sin w\cos^2 w+4u^3\cos v\cos^3 w)2(u^2\left((b^2\right. \\ & \left.+u^2(1+\varphi'^2))\cos^2 w+a^2\right))-(2a^2u+(u^3\varphi''+4u^2\varphi'^2 \\ & +4u^2+2b^2)u\cos^2 w)(u^2(a^2+(b^2+u^2)\cos^2 w)\cos v\cos w \\ & -au^3\varphi'\sin v\cos w+bu^3\varphi'\cos v\sin w\cos^2 w) \end{aligned}$$

$$\begin{aligned} A_2 = {} & 2u^2(a^2+(b^2+(1+\varphi'^2)u^2)\cos^2 w)(4u^3\cos^3 w\sin v+2a^2u\cos w\sin v \\ & +a\cos v\cos w(\varphi''u^3+3\varphi'u^2)+2b^2u\cos^3 w\sin v \\ & +b\cos^2 w\sin v\sin w(u^3\varphi''+3u^2\varphi'))-(2ua^2+u(u^3\varphi''+4u^2\varphi'^2 \\ & +2b^2+4u^2)\cos^2 w)(u^2\cos w\sin v(a^2+(b^2+u^2)\cos^2 w) \\ & +au^3\varphi'\cos v\cos w+bu^3\varphi'\cos^2 w\sin v\sin w) \end{aligned}$$

$$\begin{aligned} A_3 = {} & 2u^2(a^2+(b^2+(1+\varphi'^2)u^2)\cos^2 w)(2a^2u\sin w-b\cos^3 w(u^3\varphi'' \\ & +3u^2\varphi')+4u^3\cos^2 w\sin w+2b^2u\cos^2 w\sin w)-(2ua^2+(uu^3\varphi'' \\ & +4u^2\varphi^{2\prime}+2b^2+4u^2)\cos^2 w)(u^2\sin w(a^2+(b^2+u^2)\cos^2 w) \\ & -bu^3\varphi'\cos^3 w) \end{aligned}$$

$$\begin{aligned} A_4 = {} & 2((4u^3\varphi+u^4\varphi')\cos^2 w+2a^2(2u\varphi'+u^2\varphi''))(u^2((b^2 \\ & +u^2(1+\varphi'^2))\cos^2 w+a^2))-(2a^2u+(u^3\varphi''+4u^2\varphi'^2 \\ & +4u^2+2b^2)u\cos^2 w)(u^4\varphi\cos^2 w+2a^2u^2\varphi'). \end{aligned}$$

$$\frac{\partial}{\partial v}\mathbf{V}=\frac{1}{2\left(\det I\right)^{3/2}}\mathbf{B}=\frac{1}{2\left(\det I\right)^{3/2}}\begin{pmatrix}B_1\\B_2\\B_3\\B_4\end{pmatrix}$$

where

$$\begin{aligned}B_1 &= 2(au^2\varphi'\sin v\cos w+u(b^2+(1+\varphi'^2)u^2)\cos v\cos w-abu\sin v\sin w)\\&\quad .(u^2((b^2+u^2(1+\varphi'^2))\cos^2 w+a^2))\\B_2 &= 2(-au^2\varphi'\cos v\cos w+u(b^2+(1+\varphi'^2)u^2)\sin v\cos w+abu\cos v\sin w)\\&\quad .(u^2((b^2+u^2(1+\varphi'^2))\cos^2 w+a^2))\\B_3 &= B_4=0.\end{aligned}$$

Now

$$\frac{\partial}{\partial w}\mathbf{W}=\frac{1}{2\left(\det I\right)^{3/2}}\mathbf{C}=\frac{1}{2\left(\det I\right)^{2}}\begin{pmatrix}C_1\\C_2\\C_3\\C_4\end{pmatrix}$$

where

$$\begin{aligned}C_1 &= 2(-a^2u\cos v\cos w-(2\cos w\sin w+\cos^3 wu^3\cos v\\&\quad +3bu^2\varphi'\cos v\sin w\cos^2 w-u^3\varphi'^2\cos v(2\cos w\sin w+\cos^3 w)\\&\quad +abu\sin v\sin w)(a^2u^2+u^4\cos^2 wu^4\varphi'^2\cos^2 w+b^2u^2\cos^2 w)\\&\quad -(-2u^2((1+\varphi'^2))u^2+b^2)\sin w\cos w)(-a^2u\cos v\sin w\\&\quad -u^3\cos v\cos^2 w\sin w-bu^2\varphi'\cos v\cos^3 w-u^3\varphi'^2\cos v\cos^2 w\sin w\\&\quad -abu\cos w\sin v)\end{aligned}$$

$$\begin{aligned}C_2 &= 2(-abu\cos v\sin w-u^3\sin v(\cos^3 w-2\cos w\sin^2 w)\\&\quad -3bu^2\varphi'\sin v\cos^2 w\sin w-u^3\varphi'^2(\cos^3 w-2\cos w\sin^2 w)\sin v\\&\quad -a^2u\sin v\cos w)(a^2u^2+u^4\cos^2 w+u^4\varphi^{2\prime}\cos^2 w+b^2u^2\cos^2 w)\\&\quad -(-2u^2((1+\varphi'^2)u^2+b^2)\sin w\cos w)(-bu^2\varphi'\sin v\cos^3 w\\&\quad +abu\cos v\cos w-(a^2+u^2(1+\varphi'^2)\cos^2 w)u\sin v\sin w)\end{aligned}$$

$$\begin{aligned}C_3 &= (2bu^2\varphi'\cos w-3u^3\cos^2 w\sin w-3bu^2\varphi'\cos^3 w-3u^3\varphi'^2\cos^2 w\sin w\\&\quad -a^2u\sin w)(a^2u^2+u^4\cos^2 w+u^4\varphi^{2\prime}\cos^2 w+b^2u^2\cos^2 w)\\&\quad -(-2u^2((1+\varphi'^2)u^2+b^2)\sin w)\cos wu^3\cos^3 w+u^3\varphi'^2\cos^3 w\\&\quad +a^2u\cos w-bu^2\varphi'\cos^2 w\sin w)\end{aligned}$$

$$C_4 = 2a^2bu^2\left(2u^2\varphi'^2+2b^2+u^2\right)\cos w\sin w.$$

Hence, we have

$$\begin{aligned}
\Delta^I \mathbf{H} &= \frac{1}{\sqrt{\det I}}\left(\frac{\partial}{\partial u}\mathbf{U} - \frac{\partial}{\partial v}\mathbf{V} + \frac{\partial}{\partial w}\mathbf{W}\right) \\
&= \frac{1}{\sqrt{\det I}}\left(\left(\frac{\mathbf{A}}{2\,(\det I)^{3/2}}\right) - \left(\frac{\mathbf{B}}{2\,(\det I)^{3/2}}\right) + \left(\frac{\mathbf{C}}{2\,(\det I)^{3/2}}\right)\right) \\
&= \frac{1}{2\,(\det I)^2}\begin{pmatrix} A_1 - B_1 + C_1 \\ A_2 - B_2 + C_2 \\ A_3 - B_3 + C_3 \\ A_4 - B_4 + C_4 \end{pmatrix}
\end{aligned}$$

where

$$\det I = u^2\left((b^2 + u^2(1+\varphi'^2))\cos^2 w + a^2\right)$$

$$A_1 - B_1 + C_1 = J_1\varphi'' + J_2\varphi'^2\varphi'' + J_3\varphi'\varphi'' + J_4\varphi'^3 + J_5\varphi'^2 + J_6\varphi' + J_7$$

with

$$\begin{aligned}
J_1 &= 2u^3\cos w(-a^2u^3\cos v\cos^2 w - 2ab^2u^2\cos^2 w\sin v - b^2u^3\cos v\cos^4 w \\
&\quad -2a^3u^2\sin v - 2au^4\cos^2 w\sin v + 2b^3u^2\cos v\cos^3 w\sin w \\
&\quad +2bu^4\cos v\cos^3 w\sin w - u^5\cos v\cos^4 w + a^2bu^2\cos v\cos w\sin w) \\
J_2 &= 4u^7\cos^3 w(b\cos v\cos w\sin w - a\sin v) \\
J_3 &= -2u^7\cos^3 w(b\cos v\cos w\sin w - a\sin v) \\
J_4 &= 4u^6\cos^3 w(b\cos v\cos w\sin w - a\sin v) \\
J_5 &= 8u^7\cos v\cos^5 w \\
J_6 &= 4u^4\cos w((-2a^3 - 2ab^2\cos^2 w - au^2\cos^2 w)\sin v + ((bu^2 + 2b^3)\cos^2 w \\
&\quad +2a^2b)\cos w\cos v\sin w) \\
J_7 &= 2u^3\cos w((4a^2b^2 + 6a^2u^2 + 2a^4 + \cos^2 w(6b^2u^2 + 2b^4 + 4u^4))\cos v\cos^2 w)
\end{aligned}$$

$$A_2 - B_2 + C_2 = \mu_1\varphi'' + \mu_2\varphi'^2\varphi'' + \mu_3\varphi'\varphi'' + \mu_4\varphi'^4 + \mu_5\varphi'^3 + \mu_6\varphi'^2 + \mu_7\varphi' + \mu_8$$

where

$$\begin{aligned}
\mu_1 &= u^3\cos w(-b^2u^3\cos^4 w\sin v + 2a^2bu^2\cos w\sin v\sin w + 2au^4\cos v\cos^2 w \\
&\quad -u^5\cos^4 w\sin v + 2a^3u^2\cos v + 2bu^4\cos^3 w\sin v\sin w) \\
&\quad +2b^3u^2\cos^3 w\sin v\sin w + 2ab^2u^2\cos v\cos^2 w - a^2u^3\cos^2 w\sin v) \\
\mu_2 &= 2u^7\cos^3 w\,(2a\cos v + b\sin v\cos w\sin w) \\
\mu_3 &= -au^7\cos v\cos^3 w - bu^7\cos^4 w\sin v\sin w \\
\mu_4 &= -u^7\sin v\cos^3 w(2+\cos^2 w) \\
\mu_5 &= (-3bu^6\cos^4 w\sin v\sin w + 4au^6\cos v\cos^3 w) \\
\mu_6 &= -2a^2u^5\sin v\cos^3 w + 2u^7\sin v\cos^7 w - 4u^7 - 2u^7\sin^2 v\cos^6 w \\
&\quad -abu^5\cos v\sin w\cos^2 w - 4b^2u^5\sin v\cos^2 w - b^2u^5\sin v\cos^5 w \\
&\quad +2a^2u^5\sin v\cos w
\end{aligned}$$

$$\begin{aligned}
\mu_7 &= u^3\cos w(4au^3\cos v\cos^2 w + 6ab^2u\cos v\cos^2 w - 3bu^3\cos^3 w\sin v\sin w \\
&\quad -b^3u\cos^3 w\sin v\sin w + 6a^3u\cos v + a^2bu\cos w\sin v\sin w) \\
\mu_8 &= u^3\cos w(4a^2u^2\cos^2 w\sin v - 4b^2u^2\cos^2 w\sin v + 5b^2u^2\cos^4 w\sin v \\
&\quad -4a^2b^2\sin v - 2a^2u^2\sin v - ab^3\cos v\cos w\sin w + a^4\sin v \\
&\quad -abu^2\cos v\cos w\sin w - 2b^4\cos^2 w\sin v + 2b^4\cos^4 w\sin v \\
&\quad +3u^4\cos^4 w\sin v + 5a^2b^2\cos^2 w\sin v - 3a^3b\cos v\tan w)
\end{aligned}$$

$$A_3 - B_3 + C_3 = \theta_1\varphi'' + \theta_2\varphi'^2\varphi'' + \theta_3\varphi'\varphi'' + \theta_4\varphi'^4 + \theta_5\varphi'^3 + \theta_6\varphi'^2 + \theta_7\varphi' + \theta_8.$$

Here

$$\begin{aligned}
\theta_1 &= u^5\cos^2 w(-u^3\cos^2 w\sin w - 2a^2b\cos w - 2bu^2\cos^3 w \\
&\quad -2b^3\cos^3 w - a^2u\sin w - b^2u\cos^2 w\sin w) \\
\theta_2 &= -2bu^7\cos^5 w, \qquad \theta_3 = bu^7\cos^5 w \\
\theta_4 &= -u^7\cos^4 w\sin w, \qquad \theta_5 = -3bu^6\cos^5 w \\
\theta_6 &= u^5\cos^4 w\sin w\left(\left(2u^2-b^2\right)-2a^2\right) \\
\theta_7 &= bu^4\cos w\left(\left(-3u^2-5b^2\right)\cos^4 w + a^2\left(2-7\cos^2 w\right)\right) \\
\theta_8 &= u^3\cos w((3u^4+5b^2u^2+a^4+2b^4)\cos^4 w + (4u^2+5b^2)a^2\cos^2 w)\sin w.
\end{aligned}$$

Finally

$$\begin{aligned} A_4 - B_4 + C_4 = & \rho_1 \varphi'^2 \varphi'' + \rho_2 \varphi' \varphi'' + \rho_3 + \varphi \varphi'' \\ & + \rho_4 \varphi'' + \rho_5 \varphi'^3 + \rho_6 \varphi \varphi'^2 + \rho_7 \varphi'^2 + \rho_8 \varphi' + \rho_9 \varphi + \rho_{10} \end{aligned}$$

where

$$\begin{aligned} \rho_1 &= 4a^2 u^6 \cos^2 w, \qquad \rho_2 = -2a^2 u^6 \cos^2 w, \qquad \rho_3 = -u^8 \cos^4 w \\ \rho_4 &= (4a^2 u^4 (a^2 + \left(u^2 + b^2\right)) \cos^2 w) \\ \rho_5 &= 2u^8 \cos^4 w, \qquad \rho_6 = 4u^6 \cos^4 w, \qquad \rho_7 = 4a^2 b u^4 \sin w \cos w \\ \rho_8 &= 2u^3 \cos w (2a^2 b^2 \cos w + u^5 \cos^3 w + 2a^4 + a^2 u^3 \cos w + b^2 u^3 \cos^3 w) \\ \rho_9 &= 2u^5 \cos^2 w ((2u^2 + 3b^2) \cos w + 3a^2) \\ \rho_{10} &= 2a^2 b u^2 \left(2b^2 + u^2\right) \sin w \cos w. \end{aligned}$$

Corollary 5. *When* $\varphi = \text{const}$, *then we get*

$$\Delta^I \mathbf{H} = \frac{1}{2\left(u^2\left(\left(b^2+u^2\right)\cos^2 w + a^2\right)\right)^2} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

where

$$\begin{aligned} \alpha_1 = & 2u^3 (4a^2 b^2 + 6a^2 u^2 + 2a^4 + \cos^2 w (6b^2 u^2 + 2b^4 + 4u^4)) \cos v \cos^3 w \\ \alpha_2 = & 4a^2 u^5 \cos^3 w \sin v + a^4 u^3 \sin v \cos w + 2b^4 u^3 \cos^5 w \sin v \\ & + 5b^2 u^5 \cos^5 w \sin v + 5a^2 b^2 u^3 \cos^3 w \sin v - ab^3 u^3 \cos v \cos^2 w \sin w \\ & - 2a^2 u^5 \sin v \cos w - 4b^2 u^5 \cos^3 w \sin v - 2b^4 u^3 \sin v \cos^3 w \\ & + 3u^7 \cos^5 w \sin v - 3a^3 b u^3 \cos v \sin w - abu^5 \cos v \cos^2 w \sin w \\ & - 4a^2 b^2 u^3 \sin v \cos w \\ \alpha_3 = & u^3 (\left(3u^4 + 5b^2 u^2 + a^4 + 2b^4\right) \cos^5 w + \left(4u^2 + 5b^2\right) a^2 \cos^2 w) \sin w \cos w \\ \alpha_4 = & u^2 \cos w (2cu^3 \cos w \left(\left(2u^2 + 3b^2\right) \cos w + 3a^2\right) + 2a^2 b \left(2b^2 + u^2\right) \sin w). \end{aligned}$$

Corollary 6. *When* $w = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$ *then we get*

$$\Delta^I \mathbf{H} = \mathbf{0}.$$

Remark 7. *Finding the function* $\varphi = \varphi(u)$ *is a future problem for us when the helicoidal hypersurface* $\mathbf{H}$ *has the equation* $\Delta^I \mathbf{H} = \mathbf{0}$.

6. Helicoidal Hypersurface with $\Delta^I \mathbf{H} = A\mathbf{H}$ in $\mathbb{E}^4$

The Gauss map of the helicoidal hypersurface (9) is

$$e = \frac{1}{W}\begin{pmatrix} (u\varphi' \cos w - b\sin w)\cos v \cos w - a\sin v \\ (u\varphi' \cos w - b\sin w)\sin v \cos w + a\cos v \\ (u\varphi' \sin w + b\cos w)\cos w \\ -u\cos w \end{pmatrix}$$

where

$$W = \sqrt{(b^2 + u^2(1+\varphi'^2))\cos^2 w + a^2}.$$

We use

$$-3He = A\mathbf{H} \tag{14}$$

and we get the column matrix e whose entries are

$$\begin{aligned} e_1 &= \big(\Theta(u\varphi' \cos w) - ua_{11}\big)\cos v \cos w - (a\Theta + u\cos w a_{12})\sin v \\ &\quad - (b\Theta \cos v \cos w + ua_{13})\sin w \\ e_2 &= \big(\Theta(u\varphi' \cos w) - ua_{22}\big)\sin v \cos w + (a\Theta - u\cos w a_{21})\cos v \\ &\quad - (b\Theta \sin v \cos w + ua_{23})\sin w \\ e_3 &= \big(u\varphi'\Theta \cos w - ua_{33}\big)\sin w + (b\Theta\cos w - u\sin v a_{32} - u\cos v a_{31})\cos w \\ e_4 &= -u\Theta\cos w \end{aligned}$$

which yields

$$\begin{pmatrix} (\varphi + av + bw)\,a_{14} \\ (\varphi + av + bw)\,a_{24} \\ (\varphi + av + bw)\,a_{34} \\ u\cos v\cos w a_{41} + u\sin v\cos w a_{42} + ua_{43}\sin w + (\varphi + av + bw)\,a_{44} \end{pmatrix}$$

and where A is a 4×4 matrix

$$\Theta(u,w) = \frac{3H}{W}.$$

The equation $\Delta^I \mathbf{H} = A\mathbf{H}$ by means of

$$I = \begin{pmatrix} 1+\varphi'^2 & a\varphi' & b\varphi' \\ a\varphi' & a^2 + u^2\cos^2 w & ab \\ b\varphi' & ab & b^2 + u^2 \end{pmatrix}$$

and $\Delta^I \mathbf{H} = -3He$ leads to the following system of ODEs

$$\begin{aligned}\left(\Theta u\varphi' \cos w - ua_{11}\right) \cos v \cos w - (a\Theta + u \cos w a_{12}) \sin v \\ - (b\Theta \cos v \cos w + ua_{13}) \sin w = (\varphi + av + bw)\, a_{14}\end{aligned}$$

$$\begin{aligned}\left(\Theta u\varphi' \cos w - ua_{21}\right) \sin v \cos w + (a\Theta - u \cos w a_{22}) \cos v \\ - (b\Theta \sin v \cos w + ua_{23}) \sin w = (\varphi + av + bw)\, a_{24}\end{aligned}$$

$$\begin{aligned}\left(\Theta u\varphi' \cos w - ua_{33}\right) \sin w + (b\Theta \cos w - u \sin v a_{32} - u \cos v a_{31}) \cos w \\ = (\varphi + av + bw)\, a_{34}\end{aligned}$$

$$-u\Theta \cos w = u \cos v \cos w a_{41} + u \sin v \cos w a_{42} + u \sin w a_{43} + (\varphi + av + bw)\, a_{44}.$$

Differentiating ODE's twice with respect to v, we have

$$a_{14} = a_{24} = a_{34} = a_{44} = 0, \qquad \Theta\,(u, w) = 0. \tag{15}$$

From (15) we get

$$-a_{11}u \cos v \cos w - a_{12}u \cos w \sin v - a_{13}u \sin w = 0 \tag{16}$$
$$-a_{21}u \sin v \cos w - a_{22}u \cos w \cos v - a_{23}u \sin w = 0 \tag{17}$$
$$-a_{31}u \cos v \cos w - a_{32}u \sin v - a_{33}u \sin w = 0 \tag{18}$$
$$a_{41}u \cos v \cos w + a_{42}u \sin v \cos w + a_{43}u \sin w = 0 \tag{19}$$

and because $\cos$ and $\sin$ are linearly independent functions of v, then we can calculate that $a_{ij} = 0$. From $\Theta\,(u, w) = \frac{3H}{W}$ we obtain $H = 0$. Consequently $\mathbf{H}$ is a minimal hypersurface. Finally we have the following theorem

Theorem 8. *Let* $\mathbf{H} : M^3 \longrightarrow \mathbb{E}^4$ *be an isometric immersion given by* (9). *Then* $\Delta^I \mathbf{H} = A\mathbf{H}$ *if and only if* M^3 *has zero mean curvature.*

Future Work. We will consider the Laplace-Beltrami operator of the Gauss map

$$\Delta^I e\,(u, v, w)$$

of the helicoidal hypersurface $\mathbf{H}\,(u, v, w)$ next in our work.

References

[1] Arvanitoyeorgos A., Kaimakamis G. and Magid M., *Lorentz Hypersurfaces in* $\mathbb{E}_1^4$ *Satisfying* $\Delta H = \alpha H$, Illinois J. Math. **53** (2009) 581-590.

[2] Bour E., *Théorie de la Déformation des Surfaces,* J. l'Ecole Imperiale Polytechnique **22-39** (1862) 1-148.

[3] Chen B.Y., *Total Mean Curvature and Submanifolds of Finite Type,* World Scientific, Singapore 1984.

[4] Cheng Q. and Wan Q., *Complete Hypersurfaces of* $\mathbb{R}^4$ *with Constant Mean Curvature,* Monatsh. Math. **118** (1994) 171-204.

[5] Choi M. and Kim Y., *Characterization of the Helicoid as Ruled Surfaces with Pointwise 1-type Gauss Map,* Bull. Korean Math. Soc. **38** (2001) 753-761.

[6] Dillen F., Pas J. and Verstraelen L., *On Surfaces of Finite Type in Euclidean 3-Space*, Kodai Math. J. **13** (1990) 10-21.

[7] do Carmo M. and Dajczer M., *Helicoidal Surfaces with Constant Mean Curvature,* Tohoku Math. J. **34** (1982) 351-367.

[8] Ferrandez A., Garay O. and Lucas P., *On a Certain Class of Conformally at Euclidean Hypersurfaces*, Proc. Int. Conf. in Global Analysis and Global Differential Geometry, Berlin 1990.

[9] Ganchev G. and Milousheva V., *General Rotational Surfaces in the 4-Dimensional Minkowski Space,* Turkish J. Math. **38** (2014) 883-895.

[10] Güler E., Yaylı Y. and Hacısalihoğlu H., *Bour's Theorem on the Gauss map in 3-Euclidean Space,* Hacettepe J. Math. **39** (2010) 515-525.

[11] Lawson H., *Lectures on Minimal Submanifolds*, Rio de Janeiro 1973.

[12] Magid M., Scharlach C. and Vrancken L., *Affine Umbilical Surfaces in* $\mathbb{R}^4$*,* Manuscripta Math. **88** (1995) 275-289.

[13] Moore C., *Surfaces of Rotation in a Space of Four Dimensions,* Ann. Math. **21** (1919) 81-93.

[14] Moore C., *Rotation Surfaces of Constant Curvature in Space of Four Dimensions,* Bull. Amer. Math. Soc. **26** (1920) 454-460.

[15] Moruz M. and Munteanu M., *Minimal Translation Hypersurfaces in* $\mathbb{E}^4$*,* J. Math. Anal. Appl. **439** (2016) 798-812.

[16] Scharlach C., *Affine Geometry of Surfaces and Hypersurfaces in* $\mathbb{R}^4$*,* In: Symposium on the Differential Geometry of Submanifolds 2007, pp 251-256.

[17] Senoussi B. and Bekkar M., *Helicoidal Surfaces with* $\Delta^J r = Ar$ *in 3-Dimensional Euclidean Space,* Stud. Univ. Babeş-Bolyai Math. **60** (2015) 437-448.

[18] Takahashi T., *Minimal Immersions of Riemannian Manifolds,* J. Math. Soc. Japan **18** (1966) 380-385.

[19] Verstraelen L., Walrave J. and Yaprak S., *The Minimal Translation Surfaces in Euclidean Space,* Soochow J. Math. **20** (1994) 77–82.

[20] Vlachos Th., *Hypersurfaces in* $\mathbb{E}^4$ *with Harmonic Mean Curvature Vector Field,* Math. Nachr. **172** (1995) 145-169.

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