JGSP **32** (2013) 61–86

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

HYPERBOLIC GEOMETRY

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Communicated by Ivaïlo M. Mladenov

Abstract. Relativistic hyperbolic geometry is a model of the hyperbolic geometry of Lobachevsky and Bolyai in which Einstein addition of relativistically admissible velocities plays the role of vector addition. The adaptation of barycentric coordinates for use in relativistic hyperbolic geometry results in the relativistic barycentric coordinates. The latter are covariant with respect to the Lorentz transformation group just as the former are covariant with respect to the Galilei transformation group. Furthermore, the latter give rise to hyperbolically convex sets just as the former give rise to convex sets in Euclidean geometry. Convexity considerations are important in non-relativistic quantum mechanics where mixed states are positive barycentric combinations of pure states and where barycentric coordinates are interpreted as probabilities. In order to set the stage for its application in the geometry of relativistic quantum states, the notion of the relativistic barycentric coordinates that relativistic hyperbolic geometry admits is studied.

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doi: 10.7546/jgsp-32-2013-61-86

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1. Introduction

Relativistic hyperbolic geometry is a model of hyperbolic geometry in which Einstein addition plays the role of vector addition. Relativistic hyperbolic geometry admits the notion of hyperbolic barycentric coordinates, just as Euclidean geometry admits the notion of Euclidean barycentric coordinates. The former is potentially useful in the study of the geometry of relativistic quantum states, just as the latter is useful in the study of the geometry of quantum states in [3].

It is well-known, as emphasized in [3], that Euclidean barycentric coordinates prove useful in the geometry of quantum states. Barycentric coordinate systems underlie the study of convex analysis [36], and convexity considerations are important in non-relativistic quantum mechanics where mixed states are positive barycentric combinations of pure states, and where barycentric coordinates are interpreted as probabilities [36, p.11]. The success in [3] and [9] of the study of the geometry of quantum states in terms of barycentric coordinates suggests that relativistic barycentric coordinates can prove useful in the geometry of relativistic quantum states as well.

In the non-relativistic limit of large speed of light relativistic barycentric coordinates tend to corresponding classical, Euclidean barycentric coordinates. Relativistic barycentric coordinates and classical, Euclidean barycentric coordinates share remarkable analogies. In particular, they are both covariant. Indeed, relativistic barycentric coordinate representations are covariant with respect to the Lorentz coordinate transformation group, just as classical, Euclidean barycentric coordinate representations are covariant with respect to the Galilean coordinate transformation group. The remarkable analogies suggest that relativistic barycentric coordinates can prove useful in the study of the geometry of relativistic quantum states, just as classical barycentric coordinates prove useful in the study of the geometry of non-relativistic quantum states.

Being neither commutative nor associative, Einstein's velocity addition law of relativistically admissible velocities, which Einstein introduced in his 1905 paper [11] that founded the special theory of relativity, is seemingly structureless. Precisely because the only justification for Einstein's velocity addition law appeared to be its empirical adequacy, it remained for a long time a mystery to be conquered. The mystery has been resolved in 1988 [39–42], when the elusive, most elegant algebraic structures that Einstein's velocity addition encodes, *gyrogroups* and *gyrovector spaces*, have been discovered. The emergence of gyrovector spaces, in turn, paved the way for the appearance of relativistic hyperbolic geometry, in general, and of the relativistic barycentric coordinates in [56–59], in particular. In order to set the stage for its application in the geometry of relativistic quantum states, we study the notion of relativistic barycentric coordinates in relativistic hyperbolic geometry.

2. Einstein Addition

Let c be an arbitrarily fixed positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, $n = 1, 2, 3, \ldots$, equipped with the common vector addition, +, and inner product, $\cdot$. The home of all n-dimensional Einsteinian velocities is the c-ball

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n;\ \|\mathbf{v}\| < c\}\,. \tag{1}$$

It is the open ball of radius c, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with norm smaller than c.

Einstein addition and scalar multiplication play in the ball $\mathbb{R}^n_c$ the role that vector addition and scalar multiplication play in the Euclidean n-space $\mathbb{R}^n$.

Definition 1. *Einstein addition is a binary operation, $\oplus$, in the c-ball $\mathbb{R}^n_c$ given by the equation,* [44], [37, Eq. 2.9.2], [26, p.55], [16]

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, where $\gamma_{\mathbf{u}}$ is the Lorentz gamma factor given by the equation

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}} \tag{3}$$

where $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its space $\mathbb{R}^n$.

Einstein addition (2) of relativistically admissible velocities, with $n = 3$, was introduced by Einstein in his 1905 paper [11] [12, p.141] that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (2) are presented. One has to remember here that the Euclidean three-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [11] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (2) of Einstein addition. Einstein was aware of the nonassociativity of his velocity addition law of relativistically admissible velocities that need not be collinear. He therefore emphasized in his 1905 paper that his velocity addition law of relativistically admissible *collinear velocities* forms a group operation [11, p.907].

A nonempty set with a binary operation is called a *groupoid* so that, accordingly, the pair $(\mathbb{R}^n_c, \oplus)$ is an *Einstein groupoid.*

In the Newtonian limit of large c, $c \to \infty$, the ball $\mathbb{R}^n_c$ expands to the whole of its space $\mathbb{R}^n$, as we see from (1), and Einstein addition $\oplus$ in $\mathbb{R}^n_c$ reduces to the ordinary vector addition $+$ in $\mathbb{R}^n$, as we see from (2) and (3).

In applications to velocity spaces, $\mathbb{R}^n \equiv \mathbb{R}^3$ is the Euclidean three-space, which is the space of all classical, Newtonian velocities, and $\mathbb{R}^3_c \subset \mathbb{R}^3$ is the c-ball of $\mathbb{R}^3$ of all relativistically admissible, Einsteinian velocities. The constant c represents in special relativity the vacuum speed of light. Since we are interested in geometry, we allow n to be any positive integer and, sometimes, replace c by s.

We naturally use the abbreviation $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ for Einstein subtraction, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$, $\ominus\mathbf{v} = \mathbf{0}\ominus\mathbf{v} = -\mathbf{v}$. Einstein addition and subtraction satisfy the identities

$$\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v} \tag{4}$$

and

$$\ominus\mathbf{u}\oplus(\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} \tag{5}$$

for all $\mathbf{u}, \mathbf{v}$ in the ball $\mathbb{R}^n_c$, in full analogy with vector addition and subtraction in $\mathbb{R}^n$. Identity (4) is called the *gyroautomorphic inverse property* of Einstein addition, and Identity (5) is called the *left cancellation law* of Einstein addition. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (5) since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{v} \neq \mathbf{u}\,. \tag{6}$$

However, this seemingly lack of a *right cancellation law* of Einstein addition is repaired, for instance, in [59, §1.9].

3. Einstein Addition vs Vector Addition

Vector addition, +, in $\mathbb{R}^n$ is both commutative and associative, that is,

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} && \text{Commutative Law} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} && \text{Associative Law} \end{aligned} \tag{7}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. In contrast, Einstein addition, $\oplus$, in $\mathbb{R}^n_c$ is neither commutative nor associative.

In order to measure the extent to which Einstein addition deviates from associativity we introduce *gyrations*, which are self maps of $\mathbb{R}^n$ that are *trivial* in the special cases when the application of $\oplus$ is associative. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$ the gyration $\mathrm{gyr}\,[\mathbf{u}, \mathbf{v}]$ is a map of the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$ onto itself. Gyrations $\mathrm{gyr}\,[\mathbf{u}, \mathbf{v}] \in \mathrm{Aut}(\mathbb{R}^n_c, \oplus)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, are defined in terms of Einstein addition by the equation

$$\mathrm{gyr}\,[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\} \tag{8}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$, and they turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$.

We recall that an automorphism of a groupoid $(S, \oplus)$ is a one-to-one map f of S onto itself that respects the binary operation, that is, $f(a\oplus b) = f(a)\oplus f(b)$ for all $a, b \in S$. The set of all automorphisms of a groupoid $(S, \oplus)$ forms a group, denoted $\mathrm{Aut}(S, \oplus)$. To emphasize that the gyrations of an Einstein gyrogroup $(\mathbb{R}^n_c, \oplus)$ are automorphisms of the gyrogroup, gyrations are also called *gyroautomorphisms*.

A gyration $\mathrm{gyr}\,[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, is *trivial* if $\mathrm{gyr}\,[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}^n_c$. Thus, for instance, the gyrations $\mathrm{gyr}\,[\mathbf{0}, \mathbf{v}]$, $\mathrm{gyr}\,[\mathbf{v}, \mathbf{v}]$ and $\mathrm{gyr}\,[\mathbf{v}, \ominus\mathbf{v}]$ are trivial for all $\mathbf{v} \in \mathbb{R}^n_c$, as we see from (8). In general, however, gyrations are nontrivial gyroautomorphisms.

Einstein gyrations, which possess their own rich structure, measure the extent to which Einstein addition deviates from commutativity and from associativity, as we see from the gyrocommutative and the gyroassociative laws of Einstein addition in the identities listed in (9) below [44, 48, 53].

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$, one has

$$\begin{aligned}
\mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}\,[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law}\\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}\,[\mathbf{u},\mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law}\\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}\,[\mathbf{v},\mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law}\\
\mathrm{gyr}\,[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] &= \mathrm{gyr}\,[\mathbf{u},\mathbf{v}] && \text{Gyration Left Reduction Property}\\
\mathrm{gyr}\,[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] &= \mathrm{gyr}\,[\mathbf{u},\mathbf{v}] && \text{Gyration Right Reduction Property}\\
\mathrm{gyr}\,[\ominus\mathbf{u},\ominus\mathbf{v}] &= \mathrm{gyr}\,[\mathbf{u},\mathbf{v}] && \text{Gyration Even Property}\\
(\mathrm{gyr}\,[\mathbf{u},\mathbf{v}])^{-1} &= \mathrm{gyr}\,[\mathbf{v},\mathbf{u}] && \text{Gyration Inversion Law.}
\end{aligned} \tag{9}$$

Einstein addition is thus regulated by gyrations to which it gives rise owing to its nonassociativity, so that Einstein addition and its gyrations are inextricably linked. The resulting gyrocommutative gyrogroup structure of Einstein addition was discovered in 1988 [39]. Interestingly, gyrations are the mathematical abstraction of the relativistic effect known as *Thomas precession* [53, §10.3] [61]. Accordingly, we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and nonassociative algebra. Thomas precession, in turn, is related to the *mixed state geometric phase* in quantum mechanics, as Lévay discovered in his work [23] which, according to [23], was motivated by the work of the present author [45].

The gyration left and right reduction properties in (9) trigger a remarkable reduction of complexity in various gyration identities [48, 53] and, as such, they form important gyration identities. These two gyration identities are, however, just the tip of a giant iceberg. The identities in (9) and many other useful gyration identities are studied in [34, 35, 44, 48, 53, 55, 58, 59, 63]. Related explorations are found, for instance, in [4] and [50].

Einstein addition, $\oplus$, in $\mathbb{R}_c^n$, which is gyrocommutative, is associated with a dual binary operation, $\boxplus$, in $\mathbb{R}_c^n$, which is commutative. The mutually dual binary operations $\oplus$ and $\boxplus$ in $\mathbb{R}_c^n$ are both necessary in order to capture analogies with classical results, as demonstrated in [43] and in [51].

4. From Einstein Addition to Gyrogroups

Taking the key features of the Einstein groupoid $(\mathbb{R}_c^n, \oplus)$ as axioms, and guided by analogies with groups, we are led to the formal gyrogroup definition in which gyrogroups turn out to form a most natural generalization of groups.

Definition 2. (Gyrogroups [53, p.17]). *A groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element,* 0*, called a left identity, satisfying*

(G1) $\quad 0 \oplus a = a$

for all $a \in G$*. There is an element* $0 \in G$ *satisfying axiom* (G1) *such that for each* $a \in G$ *there is an element* $\ominus a \in G$*, called a left inverse of* a*, satisfying*

(G2) $\quad \ominus a \oplus a = 0\,.$

Moreover, for any $a, b, c \in G$ *there exists a unique element* $\mathrm{gyr}\,[a, b]c \in G$ *such that the binary operation obeys the left gyroassociative law*

(G3) $\quad a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}\,[a, b]c\,.$

The map $\mathrm{gyr}\,[a, b] : G \to G$ *given by* $c \mapsto \mathrm{gyr}\,[a, b]c$ *is an automorphism of the groupoid* $(G, \oplus)$*, that is*

(G4) $\quad \mathrm{gyr}\,[a, b] \in \mathrm{Aut}(G, \oplus)$

and the automorphism $\mathrm{gyr}\,[a, b]$ *of G is called the gyroautomorphism, or the gyration, of G generated by* $a, b \in G$*. The operator* $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ *is called the gyrator of G. Finally, the gyroautomorphism* $\mathrm{gyr}\,[a, b]$ *generated by any* $a, b \in G$ *possesses the left reduction property*

(G5) $\quad \mathrm{gyr}\,[a, b] = \mathrm{gyr}\,[a \oplus b, b]\,.$

The gyrogroup axioms (G1) – (G5) in Definition 2 are classified into three classes:

1. The first pair of axioms, (G1) and (G2), is a reminiscent of the group axioms.
2. The last pair of axioms, (G4) and (G5), presents the gyrator axioms.
3. The middle axiom, (G3), is a hybrid axiom linking the two pairs of axioms in (G1) and (G2).

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$ in gyrogroup theory as well. In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 3. (Gyrocommutative Gyrogroups). *A gyrogroup* $(G, \oplus)$ *is gyrocommutative if its binary operation obeys the gyrocommutative law*

(G6) $\quad a \oplus b = \mathrm{gyr}\,[a, b](b \oplus a)$

for all $a, b \in G$*.*

Gyrogroup theorems asserting, for instance, that a gyrogroup identity 0 and a gyrogroup inverse $\ominus a$ (both left and right) uniquely exist, are found in [44, 48, 53, 55, 58, 59].

It was the study of Einstein's velocity addition law and its associated Lorentz transformation group of special relativity theory that led to the discovery of the gyrogroup structure in 1988 [39]. However, gyrogroups are not peculiar to Einstein addition [54]. Rather, they are abound in the theory of groups [13–15, 17, 18], loops [19], quasigroup [20, 22], and Lie groups [21]. Other related interesting results are found, for instance, in [46], [47] and [52]. The path from Möbius to gyrogroups is described in [54].

5. Einstein Scalar Multiplication

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Indeed, Einstein addition admits scalar multiplication, giving rise to the Einstein gyrovector space. Remarkably, the resulting Einstein gyrovector spaces form the setting for the relativistic hyperbolic geometry (also known as the Cartesian-Beltrami-Klein ball model of hyperbolic geometry), just as vector spaces form the setting for the standard Cartesian model of Euclidean geometry, as demonstrated in [34,35,44,48,53,55,58,59,63] and as indicated in the sequel.

Let $k\otimes\mathbf{v}$ be the Einstein addition of k copies of $\mathbf{v} \in \mathbb{R}^n_c$, as shown in the first equation in (12) below. Then, for $k = 1, 2, 3, \ldots$

$$k\otimes\mathbf{v} = c\frac{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^k - \left(1-\frac{\|\mathbf{v}\|}{c}\right)^k}{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^k + \left(1-\frac{\|\mathbf{v}\|}{c}\right)^k}\frac{\mathbf{v}}{\|\mathbf{v}\|}. \tag{10}$$

The definition of scalar multiplication in an Einstein gyrovector space requires analytically continuing k off the positive integers, thus obtaining from (10) the following

Definition 4. (Einstein Scalar Multiplication). *The Einstein gyrovector space* $(\mathbb{R}^n_s, \oplus, \otimes)$ *can be introduced as an Einstein gyrogroup* $(\mathbb{R}^n_s, \oplus)$ *with scalar multiplication* $\otimes$ *given by*

$$r\otimes\mathbf{v} = s\frac{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} = s\tanh(r\ \tanh^{-1}\frac{\|\mathbf{v}\|}{s})\frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{11}$$

where r is any real number, $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n_s$, $\mathbf{v} \neq \mathbf{0}$, and $r\otimes\mathbf{0} = \mathbf{0}$, and with which we use the notation $\mathbf{v}\otimes r = r\otimes\mathbf{v}$.

Einstein gyrovector spaces are studied in [44, 48, 53, 55, 58, 59]. Einstein scalar multiplication does not distribute over Einstein addition, but it possesses other properties of vector spaces. For any positive integer k, and for all real numbers $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n_s$, we have

$$\begin{aligned} k\otimes\mathbf{v} &= \mathbf{v}\oplus\ldots\oplus\mathbf{v} && k \text{ terms} \\ (r_1 + r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} && \text{Scalar Distributive Law} \\ (r_1 r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{Scalar Associative Law} \end{aligned} \tag{12}$$

in any Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$.

Additionally, Einstein gyrovector spaces possess the *scaling property*

$$\frac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{13}$$

$\mathbf{a} \in \mathbb{R}^n_s,\ \mathbf{a} \neq \mathbf{0},\ r \in \mathbb{R},\ r \neq 0$, the *gyroautomorphism property*

$$\mathrm{gyr}\,[\mathbf{u}, \mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}\,[\mathbf{u}, \mathbf{v}]\mathbf{a} \tag{14}$$

$\mathbf{a}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s$, $r \in \mathbb{R}$, and the identity gyroautomorphism

$$\mathrm{gyr}\,[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I \tag{15}$$

$r_1, r_2 \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n_s$.

Any Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ inherits an inner product and a norm from its vector space $\mathbb{R}^n$. These turn out to be invariant under gyrations, that is,

$$\begin{aligned} \mathrm{gyr}\,[\mathbf{a}, \mathbf{b}]\mathbf{u}\cdot\mathrm{gyr}\,[\mathbf{a}, \mathbf{b}]\mathbf{v} &= \mathbf{u}\cdot\mathbf{v} \\ \|\mathrm{gyr}\,[\mathbf{a}, \mathbf{b}]\mathbf{v}\| &= \|\mathbf{v}\| \end{aligned} \tag{16}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s$.

6. From Einstein Scalar Multiplication to Gyrovector Spaces

Taking the key features of Einstein scalar multiplication as axioms, and guided by analogies with vector spaces, we are led to the formal gyrovector space definition in which gyrovector spaces turn out to form a most natural generalization of vector spaces.

Definition 5. (Real Inner Product Gyrovector Spaces [53, p.154]). *A real inner product gyrovector space* $(G, \oplus, \otimes)$ *(gyrovector space, in short) is a gyrocommutative gyrogroup* $(G, \oplus)$ *that obeys the following axioms:*

(1) *G is a subset of a real inner product vector space $\mathbb{V}$ called the carrier of G, $G \subset \mathbb{V}$, from which it inherits its inner product, $\cdot$, and norm, $\|\cdot\|$, which are invariant under gyroautomorphisms, that is*

(V1)	$\mathrm{gyr}\,[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}\,[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$	*Inner Product Gyroinvariance*

for all points $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v} \in G$.

(2) *G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$*

(V2)	$1\otimes\mathbf{a} = \mathbf{a}$	*Identity Scalar Multiplication*
(V3)	$(r_1 + r_2)\otimes\mathbf{a} = r_1\otimes\mathbf{a}\oplus r_2\otimes\mathbf{a}$	*Scalar Distributive Law*
(V4)	$(r_1 r_2)\otimes\mathbf{a} = r_1\otimes(r_2\otimes\mathbf{a})$	*Scalar Associative Law*
(V5)	$\dfrac{\|r\|\otimes\mathbf{a}}{\\|r\otimes\mathbf{a}\\|} = \dfrac{\mathbf{a}}{\\|\mathbf{a}\\|}, \quad \mathbf{a} \neq \mathbf{0},\ r \neq 0$	*Scaling Property*
(V6)	$\mathrm{gyr}\,[\mathbf{u},\mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}\,[\mathbf{u},\mathbf{v}]\mathbf{a}$	*Gyroautomorphism Property*
(V7)	$\mathrm{gyr}\,[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I$	*Identity Gyroautomorphism.*

(3) *Real, one-dimensional vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"*

(V8)	$\\|G\\| = \{\pm\\|\mathbf{a}\\|\ ;\ \mathbf{a} \in G\} \subset \mathbb{R}$	*Vector Space*

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a},\mathbf{b} \in G$,

(V9)	$\\|r\otimes\mathbf{a}\\| = \|r\|\otimes\\|\mathbf{a}\\|$	*Homogeneity Property*
(V10)	$\\|\mathbf{a}\oplus\mathbf{b}\\| \leq \\|\mathbf{a}\\|\oplus\\|\mathbf{b}\\|$	*Gyrotriangle Inequality.*

Einstein addition and scalar multiplication in $\mathbb{R}^n_s$ thus give rise to the Einstein gyrovector spaces $(\mathbb{R}^n_s, \oplus, \otimes)$, $n \geq 2$.

7. Gyrolines – The Hyperbolic Lines

Let $A, B \in \mathbb{R}^n_s$ be two distinct points of the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, and let $t \in \mathbb{R}$ be a real parameter. Then, the graph of the set of all points

$$A\oplus(\ominus A\oplus B)\otimes t \tag{17}$$

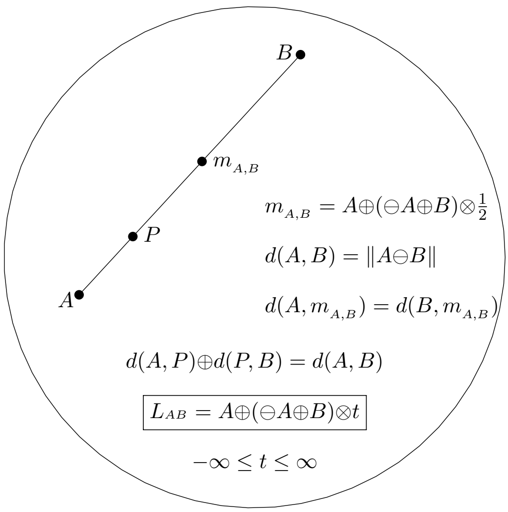

Figure 1. Gyrolines, the hyperbolic lines L_{AB} in Einstein gyrovector spaces, are fully analogous to lines in Euclidean spaces.

$t \in \mathbb{R}$, in the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is a chord of the ball $\mathbb{R}^n_s$. As such, it is a geodesic line of the Beltrami-Klein ball model of hyperbolic geometry, shown in Fig. 1 for $n = 2$. The geodesic line (17) is the unique gyroline that passes through the points A and B. It passes through the point A when $t = 0$ and, owing to the left cancellation law, (5), it passes through the point B when $t = 1$. Furthermore, it passes through the midpoint $m_{A,B}$ of A and B when $t = 1/2$. Accordingly, the *gyrosegment* AB that joins the points A and B in Fig. 1 is obtained from gyroline (17) by restricting the gyroline parameter t to the unit interval $0 \leq t \leq 1$.

Gyrolines (17) are the geodesics of relativistic hyperbolic geometry. Similarly, gyrolines (17) with Einstein addition $\oplus$ replaced by Möbius addition $\oplus_{\text{M}}$ are the geodesics of the Poincaré ball model of hyperbolic geometry. These interesting results are established by methods of differential geometry in [49].

Each point of (17) with $0 < t < 1$ is said to lie *between* A and B. Thus, for instance, the point P in Fig. 1 lies between the points A and B. As such, the points A, P and B obey the (degenerate gyrotriangle inequality, that is, the) *gyrotriangle equality*, according to which

$$d(A,P)\oplus d(P,B) = d(A,B) \tag{18}$$

in full analogy with Euclidean geometry. Here, as shown in Fig. 1

$$d(A,B) = \|\ominus A\oplus B\| \tag{19}$$

$A, B \in \mathbb{R}_s^n$, is the Einstein *gyrodistance function*, also called the Einstein *gyrometric*. This gyrodistance function in Einstein gyrovector spaces corresponds bijectively to a standard hyperbolic distance function, as demonstrated in [53, §6.19], and it gives rise to the well-known Riemannian line element of the Beltrami-Klein ball model of hyperbolic geometry, as demonstrated in [49].

8. Euclidean and Relativistic-Hyperbolic Barycentric Coordinates

In order to set the stage for the presentation of relativistic-hyperbolic barycentric coordinates, we present the Euclidean barycentric coordinates in the following two definitions.

Definition 6. (Barycentric Independence). *Some set S of $N \geq 2$ points $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}^n$, $n \geq N-1$, is barycentrically independent if the $N-1$ vectors $-A_1 + A_k$, $k = 2, \ldots, N$, are linearly independent in $\mathbb{R}^n$.*

Definition 7. (Euclidean Barycentric Coordinates). *Let $S = \{A_1, \ldots, A_N\}$ be a set of N barycentrically independent points in $\mathbb{R}^n$, $n \geq N-1$. Then, the real numbers $m_1, \ldots, m_N$, satisfying*

$$\sum_{k=1}^{N} m_k \neq 0 \tag{20}$$

are barycentric coordinates of a point $P \in \mathbb{R}^n$ with respect to the set S if

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k}. \tag{21}$$

Equation (21) *is said to be a barycentric representation of P with respect to the set $S = \{A_1, \ldots, A_N\}$.*

Barycentric coordinates are homogeneous in the sense that the barycentric coordinates $(m_1, \ldots, m_N)$ of the point P in (21) *are equivalent to the barycentric coordinates $(\lambda m_1, \ldots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since in barycentric coordinates only ratios of coordinates are relevant, the barycentric coordinates $(m_1, \ldots, m_N)$ are also written as $(m_1 \colon \ldots \colon m_N)$ so that*

$$(m_1 : m_2 : \ \ldots \ : m_N) = (\lambda m_1 : \lambda m_2 : \ \ldots \ : \lambda m_N) \tag{22}$$

for any real $\lambda \neq 0$.

In a classical mechanical interpretation of (21), i) $m_k > 0$ is the (Newtonian) mass of the k-th particle, $k = 1, \ldots, N$, of a particle system of N particles, ii) A_k is the velocity of the k-th particle relative to a given inertial rest frame, and iii) P is the center of momentum of the particle system.

It is easy to see from (21) that barycentric coordinates are independent of the choice of the origin of their vector space, that is

$$X + P = \frac{\sum_{k=1}^{N} m_k(X + A_k)}{\sum_{k=1}^{N} m_k} \tag{23}$$

for all $X \in \mathbb{R}^n$. The trivial proof that (23) follows from (21) rests on the result that scalar multiplication in vector spaces distributes over vector addition. Interestingly, however, the hyperbolic counterpart, (27) below, of (23) is far away from being trivial because it involves both ordinary vector addition, + (implicit in the Σ notation for vector summation), and Einstein vector addition, $\oplus$.

It follows from (23) that the barycentric representation (21) of a point P is *covariant* with respect to translations of $\mathbb{R}^n$ in the sense that the point P and the points A_k, $k = 1, \ldots, N$, of its generating set $S = \{A_1, \ldots, A_N\}$ vary in (23) together under translations.

Hyperbolic barycentric coordinates in Einstein gyrovector spaces, fully analogous to Euclidean barycentric coordinates, are called *gyrobarycentric coordinates*. These are defined in the following two definitions.

Definition 8. (Gyrobarycentric Independence). *A set S of $N \geq 2$ points $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}_s^n$, $n \geq N-1$, is gyrobarycentrically independent if the $N-1$ gyrovectors in $\mathbb{R}_s^n$, $\ominus A_1 \oplus A_k$, $k = 2, \ldots, N$, considered as vectors in $\mathbb{R}^n$, are linearly independent in $\mathbb{R}^n$.*

Definition 9. (Gyrobarycentric Coordinates in Einstein Gyrovector Spaces [58, p.179] [59, p.89] [62]). *Let $S = \{A_1, \ldots, A_N\}$ be a gyrobarycentrically independent set of $N \geq 2$ points in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, $n \geq N-1$. The N real numbers $m_1, \ldots, m_N$ are gyrobarycentric coordinates of a point $P \in \mathbb{R}_s^n$ with respect to S if*

$$\sum_{k=1}^{N} m_k \gamma_{A_k} \neq 0 \tag{24}$$

and

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}}. \tag{25}$$

Gyrobarycentric coordinates are homogeneous in the sense that the gyrobarycentric coordinates $(m_1, \dots, m_N)$ *of the point* P *in* (25) *are equivalent to the gyrobarycentric coordinates* $(\lambda m_1, \dots, \lambda m_N)$ *for any* $\lambda \neq 0$*. Since in gyrobarycentric coordinates only ratios of coordinates are relevant, the gyrobarycentric coordinates* $(m_1, \dots, m_N)$ *are also written as* $(m_1\!:, \dots, \!:m_N)$ *so that*

$$(m_1 : m_2 : \ \dots \ : m_N) = (\lambda m_1 : \lambda m_2 : \ \dots \ : \lambda m_N) \tag{26}$$

for any real $\lambda \neq 0$*.*

The point P *given by* (25) *is said to be a gyrobarycentric combination of the points of the set* S*, possessing the gyrobarycentric representation* (25).

In a relativistic mechanical interpretation of (25), i) $m_k \gamma_{A_k} > 0$ is the relativistic, velocity dependent mass [60] of the k-th particle, $k = 1, \dots, N$, of a particle system of N particles, ii) A_k is the velocity of the k-th particle relative to a given inertial rest frame, and iii) P is the center of momentum of the particle system.

Surprisingly, the analogies that barycentric coordinates and gyrobarycentric coordinates share include covariance. In full analogy with the covariance under translations of barycentric coordinates, gyrobarycentric coordinates are covariant under left gyrotranslations a property called *gyrocovariance*.

Indeed, if a point $P \in \mathbb{R}_s^n$ possesses the gyrobarycentric representation (25), then it obeys the identity

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \tag{27}$$

for any $X \in \mathbb{R}_s^n$, where the point P and the points A_k, $k = 1, \dots, N$, of its generating set $S = \{A_1, \dots, A_N\}$ vary together under left gyrotranslations.

The proof of the gyrocovariance of gyrobarycentric coordinates (27), in a more general context is based on the linearity of Lorentz transformations, and is found in [59, Theorem 4.6, p.90] and in [58, §4.3]. The study of gyrobarycentric coordinates along with their use in hyperbolic geometry is presented in [59] for Einstein gyrovector spaces, and in [58] for both Einstein and Möbius gyrovector spaces.

9. Example I – The Euclidean Segment

The aim of this section, Example I, is to set the stage for the presentation of its hyperbolic counterpart, Example II, of Section 10.

Let $A, B \in \mathbb{R}^2$ be two distinct points of the Euclidean plane $\mathbb{R}^2$, and let $P \in \mathbb{R}^2$ be a point on the segment AB that joins A to B. Then, the barycentric representation of P with respect to the points A and B is

$$P = \frac{m_1 A + m_2 B}{m_1 + m_2} \tag{28}$$

with barycentric coordinates m_1 and m_2 satisfying $m_1 \geq 0$, $m_2 \geq 0$ and $m_1 + m_2 \neq 0$.

1. If $m_1 = 0$, then $P = B$.
2. If $m_2 = 0$, then $P = A$.
3. If $m_1, m_2 > 0$, then P lies on the interior of segment AB, that is, between A and B.

Owing to the homogeneity of barycentric coordinates, these can be normalized by the condition

$$m_1 + m_2 = 1 \tag{29}$$

so that, for instance, we can use the notation $m_1 = t$ and $m_2 = 1 - t$, $0 < t < 1$. In that case, the point P possesses the barycentric representation

$$P = tA + (1 - t)B. \tag{30}$$

Finally, owing to the covariance of barycentric representations with respect to translations, the barycentric representation (30) of P obeys the identity

$$X + P = t(X + A) + (1 - t)(X + B) \tag{31}$$

for all $X \in \mathbb{R}^2$. The derivation of Identity (31) from (30) is trivial. However, identity (31) serves as an illustration of its hyperbolic counterpart in (35) below, which is far away from being trivial.

10. Example II – The Hyperbolic Segment

The aim of this section, Example II, is to illustrate Definition 9 of gyrobarycentric coordinates in a form analogous to Example I [38].

Let $A, B \in \mathbb{R}^2_s$ be two distinct points of the Einstein gyrovector plane $\mathbb{R}^2_s = (\mathbb{R}^2_s, \oplus, \otimes)$, and let $P \in \mathbb{R}^2_s$ be a point on the gyrosegment AB that joins A to B, as shown in Fig. 1. Then, the gyrobarycentric representation of P with respect to the points A and B is

$$P = \frac{m_1 \gamma_A A + m_2 \gamma_B B}{m_1 \gamma_A + m_2 \gamma_B} \tag{32}$$

with gyrobarycentric coordinates m_1 and m_2 satisfying $m_1 \geq 0$, $m_2 \geq 0$ and $m_1\gamma_A + m_2\gamma_B \neq 0$.

1. If $m_1 = 0$, then $P = B$.
2. If $m_2 = 0$, then $P = A$.
3. If $m_1, m_2 > 0$, then P lies on the interior of gyrosegment AB, that is, between A and B.

Owing to the homogeneity of gyrobarycentric coordinates, these can be normalized by the condition

$$m_1 + m_2 = 1 \tag{33}$$

so that, for instance, we can use the notation $m_1 = t$ and $m_2 = 1 - t$, $0 \leq t \leq 1$. In that case, the point P possesses the gyrobarycentric representation

$$P = \frac{t\gamma_A A + (1-t)\gamma_B B}{t\gamma_A + (1-t)\gamma_B}. \tag{34}$$

Finally, owing to the gyrocovariance of gyrobarycentric representations with respect to left gyrotranslations, the gyrobarycentric representation (34) of P obeys the identity

$$X \oplus P = \frac{t\gamma_{X\oplus A}(X\oplus A) + (1-t)\gamma_{X\oplus B}(X\oplus B)}{t\gamma_{X\oplus A} + (1-t)\gamma_{X\oplus B}} \tag{35}$$

for all $X \in \mathbb{R}_s^2$. Unlike its Euclidean counterpart (31), identity (35) is, indeed, far away from being trivial, and it involves an elegant harmonious interplay between the two binary operations $+$, vector addition in $\mathbb{R}^n$, and $\oplus$, Einstein addition in $\mathbb{R}_s^n$.

11. On the Use of Gyrobarycentric Coordinates

Applying techniques of hyperbolic barycentric (gyrobarycentric) coordinates in full analogy with the application of classical techniques of barycentric coordinates, one obtains novel results in relativistic hyperbolic geometry, some of which are presented in this section while others are found, for instance, in [59] and in [1, 2, 33].

Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ that possesses a circumgyrocircle, and let O be the circumgyrocenter of the gyrotriangle, as shown in Fig. 2. Then O possesses the gyrobarycentric representation

$$O = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2 + m_3\gamma_{A_3}A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}} \tag{36}$$

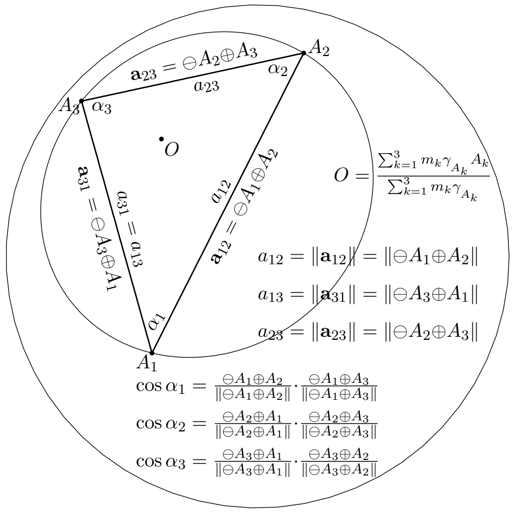

Figure 2. The circumgyrocircle, and the circumgyrocenter O, of gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n = 2$, is shown along with its associated index notation.

with respect to the gyrobarycentrically independent set $S = \{A_1, A_2, A_3\}$, where the gyrobarycentric coordinates m_1, m_2 and m_3 are given by (37) in terms of gamma factors of the gyrotriangle sides

$$\begin{aligned} m_1 &= \frac{1}{D}(\ \ \gamma_{12} + \gamma_{13} - \gamma_{23} - 1)(\gamma_{23} - 1) \\ m_2 &= \frac{1}{D}(\ \ \gamma_{12} - \gamma_{13} + \gamma_{23} - 1)(\gamma_{13} - 1) \\ m_3 &= \frac{1}{D}(-\gamma_{12} + \gamma_{13} + \gamma_{23} - 1)(\gamma_{12} - 1). \end{aligned} \tag{37}$$

Here D is given by

$$D = 1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2 - 2(\gamma_{12} - 1)(\gamma_{13} - 1)(\gamma_{23} - 1) \tag{38}$$

and for $1 \le i, j \le 3$, we use the convenient notation

$$\gamma_{ij} = \gamma_{\ominus A_i \oplus A_j} = \gamma_{\|\ominus A_i \oplus A_j\|}. \tag{39}$$

In the theorems below we ambiguously use the notation $|AB| = \|\ominus A \oplus B\|$ for points A and B in an Einstein gyrovector space, and $|AB| = \|-A + B\|$ for points A and B in a Euclidean space.

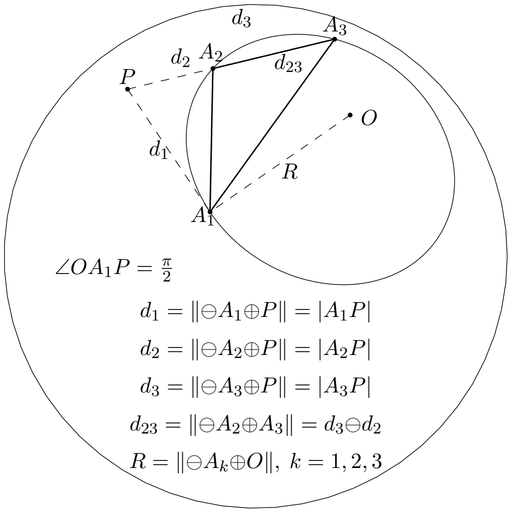

Figure 3. Illustrating the Gyrotangent–Gyrosecant Theorem 10, a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ is shown for $n = 2$, along with its circumgyrocircle and its circumgyrocenter O and circumgyroradius $R = |OA_1| = \|\ominus O \oplus A_1\|$.

Theorem 10. (The Gyrotangent–Gyrosecant Theorem). *If a gyrotangent of a gyrocircle from an external point P meets the gyrocircle at A_1, and a gyrosecant from P meets the gyrocircle at A_2 and A_3, as shown in* Fig. 3*, then*

$$\gamma_{|PA_1|}^2 |PA_1|^2 = \frac{2}{\gamma_{|A_2A_3|} + 1} \gamma_{|PA_2|} |PA_2| \gamma_{|PA_3|} |PA_3|. \tag{40}$$

In the Euclidean limit, $s \to \infty$, gyrolengths of gyrosegments tend to lengths of corresponding segments and gamma factors tend to 1. Hence, in that limit, the Gyrotangent–Gyrosecant Theorem 10 reduces to the following well-known Tangent–Secant Theorem of Euclidean geometry

Theorem 11. (The Tangent–Secant Theorem). *If a tangent of a circle from an external point P meets the circle at A_1, and a secant from P meets the circle at A_2 and A_3, then*

$$|PA_1|^2 = |PA_2||PA_3|\,. \tag{41}$$

As an obvious corollary of the Gyrotangent–Gyrosecant Theorem 10 we have the following theorem for intersecting gyrosecants of a gyrocircle

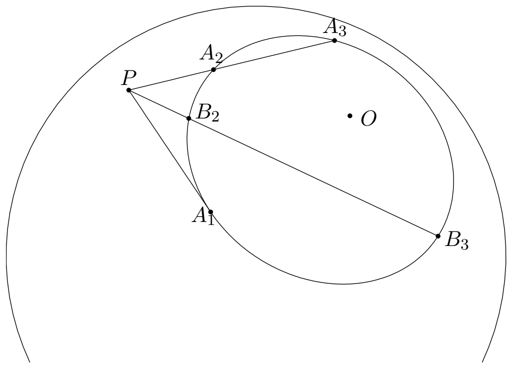

Figure 4. Illustrating the Intersecting Gyrosecants Theorem 12, two intersecting gyrosecants PA_3 and PB_3 of a gyrocircle are shown. They, respectively, intersect the gyrocircle at the points A_2, A_3 and at the points B_2, B_3.

Theorem 12. (The Intersecting Gyrosecants Theorem). *If two gyrosecants of a gyrocircle in an Einstein gyrovector space* $(\mathbb{R}^n_s, \oplus, \otimes)$*, drawn to the gyrocircle from an external point* P*, meet the gyrocircle at points* A_2, A_3 *and at points* B_2, B_3*, respectively, as shown in* Fig. 4*, then*

$$\frac{\gamma_{|PA_2|}|PA_2|\gamma_{|PA_3|}|PA_3|}{\gamma_{|A_2A_3|}+1} = \frac{\gamma_{|PB_2|}|PB_2|\gamma_{|PB_3|}|PB_3|}{\gamma_{|B_2B_3|}+1}. \tag{42}$$

Proof: Let PA_1 be a gyrotangent gyrosegment of the gyrocircle drawn from P and meeting the gyrocircle at A_1, as shown in Fig. 4. Then, by Theorem 10, each of the two sides of (42) equals half the left-hand side of (40) thus verifying (42). ■

In the Euclidean limit, $s \to \infty$, gyrolengths of gyrosegments tend to lengths of corresponding segments and gamma factors tend to one. Hence, in that limit, the Intersecting Gyrosecants Theorem 12 of hyperbolic geometry reduces to the following well-known Intersecting Secants Theorem of Euclidean geometry

Theorem 13. (The Intersecting Secants Theorem). *If two secants of a circle in a Euclidean vector space* $\mathbb{R}^n$*, drawn to the circle from an external point* P*, meet the circle at points* A_2, A_3 *and at points* B_2, B_3*, respectively, then*

$$|PA_2||PA_3| = |PB_2||PB_3|\,. \tag{43}$$

Theorem 14. (The Inscribed Gyroangle Theorem). *Let* θ *be a gyroangle inscribed in a gyrocircle with gyrocenter* O*, and let* 2ϕ *be the gyrocentral gyroangle of the*

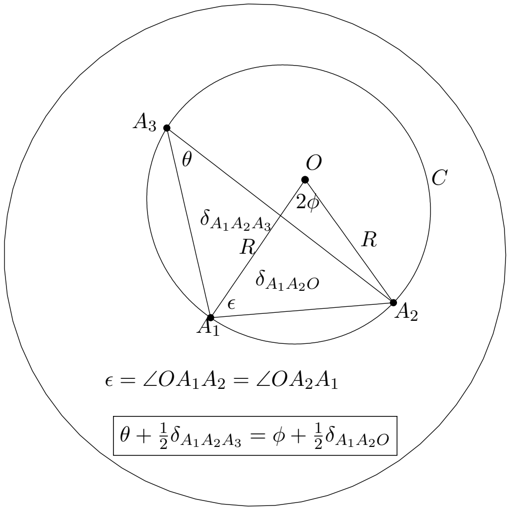

Figure 5. The Inscribed Gyroangle Theorem. The relation between the inscribed gyroangle θ and the gyrocentral gyroangle 2ϕ is expressed in terms of gyrotriangle defects. The latter vanish in Euclidean geometry, reducing the relation between θ and ϕ to the Euclidean geometric equation $\theta = \phi$.

gyrocircle such that both θ and 2ϕ subtend on the same gyroarc $\widehat{A_1A_2}$ on the gyrocircle, as shown in Fig. 5*. Furthermore, in the notation in* Fig. 5*, let $\delta_{A_1A_2A_3}$ be the defect of gyrotriangle $A_1A_2A_3$ and, similarly, let $\delta_{A_1A_2O}$ be the defect of gyrotriangle A_1A_2O. Then*

$$\sin(\theta + \tfrac{1}{2}\delta_{A_1A_2A_3}) = \sin(\phi + \tfrac{1}{2}\delta_{A_1A_2O}) \tag{44}$$

that is, either

$$\theta + \tfrac{1}{2}\delta_{A_1A_2A_3} = \phi + \tfrac{1}{2}\delta_{A_1A_2O} \tag{45a}$$

as in Fig. 5*, or*

$$\theta + \tfrac{1}{2}\delta_{A_1A_2A_3} = \pi - (\phi + \tfrac{1}{2}\delta_{A_1A_2O}) \tag{45b}$$

as in Fig. 6*.*

12. Relativistic-Hyperbolic Barycentric Coordinates and the Geometry of Relativistic Quantum States

In this section we present evidence that supports the idea that relativistic-hyperbolic barycentric (gyrobarycentric) coordinates in relativistic hyperbolic geometry form the right tool in the study of the geometry of relativistic quantum states, just as

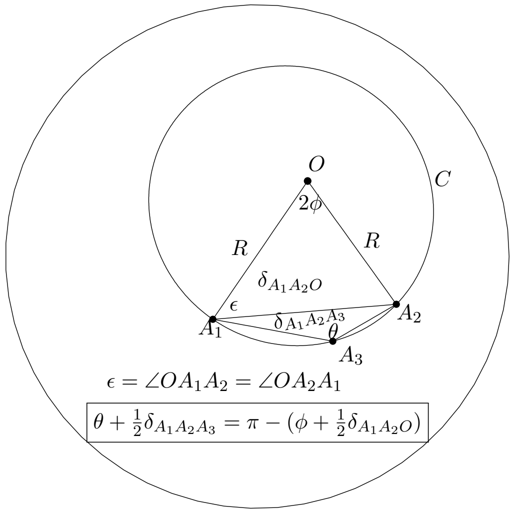

Figure 6. The Inscribed Gyroangle Theorem. This figure is similar to Fig. 5 except that here the points O and A_3 lie on opposite sides of the chord A_1A_2 of gyrocircle C in the gyroplane through the points A_1, A_2 and A_3. As in Fig. 5 the relation between the inscribed gyroangle θ and the gyrocentral gyroangle 2ϕ is expressed in terms of gyrotriangle defects. The latter vanish in Euclidean geometry, reducing the relation between θ and ϕ to the Euclidean geometric equation $\theta = \pi - \phi$.

Euclidean barycentric coordinates form, as emphasized in [3], the right tool in the study of the geometry of non-relativistic quantum states.

Bures fidelity has particularly wide currency today in quantum computation and quantum information geometry. However, Nielsen and Chuang inform that "Unfortunately, no similarly [alluding to the *trace distance*] clear geometric interpretation is known for the fidelity between two states of a qubit" [27, p.410]. The search for the missing geometric interpretation led Chen, Fu, Ungar, and Zhao to uncover a link between Bures fidelity and Einstein addition in [5] and [6]. The related relationship between Bloch vectors and Einstein addition is studied in [7], [45] and [48, Ch.9].

Supporting evidence that relativistic hyperbolic geometry in general, and relativistic barycentric coordinates, in particular, can be useful in the study of the geometry of relativistic quantum mechanics is presented in [8]. According to [8], the Bures fidelity between two states of a qubit quantifies the extent of which the two states are distinguished from one another. It is generated by the so-called Bloch vectors, which are elements of the closed unit ball of the Euclidean three-space. A link is uncovered between the Bures fidelity and Einstein addition in the ball. It is shown that in terms of Einstein addition of relativistically admissible velocities, the Bures fidelity takes a simple, elegant form. This, in turn, demonstrates that the Bures fidelity is regulated by the relativistic (Beltrami-Klein) model of the hyperbolic geometry of Bolyai and Lobachevsky. Thus, Einstein addition is already well involved in the geometry of quantum computation and information.

Particularly strong supporting evidence stems from Lévay's discovery in [23] that the special relativistic phenomenon of Thomas precession (and, hence, gyration) coincides with the quantum mechanical phenomenon known as *mixed state geometric phase*. The outstanding book [3] studies the geometry of quantum states by means of barycentric coordinates. As such, the book hardly encounters the presence of the geometric phase in quantum mechanics. In contrast, the outstanding book [10], which studies the "Geometric Phases in Classical and Quantum Mechanics", makes no use of barycentric coordinates. The reason is clear. Barycentric coordinates are admitted by Euclidean geometry and, as such, they are insensitive to geometric phases. In contrast, relativistic barycentric coordinates are admitted by relativistic hyperbolic geometry and, as such, they are sensitive to geometric phases, thus forming an adequate tool for the study of the geometry of quantum states in quantum mechanics, where geometric phases take place. Some other references that provide supporting evidence about the role of special relativity in relativistic quantum mechanics are [3, 9, 23, 24, 28–32].

References

[1] Barbu C., *Menelaus's Theorem for Hyperbolic Quadrilaterals in the Einstein Relativistic Velocity Model of Hyperbolic Geometry*, Sci. Magna **6** (2010) 19–24.

[2] Barbu C., *The Hyperbolic Stewart Theorem in the Einstein Relativistic Velocity Model of Hyperbolic Geometry*, An. Univ. Oradea Fasc. Mat. **18** (2011) 133–138.

[3] Bengtsson I. and Życzkowski., *Geometry of Quantum States*: *An Introduction to Quantum Entanglement.*, Cambridge University Press, Cambridge 2006.

[4] Brezov D., Mladenova C. and Mladenov I., *Vector Parameters in Classical Hyperbolic Geometry*, J. Geom. Symmetry Phys. **30** (2013) 19–48.

[5] Chen J., Fu L., Ungar A. and Zhao X., *Alternative Fidelity Measure Between Two States of an N-State Quantum System*, Phys. Rev. A **65** (2002) 054303.

[6] Chen J., Fu L., Ungar A. and Zhao X., *Geometric Observation of Bures Fidelity Between Two States of a Qubit*, Phys. Rev. A **65** (2002) 024303.

[7] Chen J. and Ungar A., *The Bloch Gyrovector*, Found. Phys. **32** (2002) 531–565.

[8] Chen J. and Ungar A., *Introducing the Einstein Metric to Quantum Computation and Quantum Information Geometry*, Found. Phys. Lett. **15** (2002) 189–197.

[9] Chiribella G., D'Ariano G. and Schlingemann D., *Barycentric Decomposition of Quantum Measurements in Finite Dimensions*, J. Math. Phys. **51** (2010) 022111.

[10] Chruściński D. and Jamiołkowski A., *Geometric Phases in Classical and Quantum Mechanics*, Progress in Mathematical Physics vol. 36, Birkhäuser, Boston 2004.

[11] Einstein A., *Zur Elektrodynamik Bewegter Körper* [On the Electrodynamics of Moving Bodies] (We use the English translation in [12] or in [25], or in http://www.fourmilab.ch/etexts/einstein/specrel/www/), Ann. Physik (Leipzig) **17** (1905) 891–921.

[12] Einstein A., *Einstein's Miraculous Years*: *Five Papers that Changed the Face of Physics*, Princeton, 1998, Edited and introduced by John Stachel. Includes bibliographical references. Einstein's dissertation on the Determination of Molecular Dimensions – Einstein on Brownian Motion – Einstein on the Theory of Relativity – Einstein's Early Work on the Quantum Hypothesis. A new English translation of Einstein's 1905 paper on pp. 123–160.

[13] Feder T., *Strong Near Subgroups and Left Gyrogroups*, J. Algebra **259** (2003) 177–190.

[14] Ferreira M., *Factorizations of Möbius Gyrogroups*, Adv. Appl. Clifford Algebr. **19** (2009) 303–323.

[15] Ferreira M. and Ren G., *Möbius Gyrogroups*: *A Clifford Algebra Approach*, J. Algebra **328** (2011) 230–253.

[16] Fock V., *The Theory of Space, Time and Gravitation*, Macmillan, New York 1964.

[17] Foguel T. and Ungar A., *Involutory Decomposition of Groups into Twisted Subgroups and Subgroups*, J. Group Theory **3** (2000) 27–46.

[18] Foguel T. and Ungar A., *Gyrogroups and the Decomposition of Groups into Twisted Subgroups and Subgroups*, Pac. J. Math **197** (2001) 1–11.

[19] Issa A., *Gyrogroups and Homogeneous Loops*, Rep. Math. Phys. **44** (1999) 345–358.

[20] Issa A., *Left Distributive Quasigroups and Gyrogroups*, J. Math. Sci. Univ. Tokyo **8** (2001) 1–16.

[21] Kasparian K. and Ungar A., *Lie Gyrovector Spaces*, J. Geom. Symmetry Phys. **1** (2004) 3–53.

[22] Kuznetsov E., *Gyrogroups and Left Gyrogroups as Transversals of a Special Kind*, Algebra Discrete Math. (2003) 54–81.

[23] Lévay P., *Thomas Rotation and the Mixed State Geometric Phase*, J. Phys. A **37** (2004) 4593–4605.

[24] Lindner N., Peres A. and Terno D., *Wigner's Little Group and Berry's Phase for Massless Particles*, J. Phys. A **36** (2003) L449–L454.

[25] Lorentz H., Einstein A., Minkowski H. and Weyl H., *The Principle of Relativity*, A collection of original memoirs on the special and general theory of relativity. Dover, New York, undated, With notes by A. Sommerfeld, translated by W. Perrett and G. Jeffery.

[26] Møller C., *The Theory of Relativity*, Clarendon Press, Oxford 1952.

[27] Nielsen M. and Chuang I., *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.

[28] Peres A., *Relativistic Quantum Measurements*, Ann. New York Acad. Sci. **755** (1995) 445–450.

[29] Peres A., Scudo P. and Terno D., *Quantum Entropy and Special Relativity*, Phys. Rev. Lett. **88** (2002) 230402.

[30] Peres A. and Terno D., *Convex Probability Domain of Generalized Quantum Measurements*, J. Phys. A **31** (1998) L671–L675.

[31] Peres A. and Terno D., *Relativistic Doppler Effect in Quantum Communication*, J. Modern Opt. **50** (2003) 1165–1173.

[32] Peres A. and Terno D., *Quantum Information and Relativity Theory*, Rev. Modern Phys. **76** (2004) 93–123.

[33] Pişcoran L. and Barbu C., *Pappus's Harmonic Theorem in the Einstein Relativistic Velocity Model of Hyperbolic Geometry*, Stud. Univ. Babeş-Bolyai Math. **56** (2011) 101–107.

[34] Rassias Th., Book Review: *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, by Abraham A. Ungar, Nonlinear Funct. Anal. Appl. **13** (2008) 167–177.

[35] Rassias Th., Book Review: *A Gyrovector Space Approach to Hyperbolic Geometry*, by Abraham A. Ungar, J. Geom. Symmetry Phys. **18** (2010) 93–106.

[36] Rockafellar R., *Convex Analysis*, Princeton Mathematical Series vol. 28, Princeton University Press, Princeton 1970.

[37] Sexl R. and Urbantke H., *Relativity, Groups, Particles*, Springer Physics, Springer, Vienna 2001, Special Relativity and Relativistic Symmetry in Field

and Particle Physics, Revised and translated from the third German (1992) edition by Urbantke.

[38] Sönmez N. and Ungar A., *The Einstein Relativistic Velocity Model of Hyperbolic Geometry and its Plane Separation Axiom*, Adv. Appl. Clifford Alg. **23** (2013) 209–236.

[39] Ungar A., *Thomas Rotation and the Parametrization of the Lorentz Transformation Group*, Found. Phys. Lett. **1** (1988) 57–89.

[40] Ungar A., *The Thomas Rotation Formalism Underlying a Nonassociative Group Structure for Relativistic Velocities*, Appl. Math. Lett. **1** (1988) 403–405.

[41] Ungar A., *The Relativistic Noncommutative Nonassociative Group of Velocities and the Thomas Rotation*, Resultate Math. **16** (1989) 168–179.

[42] Ungar A., *Thomas Precession and its Associated Grouplike Structure*, Amer. J. Phys. **59** (1991) 824–834.

[43] Ungar A., *The Bifurcation Approach to Hyperbolic Geometry*, Found. Phys. **30** (2000) 1253–1281.

[44] Ungar A., *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession*: *The Theory of Gyrogroups and Gyrovector Spaces*, Fundamental Theories of Physics vol. 117, Kluwer, Dordrecht 2001.

[45] Ungar A., *The Hyperbolic Geometric Structure of the Density Matrix for Mixed State Qubits*, Found. Phys. **32** (2002) 1671–1699.

[46] Ungar A., *The Relativistic Centre of Momentum*, Geometry, Integrability and Quantization **4** (2003) 316–325.

[47] Ungar A., *The Hyperbolic Triangle Defect*, Geometry, Integrability and Quantization **5** (2004) 225–236.

[48] Ungar A., *Analytic Hyperbolic Geometry*: *Mathematical Foundations and Applications*, World Scientific, Hackensack 2005.

[49] Ungar A., *Gyrovector Spaces and their Differential Geometry*, Nonlinear Funct. Anal. Appl. **10** (2005) 791–834.

[50] Ungar A., *The Proper-Time Lorentz Group Demystified*, J. Geom. Symmetry Phys. **4** (2005) 69–95.

[51] Ungar A., *The Relativistic Hyperbolic Parallelogram Law*, Geometry, Integrability and Quantization **7** (2006) 249–264.

[52] Ungar A., *Thomas Precession*: *A Kinematic Effect of the Algebra of Einstein's Velocity Addition Law. Comments on*: "Deriving Relativistic Momentum and Energy. II. Three-Dimensional Case" [European J. Phys. **26** (2005), 851–856, MR2227176] by S. Sonego and M. Pin, European J. Phys. **27** (2006) L17–L20.

[53] Ungar A., *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, World Scientific, Hackensack 2008.

[54] Ungar A., *From Möbius to Gyrogroups*, Amer. Math. Monthly **115** (2008) 138–144.

[55] Ungar A., *A Gyrovector Space Approach to Hyperbolic Geometry*, Morgan & Claypool, San Rafael 2009.

[56] Ungar A., *Hyperbolic Barycentric Coordinates*, Aust. J. Math. Anal. Appl. **6** (2009) 1–35.

[57] Ungar A., *The Hyperbolic Triangle Incenter. Dedicated to the 30th Anniversary of Themistocles M. Rassias' Stability Theorem*, Nonlinear Funct. Anal. Appl. **14** (2009) 817–841.

[58] Ungar A., *Barycentric Calculus in Euclidean and Hyperbolic Geometry*: *A Comparative Introduction*, World Scientific, Hackensack 2010.

[59] Ungar A., *Hyperbolic Triangle Centers*: *The Special Relativistic Approach*, Springer, New York 2010.

[60] Ungar A., *When Relativistic Mass Meets Hyperbolic Geometry*, Commun. Math. Anal. **10** (2011) 30–56.

[61] Ungar A., *Gyrations*: *The Missing Link Between Classical Mechanics with its Underlying Euclidean Geometry and Relativistic Mechanics with its Underlying Hyperbolic Geometry*, Essays in Mathematics and its Applications in Honor of Stephen Smale's 80th Birthday, Springer, Heidelberg 2012, arXiv 1302.5678 (math-ph), pp.463–504.

[62] Ungar A., *An Introduduction to Hyperbolic Barycentric Coordinates and their Applications*, 2013 - arXiv 1304.0205 (math-ph).

[63] Walter S., Book Review: *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession*: *The Theory of Gyrogroups and Gyrovector Spaces*, by Abraham A. Ungar, Found. Phys. **32** (2002) 327–330.

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