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ON THE CONSTRUCTION OF RECURSION OPERATORS FOR THE KERR-NEWMAN AND FRLW METRICS

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Abstract. We consider complete integrability of the Hamiltonian of the geodesic flow of two particular solutions, the Kerr-Newman and the FRLW metrics of the Einstein equations in the sense of Liouville. We construct recursion operators using first integrals, and then obtain constants of motion of the geodesic flows by using the recursion operators.

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1. Introduction

We consider two particular solutions of the Einstein equations, the Kerr-Newman and the FRLW metrics. The Kerr-Newman metric is a metric of space-time symmetry axis representing the black hole that was charged to rotation (see e.g. [6]). The FRLW metric stands for the Friedmann-Lemaître-Robertson-Walker metric, which is widely used as a first approximation of the expanding universe model (see e.g., [8]). These metrics are well-known as the exact solutions of the Einstein equations.

In [10], we get complete integrability of the Hamiltonian of the geodesic flows of four solutions of the Einstein equations: Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman metrics. In this paper, we show the Hamiltonian function of the geodesic flow of the Kerr-Newman metric and the FRLW metric are system of separation of variables, and then we get complete integrability of the Hamiltonian of the geodesic flow of the Kerr-Newman metric and the FRLW metric in the sense of Liouville, respectively.

In [1] and [2] the authors proposed a new characterization of integrable systems, which is called a recursion operator. A recursion operator is a $(1,1)$-tensor field which satisfies the conditions: $1)$ Lie derivative is zero under a dynamical vector

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field, $2)$ it has a vanishing Nijenhuis torsion, $3)$ it has doubly degenerate eigenvalues with nowhere vanishing differentials. Also, it is known that traces of the recursion operators are the constants of motions (see e.g., [4], [9], [11], [12]). However, up to now, a few concrete examples of a recursion operator are known. The Kepler dynamics is well known integrable systems. In [5], the Kepler dynamics get functionally independent constants of motion by constructing a recursion operator. Using the complete integrability, they have considered a quantization-problem of the Kepler problem (see also e.g., [3], [7], [13]). In [12], a construction of a recursion operator of the rigid body shows the integrability of the systems. In [10], we construct recursion operators for the geodesic flows of four solutions of the Einstein equations.

In this paper, we construct recursion operators of the Hamiltonian of the geodesic flow of two particular solutions of the Einstein equations using the first integrals. And then we obtain constants of motion of the geodesic flows by using that recursion operators.

We introduce the definition and the Minkowski metric of the simplest example among the pseudo-Riemannian metrics of the recursion operator in Section 2. In Section 3, we construct recursion operators of the geodesic flow for the Kerr-Newman and the FRLW metrics. Using the recursion operators, we see that the geodesic flows for the Kerr-Newman and the FRLW metrics are integrable systems and we obtain the respective constants of motion.

2. Recursion Operator

The recursion operator is introduced in [1] and [2] as a new characterization of integrable systems. In this section, we describe the definition of the recursion operator, and the theorem concerning the separability and complete integrability of the recursion operator (Theorem 1), and prove a lemma about construction of the recursion operator (Lemma 3). Then, we construct a recursion operator for the Hamiltonian of the geodesic flow of the Minkowski metric using the first integral. This construction is a simple example of a geodesic flow of a pseudo-Riemannian metric.

For that purpose let us consider a vector field on $2n$-dimensional manifold $\mathcal{M}^{2n}$. Then, the following definition and theorem are given in [12].

Theorem 1. *A vector field X is separable, integrable and Hamiltonian for certain symplectic structure when X admits an invariant, mixed, diagonalizable $(1,1)$-tensor field T with vanishing Nijenhuis torsion and doubly degenerate eigenvalues without stationary points. Then, the vector field X is a separable and completely* *integrable Hamiltonian system with respect to the symplectic structure in the sense of Liouville.*

Definition 2. *A $(1,1)$-tensor field in above theorem is called a recursion operator.*

In a particular case, a recursion operator can be constructed in [10] as follows

Lemma 3. *Let us consider vector fields*

$$X_j = -\frac{\partial}{\partial x_{n+j}}, \qquad j = 1, \ldots, n$$

on $\mathbb{R}^{2n}$ and let U be a $(1,1)$-tensor field on $\mathbb{R}^{2n}$ given by

$$U = \sum_{i=1}^{n} x_i \left(\frac{\partial}{\partial x_i} \otimes \mathrm{d}x_i + \frac{\partial}{\partial x_{n+i}} \otimes \mathrm{d}x_{n+i} \right).$$

Then we have vanishing Nijenhuis torsion $\mathcal{N}_U = 0$ and $\mathcal{L}_{X_j} U = 0$. That is, a $(1,1)$-tensor field U is a recursion operator for X_j.

Next, we introduce an example using pseudo-Riemannian metrics.

2.1. The Geodesic Flow for the Minkowski Metric

Now, we consider geodesic flows in pseudo-Riemannian metrics. In particular, we consider the Hamiltonian of the geodesic flow of the Minkowski metric. For details we refer to [10].

First, we construct a vector field X on the phase space for the geodesic flow for the Minkowski metric. A matrix g_{ij} of the Minkowski metric is

$$g_{ij} = g^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the equation of geodesics is

$$\frac{\mathrm{d}^2 q^\kappa}{\mathrm{d}t^2} + \Gamma^{\,\kappa}_{\mu\nu} \frac{\mathrm{d}q^\mu}{\mathrm{d}t} \frac{\mathrm{d}q^\mu}{\mathrm{d}t} = \frac{\mathrm{d}^2 q^\kappa}{\mathrm{d}t^2} = 0, \qquad \kappa = 1, 2, 3, 4.$$

If we put $v^\kappa = \dfrac{\mathrm{d}q^\kappa}{\mathrm{d}t}$ then we have a first order differential equation on TM

$$\dot{q}^\kappa = v^\kappa, \qquad \dot{v}^\kappa = -\Gamma^{\,\kappa}_{\mu\nu} v^\mu v^\nu = 0.$$

From the above equations, we get a geodesic spray

$$X = v^{\kappa}\frac{\partial}{\partial q^{\kappa}} - \Gamma^{\ \kappa}_{\mu\nu} v^{\mu} v^{\nu}\frac{\partial}{\partial v^{\kappa}} = v^{\kappa}\frac{\partial}{\partial q^{\kappa}}.$$

By setting $p_{\kappa} = g_{\kappa\varepsilon}v^{\varepsilon}$, the vector field X is equivalently transformed to the vector field X on T^*M such that

$$X = \sum_{k=1}^{4}\left(\dot{q}_k\frac{\partial}{\partial q_k} - \dot{p}_k\frac{\partial}{\partial p_k}\right) = -p_1\frac{\partial}{\partial q_1} + \sum_{k=2}^{4} p_k\frac{\partial}{\partial q_k}.$$

The vector field X is a Hamiltonian vector field of a certain Hamiltonian function. We put a symplectic form ω as

$$\omega = \sum_{k=1}^{4} \mathrm{d}p_k \wedge \mathrm{d}q_k$$

and a function H as

$$H = \frac{1}{2}\left(-p_1^2 + \sum_{k=2}^{4} p_k^2\right). \tag{1}$$

Then, we have

$$i_X\omega = -\mathrm{d}H.$$

The vector field X is a Hamiltonian vector field of the Hamiltonian function H which will be denoted by X_H. Next, we consider the Hamilton-Jacobi equation with the Hamiltonian function (1). The Hamiltonian function (1) does not include q_k $(k = 1, 2, 3, 4)$, therefore p_k $(k = 2, 3, 4)$ are circular coordinates. Let us consider the respective Hamilton-Jacobi equation

$$E = H\left(q, \frac{\partial W}{\partial q}\right)$$

where E is a constant. We set a generating function as

$$W = \sum_{i=1}^{4} W_i(q_i).$$

Since $p_k = \dfrac{\partial W_k(q_k)}{\partial q_k}$ $(k = 2, 3, 4)$ are first integrals, we set $a_k = \dfrac{\partial W_k(q_k)}{\partial q_k}$. Then we have

$$2E = -\left(\frac{\partial W_1(q_1)}{\partial q_1}\right)^2 + \sum_{k=2}^{4} a_k^2.$$

Thus, the generating function W is

$$W = q_1\sqrt{\sum_{k=2}^{4} a_k^2 - 2E} + \sum_{k=2}^{4} a_k q_k.$$

In addition, we determine a canonical coordinate system using the generating function W. We put

$$Q_1 = H, \qquad Q_k = \frac{\partial W_k(q_k)}{\partial q_k}, \qquad k = 2, 3, 4$$

and then a canonical coordinate system (P, Q) is given by

$$Q_1 = H, \quad Q_k = \frac{\partial W_k}{\partial q_k}, \quad P_1 = -\frac{\partial W}{\partial Q_1} = \frac{q_1}{p_1}, \quad P_k = -\frac{\partial W}{\partial Q_k} = -\frac{q_1 p_k}{p_1} - q_k.$$

Here we regard parameters Q_k $(k = 1, 2, 3, 4)$ variables. Hence, the relationship between a canonical coordinate system (P, Q) and the original coordinate system (p, q) is

$$p_1 = \sqrt{\sum_{k=2}^{4} Q_k^2 - 2Q_1}, \; q_1 = P_1\sqrt{\sum_{k=2}^{4} Q_k^2 - 2Q_1}, \; p_k = Q_k, \; q_k = -P_k - Q_k P_1.$$

We put a tensor field T of $(1, 1)$ type as

$$T = \sum_{i=1}^{4} Q_i \left(\frac{\partial}{\partial P_i} \otimes \mathrm{d}P_i + \frac{\partial}{\partial Q_i} \otimes \mathrm{d}Q_i \right). \tag{2}$$

From Lemma 3, we have that $\mathcal{L}_{X_H} T = 0$, $\mathcal{N}_T = 0$ and $\deg Q_i = 2$ for the equation (2). Thus, the $(1, 1)$-tensor field T is a recursion operator for X_H.

It is known the traces $\mathrm{Tr}(T)$, $\mathrm{Tr}(T^2)$, $\mathrm{Tr}(T^3)$ and $\mathrm{Tr}(T^4)$ are constants of motion (see, [1]). If we express in the original coordinate system (q, p), T and $\mathrm{Tr}(T^\ell)$ are written respectively as

$$T = \sum_{i,j=1}^{4} \left(\left({}^t A\right)^i{}_j \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_j + B^i{}_j \frac{\partial}{\partial q_i} \otimes \mathrm{d}p_j + A^i{}_j \frac{\partial}{\partial q_i} \otimes \mathrm{d}q_j \right)$$

$$\mathrm{Tr}(T^\ell) = \frac{1}{2^{\ell-1}} \left(-p_1^2 + p_2^2 + p_3^2 + p_4^2\right)^\ell + 2\left(p_2^\ell + p_3^\ell + p_4^\ell\right), \qquad \ell = 1, 2, 3, 4$$

where $A = \begin{pmatrix} H & 0 & 0 & 0 \\ \frac{p_2}{p_1}(p_2 - H) & p_2 & 0 & 0 \\ \frac{p_3}{p_1}(p_3 - H) & 0 & p_3 & 0 \\ \frac{p_4}{p_1}(p_4 - H) & 0 & 0 & p_4 \end{pmatrix}, \qquad B = \frac{q_1}{p_1}({}^t A - A).$

Thus, we get a recursion operator of the simplest example among the pseudo-Riemannian metrics.

3. Geodesic Flows for Two Types Solutions of Einstein Equations

In this section, we consider geodesic flows for two particular solutions of Einstein equations, and we construct recursion operators. These solutions of Einstein equations are the Kerr-Newman and the FRLW metrics. We describe a construction of a recursion operator for the solution of Einstein equations. And we get constants of motions with recursion operators.

3.1. The Geodesic Flow for the Kerr-Newman Metric

The Kerr-Newman metric is one of the exact solution of Einstein equations of general relativity, it is a metric of space-time symmetry axis representing the black hole that was charged to rotation. If the charge is equal to zero, the Kerr-Newman metric is the Kerr metric. At very large radii, the curvature and dragging effects of the central object are negligible, so the Kerr metric becomes flat as can be seen by letting $t \to \infty$ (see [6]). Of the several forms of the Kerr-Newman metric, the most useful expression for our purpose is given by the Boyer-Lindquist coordinates.

We consider the Kerr-Newman metric by the Boyer-Lindquist coordinates

$$\mathrm{d}s^2 = -\frac{\kappa}{\rho^2}\left(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left(\left(r^2+a^2\right)\mathrm{d}\phi - a\mathrm{d}t\right)^2 + \frac{\rho^2}{\kappa}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2$$

where $t \in (-\infty,\infty)$, $r \in (2M,\infty)$, $\theta \in (0,\pi)$, $\phi \in (0,2\pi)$, $\kappa \equiv r^2 - 2rM + a^2 + Q^2$, $\rho^2 \equiv r^2 + a^2\cos^2\theta$ and $aM = J$. M is the mass of the black hole, Q is the electric charge and J is the angular momentum. In addition, the Kerr-Newman geometry has a horizon, and therefore describes a black hole, if and only if $M^2 \geq Q^2 + a^2$.

Here, for simplicity of notation, we put $t = q_1$, $r = q_2$, $\theta = q_3$ and $\phi = q_4$

$$\begin{aligned}\mathrm{d}s^2 = &-\frac{\kappa}{\rho^2}\left(\mathrm{d}q_1 - a\sin^2 q_3\mathrm{d}q_4\right)^2 \\ &+ \frac{\sin^2 q_3}{\rho^2}\left(\left(q_2^2+a^2\right)\mathrm{d}q_4 - a\mathrm{d}q_1\right)^2 + \frac{\rho^2}{\kappa}\mathrm{d}q_2^2 + \rho^2\mathrm{d}q_3^2.\end{aligned} \tag{3}$$

For the canonical symplectic structure, we have the Hamiltonian vector field X_H of the geodesic flow for the Kerr-Newman metric

$$X_H = \sum_{k=1}^{4}\left(U_k\frac{\partial}{\partial q_k} + V_k\frac{\partial}{\partial p_k}\right)$$

where

$$U_1 = \frac{2}{\rho^2}\left(aB\sin q_3 - \frac{A}{\kappa}(Ap_1 + ap_4)\right), \qquad U_2 = \frac{2\kappa p_2}{\rho^2}, \qquad U_3 = \frac{2p_2}{\rho^2}$$
$$U_4 = \frac{2}{\kappa}\left(\frac{C(\kappa - \rho^2 + A)}{\rho^2} - ap_1\right)$$
$$V_1 = 0, \quad V_2 = \frac{2q_2}{\rho^4}(C^2 - q_3^2 + \kappa p_2^2) - \frac{4q_2p_1}{\kappa\rho^2}(Ap_1 + ap_4) - (M - q_2)p_2^2$$
$$V_3 = \frac{2\sin q_3\cos q_3}{\rho^2}\left(\frac{a^2}{\rho^2}\left(B^2 + \kappa p_2^2 - \frac{2p_4^2}{\sin^2 q_3}\right) + B^2 - \frac{2ap_1p_4}{\sin^2 q_3}\right), \qquad V_4 = 0$$
$$A = a^2 + q_2^2, \qquad B = ap_1\sin q_3 + \frac{p_4}{\sin q_3}, \qquad C = ap_1\cos q_3 + \frac{p_4}{\sin q_3}.$$

The Hamiltonian function H of the vector field X_H is

$$H = \frac{1}{2}\left[\left(\frac{a^2}{\rho^2}\sin^2 q_3 - \frac{(q_2^2 + a^2)^2}{\kappa\rho^2}\right)p_1^2 + \frac{\kappa}{\rho^2}p_2^2 \right.$$
$$\left. + \frac{1}{\rho^2}p_3^2 + \left(\frac{a^2}{\kappa\rho^2} - \frac{1}{\rho^2\sin^2 q_3}\right)p_4^2 + 2\left(\frac{a}{\rho^2} - \frac{a(q_2^2 + a^2)}{\kappa\rho^2}\right)p_1p_4\right].$$

We see that the Hamiltonian function H does not include q_1 and q_4. Hence, p_1 and p_4 are first integrals, and we put $p_1 = \alpha$, $p_4 = \beta$. Then, we consider the Hamilton-Jacobi equation

$$\begin{aligned} 2Eq_2^2 + \frac{(q_2^2 + a^2)^2}{\kappa}\alpha^2 - \kappa\left(\frac{\mathrm{d}W_2}{\mathrm{d}q_2}\right)^2 + \frac{a^2}{\kappa}\beta^2 + \frac{2a(q_2^2 + a^2)}{\kappa}\alpha\beta \\ = -2Ea^2\cos^2 q_3 + a^2\alpha^2\sin^2 q_3 + \left(\frac{\mathrm{d}W_3}{\mathrm{d}q_3}\right)^2 - \frac{\beta^2}{\sin^2 q_3} + 2a\alpha\beta \end{aligned} \tag{4}$$

where $W = \sum_{k=1}^{4} W_k(q_k)$ is the generating function. Since the equation (4) is a type of separation of variables, we put K as

$$K = -2Ea^2\cos^2 q_3 + a^2\alpha^2\sin^2 q_3 + \left(\frac{\mathrm{d}W_3}{\mathrm{d}q_3}\right)^2 - \frac{\beta^2}{\sin^2 q_3} + 2a\alpha\beta$$

where K is the third integral. Therefore, we have a generating function

$$W = \alpha q_1 + \int \frac{\mathrm{d}W_2}{\mathrm{d}q_2}\mathrm{d}q_2 + \int \frac{\mathrm{d}W_3}{\mathrm{d}q_3}\mathrm{d}q_3 + \beta q_4 = \alpha q_1 + W_2 + W_3 + \beta q_4.$$

Next, we determine the canonical coordinate system (P, Q) using a generating function W. Thus, we get

$$
\begin{aligned}
&Q_1 = E, \qquad Q_2 = K, && Q_3 = \frac{\mathrm{d}W_1}{\mathrm{d}q_1}, \qquad Q_4 = \frac{\mathrm{d}W_4}{\mathrm{d}q_4} \\
&P_1 = -\frac{\partial W_2}{\partial Q_1} - \frac{\partial W_3}{\partial Q_1}, && P_2 = -\frac{\partial W_2}{\partial Q_2} - \frac{\partial W_3}{\partial Q_2} \\
&P_3 = -q_1 - \frac{\partial W_2}{\partial Q_3} - \frac{\partial W_3}{\partial Q_3}, && P_4 = -\frac{\partial W_2}{\partial Q_4} - \frac{\partial W_3}{\partial Q_4} - q_4
\end{aligned}
$$

by considering a canonical coordinates in the same manner as the geodesic flow for the Minkowski metric. We consider that Q_k $(k = 1, 2, 3, 4)$ are variables. In terms of the canonical coordinate system, a vector field X_H and symplectic form ω are written as

$$
X_H = \{H, E\} = -\frac{\partial}{\partial P_1}, \qquad \omega = \sum_{k=1}^{4} \mathrm{d}P_k \wedge \mathrm{d}Q_k.
$$

We put a $(1, 1)$-tensor field T as

$$
T = \sum_{i=1}^{4} Q_i \left(\frac{\partial}{\partial P_i} \otimes \mathrm{d}P_i + \frac{\partial}{\partial Q_i} \otimes \mathrm{d}Q_i \right).
$$

Then, from Lemma 3, T is a recursion operator for X_H. In additon, the constants of motion $\mathrm{Tr}(T^\ell)$ $(\ell = 1, 2, 3, 4)$ of the geodesic flow of the Kerr-Newman metric is

$$
\mathrm{Tr}(T^\ell) = 2\left(E^\ell + K^\ell + \alpha^\ell + \beta^\ell\right), \qquad \ell = 1, 2, 3, 4.
$$

Now, if $Q = 0$, (3) is the Kerr metric. If $J = 0$, (3) is the Reissner-Nordström metric. And if $Q = 0$ and $J = 0$, (3) is the Schwarzschild metric. Then it enables us to get the other three respective recursion operators for the Kerr metric, the Reissner-Nordström metric and the Schwarzschild metric.

3.2. The Geodesic Flow for the FRLW Metric

Now, we consider the following metric

$$
\mathrm{d}s^2 = -\mathrm{d}t^2 + R(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right) \tag{5}
$$

where $R(t)$ is a scale factor and k is a constant representing the curvature of the space. The metric (5) is called the Friedmann-Lemaître-Robertson-Walker metric. Also called simply the FRLW metric. This metric is widely used as a first approximation of the expanding universe model.

Notice that we can, without loss of generality, scale the coordinate r in such a way as to make k take one of the three values $+1$, 0, -1 ([8]). That is, if $R(t)$ becomes to c times, then the curvature becomes $1/c^2$ times. In other words, radius of the universe swells to c times. Let t_0 be the present time. And we assume that $R(t_0) = 1$. Then, k will be the curvature of present universe.

Also, if $k = 0$, (5) is called the flat FRLW metric. And if $k = +1$, (5) is called the closed, or spherical FRLW metric. Moreover, if $k = -1$, (5) is called the hyperbolic, or open FRLW metric.

Here, for simplicity of notation, we put $t = q_1$, $r = q_2$, $\theta = q_3$ and $\phi = q_4$, the metric (5) becomes

$$\mathrm{d}s^2 = -\mathrm{d}q_1^2 + R^2(q_1)\left(\frac{\mathrm{d}q_2^2}{1-kq_2^2} + q_2^2\left(\mathrm{d}q_3^2 + \sin^2 q_3 \mathrm{d}q_4^2\right)\right)$$

where $q_1 \in (-\infty, \infty)$, $q_2 \in (2M, \infty)$, $q_3 \in (0, \pi)$, $q_4 \in (0, 2\pi)$.

For the canonical symplectic structure, we have the Hamiltonian vector field X_H of the geodesic flow for the FRLW metric

$$X_H = \sum_{i=1}^{4}\left(U_i \frac{\partial}{\partial q_i} + V_i \frac{\partial}{\partial p_i}\right)$$

where

$$U_1 = -p_1, \quad U_2 = \frac{1-kq_2^2}{R^2(q_1)} p_2, \quad U_3 = \frac{1}{R^2(q_1)q_2^2} p_3, \quad U_4 = \frac{1}{R^2(q_1)q_2^2 \sin^2 q_3} p_4$$

$$V_1 = \frac{1}{R^3(q_1)}\left(\left(1-kq_2^2\right)p_2^2 + \frac{p_3^2}{q_2^2} + \frac{p_4^2}{q_2^2 \sin^2 q_3}\right)\frac{\mathrm{d}R(q_1)}{\mathrm{d}t}$$

$$V_2 = \frac{q_2}{R^2(q_1)}\left(kp_2^2 + \frac{p_3^2}{q_2^4} + \frac{p_4^2}{q_2^4 \sin^2 q_3}\right), \quad V_3 = \frac{\cos q_3}{R^2(q_1)q_2^2 \sin^3 q_3} p_4^2, \quad V_4 = 0.$$

Then, the Hamiltonian function H of the vector field X_H is

$$H = -\frac{1}{2}p_1^2 + \frac{1-kq_2^2}{2R^2(q_1)} p_2^2 + \frac{1}{2R^2(q_1)q_2^2} p_3^2 + \frac{1}{2R^2(q_1)q_2^2 \sin^2 q_3} p_4^2.$$

We see that the Hamiltonian function H does not include q_4. Hence, p_4 is first integrals, and we put $p_4 = \alpha$. Then, we consider the Hamilton-Jacobi equation

$$\begin{aligned} 2E = &-\left(\frac{\mathrm{d}W_1}{\mathrm{d}q_1}\right)^2 + \frac{1-kq_2^2}{R^2(q_1)}\left(\frac{\mathrm{d}W_2}{\mathrm{d}q_2}\right)^2 \\ &+ \frac{1}{R^2(q_1)q_2^2}\left(\frac{\mathrm{d}W_3}{\mathrm{d}q_3}\right)^2 + \frac{\alpha^2}{2R^2(q_1)q_2^2\sin^2 q_3} \end{aligned} \tag{6}$$

where $W = \sum_{k=1}^{4} W_k(q_k)$ is the generating function. Since the equation (6) is a type of separation of variables, we put K and L as

$$\begin{aligned} K &= (1-kq_2^2)\left(\frac{\mathrm{d}W_2}{\mathrm{d}q_2}\right)^2 + \frac{1}{q_2^2}\left(\frac{\mathrm{d}W_3}{\mathrm{d}q_3}\right)^2 + \frac{1}{q_2^2\sin^2 q_3}\alpha^2 \\ L &= \left(\frac{\mathrm{d}W_3}{\mathrm{d}q_3}\right)^2 + \frac{1}{\sin^2 q_3}\alpha^2. \end{aligned}$$

K and L are the third integral. Therefore, we have a generating function

$$W = \int \frac{\mathrm{d}W_1}{\mathrm{d}q_1}\mathrm{d}q_1 + \int \frac{\mathrm{d}W_2}{\mathrm{d}q_2}\mathrm{d}q_2 + \int \frac{\mathrm{d}W_3}{\mathrm{d}q_3}\mathrm{d}q_3 + \alpha q_4 = W_1 + W_2 + W_3 + \alpha q_4.$$

Next, we determine the canonical coordinate system (P, Q) using a generating function W. Thus, we get

$$\begin{aligned} &Q_1 = E, \qquad Q_2 = K, && Q_3 = L, \qquad Q_4 = \alpha \\ &P_1 = -\frac{\partial W_1}{\partial Q_1}, && P_2 = -\frac{\partial W_1}{\partial Q_2} - \frac{\partial W_2}{\partial Q_2} \\ &P_3 = -\frac{\partial W_2}{\partial Q_3} - \frac{\partial W_3}{\partial Q_3}, && P_4 = -\frac{\partial W_3}{\partial Q_4} - q_4 \end{aligned}$$

by considering a canonical coordinates in the same manner as the geodesic flow for the Minkowski metric. In terms of the canonical coordinate system, a vector field X_H and symplectic form ω are written as

$$X_H = \{H, E\} = -\frac{\partial}{\partial P_1}, \qquad \omega = \sum_{k=1}^{4} \mathrm{d}P_k \wedge \mathrm{d}Q_k.$$

We put a $(1,1)$-tensor field T as

$$T = \sum_{i=1}^{4} Q_i \left(\frac{\partial}{\partial P_i} \otimes \mathrm{d}P_i + \frac{\partial}{\partial Q_i} \otimes \mathrm{d}Q_i\right).$$

Then, from the Lemma 3, T is a recursion operator for X_H. In additon, the constants of motion $\mathrm{Tr}(T^\ell)$ $(\ell = 1, 2, 3, 4)$ of the geodesic flow of the FRLW metric is

$$\mathrm{Tr}(T^\ell) = 2\left(E^\ell + K^\ell + \alpha^\ell + \beta^\ell\right), \qquad \ell = 1, 2, 3, 4.$$

These results provide new examples of recursion operators of the geodesic flow for pseudo-Riemannian metrics.

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