THE PTOLEMAEAN INEQUALITY IN H–TYPE GROUPS

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Abstract. We prove the Ptolemaean inequality and the Theorem of Ptolemaeus in the setting of H–type Iwasawa groups.

1. Introduction

The purpose of this short note is to give an elementary proof of a generalization of the Ptolemaean inequality in the context of H–type Iwasawa groups. These groups are precisely the Iwasawa $\mathfrak{n}$–components of simple Lie algebras of real rank one [1, p.704], thus we recover the results in [7] with a considerably simplified proof.

The Ptolemaean inequality in planar Euclidean geometry states that given a quadrilateral, then the product of the lengths of the diagonals is less or equal to the sum of the products of the lengths of its opposite sides. Moreover, equality holds if and only if the quadrilateral is inscribed in a circle. Many authors have proved generalization of the Ptolemaean inequality in various settings (e.g. normed spaces [8], CAT(0) spaces [2], Möbius spaces [3]).

Let (X, d) be a metric space. The metric d is called *Ptolemaean* if any four distinct points p_1, p_2, p_3 and p_4 in X satisfy the Ptolemaean inequality; that is, for any permutation (i, j, k, l) in the permutation group S_4 we have

$$d(p_i, p_k) \cdot d(p_j, p_l) \leq d(p_i, p_j) \cdot d(p_k, p_l) + d(p_j, p_k) \cdot d(p_l, p_i). \qquad (1)$$

In a Ptolemaean space (X, d), we are most interested in the sets where Ptolemaean inequality holds as an equality (Ptolemaeus' Theorem). A subset Σ of X is called a *Ptolemaean circle* if for any four distinct points $p_1, p_2, p_3, p_4 \in \Sigma$ such that p_1 and p_3 separate p_2 and p_4 we have

$$d(p_1, p_3) \cdot d(p_2, p_4) = d(p_1, p_2) \cdot d(p_3, p_4) + d(p_2, p_3) \cdot d(p_4, p_1).$$

Our main theorem is the following.

Theorem 1. *Let G be an $H-$type Iwasawa group. Then the metric d defined in (9) is Ptolemaean and its Ptolemaean circles are $\mathbb{R}-$circles.*

The key ingredient of our proof is that under an appropriate normalization the Ptolemaean inequality reduces to the triangle inequality.

2. $H-$Type Iwasawa Groups

In this section we briefly recall the definition and some basic properties of $H-$type Iwasawa groups.

Let $\mathfrak{g}$ be a finite–dimensional Lie algebra endowed with a left invariant inner product $\langle , \rangle$. Let $\mathfrak{j}$ be the center of $\mathfrak{g}$ and let $\mathfrak{b}$ be the orthogonal completion of $\mathfrak{j}$ in $\mathfrak{g}$. For fixed $t \in \mathfrak{j}$ consider the map $J_t \colon \mathfrak{b} \to \mathfrak{b}$ defined by $\langle J_t(x), y \rangle = \langle t, [x, y] \rangle$, where $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{g}$. Then, $\mathfrak{g}$ is called an H–*type algebra* if $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{j}$ and moreover J_t is an orthogonal map whenever $\langle t, t \rangle = 1$. An H–*type group G* is a connected and simply connected Lie group whose Lie algebra is an H–type algebra.

H–type groups is a class of step 2 Carnot groups. If $m = \dim \mathfrak{b}, n = \dim \mathfrak{j}$ and $N = m + n$, then G is a homogeneous Carnot group on $\mathbb{R}^N$ with dilations $\delta_\lambda(x, t) = (\lambda x, \lambda^2 t)$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, where $\lambda > 0$.

An H–type group G is called an *Iwasawa group* if for every $x \in \mathfrak{b}$ and for every $t, t^{'} \in \mathfrak{j}$ with $\langle t, t^{'} \rangle = 0$, there exists a $t^{''} \in \mathfrak{j}$ such that $J_t(J_{t'}(x)) = J_{t''}(x)$, [4, p.23].

From now on G shall always denote an H–type Iwasawa group. Since the exponential mapping is a bijection of $\mathfrak{g}$ onto G, we shall parametrize p in G by $(x, t) \in \mathfrak{b} \oplus \mathfrak{j} = \mathfrak{g}$, where $p = \exp(x, t)$. Multiplication in G is of a special form, see [1, p.687]: there exist $m \times m$ skew symmetric and orthogonal matrices $U^1, \ldots, U^n$ such that

$$
\begin{aligned}
(x, t)(x^{'}, t^{'}) &= \left(x + x', t + t' + \frac{1}{2}[x, x'] \right) \\
&= \left(x + x^{'}, t^1 + t^{'1} + \langle U^1 x, x^{'} \rangle, \ldots, t^n + t^{'n} + \langle U^n x, x^{'} \rangle \right)
\end{aligned}
$$

for all $(x, t), (x^{'}, t^{'}) \in G$.

We note that $(x, t)^{-1} = (-x, -t)$ and also that the matrices $U^1, \ldots, U^n$ have the following property

$$
U^i U^j + U^j U^i = 0, \quad \text{for every} \quad i, j \le n \quad \text{with} \quad i \ne j.
$$

The distance d in G is defined via a gauge $\widetilde{d}$. If $p \in G$ is parametrized by $(x, t) \in \mathbb{R}^n \times \mathbb{R}^n$ we set

$$
\widetilde{d}(p) = \left(|x|^4 + 16|t|^2 \right)^{\frac{1}{4}}. \tag{2}
$$

For clarity we shall prove the following proposition.

Proposition 2. *For every $p, q \in G$ we have*

$$\widetilde{d}(p^{-1}q) \leq \widetilde{d}(p) + \widetilde{d}(q). \tag{3}$$

Equality holds if and only if $p = \exp(x, 0)$, $q = \exp(\lambda x, 0)$ for some $\lambda \in \mathbb{R}$.

The above inequality has been proved by Cygan in [5] and we shall only sketch the proof.

Proof: We parametrize p^{-1} by (x, t) and q by (x', t). Then,

$$\widetilde{d}(p^{-1}q)^4 = |x + x'|^4 + 16 \left| t + t' + \frac{1}{2}[x, x'] \right|^2$$

which is equal to

$$\begin{aligned}
\widetilde{d}(p)^4 + \widetilde{d}(q)^4 \ &+ \ 4 \left(\langle x, x' \rangle^2 + |[x, x']|^2 \right) \\
&+ \ 4 \left(|x|^2 \langle x, x' \rangle + 4 \langle t, [x, x'] \rangle \right) \\
&+ \ 4 \left(|x'|^2 \langle x, x' \rangle + 4 \langle t', [x, x'] \rangle \right) \\
&+ \ 2 \left(|x|^2 |x'|^2 + 16 \langle t, t' \rangle \right).
\end{aligned}$$

The following inequalities hold.

$$\left(|x|^2 |x'|^2 + 16 \langle t, t' \rangle \right) \leq \widetilde{d}(p)^2 \widetilde{d}(q)^2 \tag{4}$$

$$\left(|x|^2 \langle x, x' \rangle + 4 \langle t, [x, x'] \rangle \right) \leq \widetilde{d}(p)^2 \left(\langle x, x' \rangle^2 + |[x, x']|^2 \right)^{1/2} \tag{5}$$

$$\left(|x'|^2 \langle x, x' \rangle + 4 \langle t', [x, x'] \rangle \right) \leq \widetilde{d}(q)^2 \left(\langle x, x' \rangle^2 + |[x, x']|^2 \right)^{1/2} \tag{6}$$

$$\left(\langle x, x' \rangle^2 + |[x, x']|^2 \right) \leq |x|^2 |x'|^2 \leq \widetilde{d}(p)^2 \widetilde{d}(q)^2. \tag{7}$$

The first three inequalities are immediate. The last inequality follows by applying the Cauchy–Schwarz inequality to Cygan's hermitian form, [5, p.70]

$$h(x, x') = \langle x, x' \rangle + \mathrm{i}[x, x'].$$

Combining the above we have

$$\begin{aligned}
\widetilde{d}(p^{-1}q)^4 \leq &\widetilde{d}(p)^4 + \widetilde{d}(q)^4 + 6\widetilde{d}(p)^2 \widetilde{d}(q)^2 + 4\widetilde{d}(p)^3 \widetilde{d}(q) + 4\widetilde{d}(p)\widetilde{d}(q)^3 \\
= &\left(\widetilde{d}(p) + \widetilde{d}(q) \right)^4
\end{aligned} \tag{8}$$

thus we obtain the desired triangle inequality.

Observe now that equality in (3) holds if and only if (4), (5), (6) and (7) hold simultaneously as equalities. Therefore, from the last inequality considered as an equality we have $x' = \lambda x$. Hence all other inequalities hold as equalities if and only if $t = t' = 0$.

∎

The distance d in G is defined by

$$d(p, q) = \widetilde{d}(p^{-1}q). \tag{9}$$

Then, we have that $d(p, q) = 0$ if and only if $p = q$ and $d(p, q) = d(q, p)$. Also we observe that d is invariant by left translations and is scaled up to a factor λ when we apply a dilation δ_λ, [1, p.705]. From (3) we have that

$$d(p, q) \leq d(p, 0) + d(0, q). \tag{10}$$

Thus, if $p, r, q \in G$ then by invariance we deduce that

$$d(p, q) \leq d(p, r) + d(r, q)$$

i.e., the triangle inequality for the distance function d.

It is worth remarking that the definition of the gauge function $\widetilde{d}$ generalizes to arbitrary Carnot groups but the corresponding function $d = d(\cdot, \cdot)$ is not in general a distance but only a pseudo–distance, [6, p.300], [1, p.231].

The key feature of the class of $H-$type Iwasawa groups is the existence of a natural inversion, which generalizes the inversion $\sigma(x) = -\frac{x}{|x|^2}$ of $\mathbb{R}^N \setminus \{0\}$.

The inversion map $\sigma \colon G \setminus \{0\} \to G \setminus \{0\}$ is defined by (cf. [1, p.705])

$$\sigma(x, t) = \left(-\frac{|x|^2 x - 4\sum_{k=1}^n U^k x}{|x|^4 + 16|t|^2}, -\frac{t}{|x|^4 + 16|t|^2} \right)$$

and satisfies $\sigma^2 = \mathrm{id}$.

We shall consider the one point compactification $\widehat{G}$ of G by adding the point ∞ at infinity. The distance d is extended in $\widehat{G}$ in the obvious way and is denoted by the same symbol: for every $p \in G$

$$d(p, \infty) = +\infty, \qquad d(\infty, \infty) = 0.$$

The actions of left translations and dilations are also extended naturally in $\widehat{G}$: left translation of any element of $\widehat{G}$ by ∞ maps it to ∞ and the image ∞ by any $\lambda-$dilation is again ∞. For the inversion σ we set $\sigma(0) = \infty$ and $\sigma(\infty) = 0$; the following holds [1, p.706]

$$d(\sigma(p), 0) = \frac{1}{d(p, 0)}, \qquad d(\sigma(p), \sigma(q)) = \frac{d(p, q)}{d(p, 0)\, d(0, q)}.$$

Note that the last equality holds in an H–type group G if and only if G is Iwasawa [4, p.23].

Finally, for every $x \in \mathbb{R}^m \backslash \{0\}$ we define the *standard $\mathbb{R}$–circle* R_x passing through 0 and ∞ as the set

$$R_x = \{(\lambda x, 0)\,;\,\lambda \in \mathbb{R}\}.$$

An $\mathbb{R}$–*circle* is the image of some R_x under the action of the *similarity* group $\widehat{G}$: this group comprises maps which are composites of left translations, dilations and the inversion σ, [4, p.9].

3. Proof of Theorem 1

Given four distinct points $p_1, p_2, p_3, p_4 \in G$ we define their *cross–ratio* $X(p_1, p_2, p_3, p_4)$ by

$$X^{1/2}(p_1, p_2, p_3, p_4) = \frac{d(p_1, p_3)d(p_2, p_4)}{d(p_1, p_4)d(p_2, p_3)} \tag{11}$$

and the definition is extended in the obvious way if one of the points is ∞. From the properties of left translations, dilations and inversion one may verify that the cross–ratio $X(p_1, p_2, p_3, p_4)$ is invariant under the action of the similarity group of $\widehat{G}$. This allows us to normalize so that two of the points of a given quadruple are 0 and ∞.

Given a quadruple p_1, p_2, p_3, p_4 of distinct points in $\widehat{G}$, let p_i, p_i, p_k, p_l a permutation of these points. Let also

$$X_1 = X(p_i, p_j, p_k, p_l), \qquad X_2 = X(p_i, p_k, p_j, p_l).$$

From (1) and (11) it follows that the Ptolemaean inequality is satisfied if and only if

$$X_1^{1/2} + X_2^{1/2} \geq 1. \tag{12}$$

We now normalise so that $p_i = \infty$, $p_l = 0$. Thus, we have

$$X_1^{1/2} = \frac{d(p_j, 0)}{d(p_j, p_k)}, \qquad \text{and} \qquad X_2^{1/2} = \frac{d(p_k, 0)}{d(p_j, p_k)}.$$

Consequently, (12) is equivalent to

$$d(p_j, 0) + d(0, p_k) \geq d(p_j, p_k) \tag{13}$$

i.e., the triangle inequality for the distance function d. Since the permutation was arbitrary, this proves the Ptolemaean inequality (1).

Moreover, if equality is valid then

$$X_1^{1/2} + X_2^{1/2} = 1 \tag{14}$$

which is equivalent to

$$d(p_j, 0) + d(0, p_k) = d(p_j, p_k). \tag{15}$$

Thus, as we have seen in Section 2, this happens if and only if $p_j = (x, 0)$ and $p_k = (\lambda x, 0)$ for some $\lambda \in \mathbb{R}$. Since $p_i = \infty$ and $p_l = 0$ we conclude that p_i, p_j, p_k, p_l are in a standard $\mathbb{R}$–circle and the proof of the result is complete.

Finally, let us observe that there is at least one Ptolemaean *pseudo-metric* which has the same isometries with the Korányi metric.

More precisely, let G be an $H-$type group and $\Pi : G \to \mathbb{R}^n, (x, t) \to x$, be the projection to $\mathbb{R}^n$. Then the pseudo-metric defined by

$$d_1((x, t), (x', t')) = d_e(x, x')$$

where d_e is the Euclidean metric, is Ptolemaean (since the Euclidean metric is). It is elementary to prove that this pseudo-metric is invariant under the action of the group by left translations as well as rotations.

Acknowledgements

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