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NONLINEAR 1-D OSCILLATIONS OF A CHARGE PARTICLE UNDER COULOMB FORCES AND DRY FRICTION

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Abstract. We study the 1-D motions of a charge under Coulomb force, within the electrostatic approximation. When the electrostatic force is attractive, no oscillating motion takes place. When repulsive, nonlinear oscillations will arise. In both cases dry friction has been taken into account and time equations have been solved providing time as elliptic integrals of first and second kind. A short phase plane analysis has been included. The oscillation period has been exactly computed and found to increase versus the initial speed of the mobile.

1. Introduction

Most physical, biological, economic systems are inherently not linear so that they lead to nonlinear ordinary differential equations. As a consequence a branch of applied research is looking for exact solutions, by means of either special functions of the Mathematical Physics, or iterative approaches or perturbations, and so on. For instance, A. More [3] highlights that a closed form solution makes the analysis by far more elaborate and easier as its behavior becomes at once clear, when expressed in closed form in terms of known functions.

Our Problem

A mobile M-point mass, always constrained on a straight trajectory, undergoes dry friction, say $\mu > 0$ its dynamic coefficient, having as propelling cause an electrostatic force. The motion time law is required.

In fact, two invariable electric charges of *opposite signs* are placed: q_0, at a certain fixed point; the latter, say q, on the moving particle of mass $M > 0$. Let be L the initial distance between them: we put the origin O of the reference at the start of the mobile: so that the resting charge stays at $x = -L$ and the Coulomb force will be directed as the negative sense of x with a intensity $F(x) = kqq_0/(L + x)^2$, being $x = x(t) \geq 0$ the (unknown) particle's position at time t, and $k = 9 \times 10^9 N \times$

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m^2/C^2 the Coulomb's constant. Considering, as before, the forces equation

$$\ddot{x} = -\mu g - \frac{kqq_0}{M(L+x)^2}, \qquad x(0) = 0, \qquad \dot{x}(0) = v_0 > 0 \tag{1}$$

we scale from x to the non dimensional co-ordinate $\xi = x/L$ and put: $a = 2\mu g/L$, $b = 2kqq_0/ML^2$, $\sqrt{c} = v_0/L$. In such a way the motion initial value problem for the case of opposite charges takes the form of equation (2) below. We are going to analyze the particle dynamics within the so called electrostatic approximation[1].

2. Solution with Electric Opposite Charges

The former discussion leads us to study the initial value problem for the second order nonlinear autonomous differential equation:

$$\ddot{\xi} = -\frac{a}{2} - \frac{b}{2(1+\xi)^2}, \qquad \xi(0) = 0, \qquad \dot{\xi}(0) = \sqrt{c} \tag{2}$$

where a, b, $c > 0$. The first remark is that the relevant differential equation has no equilibria. The trajectories in the phase plan are divided by the (forbidden) straight line $\xi = -1$. A phase portrait of (2), obtained using the `VisualDSolve` Mathematica® package [5] is represented in Fig. 1.

Let us approach (2) by the Weierstrass method. First we remark that we are interested only in the half plane $\xi > -1$. We see the sign of acceleration is negative for any ξ. The Weierstrass function relevant to (2) is

$$\Phi(\xi) = 2\int_0^{\xi} \left(-\frac{a}{2} - \frac{b}{2(1+u)^2}\right) \mathrm{d}u + c = \frac{c - (a+b-c)\xi - a\xi^2}{1+\xi}$$

so that its roots are

$$\xi_{1,2} = \frac{-a-b+c \pm \sqrt{(a+b-c)^2 + 4ac}}{2a}. \tag{3}$$

The singularity $\xi = -1$ lies inside the ξ-segment joining the roots $\xi_1 < \xi_2$: in such a way the mass instantaneous ξ-position moves from the origin outward the major

[1]Coulomb's law holds when the objects remain at rest, and has been found approximately correct only for slow movements. Such all conditions constitute the so called *electrostatic approximation*. When a movement takes place, some magnetic fields are produced altering the force between the objects. A Gauss's assistant and leading experimental collaborator, Wilhelm Weber, established in 1846–1848 an electrodynamics which would predict some velocity-dependent corrections to Coulomb's law. Nevertheless in what follows we will not take into account any change on dependence of the mobile charge velocity.

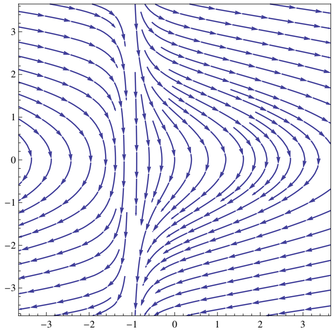

Figure 1. Phase portrait $(\dot{\xi}, \xi)$ of (2) with $a = b = c = 1$.

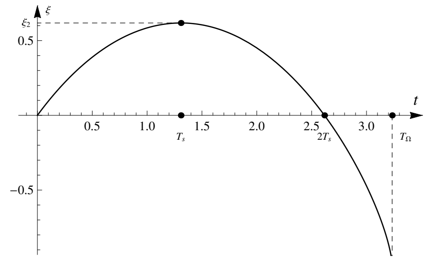

Figure 2. Non-oscillatory motion: inversion of (7) with $a = b = c = 1$.

root ξ_2 and and, after having caught the stopping point, is restored back towards the minor root ξ_1, without getting it because the motion meets its singular point at $\xi = -1$. About this, Stratton, [6, p. 170] writes

> In this fashion a point singularity is generated and the Coulomb law is then valid at all points except $r = 0$. There is no reason to believe that such singularities exist in nature, but it is convenient to interpret a field at sufficient distances as that which might be generated by systems of mathematical point charges.

When the point charge is going away from the start up position, $0 \leq \xi \leq \xi_2$, time equation becomes

$$t = \frac{1}{\sqrt{a}} \int_0^{\xi} \sqrt{\frac{1+x}{(\xi_2 - x)(x - \xi_1)}}\, \mathrm{d}x. \tag{4}$$

Stopping time T_s can then be found by taking $\xi = \xi_2$ in (4) and thanks to formula 3.141.10 p. 263 of [2]. We get

$$T_s = \frac{2\sqrt{\xi_2 - \xi_1}}{\sqrt{a}} E\left(\varphi(0, \xi_2), k(\xi_1, \xi_2)\right) + \frac{2\,(1+\xi_1)}{\sqrt{a}\sqrt{\xi_2 - \xi_1}} F\left(\varphi(0, \xi_2), k(\xi_1, \xi_2)\right) \tag{5}$$

with

$$\varphi(\xi, \xi_2) = \arcsin\sqrt{\frac{\xi_2 - \xi}{\xi_2 + 1}}, \qquad k(\xi_1, \xi_2) = \sqrt{\frac{\xi_2 + 1}{\xi_2 - \xi_1}}.$$

Solution to (2) for $0 \leq t \leq T_s$ is defined by

$$t = T_s - \frac{2\sqrt{\xi_2 - \xi_1}}{\sqrt{a}} E\left(\varphi(\xi, \xi_2), k(\xi_1, \xi_2)\right) + \frac{2\,(1+\xi_1)}{\sqrt{a}\sqrt{\xi_2 - \xi_1}} F\left(\varphi(\xi, \xi_2), k(\xi_1, \xi_2)\right).$$

In what above $F(\varphi,k)$ and $E(\varphi,k)$ are the incomplete elliptic integrals of first and second kind of amplitude φ and modulus k.

After T_s, in order to go on with integration, the initial conditions of (2) have to be modified conveniently, so that

$$\ddot{\xi} = -\frac{a}{2} - \frac{b}{2(1+\xi)^2}, \qquad \xi(T_s) = \xi_2, \qquad \dot{\xi}(T_s) = 0. \tag{6}$$

Weierstrass function relevant to (6) is

$$\Phi(\xi) = a\frac{(\xi_2 - \xi)(\xi - \xi_3)}{1+\xi} \qquad \text{where} \qquad \xi_3 = -\frac{a\xi_2 + a + b}{a\,(1+\xi_2)}.$$

Notice that $\xi_3 < -1$ since

$$-\frac{a\xi_2 + a + b}{a\,(1+\xi_2)} < -1 \iff a\xi_2 + a + b > a\,(1+\xi_2) \iff b > 0$$

Time equation is then, in force of 3.141.10 p. 263 of [2]

$$\begin{aligned} t =& T_s + \frac{2}{\sqrt{a}} \\ &\times \left(\frac{1+\xi_3}{\sqrt{\xi_2 - \xi_3}}\, F(\varphi(\xi,\xi_2), k(\xi_3,\xi_2)) + \sqrt{\xi_2 - \xi_3}\, E\left(\varphi(\xi,\xi_2), k(\xi_3,\xi_2)\right) \right) \end{aligned} \tag{7}$$

Time T_Ω when the singularity would be theoretically met, can be set putting $\xi = -1$ in (7) so that

$$T_\Omega = T_s + \frac{2}{\sqrt{a}} \left(\frac{1+\xi_3}{\sqrt{\xi_2 - \xi_3}}\, \boldsymbol{K}(k(\xi_3,\xi_2)) + \sqrt{\xi_2 - \xi_3}\, \boldsymbol{E}(k(\xi_3,\xi_2)) \right)$$

being $\boldsymbol{K}$ and $\boldsymbol{E}$ the *complete* determinations of F and E.

Fig. 2 shows the solution $\xi = \xi(t)$ in the special case $a = b = c = 1$, so that motion can be seen as it follows. By equation (2) the particle undergoes always a negative acceleration. At the start-up, as a consequence of the initial speed, it goes away from the origin, but at ξ_1 it will be stopped. The motion sense is then inverted and the electrostatic attraction will move it towards the fixed charge, which cannot be raised due to a singularity at $\xi = -1$ where the mobile particle will never arrive. The motion then will terminate just before there: no oscillation can arise.

3. Solution with Concordant Electric Charges

The only change we make to the preceding model is the polarity of charges so that, *cœteris paribus*, an electrostatic repulsion will act. Accordingly, the new Cauchy problem is

$$\ddot{\xi} = -\frac{a}{2} + \frac{b}{2(1+\xi)^2}, \qquad \xi(0) = 0, \qquad \dot{\xi}(0) = \sqrt{c} \tag{8}$$

where $a,\ b,\ c > 0$. First of all, we see the sign of acceleration is *a priori* not fixed, depending on ξ time evolution and on relationship between a and b. The Weierstrass function of (8) is

$$\Phi(\xi) = 2\int_0^{\xi}\left(-\frac{a}{2} + \frac{b}{2(1+u)^2}\right)\mathrm{d}u + c = \frac{c-(a-b-c)\xi - a\xi^2}{1+\xi}$$

whose roots are

$$\xi_{1,2} = \frac{-a+b+c \pm \sqrt{(-a+b+c)^2+4ac}}{2a}. \tag{9}$$

The singularity $\xi = -1$ lies outside the roots $\xi_1 < \xi_2$, therefore the solution to (8) is periodic[2] and moves between them. Accordingly, the mobile will go towards ξ_2 and, after having raised the stopping point, it goes back towards ξ_1. Notice that in such a case we have: $-1 < \xi_1 < 0 < \xi_2$. If we study the differential equation (8) in the phase plane, introducing the Hamiltonian

$$H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\left(aq + \frac{b}{1+q}\right)$$

and the phase plan equivalent system in the (q,p) plane

$$\dot{q} = \frac{\partial H}{\partial p} = p, \qquad \dot{p} = -\frac{\partial H}{\partial q} = \frac{1}{2}\left(-a + \frac{b}{(1+q)^2}\right)$$

we find a center C and a saddle S where

$$C = \left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}}, 0\right), \qquad S = \left(-\frac{\sqrt{b}+\sqrt{a}}{\sqrt{a}}, 0\right).$$

The system orbits are separated again by the straight line $q = -1$. Center C lies in the half plane $q > -1$ which is of interest for the model, while saddle S lies in the half plane $q < -1$ not interesting in our analysis. The phase portrait, again obtained using [5] is shown in Fig. 3.

Time equation, from the motion start-up is

$$t = \int_0^{\xi}\frac{\mathrm{d}x}{\sqrt{\Phi(x)}} = \frac{1}{\sqrt{a}}\int_0^{\xi}\sqrt{\frac{1+x}{(\xi_2-x)(x-\xi_1)}}\,\mathrm{d}x. \tag{10}$$

[2] This means that, notwithstanding the physical energy dissipation due to the friction, the oscillation amplitude is not damped thanks to the energy provided by the electric field of source.

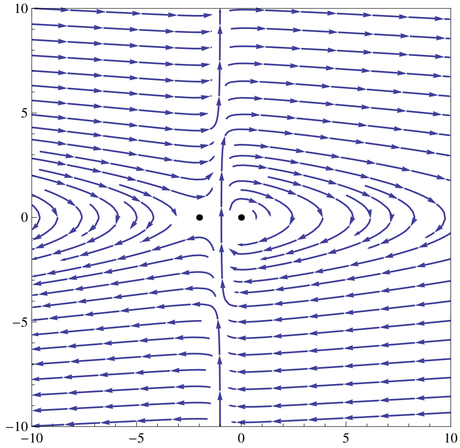

Figure 3. Phase portrait of $(\dot{\xi}, \xi)$, generated via (8) with $a = b = c = 1$.

The stop-time T_s can be found by taking $\xi = \xi_2$ in (10): we compute by means of formula 3.141.17, p. 263 of [2] or through the entry 236.01 of [1]

$$T_s = \frac{2\sqrt{\xi_2 + 1}}{\sqrt{a}}\, E\left(\hat{\varphi}(0, \xi_2), \hat{k}(\xi_1, \xi_2)\right) \tag{11}$$

with

$$\hat{\varphi}(\xi, \xi_2) = \arcsin\sqrt{\frac{\xi_2 - \xi}{\xi_2 - \xi_1}}, \qquad \hat{k}(\xi_1, \xi_2) = \sqrt{\frac{\xi_2 - \xi_1}{\xi_2 + 1}}.$$

The oscillation period is provided by the complete elliptic integral of second kind, see [2, entry 3.141.17, p. 263]

$$T = \frac{2}{\sqrt{a}} \int_{\xi_1}^{\xi_2} \sqrt{\frac{1 + x}{(\xi_2 - x)(x - \xi_1)}}\, \mathrm{d}x = \frac{4}{\sqrt{a}} \sqrt{1 + \xi_2}\, \boldsymbol{E}\left(k(\xi_1, \xi_2)\right). \tag{12}$$

Integrating (10), one finds time equation giving time as a second kind incomplete elliptic integral of position, $0 \le t \le T_s$

$$t = T_s - \frac{\sqrt{1 + \xi_2}}{\sqrt{a}}\, E\left(\hat{\varphi}(\xi, \xi_2), \hat{k}(\xi_1, \xi_2)\right). \tag{13}$$

Going ahead, if $T_s < t \le T_s + \frac{T}{2}$ we get

$$t = T_s + \frac{\sqrt{1 + \xi_2}}{\sqrt{a}}\, E\left(\hat{\varphi}(\xi, \xi_2), \hat{k}(\xi_1, \xi_2)\right). \tag{14}$$

Finally, for $T + \frac{T}{2} < t \leq T$

$$t = T_s + T - \frac{\sqrt{1+\xi_2}}{\sqrt{a}}\, E\left(\hat{\varphi}(\xi, \xi_2), \hat{k}(\xi_1, \xi_2)\right). \tag{15}$$

Each of equations (13), (14), (15) cannot be explicitly inverted by means of known functions. Nevertheless they allow the solution's computation. Accordingly, doing this and the relevant welding, we get a time-sketch of oscillations.

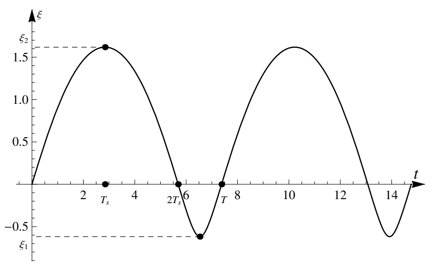

Figure 4. Oscillatory motion (libration) sample case $a = b = c = 1$.

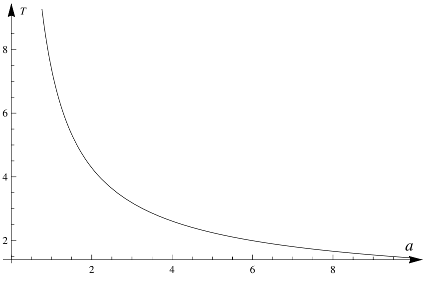

Figure 5. The oscillation period T special case $c = b = 1$, $0 \leq a \leq 10$, as a function of the a-friction parameter.

The motion can then be described as follows. By (8) the particle undergoes a variable-sign-acceleration as a consequence of the friction. The motion with $\mu > 0$ is then foreseen to be periodic, between the positions ξ_1 and ξ_2 with $\xi_2 < 0$. The integration highlights that, at the start-up, as a consequence of its initial speed, the particle goes away from the origin: but at ξ_1 it will be stopped at time T_s. The motion sense is then inverted and the point moves beyond the origin in the region of negative ξ till to the most far, say ξ_1. After that, the electrostatic repulsion prevails and the particle is pushed back to the origin. The complete oscillation period $2T_s$ is over when the mass crosses the origin again, according to the sketch of Fig. 4. Notice that such a oscillation is not symmetric with respect to $\xi = 0$. Following Pars [4], it could be more appropriately named as *libration*. Looking at the period expression (12), we check that it (obviously) goes inversely with the friction parameter a, as shown in Fig. 5.

Furthermore, the modulus $\hat{k}$ depends on ξ_1 and ξ_2 which depend on a, b, c. For instance fixing the values $a = 1, b = 1$ there will survive a dependence on c only, with c linked to the kinetic energy E by $c = \frac{2E}{mL^2}\cdot$ We can then obtain a plot of such a period-energy function $T = T(c)$, see Fig. 6.

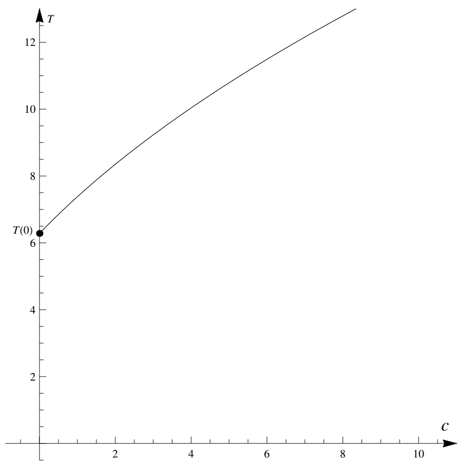

Figure 6. The oscillation period-energy function $T = T(c)$, special case $a = b = 1$, $0 \leq c \leq 10$.

The period-energy $T = T(c)$ function's growth with c can analytically be inspected by inserting the values of ξ_1 and ξ_2, as given by (9), in its c-derivative obtained from (12). Notice that the $T(0)$ value is

$$T(0) = \frac{4\sqrt{b}}{a}\,\boldsymbol{E}\left(\sqrt{\frac{b-a}{b}}\right)$$

which in the computational example $a = b = 1$ reduces to 2π. By "energy" we refer only to initial *kinetic energy* content of the mobile, with no reference to its *electrostatic energy* which is a deeply different object concerning a system of charges at rest in space.

About the function period-energy, we observe what follows. The greater is such a energy, the greater will be the leak space the particle will sweep before its stop due to loss of energy. In such a way its motion reversion will start from a more long distance: so that the total time it will spend for a go/return cycle will be greater.

Conclusions

Two kinds of one dimensional motion have been comparatively computed within the electrostatic approximation and with different signs of the charges. When they are opposite, the motion, first flowing outwards, is inverted being the particle necessarily attracted by the source of the field: in such a way the motion terminates just for not falling into the singularity.

If the charges are concordant, the mobile, after a initial leak away, inverts its motion again, but the repulsion makes it to oscillate around its start up position $\xi = 0$. The period of such asymmetric oscillation is computed by a complete elliptic integral of second kind and has been found to increase for increasing initial speeds of the mobile charge.

In both movements time is obtained as a function of position ξ through incomplete elliptic integrals of first and second kind.

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