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DRAWING THE FREE RIGID BODY DYNAMICS ACCORDING TO JACOBI

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Abstract. Guided by the Jacobi's work published one year before his death about the rotation of a rigid body, the behavior of the rotation matrix describing the dynamics of the free rigid body is studied. To illustrate this dynamics one draws on a unit sphere the trace of the three unit vectors, in the body system along the principal directions of inertia. A minimal set of properties of Jacobi's elliptic functions are used, those which allow to compute with the necessary precision the dynamics of the rigid body without torques, the so called Euler's top. Emphasis is on the paper published by Jacobi in 1850 on the explicit expression for the components of the rotation matrix. The tool used to compute the trajectories to be drawn are the Jacobi's Fourier series for *theta* and *eta* functions with extremely fast convergence. The Jacobi's sn, cn and dn functions, which are better known, are used also as ratios of *theta* functions which permit quick and accurate computation. Finally the main periodic part of the herpolhode curve was computed and graphically represented.

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Contents

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1. Introduction

The history of the dynamics of the free asymmetric rigid body started with Euler. In the Whittaker's book on mechanics one finds several references to the original papers [14].

At present time, the dominant source for a detailed and profound treatment of the dynamics of the rigid body is the Klein and Sommerfeld treatise, recently translated into English [8].

The purpose to contributing to this venerable subject is the drawing of the predicted motion of the body represented by the traces of the three unit vectors, forming the rows of the rotation matrix, portraying on the unit sphere close trajectories representing the main periodic behavior of the rigid body. Actually, the third row coincides with the representation of the angular momentum vector as is seen from the body system in the intersection of the sphere of angular momentum, and the ellipsoid of constant energy. This trajectory of the angular momentum vector has been graphically represented in different publications, see for example the Bender and Orzag book [4, p 203].

On the contrary, to our knowledge, the other two rows of the rotation matrix, have not received the pictorial representation of its motion, notwithstanding this motion for the second row has similar properties of symmetry as that of the angular momentum, whereas the first row has different symmetries from the other rows. In any case each of the three trajectories are periodic of the same period. The second and third rows are symmetric with respect to the same coordinate planes in the body system of principal moments of inertia. The first row, on the contrary, is skew symmetric with respect to the same planes. In my opinion this behavior is an essential knowledge of the physics of the free rotation in space far from acting torques, which deserves to be known and taken in account.

In order to present those facts and draw accurately as needed those curves, one uses a minimum of properties of elliptic functions, just those which are efficient and precise. Everything supported the magistral Jacobi's work.

Our purpose is to dilute the embarrassment expressed by the historian of mathematics Bell [2, p 399] who writes:

"The rotation of the rigid body, for example, yields numerous elegant exercises in the elliptic theta functions; but few engineers who must busy themselves with rotation have time for elegant analysis."

After Jacobi, Weierstrass contributed in extraordinary form to the theory of the elliptic functions. In both cited treatises both of Whittaker and Klein and Sommerfeld, the properties of the new functions introduced by Weierstrass are used. However in this paper almost nothing of the Weierstrass functions is included.

The basis to ignore the Weierstrass' work, here is justified by the following text extracted from the Whittaker and Watson treatise on analysis [15, p 523] which refers to the mathematics of the rigid body dynamics as:

"This result determines the mean precession about the invariable line in the motion of the rigid body relative to its center of gravity under forces whose resultant passes through its center of gravity. It is evident that, for purposes of computation, a result of this nature is preferable to the corresponding result in terms of sigma-functions and Weierstrassian zeta-functions, for the reason that the theta-functions have a specially simple behavior with respect to the real period–the period which is of importance in Applied Mathematics–and that the q-series are much better adapted for computation than the product by which the sigma-function is most simply defined."

The previous text from two recognized experts in elliptic functions supports the fact that for computing and also for drawing accurately our tools are the best. One can be more explicit: the *theta* functions used here are written in terms of Fourier series with n-coefficients which are proportional to a number q, smaller than one, with an exponent n^2. In such computations less than ten terms are necessary to obtain a precision of 15 places, for numbers q different from one in one over a million, one exceptional case, without general interest. One should use the *theta* functions because of their extremely fast convergence.

Richard Bellman confirms this point without any doubt in his brief book on *theta* functions [3, p 12].

The main objective of present paper is to represent by drawings the motion of a rigid body formed by particles which relative mutual distances do not change in time. The center of mass is assumed fixed at the origin of coordinates in the inertial system. The position of particle i of the body in the inertial system $\mathbf{r}_i$ is known in terms of the entries of the rotation matrix, because the existence of a coordinate system fixed to the body which will be named the body system, with the same origin of coordinates as the inertial system, at the rest center of mass, where all the coordinates of the rigid body $\mathbf{a}_i$ are constants of motion. The rotation matrix $\mathbf{R}$ transforms the coordinates of the i-th particle from the body system to the inertial system

$$\mathbf{r}_i = \mathbf{R}\mathbf{a}_i. \tag{1}$$

In what follows we will be concentrated in the drawing of the components of the rotation matrix, which preserves the constant distance between particles, requires its rows to be formed by the components of three mutually orthogonal, unit vectors, that form a right tern, namely, the $\times$ product of the two first vectors of $\mathbf{R}$ is equal to the third row of the same. The ortho-normality of the rows of the rotation matrix is represented in matrix notation as

$$\mathbf{R}^t\mathbf{R} = \mathbf{R}\mathbf{R}^t = \mathbf{E} \tag{2}$$

where $\mathbf{E}$ is the unit matrix. The super-index t to the right of a vector or matrix denotes the transposed vector or matrix. Therefore one has the inverse matrix of the rotation matrix is equal to its transposed matrix.

The derivative with respect to time is denoted with a point on the function to be derived. The derivative with respect to time of the equations in (2), uncovers the existence of the antisymmetric matrices defining the angular velocity

$$\begin{aligned}
\boldsymbol{\omega}^\times &= \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \equiv \mathbf{R}^t\dot{\mathbf{R}} \\
\boldsymbol{\Omega}^\times &= \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \equiv \dot{\mathbf{R}}\mathbf{R}^t.
\end{aligned} \tag{3}$$

The first definition corresponds to the angular velocity in the body system

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{4}$$

which gives the derivative with respect to time of the rotation matrix in the useful form

$$\dot{\mathbf{R}} = \mathbf{R}\,\boldsymbol{\omega}^\times. \tag{5}$$

The notation with the $\times$ product is used because the product by the right of the antisymmetric matrix with any vector is equal to the $\times$ product of the vector of the matrix with the same vector

$$\boldsymbol{\omega}^\times \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \tag{6}$$

The angular velocity in the inertial system $\boldsymbol{\Omega}$ will be important in the last part of this work because one pretends to draw the curve, called herpolhode, portrayed by this vector when one projects it on the orthogonal plane to the angular momentum vector.

From their definitions (3) the angular velocities vectors are related by the equation

$$\boldsymbol{\Omega}^\times = \mathbf{R}\boldsymbol{\omega}^\times\mathbf{R}^t = (\mathbf{R}\boldsymbol{\omega})^\times, \qquad \boldsymbol{\Omega} = \mathbf{R}\boldsymbol{\omega} \tag{7}$$

where we have used the geometric theorem: the $\times$ product of rotated vectors is equal to the rotation of the $\times$ product of the vectors.

To know the rotation matrix one starts using the conservation of the angular momentum vector. The angular momentum vector (with respect to the center of mass) is defined by the sum vector of the mass by the $\times$ product of the position and the velocity

$$\begin{aligned}\mathbf{J} &= \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \sum_i m_i (\mathbf{R}\mathbf{a}_i) \times (\dot{\mathbf{R}}\mathbf{a}_i) \\ &= \sum_i m_i (\mathbf{R}\mathbf{a}_i) \times (\mathbf{R}\boldsymbol{\omega} \times \mathbf{a}_i) = \mathbf{R}\sum_i m_i \mathbf{a}_i \times (\boldsymbol{\omega} \times \mathbf{a}_i)\end{aligned} \tag{8}$$

where the positions were used in terms of the rotation matrix. The time derivative of the rotation matrix as a function of the angular velocity, and the geometric theorem: the $\times$ product of the rotated vectors is the rotation of the $\times$ product of the vectors. Next the double $\times$ product is used to obtain

$$\mathbf{J} = \mathbf{R}\sum_i m_i \left(\mathbf{a}_i^t \mathbf{a}_i \boldsymbol{\omega} - \mathbf{a}_i \mathbf{a}_i^t \boldsymbol{\omega}\right) = \mathbf{R}\sum_i m_i \left(\mathbf{a}_i^t \mathbf{a}_i \mathbf{E} - \mathbf{a}_i \mathbf{a}_i^t\right)\boldsymbol{\omega} \tag{9}$$

where one has introduced the unit matrix $\mathbf{E}$ to take out as a common factor the angular velocity vector. In this way one finds the angular momentum vector of the inertial system as the product of the rotation matrix, the inertia matrix of inertia in the body system, and the angular velocity of the same system

$$\mathbf{J} = \mathbf{R}\boldsymbol{\mathcal{I}}\boldsymbol{\omega}. \tag{10}$$

The inertia matrix $\boldsymbol{\mathcal{I}}$ is the following constant symmetric matrix, which is assumed diagonal since the body system could be selected in such a way. i.e.,

$$\boldsymbol{\mathcal{I}} = \sum_i m_i \left(\mathbf{a}_i^t \mathbf{a}_i \mathbf{E} - \mathbf{a}_i \mathbf{a}_i^t\right) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{11}$$

Quantities I_j are positive and are called principal moments of inertia. The body system where the inertia matrix is diagonal is called the system of the principal moments of inertia. In what follows all the vectors of the body system have components in this system of coordinates.

2. Computing the Angular Momentum in the Body System

The angular momentum vector is conserved. This is a general theorem of mechanics: if no external torques are present, the angular momentum vector is conserved.

The inertial system of coordinates is selected so that the angular momentum vector is directed along the third coordinate axis. The constant magnitude of this vector is J

$$\mathbf{J}^t = J(0,0,1). \tag{12}$$

According to equation (10) the components of the angular momentum in the body system are $\mathbf{L} = \boldsymbol{\mathcal{I}}\boldsymbol{\omega}$. The magnitude of this vector is the constant J as it does not change with the rotation, but its direction changes with time. One denotes by $\mathbf{u}$ the unit vector in the body system in the direction of this vector

$$\mathbf{L} = \boldsymbol{\mathcal{I}}\boldsymbol{\omega} = J\mathbf{u}. \tag{13}$$

Hence the vector $\mathbf{L}$ is rotated into the vector $\mathbf{J}$ and the vector $\mathbf{u}$ is rotated into the vector $\mathbf{k}$

$$\mathbf{J} = \mathbf{RL}, \qquad \mathbf{Ru} = \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{14}$$

As the rows of the rotation matrix are mutually orthogonal the second equation in (14) implies that $\mathbf{u}^t$ is the vector forming the third row of the rotation matrix. One proceeds to compute this vector.

The equation of motion for $\mathbf{L}$ is obtained from the derivative with respect to time of the previous equation

$$\mathbf{0} = \mathbf{R}(\dot{\mathbf{L}} + \boldsymbol{\omega} \times \mathbf{L}), \qquad \dot{\mathbf{u}} = \mathbf{u} \times \boldsymbol{\omega} \tag{15}$$

where it has been written the time derivative of the matrix $\mathbf{R}$ in terms of the angular velocity. This is the Euler equation for the free rigid body.

In other hand, according to (13), the angular velocity is written in terms of the vector $\mathbf{L}$ as

$$\boldsymbol{\omega} = \boldsymbol{\mathcal{I}}^{-1}\mathbf{L}. \tag{16}$$

We find the equation of motion for the components of the angular momentum vector in the body system

$$\dot{\mathbf{L}} = \mathbf{L} \times \boldsymbol{\mathcal{I}}^{-1}\mathbf{L}. \tag{17}$$

Which has as constants of integration the energy E and the magnitude of the angular momentum vector

$$\mathbf{L}^t\boldsymbol{\mathcal{I}}^{-1}\mathbf{L} = 2E, \qquad \mathbf{L}^t\mathbf{L} = J^2. \tag{18}$$

We turn our attention to the unit vector $\mathbf{u}$ in the direction of the angular momentum, dividing the angular momentum vector by its magnitude.

One writes the inverses of the principal moments of inertia and the constant $2E/J^2$ in the form

$$\frac{1}{I_j} = H + Pe_j, \qquad \frac{2E}{J^2} = H + Pe_0 \tag{19}$$

where H and P are defined in order to diminish the number of independent parameters

$$\begin{aligned} H &= \frac{1}{3}\left(\frac{1}{I_1} + \frac{1}{I_2} + \frac{1}{I_3}\right), \qquad e_1 + e_2 + e_3 = 0 \\ P^2 &= \frac{4}{9}\left(\frac{1}{I_1^2} + \frac{1}{I_2^2} + \frac{1}{I_3^2} - \frac{1}{I_1 I_2} - \frac{1}{I_2 I_3} - \frac{1}{I_3 I_1}\right), \qquad e_1^2 + e_2^2 + e_3^2 = 3/2 \end{aligned} \tag{20}$$

which implies that the parameters e_j can be expressed in terms of just one angle γ

$$e_1 = \cos\gamma, \qquad e_2 = \cos(\gamma - 2\pi/3), \qquad e_3 = \cos(\gamma + 2\pi/3). \tag{21}$$

While the constants of motion of the angular momentum vector are functions of the five parameters E, J, I_1, I_2, I_3, the components u_j of the unit vector $\mathbf{u}$ in the direction of the angular momentum can be written just in terms of the two parameters γ and e_0

$$u_1^2 + u_2^2 + u_3^2 = 1, \qquad e_1 u_1^2 + e_2 u_2^2 + e_3 u_3^2 = e_0. \tag{22}$$

These equations suggest to utilize the spheroconical coordinates α_1 and α_2 for the components of this vector

$$\begin{aligned} u_1 &= \mathrm{cn}(\alpha_2, k_2)\mathrm{cn}(\alpha_1, k_1) \\ u_2 &= \mathrm{dn}(\alpha_2, k_2)\mathrm{sn}(\alpha_1, k_1) \\ u_3 &= \mathrm{sn}(\alpha_2, k_2)\mathrm{dn}(\alpha_1, k_1). \end{aligned} \tag{23}$$

According to the NIST Handbook of Mathematical Functions [13, p 693], the parameter β in 29.18.2 corresponds in our notation to $K(k_1) + \mathrm{i}K(k_2) - \mathrm{i}\alpha_2$, the function K and its arguments will be defined in this section. One prefers the form (23) since it is not necessary to use complex variables. In the quantum case of the same Euler free rigid body motion the Schrödinger equation is separated in spheroconical coordinates [12]. (See for example the Méndez-Fragoso and Ley-Koo [10] review paper on quantum rotations.) The spheroconical coordinates in the form (23) are also found in the Morse and Feshbach book [11, p 659], as conical coordinates. The stereoscopic view in three dimensions of some of the conical coordinate curves have been drawn in this reference. In our forward Fig. 1 some coordinate curves (α_2 = constant) are drawn on the sphere as the trajectories followed by the angular momentum vector in the system of the principal moments of inertia.

The first equation of (22) is satisfied identically if we use the properties of the Jacobi elliptic functions

$$\mathrm{sn}^2(\beta,k)+\mathrm{cn}^2(\beta,k)=1, \qquad k^2\mathrm{sn}^2(\beta,k)+\mathrm{dn}^2(\beta,k)=1 \tag{24}$$

and the relation for the constants k_i

$$k_1^2+k_2^2=1. \tag{25}$$

The second equation in (22) is satisfied identically if α_2, k_1, k_2 are constants chosen with the restrictions

$$\mathrm{sn}(\alpha_2,k_2)=\sqrt{\frac{e_1-e_0}{e_1-e_3}}, \quad \mathrm{cn}(\alpha_2,k_2)=\sqrt{\frac{e_0-e_3}{e_1-e_3}}, \quad \mathrm{dn}(\alpha_2,k_2)=\sqrt{\frac{e_0-e_3}{e_2-e_3}}$$

$$k_1=\sqrt{\frac{(e_1-e_2)(e_0-e_3)}{(e_2-e_3)(e_1-e_0)}}, \qquad k_2=\sqrt{\frac{(e_1-e_3)(e_2-e_0)}{(e_2-e_3)(e_1-e_0)}} \tag{26}$$

which are redundant because the equations (24) and (25) are identically satisfied. Note that these constants are functions only of the parameters γ of asymmetry and e_0 of energy. For the components u_1, u_2 and u_3 of the vector $\mathbf{u}$, and for the third row of the rotation matrix $\mathbf{R}$, it is preferable to use instead of (23) the coordinates

$$\begin{aligned} u_1 &= \sqrt{\frac{e_0-e_3}{e_1-e_3}}\mathrm{cn}(\alpha_1,k_1) \\ u_2 &= \sqrt{\frac{e_0-e_3}{e_2-e_3}}\mathrm{sn}(\alpha_1,k_1) \\ u_3 &= \sqrt{\frac{e_1-e_0}{e_1-e_3}}\mathrm{dn}(\alpha_1,k_1). \end{aligned} \tag{27}$$

The argument α_1 is a function of time. To know its behavior we must know the derivatives of the Jacobi functions. Owed to the quadratic relations (24), it is sufficient to know one of them, but one includes the three to stress on their simplicity and similarity

$$\begin{aligned} \frac{\mathrm{d}\,\mathrm{sn}(\beta,k)}{\mathrm{d}\,\beta} &= \mathrm{cn}(\beta,k)\mathrm{dn}(\beta,k) \\ \frac{\mathrm{d}\,\mathrm{cn}(\beta,k)}{\mathrm{d}\,\beta} &= -\mathrm{sn}(\beta,k)\mathrm{dn}(\beta,k) \\ \frac{\mathrm{d}\,\mathrm{dn}(\beta,k)}{\mathrm{d}\,\beta} &= -k^2\mathrm{sn}(\beta,k)\mathrm{cn}(\beta,k). \end{aligned} \tag{28}$$

We need to know also the equation of the motion for the coordinate α_1. Substituting equation (13) in the Euler equation of motion, with the explicit form (27) for the components of $\mathbf{u}$ and the principal moments of inertia in the form (19) one finds the constant derivative

$$\dot{\alpha_1} = PJ\sqrt{(e_1 - e_0)(e_2 - e_3)}. \tag{29}$$

Note that besides the used parameters γ and e_0, the time appears without dimensions in the combination JPt.

To draw the curve $\mathbf{u}$, going over the unit sphere, we should know how to compute the Jacobi elliptic functions. As functions of the real variable α_1 they are periodic. The period of the vector $\mathbf{u}$ is four times the function $K(k_1)$. The function $K(k)$ is the complete elliptic integral of the first kind. To compute it we have the algorithm of the arithmetic-geometric mean [1] allowing to calculate it with rapidity which can be described as follows: Take the initial values $x_0 = 1 + k$, $y_0 = 1 - k$. Iterate $x_n, y_n \to x_{n+1} = (x_n + y_n)/2, y_{n+1} = \sqrt{x_n y_n}$, up to the desired precision when the two averages are equal to a certain value x. Then $K(k) = \pi/(2x)$. To draw the vector $\mathbf{u}$ one divides the period in equal parts and one draws the components of $\mathbf{u}$ calculated in all the points corresponding to the increments of the division of the period.

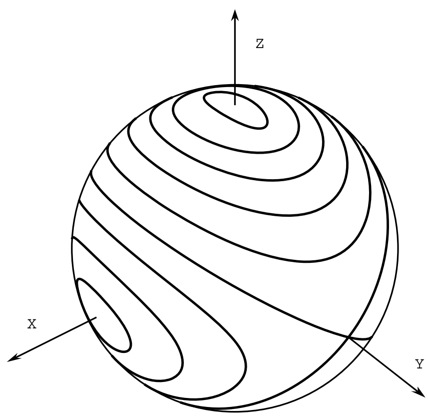

Figure 1. The sphere of angular momentum intersected by ellipsoids of constant energy. The curves follow the motion of the angular momentum vector on the sphere of angular momentum for different values of the energy. These curves are symmetric with respect to two of the coordinate planes.

We have several efficient algorithms to compute numerically the value of the Jacobian functions [1]. Here we make use of the Jacobi formula [7] as the ratio of

$\Theta(\beta, k)$ and $H(\beta, k)$ functions, which has the advantage of a very fast convergence, comparable in precision and fastness with other methods, and the bonus of being one of them is indispensable to the calculus of the other six entries of the rotation matrix $\mathbf{R}$.

The *theta* Jacobi functions require other function of the parameter k defined in terms of $K(k)$ by

$$q(k) = \exp(-\pi K(\sqrt{1-k^2})/K(k) \tag{30}$$

which is a positive number lower than one, which is shortened as q. The *theta* functions Θ and H are computed then as Fourier [7] series with real period $2K(k)$

$$\Theta(\beta, k) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos\frac{\pi n \beta}{K(k)} \tag{31}$$

and respectively $4K(k)$

$$H(\beta, k) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2} \sin\frac{\pi(2n+1)\beta}{2K(k)}. \tag{32}$$

These series should be computed up to the necessary precision when the next term is negligible, because of the fast convergency to zero of the factor q^{n^2}. These *theta* functions are the algorithms to compute other elliptic Jacobi functions by means of the equations [7]

$$\begin{gathered}
\mathrm{sn}(\beta, k) = \frac{H(\beta, k)}{\sqrt{k}\,\Theta(\beta, k)}, \qquad \mathrm{cn}(\beta, k) = \frac{\sqrt[4]{1-k^2}}{\sqrt{k}}\frac{H(\beta+K(k), k)}{\Theta(\beta, k)} \\
\mathrm{dn}(\beta, k) = \sqrt[4]{1-k^2}\,\frac{\Theta(\beta+K(k), k)}{\Theta(\beta, k)}.
\end{gathered} \tag{33}$$

Jacobi [7] provides the entries of the rotation matrix as ratios of these *theta* functions. He understood the computational advantage of their use. In this work we give a priority to the Jacobi's elliptic functions sn, cn, dn whenever it is possible because their properties are better known. But the Jacobi's *theta* functions are actually used as algorithms for their evaluation. Comparing the Jacobi's expressions [7] for the third row of the rotation matrix we can observe the same components with the same functions. We are following Jacobi very closely.

As an example of the use of this algorithm we draw on the angular momentum sphere the intersections with ellipsoids of constant energy one has Fig. 1, in which the asymmetry parameter was selected as $\gamma = \pi/5$ and several parameters of energy e_0. This is the trajectory followed by the vector along the third row of the rotation matrix on the sphere of unit radius. The observer of the picture has the spherical coordinates $2\pi/7, 2\pi/7$ with respect to the principal inertia directions.

3. The Entries Perpendicular to the Body Angular Momentum in the Rotation Matrix

In this section one analyses the paleography of the rotation matrix as a function of time, of the motion without torques of a rigid body, as published by Jacobi [7] at 1850. Understood paleography as the study of an old text and its traduction or explanation in modern terms. The nine components of the rotation matrix were written by Jacobi by means of a set of *theta* functions, the same functions used in the previous section. The third row of the rotation matrix, as pointed before, was written in spheroconical coordinates in terms of the H and Θ *theta* functions. Jacobi finds the other six components of the same matrix expressed also by using ratios of the same functions, although now with complex arguments. The corresponding entries in the two first rows of the rotation matrix involves three *theta* functions with different arguments, and multiplied, by the real and the other by the imaginary part of a *theta* function with the complex argument $\alpha_1 + \mathrm{i}\alpha_2$. Remember that α_1 is proportional to time, and that α_2 is a constant coordinate satisfying the three equations in (26), but its numerical value should be determined now.

Since α_2 is an elliptic integral of the first kind

$$\alpha_2 = \int_0^{\sqrt{\frac{e_1-e_0}{e_1-e_3}}} \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k_2^2x^2)}}. \tag{34}$$

To attain a more precise value for α_2, one introduces the numerical value of this integral as approximated root and look for the root α_2 of the equation

$$\mathrm{sn}(\alpha_2, k_2) - \sqrt{\frac{e_1 - e_0}{e_1 - e_3}} = 0. \tag{35}$$

Actually one finds two roots for α_2 in the interval $[0, 2K(k_2)]$. The smaller of the two, which has to be selected, corresponds to the choice of positive values in (26).

The theta functions with complex argument are defined by the same Fourier series as before in (31) and (32). The modification due to the imaginary argument changes the coefficient of the Fourier series, however the convergence remains very fast.

Observing the two first rows of the rotation Jacobi's matrix one deduces the convenience of combining the vector $\mathbf{s}$ of the first row as the real part, and vector $\mathbf{t}$ of the second row as minus the imaginary part of a complex vector, function of a complex variable. The two vectors are orthogonal to the third row, which is a complex vector too. The scalar product of the complex vector with itself should be zero as a consequence of the orthogonality of the real and imaginary parts of the same magnitude.

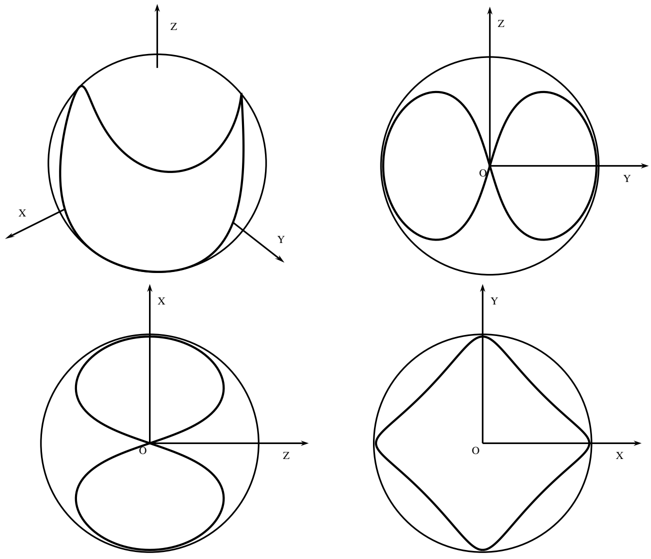

Figure 2. The trajectory followed by the vector forming the first row of the rotation matrix in a period. Perspective view and projection on the coordinate planes in the system of principal moments of inertia. The curve is antisymmetric with respect to the two coordinate planes crossing at the OZ axis of coordinates.

In our notation the complex vector is

$$\mathbf{s} - \mathrm{i}\mathbf{t} = \frac{1}{H(\mathrm{i}\alpha_2 + K(k_1))\Theta(\alpha_1)} \begin{pmatrix} \Theta(K(k_1))H(\alpha_1 + \mathrm{i}\alpha_2) \\ -\Theta(0)H(\alpha_1 + \mathrm{i}\alpha_2 + K(k_1)) \\ -\mathrm{i}H(K(k_1))\Theta(\alpha_1 + \mathrm{i}\alpha_2) \end{pmatrix} \tag{36}$$

where one has suppressed the second argument of the *theta* functions because it is always the same: k_1. We have changed also the sign of the first component because one assumes the Jacobi's mistake to be verified in the sequel.

Let us notes that components 1 and 2 divided by the third allows to introduce the functions sn and cn of complex argument as the ratio of $theta$ functions

$$\frac{\Theta(K(k_1))\,H(\alpha_1 + \mathrm{i}\alpha_2)}{H(K(k_1))\,\Theta(\alpha_1 + \mathrm{i}\alpha_2)} = \frac{\mathrm{sn}\,(\alpha_1 + \mathrm{i}\alpha_2, k_1)}{\mathrm{sn}\,(K(k_1), k_1)} = \mathrm{sn}\,(\alpha_1 + \mathrm{i}\alpha_2, k_1)$$

$$\frac{\Theta(0)\,H(\alpha_1 + \mathrm{i}\alpha_2) + K(k_1))}{H(K(k_1))\,\Theta(\alpha_1 + \mathrm{i}\alpha_2)} = \frac{\mathrm{cn}\,(\alpha_1 + \mathrm{i}\alpha_2, k_1)}{\mathrm{cn}\,(0, k_1)} = \mathrm{cn}\,(\alpha_1 + \mathrm{i}\alpha_2, k_1)$$

and therefore the complex vector can be written as

$$\mathbf{s} - \mathrm{i}\mathbf{t} = \frac{H(K(k_1))\Theta(\alpha_1 + \mathrm{i}\alpha_2)}{H(K(k_1) + \mathrm{i}\alpha_2)\Theta(\alpha_1)} \begin{pmatrix} \mathrm{sn}(\alpha_1 + \mathrm{i}\alpha_2, k_1) \\ -\mathrm{cn}(\alpha_1 + \mathrm{i}\alpha_2, k_1) \\ -\mathrm{i} \end{pmatrix}. \tag{37}$$

Let us notice also that the vector on the right is a null vector as a consequence of the first quadratic identity in (24), although the magnitudes of the real and imaginary parts of this vector are not equal to one.

One substitutes the equations in the Abramowitz and Stegun handbook [1, p 575, equation(16.21)] for functions sn and cn of complex argument, which have been written in function of the three components of the unit vector along the angular momentum. These formula were not found in other classical references and therefore were verified starting from the addition formulae for the Jacobi elliptic functions and the Jacobi equations for the same functions of imaginary argument which are reproduced in many references

$$\mathbf{s} - \mathrm{i}\mathbf{t} = \frac{H(K(k_1))\Theta(\alpha_1 + \mathrm{i}\alpha_2)}{H(K(k_1) + \mathrm{i}\alpha_2)\Theta(\alpha_1)} \begin{pmatrix} \dfrac{u_2 + \mathrm{i}u_3u_1}{1 - u_3^2} \\ \dfrac{-u_1 + \mathrm{i}u_2u_3}{1 - u_3^2} \\ -\mathrm{i} \end{pmatrix}. \tag{38}$$

We verify in this way the misprint in the first component of the complex vector which was corrected to be orthogonal to the vector $\mathbf{u}$, as it should be.

To establish clearly these equations one rewrites them in terms of Euler angles as defined by Goldstein [6, p 147]. Our matrix $\mathbf{R}$ corresponds to his $\tilde{\mathbf{A}}$. The third row is the unit vector in the angular momentum direction which components are written in terms of two Euler angles

$$\mathbf{u}^t = (u_1, u_2, u_3) = (\sin\theta\sin\psi, \sin\theta\cos\psi, \cos\theta). \tag{39}$$

Adding the first row to the second multiplied by the square root of minus one of the rotation matrix, it is possible to extract a common factor in the three complex components and to find the null vector

$$\mathrm{e}^{\mathrm{i}\phi}(\cos\psi + \mathrm{i}\sin\psi\cos\theta, -\sin\psi + \mathrm{i}\cos\psi\cos\theta, -\mathrm{i}\sin\theta). \tag{40}$$

It is interesting to observe that if the vector between parentheses is divided by $\sin\theta$ it appears another vector which can be found in the Jacobi's publication

$$\begin{pmatrix} \dfrac{\cos\psi}{\sin\theta} + \mathrm{i}\sin\psi\cot\theta \\ -\dfrac{\sin\psi}{\sin\theta} + \mathrm{i}\cos\psi\cot\theta \\ -\mathrm{i} \end{pmatrix} = \begin{pmatrix} \dfrac{u_2 + \mathrm{i}u_3u_1}{1 - u_3^2} \\ \dfrac{-u_1 + \mathrm{i}u_2u_3}{1 - u_3^2} \\ -\mathrm{i} \end{pmatrix} = \begin{pmatrix} \mathrm{sn}(\alpha_1 + \mathrm{i}\alpha_2, k_1) \\ -\mathrm{cn}(\alpha_1 + \mathrm{i}\alpha_2, k_1) \\ -\mathrm{i} \end{pmatrix}. \tag{41}$$

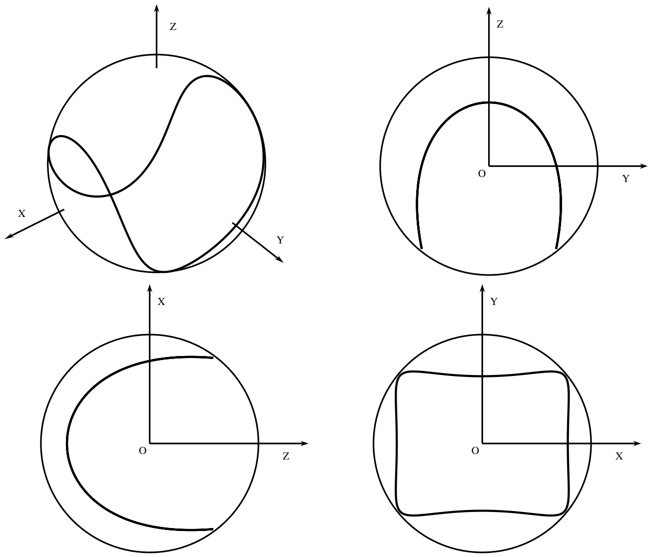

Figure 3. The trajectory followed by the vector forming the second row of the rotation matrix in a period. The curve is symmetric with respect to the same two coordinate planes as of the body angular momentum vector. Perspective view and projection on the coordinate planes in the system of principal moments of inertia.

For computations we should use the middle form because it has been written in terms of the components of the unit vector $\mathbf{u}$. And this means an economy in machine time.

The factor that should multiply this vector to obtain the first two rows of the rotation matrix as its real and imaginary parts of the vector (40) is $\mathrm{e}^{\mathrm{i}\phi_1}\sin\theta$, where ϕ_1 is the main periodic part of the third Euler angle ϕ. According to Jacobi that factor is

$$\mathrm{e}^{\mathrm{i}\phi_1}\sqrt{1-u_3^2} = \frac{H(K(k_1))\Theta(\alpha_1+\mathrm{i}\hat{\alpha}_2)}{H(K(k_1)+\mathrm{i}\hat{\alpha}_2)\Theta(\alpha_1)} = \mathrm{cn}(\hat{\alpha}_2,k_2)\frac{\Theta(0)\Theta(\alpha_1+\mathrm{i}\hat{\alpha}_2)}{\Theta(\alpha_1)\Theta(\mathrm{i}\hat{\alpha}_2)}.$$

Actually plotting the real *vs.* the imaginary part of the $\exp(\mathrm{i}\phi_1$, the points do not fall on a unit circle as it should be. The reason seams to be subtle to me. A graphical solution was to use a the other α_2 root in the right hand side of this equation, which was denoted as $\hat{\alpha}_2$ which is equal to $2K(k_2)-\alpha_2$. Modification of the right hand side is performed by prescription 16.33.4 in reference [1] who adds a constant angular velocity along the angular momentum vector with the same period that vector $\mathbf{u}$

$$\begin{aligned} e^{i\phi_1}\sqrt{1-u_3^2} &= \frac{H(K(k_1))\Theta(\alpha_1+i\alpha_2)}{H(K(k_1)+i\alpha_2)\Theta(\alpha_1)}\exp\left(\frac{-i\pi\alpha_1}{2K(k_1)}\right) \\ &= \mathrm{cn}(\alpha_2,k_2)\frac{\Theta(0)\Theta(\alpha_1+i\alpha_2)}{\Theta(\alpha_1)\Theta(i\alpha_2)}\exp\left(\frac{-i\pi\alpha_1}{2K(k_1)}\right). \end{aligned} \tag{42}$$

Jacobi's equation allows to obtain the main periodic part of the third Euler angle which remains to complete the three Euler angles as a function of time. In fact one has the real quantities $\sqrt{1-u_3^2}$, $\mathrm{cn}(\alpha_2,k_2)$, $\Theta(0)$, $\Theta(i\alpha_2)$, $\Theta(\alpha_1)$, therefore taking the logarithm of (42) and subtracting its complex conjugated expression give us

$$2i\phi_1 = \ln\frac{\Theta(\alpha_1+i\alpha_2)}{\Theta(\alpha_1-i\alpha_2)} - i\frac{\pi\alpha_1}{K(k_1)} \tag{43}$$

as found in the references [14] and [9, p 119], as the periodic part with the same period of the $\mathbf{u}$ vector. But Jacobi's equation (42) contains more information since provides economy of computation, the factor that is needed to compute the two missing rows of the rotation matrix.

The difference of the two angles ϕ and ϕ_1 is an angle with constant angular velocity. This was neglected by Jacobi due to its simplicity. One draws the figures of the unit vectors forming the Jacobi's rotation matrix neglecting the same constant angular velocity around the constant angular momentum.

The curves that follow these two rows of the rotation matrix on the unit sphere with period $4K(k_1)$ of α_1 are drawn in Fig. 2 and Fig. 3, for a typical particular case, computed from equation (42).

4. Recovering the Jacobi's Expression for the Components of the Rotation Matrix

In this Section we prove the Jacobi's equation (42) of the precedent section which could be used as an efficient algorithm to compute and draw the curves associated with the other two rows of the rotational matrix. The full rotation matrix including the missing constant angular velocity rotation around the constant angular momentum vector is fully reincorporated here.

One presents a simplified mathematical proof to verify equation (42). As an alternative one could use equation (43) that seems to be more elegant and simple. However, as has been pointed out, it hides information, as it implies first to compute the angle, and then to compute the trigonometric functions and replace them in the rotation matrix. The Jacobi's equation (42) give directly the trigonometric functions and permits "quick and accurate computation" [3, p 12].

The derivative of the angle ϕ is obtained by Landau and Lifshitz [9, p 119] from the angular velocity and angular momentum in terms of the Euler angles (the same convention for the Euler angles used by Goldstein [6, p 147] and is adopted in this work)

$$\begin{aligned} \dot{\phi} &= \frac{\omega_1 u_1 + \omega_2 u_2}{1-u_3^2} = \frac{\mathbf{u}^t \boldsymbol{\omega} - u_3\omega_3}{1-u_3^2} = \frac{2E/J - u_3^2 J/I_3}{1-u_3^2} = \frac{J}{I_3} + \frac{JP(e_0-e_3)}{1-u_3^2} \\ &= \frac{J}{I_3} + JP\sqrt{(e_1-e_0)((e_2-e_3)}\frac{e_0-e_3}{\sqrt{(e_1-e_0)((e_2-e_3)}}\frac{1}{1-u_3^2}. \end{aligned} \tag{44}$$

The first term in the right hand side of this equation is a constant, with an integral linear in time. The second summand is a periodic function of the variable α_1 with period $4K(k_1)$, considered by Jacobi. The integral with respect to time of a periodic function of another function which is linear in time plus a periodic function of the same period. The behavior of ϕ is then separated in two terms, one periodic function of the variable α_1 of period $4K(k_1)$ (the main periodic term) plus other function that is linear in time, where the two linear in time contributions are added. The last is a motion with constant angular velocity, with other period generally incommensurable to the previous one. This last with a constant angular velocity is separated and neglected by Jacobi due to its relative simplicity. In what follows these terms are not longer ignored.

From the two terms in the right hand side of (44), the second which will be denoted $\mathrm{d}\phi_0/\mathrm{d}t$, is convenient to express in terms of the coordinate α_1

$$\frac{\mathrm{d}\phi_0}{\mathrm{d}\alpha_1} = \frac{e_0-e_3}{\sqrt{(e_1-e_0)((e_2-e_3)}}\frac{1}{1-\mathrm{sn}^2(\alpha_2,k_2)\mathrm{dn}^2(\alpha_1,k_1)}. \tag{45}$$

The same derivative is now transformed replacing the constants in terms of elliptic functions of the imaginary value $\mathrm{i}\alpha_2$, which is accomplished by means of our equations (26), and Jacobi's formulas for imaginary argument [13, p 693], [1]

$$\frac{\mathrm{d}\phi_0}{\mathrm{d}\alpha_1} = \mathrm{i}\frac{\mathrm{cn}(\mathrm{i}\alpha_2,k_1)\mathrm{dn}(\mathrm{i}\alpha_2,k_1)}{\mathrm{sn}(\mathrm{i}\alpha_2,k_1)}\frac{1}{1-k_1^2\mathrm{sn}^2(\mathrm{i}\alpha_2,k_1)\mathrm{sn}^2(\alpha_1,k_1)}. \tag{46}$$

In what follows the argument k_1 is suppressed since it is the same for all functions.

It is customary to present the elliptic integral of the third kind in the standard form [15, p 523]

$$\Pi(\alpha_1,\mathrm{i}\alpha_2) = \int_0^{\alpha_1} \mathrm{d}\alpha_1 \frac{k_1^2\mathrm{sn}(\mathrm{i}\alpha_2)\mathrm{cn}(\mathrm{i}\alpha_2)\mathrm{dn}(\mathrm{i}\alpha_2)\mathrm{sn}^2(\alpha_1)}{1-k_1^2\mathrm{sn}^2(\mathrm{i}\alpha_2)\mathrm{sn}^2(\alpha_1)} \tag{47}$$

which is related to the angle $\phi_0(\alpha_1)$ because

$$\Pi(\alpha_1, \mathrm{i}\alpha_2) = \frac{\mathrm{cn}(\mathrm{i}\alpha_2)\mathrm{dn}(\mathrm{i}\alpha_2)}{\mathrm{sn}(\mathrm{i}\alpha_2)} \left[\int_0^{\alpha_1} \mathrm{d}\alpha_1 \frac{1}{1 - k_1^2 \mathrm{sn}^2(\mathrm{i}\alpha_2)\mathrm{sn}^2(\alpha_1)} - \alpha_1 \right]. \quad (48)$$

One finds in this way the formula

$$\phi_0(\alpha_1) = \mathrm{i}\Pi(\alpha_1, \mathrm{i}\alpha_2) + \mathrm{i}\frac{\mathrm{cn}(i\alpha_2)\mathrm{dn}(\mathrm{i}\alpha_2)}{\mathrm{sn}(\mathrm{i}\alpha_2)}\alpha_1 \quad (49)$$

where the constant of integration is assumed to be $\phi_0(0) = 0$. However instead of using the elliptic integral of the third kind only the Jacobi's theta functions are used here.

To obtain the expression for the factor used by Jacobi in his rotational matrix one can use a relation borrowed from the Whittaker and Watson book on analysis [15, p 518]

$$\frac{\Theta'(\alpha_1 + \mathrm{i}\alpha_2)}{\Theta(\alpha_1 + \mathrm{i}\alpha_2)} - \frac{\Theta'(\alpha_1)}{\Theta(\alpha_1)} - \frac{\Theta'(\mathrm{i}\alpha_2)}{\Theta(\mathrm{i}\alpha_2)} = -k_1^2 \mathrm{sn}(\alpha_1)\mathrm{sn}(\mathrm{i}\alpha_2)\mathrm{sn}(\alpha_1 + \mathrm{i}\alpha_2). \quad (50)$$

The right hand side can be written with the addition formula of Jacobi's function $\mathrm{sn}(\alpha_1 + \mathrm{i}\alpha_2)$ used in (41) as

$$\begin{aligned} \frac{\Theta'(\alpha_1 + \mathrm{i}\alpha_2)}{\Theta(\alpha_1 + \mathrm{i}\alpha_2)} - \frac{\Theta'(\alpha_1)}{\Theta(\alpha_1)} - \frac{\Theta'(\mathrm{i}\alpha_2)}{\Theta(\mathrm{i}\alpha_2)} &= -\frac{k_1^2 \mathrm{sn}(\mathrm{i}\alpha_2)\mathrm{cn}(\mathrm{i}\alpha_2)\mathrm{dn}(\mathrm{i}\alpha_2)\mathrm{sn}^2(\alpha_1)}{1 - k_1^2 \mathrm{sn}^2(\alpha_1)\mathrm{sn}^2(\mathrm{i}\alpha_2)} \\ &+ \frac{k_1^2 \mathrm{sn}^2(\alpha_2, k_2)\mathrm{sn}(\alpha_1, k_1)\mathrm{cn}(\alpha_1, k_1)\mathrm{dn}(\alpha_1, k_1)}{1 - \mathrm{sn}^2(\alpha_2, k_2)\mathrm{dn}^2(\alpha_1, k_1)}. \quad (51) \end{aligned}$$

Integrating both sides of this equation from 0 to α_1 one finds

$$\begin{aligned} \ln \frac{\Theta(0)\Theta(\alpha_1 + \mathrm{i}\alpha_2)}{\Theta(\alpha_1)\Theta(i\alpha_2)} - \alpha_1 \frac{\Theta'(\mathrm{i}\alpha_2)}{\Theta(\mathrm{i}\alpha_2)} &= \int_0^{\alpha_1} \frac{-k_1^2 \mathrm{sn}(\mathrm{i}\alpha_2)\mathrm{cn}(\mathrm{i}\alpha_2)\mathrm{dn}(\mathrm{i}\alpha_2)\mathrm{sn}^2(\alpha_1)}{1 - k_1^2 \mathrm{sn}^2(\alpha_1)\mathrm{sn}^2(\mathrm{i}\alpha_2)} \mathrm{d}\alpha_1 \\ &+ \ln \frac{\sqrt{1 - \mathrm{sn}^2(\alpha_2, k_2)\mathrm{dn}^2(\alpha_1, k_1)}}{\mathrm{cn}(\alpha_2, k_2)} \quad (52) \end{aligned}$$

where one replaces the angle ϕ_0 to find a version of the Jacobi equation (42) if one suppresses the two terms linear in the time that are neglected by Jacobi

$$\ln \frac{\Theta(0)\Theta(\alpha_1 + \mathrm{i}\alpha_2)}{\Theta(\alpha_1)\Theta(\mathrm{i}\alpha_2)} = \mathrm{i}\left[\phi_1(\alpha_1) + \frac{\pi\alpha_1}{2K(k_1)}\right] + \ln \frac{\sqrt{1 - \mathrm{sn}^2(\alpha_2, k_2)\mathrm{dn}^2(\alpha_1, k_1)}}{\mathrm{cn}(\alpha_2, k_2)}$$

and where the linear terms in time are no longer present as

$$\phi_0(\alpha_1) - \phi_1(\alpha_1) = \frac{\pi\alpha_1}{2K(k_1)} \mathrm{i}\alpha_1 \left[\frac{\Theta'(\mathrm{i}\alpha_2)}{\Theta(\mathrm{i}\alpha_2)} + \frac{\mathrm{cn}(\mathrm{i}\alpha_2)\mathrm{dn}(\mathrm{i}\alpha_2)}{\mathrm{sn}(\mathrm{i}\alpha_2)} \right]. \tag{53}$$

This constant angular has been ignored in the present work, as in Jacobi's, taking in account only the main period of the rotation matrix, in which we have suppressed such constant angular velocity. One regards preferable this choice as discussed at once.

The angular velocity constant has two terms, one of them depends on the parameter H appearing in (20) and disappearing from discussion until it reappears as a summand in the velocity of the angle of Euler $\dot{\phi}$ in (44)

$$\frac{J}{I_3} = JH + JPe_3. \tag{54}$$

One adds all the linear terms in time to have the complete angle ϕ minus its periodic ϕ_1, considered by Jacobi

$$\phi - \phi_1 = JHt + \alpha_1 \left[\frac{e_0}{\sqrt{(e_1 - e_0)(e_2 - e_3)}} + \frac{\Theta'(\alpha_2, k_2)}{\Theta(\alpha_2, k_2)} + \frac{\pi(\alpha_2 + K(k_2))}{2K(k_1)K(k_2)} \right].$$

The term proportional to H could have any value and give an undetermined character to any draw of the rotation matrix, as a consequence it is a vagary to pretend to draw it. Nevertheless it deserves some extra consideration. The first term and the first inside the parentheses should be reassembled as

$$JHt + \alpha_1 \frac{e_0}{\sqrt{(e_1 - e_0)(e_2 - e_3)}} = \frac{E}{J}\, t. \tag{55}$$

In the remaining terms in that Euler angle we see the logarithmic derivative of $\Theta(\alpha_2)$ which is known in analysis as the $Z(\alpha_2)$

$$\phi - \phi_1 = \frac{E}{J}\, t + \alpha_1 \left[\frac{\Theta'(\alpha_2, k_2)}{\Theta(\alpha_2, k_2)} + \frac{\pi(\alpha_2 + K(k_2))}{2K(k_1)K(k_2)} \right]. \tag{56}$$

The logarithmic derivative $Z(\alpha_2, k_2) = \Theta'(\alpha_2, k_2)/\Theta(\alpha_2, k_2)$ is computed as the ratio of Fourier series or by using the algorithm of the arithmetic-geometric mean as can be found in the literature [1].

As one has computed the nine entries of the main period of the rotation matrix, it is possible to draw on the unit sphere three curves corresponding to the three columns of it. They give the main periodic motion in the inertial system of coordinates of the rotating rigid body with the constant angular velocity around the constant angular momentum. The resulting curves are different from the previous result: one finds one curve, almost plane, symmetric with respect to two coordinate planes. The two other curves are both symmetric with respect to one of those two coordinate planes. They transform one into the other by reflection in the other two planes.

5. Drawing the Herpolhode

The equation of energy conservation when the components of the angular momentum and angular velocity vectors are given in the inertial system, imply that the component of the angular velocity along the angular momentum is constant

$$2E/J = \mathbf{k}^t \boldsymbol{\Omega}. \tag{57}$$

The components of $\boldsymbol{\Omega}$ perpendicular to $\mathbf{k}$ move on the plane perpendicular to $\mathbf{k}$, called the invariable plane. Its trajectory on the invariable plane follow the curve called herpolhode. The vector $\boldsymbol{\Omega}$ minus the component of $\boldsymbol{\Omega}$ along the angular momentum $\mathbf{k}$ is written with the double $\times$ product as

$$(\mathbf{k} \times \boldsymbol{\Omega}) \times \mathbf{k}. \tag{58}$$

This vector in the body system is

$$(\mathbf{u} \times \boldsymbol{\omega}) \times \mathbf{u} = \dot{\mathbf{u}} \times \mathbf{u} \tag{59}$$

where equation (15) was utilized.
As a consequence the vector describing the herpolhode is

$$\mathbf{R}(\dot{\mathbf{u}} \times \mathbf{u}). \tag{60}$$

When studying the spheroconical coordinates (23) one finds the tangent vectors to the coordinate lines

$$\mathbf{e}_1 = \frac{\partial \mathbf{u}}{\partial \alpha_1} = \begin{pmatrix} -\mathrm{cn}(\alpha_2, k_2)\mathrm{sn}(\alpha_1, k_1)\mathrm{dn}(\alpha_1, k_1) \\ \mathrm{dn}(\alpha_2, k_2)\mathrm{cn}(\alpha_1, k_1)\mathrm{dn}(\alpha_1, k_1) \\ -\mathrm{sn}(\alpha_2, k_2)k_1^2\mathrm{sn}(\alpha_1, k_1)\mathrm{cn}(\alpha_1, k_1) \end{pmatrix} \tag{61}$$

and

$$\mathbf{e}_2 = \frac{\partial \mathbf{u}}{\partial \alpha_2} = \begin{pmatrix} -\mathrm{sn}(\alpha_2, k_2)\mathrm{dn}(\alpha_2, k_2)\mathrm{cn}(\alpha_1, k_1) \\ -k_2^2\mathrm{sn}(\alpha_2, k_2)\mathrm{cn}(\alpha_2, k_2)\mathrm{sn}(\alpha_1, k_1) \\ \mathrm{cn}(\alpha_2, k_2)\mathrm{dn}(\alpha_2, k_2)\mathrm{dn}(\alpha_1, k_1) \end{pmatrix}. \tag{62}$$

These vectors are perpendicular to vector $\mathbf{u}$, and mutually perpendicular. The two vectors are of the same magnitude. Hence one has

$$\mathbf{e}_1 \times \mathbf{u} = -\mathbf{e}_2. \tag{63}$$

Note vectors $\dot{\mathbf{u}}$ and $\mathbf{e}_1$ differ by the constant factor $\dot{\alpha}_1$, therefore vectors $\dot{\mathbf{u}} \times \mathbf{u}$ and $-\mathbf{e}_2$ are different by the same constant factor. Besides, because α_2 is a constant

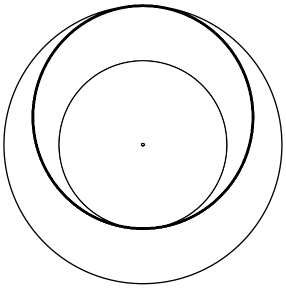

Figure 4. The herpolhode without the term of constant angular velocity of the rotation matrix.

coordinate, one makes the substitution of the constants (26) in the vector $\mathbf{e}_2$. One deduces that the entries of the herpolhode Ω_1 and Ω_2 are

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ 0 \end{pmatrix} = -\dot{\alpha_1}\mathbf{R}\mathbf{e}_2. \tag{64}$$

Eliminating the common constant factor from the components of this vector one ends with

$$\mathbf{R}\begin{pmatrix} \sqrt{\dfrac{(e_1-e_0)(e_2-e_3)}{e_2-e_0}}\,\mathrm{cn}(\alpha_1,k_1) \\ \sqrt{\dfrac{(e_1-e_3)(e_2-e_0)}{e_1-e_0}}\,\mathrm{sn}(\alpha_1,k_1) \\ -\sqrt{\dfrac{(e_0-e_3)(e_2-e_3)}{e_2-e_0}}\,\mathrm{dn}(\alpha_1,k_1) \end{pmatrix}. \tag{65}$$

We can trust to the computer about the drawing and computation of the herpolhode until the rotation is with a constant angular velocity around the angular momentum vector. In Fig. 4 one finds this curve and notes the few known properties of the herpolhode. It is symmetric with respect to one of the coordinate axis. We find the curve between two concentric circles. It is tangent to them at points on the coordinate axis of symmetry. As time increases, the tangent vector to the herpolhode rotates always in the same direction, until the value 2π is reached in a complete period. Some discrepancy is apparent when comparing with pictures of the herpolhode in the literature [5, p 119], but the explanation for to this discrepancy lays in the fact that other authors do not suppress the action of the constant angular velocity that we ignore. This is an useful addition to the existing work in the literature.

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