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ON SOLITON EQUATIONS WITH $\mathbb{Z}_h$ AND $\mathbb{D}_h$ REDUCTIONS: CONSERVATION LAWS AND GENERATING OPERATORS

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Abstract. The Lax representations for the soliton equations with $\mathbb{Z}_h$ and $\mathbb{D}_h$ reductions are analyzed. Their recursion operators are shown to possess factorization properties due to the grading in the relevant Lie algebra. We show that with each simple Lie algebra one can relate r fundamental recursion operators $\boldsymbol{\Lambda}_{m_k}$ and a master recursion operator $\boldsymbol{\Lambda}$ generating NLEEs of MKdV type and their Hamiltonian hierarchies. The Wronskian relations are formulated and shown to provide the tools to understand the inverse scattering method as a generalized Fourier transform. They are also used to analyze the conservation laws of the above mentioned soliton equations.

Contents

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1. Introduction

This paper is a natural continuation of our articles [21], where a gauge covariant approach to the generating operator was proposed for the case of the Zakharov-Shabat system and [22], where the results obtained in [21] have been generalized to the cases of the so-called Caudrey-Beals-Coifman system. Here we consider the nonlinear evolution equations (NLEEs), solvable through the inverse scattering method (ISM) for the linear system

$$\begin{gathered} L\psi(x,t,\lambda) \equiv \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + U(x,t,\lambda) \right) \psi(x,t,\lambda) = 0 \\ U(x,t,\lambda) = U_0(x,t) + \lambda U_1, \qquad U_0(x,t) = [U_1, Q(x,t)]. \end{gathered} \tag{1}$$

If we choose U_1 to be constant diagonal matrix with real eigenvalues the system (1) is the generalized Zakharov-Shabat system [42, 45]. It allows one to solve the class of N-wave equations which have important physical applications.

As has been shown by Shabat [37] the difficulties of solving the inverse scattering problem for (1) could be overcome if we are able to reduce the solution of the relevant nonlinear evolution equation (NLEE) to a Riemann-Hilbert problem (RHP)

$$\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda) G(\lambda,t), \qquad \lambda \in \mathbb{R}. \tag{2}$$

In the above $\chi^{\pm}(x,t,\lambda)$ are the fundamental analytic solutions (FAS), that is, fundamental solutions of (1) that allow analytic extension in λ for $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$ – the upper and lower half-plane.

In particular, using this fact, Zakharov and Shabat developed the well known dressing method, which became the most efficient method for constructing the soliton solutions of the NLEEs we speak about.

Later Caudrey, Beals and Coifman [3, 4, 6] studied the inverse scattering problem for a more general system which has also the form of (1) but now the eigenvalues of U_1 were assumed to be complex. They succeeded to construct the FAS which now have analyticity properties only inside certain sectors in the complex λ-plane and the RHP is formulated on the rays separating these sectors.

The results of Caudrey-Beals-Coifman were extended from $\mathfrak{sl}(n)$ to any semisimple Lie algebra in any faithful representation [22]. In [22] we have assumed that $Q(x,t)$ is an arbitrary element of a semisimple complex Lie algebra $\mathfrak{g}$ of rank r and U_1 is a regular element of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Applying the inverse scattering method (ISM) to the CBC system one can integrate generic NLEEs. The simplest of them are analogous to the N-wave equations but with complex-valued 'group velocities'. Until now we know nothing about physical applications of such NLEEs, though they are completely integrable Hamiltonian systems whose action-angle variables have been constructed in [5].

The situation changes if we apply the reduction group technique [32] and request that the Lax operator possesses $\mathbb{Z}_h$ or $\mathbb{D}_h$ symmetry. The importance of such Lax operators became clear due to the pioneer paper of Mikhailov [32] where he discovered the integrability of the two-dimensional Toda field theories (2DTFT). Mikhailov was the first to show that the ISM for $\mathbb{Z}_h$ and $\mathbb{D}_h$-reduced Lax operators can be reduced to a RHP on a set of $2h$ rays starting from the origin and closing angles π/h. Next, using the Zakharov-Shabat dressing method [45], he constructed the soliton solutions of the 2DTFT and proved that there are gaps in the sequences of integrals of motion.

Soon the results of [32] were generalized to any simple Lie algebra and draw a large attention, see [13,33–35] and the references therein. Of course, each 2DTFT is just one representative from a hierarchy of NLEEs that are related to the corresponding Lax operator.

One of the aims of the present paper is to provide a tool to study the other members of these hierarchies. To this end we have to reformulate most of the results we have for the general system to the special but important case when the Lax operator is subject to additional group of symmetries, among which are $\mathbb{Z}_h$ and $\mathbb{D}_h$ symmetries proposed by Mikhailov [32].

Our second aim is to generalize the AKNS method [1, 26] and to construct the recursion operators for the NLEE with $\mathbb{Z}_h$ and $\mathbb{D}_h$ symmetries. The recursion operators (see the monographs [7,20,27] and [1,8,12,14–16,19,21,22,24,26]) are an effective tool to generate both the NLEEs and their Hamiltonian hierarchies [15, 16, 19, 28, 29]. We confirm and generalize previous results on recursion operators of reduced systems [11, 13, 18, 23, 38, 39], especially their factorization properties.

In Section 2 we give some Lie algebraic preliminaries and introduce the notion of Mikhailov reduction group [32]. In Section 3 we outline the spectral theory for the Lax operators, introduce their FAS and the scattering data. Section 4 demonstrates that the FAS satisfy a local RHP on a set of $2h$ rays passing through the origin and closing angles π/h [32]. Section 5 is dedicated to the calculation of the recursion operators introduced by [1]. However, the AKNS method needs generalization, because we are dealing with a $\mathbb{Z}_h$-reduced Lax pair that take values in the graded algebra. The recursion operators that are obtained have substantially new structure as compared to the AKNS ones. The recursion operators now factorizes into a product of h more elementary operators. For the first time such factorization has been observed studying a particular case in [11]. We also show that to each simple Lie algebra of rank r one can relate r fundamental recursion operators $\mathbf{\Lambda}_{m_k}$ and a master recursion operators $\boldsymbol{\Lambda}$ generating both NLEEs of MKdV type and their Hamiltonian hierarchies. In Section 6 we derive three types of Wronskian relations for the Lax operator $\tilde{L}$. The first type allows one to interpret the mapping $\mathcal{F} : \mathcal{M} \to \mathcal{T}$ from the manifold of allowed potentials $\mathcal{M}$ to the set of scattering data $\mathcal{T}$ as a generalized Fourier transform (GFT). It allows also to introduce the so-called 'squared solutions' of $\tilde{L}$, called also generalized exponentials or adjoint solutions. The second type of Wronskian relations permits to establish that the same GFT allows one to analyze the mapping $\delta\mathcal{F} : \delta\mathcal{M} \to \delta\mathcal{T}$ between the variations of the potentials and the relevant variations of scattering data. The third type of Wronskian relations is useful for the analysis of the conservation laws. Section 7 briefly comments on the locality of the integrals of motion and on the Hamiltonian structures of the NLEEs.

2. Preliminaries

We assume that the reader is familiar with the basic facts from the theory of simple Lie algebras [25]. In what follows by $\mathfrak{h}$ is denoted a fixed Cartan subalgebra of $\mathfrak{g}$, Δ is the root system of $\mathfrak{g}$, $\alpha \in \Delta$ are the roots, $A \equiv \{\alpha_1, \dots, \alpha_r\}$ is the set of the simple roots – naturally, the rank of $\mathfrak{g}$ is r. Suppose that E_α, $\alpha \in \Delta$ and H_k, $k = 1, \dots, r$ denotes the standard Cartan-Weyl basis in $\mathfrak{g}$, see e.g. [25]. Then as well known the commutation relations of the Cartan-Weyl basis have the form

$$\begin{aligned} &[H, E_\alpha] = \alpha(H) E_\alpha, && [E_\alpha, E_{-\alpha}] = H_\alpha \\ &[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, && N_{\alpha,\beta} = \begin{cases} \neq 0 & \text{if } \alpha+\beta \in \Delta \\ 0 & \text{if } \alpha+\beta \notin \Delta. \end{cases} \end{aligned} \tag{3}$$

Let us outline some important facts about the graded Lie algebras and more specifically – how one can introduce bases in $\mathfrak{g}^{(k)}$, see below.

In this article we shall consider grading defined by the Coxeter automorphism, that is $C \in \operatorname{Aut}\mathfrak{g}$, $C^h = \mathbb{1}$ and h is the Coxeter number. Obviously the eigenvalues of C are ω^k, $k = 0, 1, \dots, h-1$, where $\omega = \exp(2\pi i/h)$. To each eigenvalue there corresponds a linear subspace $\mathfrak{g}^{(k)} \subset \mathfrak{g}$ determined by

$$\mathfrak{g}^{(k)} \equiv \left\{ X\,;\, X \in \mathfrak{g}, \quad C(X) = \omega^k X \right\}.$$

We than have $\mathfrak{g} = \mathop{\oplus}\limits_{k=0}^{h-1} \mathfrak{g}^{(k)}$ and $\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(n)}\right] \subset \mathfrak{g}^{(k+n)}$, where $k+n$ is taken modulo h. This of course turns $\mathfrak{g}$ into a graded algebra.

Remark 1. *In fact, as we shall see below, one can view the potentials of the Lax operators as elements of Kac-Moody type $\widehat{\mathfrak{g}}_C$ whose elements are*

$$X(\lambda) = \sum_k X_k \lambda^k, \qquad X_k \in \mathfrak{g}^{(k)}$$

where the sum runs contains only a finite number of terms. For reasons that will become clear later we shall assume that the element U_1 from (1) *belongs to* $\mathfrak{g}^{(1)}$.

Remark 2. *Note also that the Killing form between any two elements $X_{(k)} \in \tilde{\mathfrak{g}}^{(k)}$ and $Y_{(k)} \in \tilde{\mathfrak{g}}^{(m)}$ may be non-vanishing only if $k+m=0 \mod (h)$.*

2.1. The Coxeter Automorphism as Cartan Subgroup Element

We shall consider two realizations of the Coxeter automorphism and start with a realization of as element from the Cartan subgroup

$$C = \exp\left(\frac{2\pi \mathrm{i}}{h} H_{\hat{\rho}}\right), \qquad (\hat{\rho}, \alpha_j) = 1 \tag{4}$$

where

$$\hat{\rho} = \sum_{j=1}^{r} \frac{(\alpha_j, \alpha_j)}{2} \omega_j \tag{5}$$

and ω_j are the fundamental weights of $\mathfrak{g}$. For the classical Lie algebras the vectors $\hat{\rho}$take the form

$$\begin{aligned} A_r\colon \quad \hat{\rho} &= \sum_{s=0}^{r} \left(\left[\frac{r}{2}\right] - s\right) e_{s+1}, \qquad & B_r\colon \quad \hat{\rho} &= \sum_{s=0}^{r-1} (r-s) e_{s+1} \\ C_r\colon \quad \hat{\rho} &= \sum_{s=0}^{r-1} \left(r - s - \frac{1}{2}\right) e_{s+1}, \qquad & D_r\colon \quad \hat{\rho} &= \sum_{s=0}^{r-1} (r-s-1) e_{s+1} \end{aligned} \tag{6}$$

where e_k are as usual orthonormal vectors. Obviously $(\hat{\rho}, \alpha_j) = 1$ for $j = 1, \ldots, r$. With the above choices we have

$$\mathfrak{g}^{(0)} \equiv \mathfrak{h}, \qquad \mathfrak{g}^{(k)} \equiv \operatorname{span}\{E_\alpha;\, \operatorname{ht}\alpha = k \mod (h)\} \tag{7}$$

where $\operatorname{ht}\alpha$ is the height of the root α. In other words, if $\alpha = \sum_{k=1}^{r} m_k \alpha_k$, then $\operatorname{ht}\alpha = \sum_{k=1}^{r} m_k$.

In this case the matrix U_1 that appears in the Lax operator will be denoted by J and is given by

$$J = \sum_{s=0}^{r} E_{\alpha_s} \tag{8}$$

where α_k, $k = 1, \ldots, r$ are the simple roots of $\mathfrak{g}$ and α_0 is the minimal root. We will use it applying additional normalization

$$\langle J, J^T \rangle = 1, \qquad J^T = \sum_{s=0}^{r} E_{-\alpha_s} \tag{9}$$

where $\langle X, Y\rangle$ denotes the Killing form applied to X and Y. The characteristic equations for J are as follows

$$\begin{aligned} &A_r \qquad z^h - 1 = 0, &&B_r \qquad (z^h-1)z = 0 \\ &C_r \qquad z^h - 1 = 0, &&D_r \qquad (z^h-1)z^2 = 0 \end{aligned} \tag{10}$$

where h is the Coxeter number of the algebra, see Table 1. In what follows we will need not only the basis in each of $\mathfrak{g}^{(k)}$, but also the orthogonal splitting

$$X_k \in \mathfrak{g}^{(k)}, \qquad X_k = X_k^{\perp} + X_k^{\parallel} \tag{11}$$

where

$$[X_k^{\parallel}, J] = 0, \qquad \langle X_k^{\perp}, X_k^{\parallel}\rangle = 0. \tag{12}$$

Table 1. The Coxeter numbers, the exponents and the minimal roots of the classical series of Lie algebras, see [25].

Algebra	Coxeter number	exponents	minimal root
$A_r \simeq \mathfrak{sl}(r+1)$	$r+1$	$1, 2, 3, \dots, r$	$e_{r+1} - e_1$
$B_r \simeq \mathfrak{so}(2r+1)$	$2r$	$1, 3, 5, \dots, 2r-1$	$-e_1 - e_2$
$C_r \simeq \mathfrak{sp}(2r)$	$2r$	$1, 3, 5, \dots, 2r-1$	$-2e_1$
$D_r \simeq \mathfrak{so}(2r)$	$2r-2$	$1, 3, 5, \dots, 2r-3, r-1$	$-e_1 - e_2$
G_2	6	$1, 5$	
F_4	12	$1, 5, 7, 11$	

We will also need ad_J^{-1}. Since J is not diagonal this will present technical difficulty which can be overcome by using the characteristic equation for ad_J, see [21]. As for the splitting (11), for the classical series A_r, B_r and C_r it can be found using the fact that any matrix from the algebra commuting with J (9) is a polynomial of J of maximal order h. In particular, the powers of J commute with J. For A_r it is enough to use J^k, $k = 1, \dots, r$. Then for $\mathfrak{g} \simeq \mathfrak{sl}(r+1)$ we have

$$X_k = X_k^{\perp} + X_k^{\parallel}, \qquad X_k^{\parallel} = c_k^{-1} J^k \langle X, J^{h-k}\rangle, \qquad h = r+1 \tag{13}$$

where $c_k = \langle J^k, J^{h-k}\rangle$. For the classical series B_r and C_r we have

$$\begin{aligned} &X_k^{\parallel} = 0, && \text{if } k = 2s \text{ is not an exponent} \\ &X_k^{\parallel} = c_k^{-1} J^k \langle X_k, J^{h-k}\rangle && \text{if } k = 2s-1 \text{ is an exponent.} \end{aligned} \tag{14}$$

The case of the series D_r will be treated separately elsewhere, because for it there are some specifics due to the fact that $r-1$ is an exponent, see Table 1. For the algebra D_r, in case $r = 2p+1$ the even number $2p$ is as exponent while in the case $r = 2p$ the odd number $2p-1$ is a double valued exponent.

2.2. The Coxeter Automorphism as Weyl Group Element

Now we shall consider another realization of the Coxter automorphism. Technically it is easier to handle the case $\mathfrak{g} \simeq \mathfrak{sl}(r+1)$, so we shall start with it.
The Coxeter automorphism $\tilde{C}$ can be realized as

$$\tilde{C}(X) = KXK^{-1}, \qquad K = \sum_{s=1}^{h} E_{s+1,s}, \qquad (E_{k,m})_{jp} = \delta_{kj}\delta_{mp}. \tag{15}$$

Here and below all indexes are understood modulo h, so that the last term with $s = h$ in (15) equals $E_{h,1}$. The calculations are much simpler if we introduce a convenient basis in $\mathfrak{g}^{(k)}$, namely

$$J_s^{(k)} = \sum_{j=1}^{h} \omega^{kj} E_{j,j+s}, \qquad KJ_s^{(k)}K^{-1} = \omega^k J_s^{(k)}. \tag{16}$$

Obviously, $J_s^{(k)}$ satisfies the commutation relations

$$\left[J_s^{(k)}, J_l^{(m)}\right] = \left(\omega^{ms} - \omega^{kl}\right) J_{s+l}^{(k+m)}. \tag{17}$$

Another option is to use the dihedral realization of the Coxeter automorphism $\tilde{C}$

$$\tilde{C} = C_1 C_2 \tag{18}$$

where

$$C_1 = \prod_{\alpha \in A_1} S_\alpha, \qquad C_2 = \prod_{\beta \in A_2} S_\beta. \tag{19}$$

In the above A_0 and A_1 are subsets of the set of simple roots $A = \{\alpha_1, \dots \alpha_r\}$ of $\mathfrak{g}$ such that

$$A_1 \cup A_2 = A, \qquad (\alpha_j, \alpha_k) = 0, \qquad (\beta_j, \beta_k) = 0, \quad \text{for } j \neq k \tag{20}$$

for all $\alpha_j, \alpha_k \in A_0$ and $\beta_j, \beta_k \in A_1$. Also, by S_α is denoted the Weyl reflection related to the root α, i.e., $S_\alpha \vec{x} = \vec{x} - \dfrac{2(\vec{x}, \alpha)}{(\alpha, \alpha)}\alpha$, where $(\ ,\ )$ is the canonical inner product in the Euclidean space $\mathbb{E}^r$. For the classical series A_r, B_r and C_r of Lie algebras

$$\begin{aligned} &A_1 = \{\alpha_2, \alpha_4, \dots, \alpha_{2p}\}, \qquad A_2 = \{\alpha_1, \alpha_3, \dots, \alpha_{2p-1}\}, \quad \text{if } r = 2p \\ &A_1 = \{\alpha_2, \alpha_4, \dots, \alpha_{2p}\}, \qquad A_2 = \{\alpha_1, \alpha_3, \dots, \alpha_{2p+1}\}, \quad \text{if } r = 2p+1 \end{aligned} \tag{21}$$

and for the D_r series

$$\begin{aligned} &A_1 = \{\alpha_2, \alpha_4, \ldots, \alpha_{2p-2}\}, \;\; A_2 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}, \alpha_{2p}, \}, \quad \text{if } r = 2p \\ &A_1 = \{\alpha_2, \alpha_4, \ldots, \alpha_{2p}, \alpha_{2p+1}\}, \;\; A_2 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}\}, \quad \text{if } r = 2p+1. \end{aligned} \tag{22}$$

In the above we have used the standard notations for the sets of simple roots $\{\alpha_1, \ldots, \alpha_r\}$, see [25].

It is natural to call the set of roots $\mathcal{O}_\alpha \equiv \{\alpha, \tilde{C}\alpha, \ldots, \tilde{C}^{h-1}\alpha\}$ the orbit of $\tilde{C}$ passing through the root α. The special choice for $\tilde{C}$ (18) has the advantage to split the root system Δ into r orbits as follows

$$\Delta = \bigcup_{\alpha \in A_1} \mathcal{O}_\alpha \cup \bigcup_{\beta \in A_2} \mathcal{O}_{-\beta} = \bigcup_{\alpha \in A_1} \mathcal{O}_{-\alpha} \cup \bigcup_{\beta \in A_2} \mathcal{O}_\beta. \tag{23}$$

This allows us to introduce two bases in $\mathfrak{g}$ compatible with the grading

$$\begin{aligned} \tilde{\mathfrak{g}}^{(k)} &\equiv \operatorname{span}\{\mathcal{E}_\alpha^{(k)}, \; \mathcal{E}_{-\beta}^{(k)}, \; \mathcal{H}_j^{(k)}; \; \alpha \in A_0, \; \beta \in A_1\} \\ \tilde{\mathfrak{g}}^{(k)} &\equiv \operatorname{span}\{\mathcal{E}_{-\alpha}^{(k)}, \; \mathcal{E}_{\beta}^{(k)}, \; \mathcal{H}_j^{(k)}; \; \alpha \in A_0, \; \beta \in A_1\} \\ \mathcal{E}_\alpha^{(k)} &= \frac{1}{h} \sum_{s=0}^{h-1} \omega^{-sk} \tilde{C}^s(E_\alpha), \qquad \mathcal{H}_j^{(k)} = \frac{1}{h} \sum_{s=0}^{h-1} \omega^{-sk} \tilde{C}^s(H_j). \end{aligned} \tag{24}$$

Remark 3. *Note that $\mathcal{H}_j^{(k)}$ is non-vanishing only if k is an exponent of $\mathfrak{g}$. That also means that each $\tilde{\mathfrak{g}}^{(k)}$ has at most one-dimensional intersection with the Cartan subalgebra and the only exceptions take place for the algebras of the series D_{2r} which have $2r-1$ as double valued exponent. These cases will be considered elsewhere.*

We can pick U_1 to be equal to $\mathcal{J}$ where

$$\begin{aligned} &\mathcal{J} = H_{e_1} + \sum_{s=1}^{p-1} \left(\omega^s H_{e_{2s}} + \omega^{-s} H_{e_{2s+1}}\right), \qquad \text{for } A_{2p}, B_{2p+1}, C_{2p+1}, D_{2p} \\ &\mathcal{J} = H_{e_1} + \sum_{s=1}^{p-1} \left(\omega^s H_{e_{2s}} + \omega^{-s} H_{e_{2s+1}}\right) + \omega^p H_{e_{2p}}, \quad \text{for } A_{2p-1}, B_{2p}, C_{2p}, D_{2p+1}. \end{aligned} \tag{25}$$

Choosing $\mathcal{J}$ as in (25) we have

$$\tilde{C}(\mathcal{J}) = \omega \mathcal{J}, \qquad C_1(\mathcal{J}) = \mathcal{J}^*, \qquad C_2(\mathcal{J}) = \omega \mathcal{J}^* \tag{26}$$

where $\omega = \exp(2\pi \mathrm{i}/h)$. In addition

$$\begin{aligned} &C_1(\mathcal{E}_\alpha^{(0)}) = \mathcal{E}_{-\alpha}^{(0)}, && C_2(\mathcal{E}_\beta^{(0)}) = \mathcal{E}_{-\beta}^{(0)} \\ &C_2(\mathcal{E}_\alpha^{(0)}) = C_1(\mathcal{E}_\alpha^{(0)}), && C_1(\mathcal{E}_{-\beta}^{(0)}) = C_2(\mathcal{E}_{-\beta}^{(0)}) \\ &(\mathcal{E}_\alpha^{(s)})^\dagger = \mathcal{E}_{-\alpha}^{(s)}, && C_i(\mathcal{E}_\alpha^{(s)}) = \mathcal{E}_{w_i(\alpha)}^{(h-s)}, \qquad i = 1, 2. \end{aligned} \tag{27}$$

Below we will need the commutation relations between $\mathcal{J}$ and the $\mathcal{E}_{\beta}^{(s)}$. They can be derived as follows

$$
\begin{aligned}
[\mathcal{J}, \mathcal{E}_{\alpha}^{(p)}] &= \frac{1}{h}\sum_{s=0}^{h-1} \omega^{-sp}[\mathcal{J}, \tilde{C}^s(E_\alpha)] = \frac{1}{h}\sum_{s=0}^{h-1} \omega^{-sp}\tilde{C}^s\left([\tilde{C}^{-s}(\mathcal{J}), E_\alpha]\right) \\
&= \frac{1}{h}\sum_{s=0}^{h-1} \omega^{-sp-s}\tilde{C}^s\left([\mathcal{J}, E_\alpha]\right) = \frac{1}{h}\sum_{s=0}^{h-1} \omega^{-sp-s}\alpha(\mathcal{J})\tilde{C}^s\left(E_\alpha\right) \\
&= \alpha(\mathcal{J})\mathcal{E}_{\beta}^{(p+1)}
\end{aligned}
\tag{28}
$$

and similarly

$$
[\mathcal{J}, \mathcal{E}_{-\beta}^{(p)}] = -\beta(\mathcal{J})\mathcal{E}_{\beta}^{(p+1)}. \tag{29}
$$

The rest of the commutation relations between the basic elements in $\tilde{\mathfrak{g}}^{(k)}$ may be a bit complicated to derive though of course one can do it using the equations (3) and (24) and take into account the fundamental property of the coefficients $N_{\alpha,\beta}$, namely $N_{\alpha,\beta} = N_{\tilde{C}(\alpha),\tilde{C}(\beta)}$. However, one can approach the problem differently if we notice that $\tilde{C}$ has the same set of eigenvalues as C, which as we mentioned are equal to the powers of $\omega = \exp(2\pi \mathrm{i}/h)$. Hence $\tilde{C}$ and C are related by a similarity transformation

$$
\tilde{C} = u_0^{-1} C u_0 \tag{30}
$$

where u_0 is some constant element of the corresponding Lie group. Therefore the basis in $\tilde{\mathfrak{g}}^{(k)}$ (24) can be obtained from the basis in $\mathfrak{g}^{(k)}$ (7) via this transformation.

2.3. Mikhailov's Reduction Group

Mikhailov's reduction group is a finite group, which must have two realizations: i) as a subgroup of the group of automorphisms $\mathrm{Aut}\,\mathfrak{g}$ of the algebra $\mathfrak{g}$ and ii) as a subgroup of the conformal transformations Conf of the complex λ-plane. In what follows we shall use the Coxeter automorphism $C^h = 1\!\!1$ and the involutions $C_1^2 = 1\!\!1$, or $C_2^2 = 1\!\!1$ for realizations of the groups $\mathbb{Z}_2$, $\mathbb{Z}_h$, $\mathbb{D}_h$ acting on $\mathfrak{g}$. Note that for a given realization in $\mathfrak{g}$ one may have inequivalent realizations in Conf, that is why we use the indexes $1, 2$ to distinguish different cases. More specifically, the automorphisms C, C_1 and C_2 listed below lead to the following reductions for the matrix-valued functions

$$
\begin{aligned}
&1) \qquad C(U(\lambda)) = U(\omega\lambda), \qquad && C(V(\lambda)) = V(\omega\lambda) \\
&2) \qquad C_1(\tilde{U}^\dagger(\lambda^*)) = \tilde{U}(\lambda), \qquad && C_1(\tilde{V}^\dagger(\lambda^*)) = \tilde{V}(\lambda) \\
&3) \qquad C_2(\tilde{U}^\dagger((\lambda\omega)^*)) = \tilde{U}(\lambda). \qquad && C_2(\tilde{V}^\dagger((\lambda\omega)^*)) = \tilde{V}(\lambda).
\end{aligned}
\tag{31}
$$

The above restrictions naturally extend to restrictions on the FAS, the scattering matrix $T(\lambda,t)$, the spectral data of the Lax operator etc., see below.

3. Lax Pairs and NLEEs

Now we are in position to outline the Lax pairs with $\mathbb{Z}_h$- and $\mathbb{D}_h$- reductions and the relevant NLEEs that result in. First we shall use the realization of the Coxeter automorphism as an element of the Cartan subgroup. We have

$$\begin{aligned} L\chi &\equiv \mathrm{i}\frac{\partial\chi}{\partial x} + (q(x,t)-\lambda J)\chi(x,t,\lambda) = 0 \\ q(x) &= \sum_{j=1}^{r} q_j(x,t)H_j \in \mathfrak{h}, \qquad J = \sum_{\mathrm{ht}\,\alpha=1} E_\alpha . \end{aligned} \tag{32}$$

Note that the root height should be understood modulo h, so in J along with the generators corresponding to the simple roots we have to add also the generator corresponding to the minimal root whose height is $-h+1 = 1 \mod (h)$.

The best known NLEE's of the above type are the well known two-dimensional Toda field theories, discovered by Mikhailov [32]. They attracted a lot of attention, see [9, 30, 32, 34, 35] and the numerous references therein. Their Lax representation $[L, M_{\mathrm{Tft}}] = 0$ involves an L-operator as in equation (32) with $q(x,t) = 2\mathrm{i}\phi_x(x,t) \in \mathfrak{h}$ and M-operator of the form

$$\begin{aligned} M_{\mathrm{Tft}}\chi &\equiv \mathrm{i}\frac{\partial\chi}{\partial x} + \frac{\mathrm{i}}{\lambda}V_{-1}(x,t)\chi(x,t,\lambda) = 0 \\ V_{-1}(x,t) &= \sum_{k=0}^{r} \mathrm{e}^{2(\vec{\phi},\alpha_k)} E_{-\alpha_k} \in \mathfrak{g}^{(h-1)} \end{aligned} \tag{33}$$

where $\vec{\phi}(x,t)$ is the vector in the Euclidean space, that corresponds to the Cartan element $\phi(x,t)$ in $\mathfrak{h}$. The corresponding equations take the form

$$\frac{\partial^2\vec{\phi}}{\partial x\partial t} = \sum_{k=0}^{r} \alpha_k \mathrm{e}^{2(\vec{\phi},\alpha_k)}. \tag{34}$$

These equations have been studied in detail, so we will turn our attention to the other members in their hierarchies. For that reason below we will consider M-operators that are polynomial in λ of order N. The compatibility of the Lax pair requires in particular $[\mathcal{J},\mathcal{K}_N] = 0$, which is possible only if $N = m_k + ph$ where

Table 2. The orders of the NLEE's with $N = ph + m_k$ for the classical Lie algebras of rank up to 4 and $p = 1$ and $p = 2$.

N	A_2	B_2, C_2	G_2	
$h + m_k$	$4, 5$	$5, 7$	$7, 11$	
$2h + m_k$	$7, 8$	$9, 11$	$13, 17$	
	A_3	B_3, C_3		
$h + m_k$	$5, 6, 7$	$7, 9, 11$		
$2h + m_k$	$9, 10, 11$	$13, 15, 17$		
	A_4	B_4, C_4	D_4	F_4
$h + m_k$	$6, 7, 8, 9$	$9, 11, 13, 15$	$7, 9, 9, 11$	$13, 17, 19, 23$
$2h + m_k$	$11, 12, 13, 14$	$17, 19, 21, 23$	$13, 15, 15, 17$	$25, 29, 31, 35$

m_k is an exponent of the algebra $\mathfrak{g}$, see Table 1, and p is any integer. Therefore

$$\begin{aligned} M_{(k)}\chi &\equiv \mathrm{i}\frac{\partial \chi}{\partial t} + \left(\sum_{s=1}^{N} \lambda^{N-s} V_s(x,t) - \lambda^N K_N \right) \chi(x,t,\lambda) = 0 \\ V_s(x) &= \sum_{\mathrm{ht}\,\beta = N-s} v_{s;\beta}(x,t) E_\beta \in \mathfrak{g}^{(N-s)}, \qquad s > 1, \qquad K_N \in \mathfrak{g}^{(N)}. \end{aligned} \tag{35}$$

Remark 4. *As one can see, even for algebras of low rank and for small values of p the order N of the NLEE grows rather quickly, see* Table 2.

Now let us consider the Lax operators

$$\begin{aligned} \tilde{L}\tilde{\chi} &\equiv \mathrm{i}\frac{\partial \tilde{\chi}}{\partial x} + \tilde{U}(x,t,\lambda)\tilde{\chi}(x,t,\lambda) = 0 \\ \tilde{M}\tilde{\chi} &\equiv \mathrm{i}\frac{\partial \tilde{\chi}}{\partial t} + \tilde{V}(x,t,\lambda)\tilde{\chi}(x,t,\lambda) = 0 \end{aligned} \tag{36}$$

where

$$\begin{aligned} \tilde{U}(x,t,\lambda) &= Q(x,t) - \lambda \mathcal{J} \\ Q(x) &= \sum_{j=1}^{r} q_j(x,t) u_0^{-1} H_j\, u_0 = \sum_{\alpha \in A_1} q_\alpha \mathcal{E}_\alpha^{(0)} + \sum_{\beta \in A_2} q_\beta \mathcal{E}_{-\beta}^{(0)} \end{aligned} \tag{37}$$

and $C_1(Q^\dagger(x,t)) = Q(x,t)$, i.e.,

$$q_\alpha^*(x,t) = q_\alpha(x,t), \qquad q_\beta^*(x,t) = q_\beta(x,t). \tag{38}$$

The potential of the operator $\tilde{M}$ takes the form

$$\begin{aligned}
\tilde{V}(x,t,\lambda) &= \sum_{s=1}^{N} \lambda^{N-s}\tilde{V}_s(x,t,\lambda) - \lambda^N \mathcal{K}_N \\
\tilde{V}_s(x,t) &= \sum_{\alpha\in A_1} v_{s,\alpha}(x,t)\mathcal{E}_{\alpha}^{(N-s)} + \sum_{\beta\in A_2} v_{s,\beta}(x,t)\mathcal{E}_{-\beta}^{(N-s)} + \tilde{V}_s^{\|}(x,t) \\
V_N(x) &= \sum_{j=1}^{r} v_{N,j}(x,t)H_j \in \mathfrak{h} \\
\mathcal{J} &= u_0^{-1}Ju_0 \in \mathfrak{h}, \quad \mathcal{K}_N = u_0^{-1}K_N u_0 \in \mathfrak{h}, \quad s=1,\dots,N-1.
\end{aligned} \tag{39}$$

The above form of the $\tilde{L}$-$\tilde{M}$ pair is obtained from L (32) and M (35) by taking a similarity transformation with u_0, see equation (30), that diagonalizes simultaneously J and K_N. In equation (39) $\tilde{V}_s(x,t)^{\|} \in \mathfrak{h}$ and is non-vanishing only if s is an exponent (of course $\mod (h)$) of the corresponding algebra.

Now we can formulate a $\mathbb{D}_h$-reduction group. It is generated by the the involutions:

$$\begin{aligned}
&C_1(\tilde{U}^\dagger(x,t,\lambda^*)) = \tilde{U}(x,t,\lambda), \qquad C_2(\tilde{U}^\dagger(x,t,(\lambda\omega)^*)) = \tilde{U}(x,t,\lambda) \\
&C_1(\tilde{V}^\dagger(x,t,\lambda^*)) = \tilde{V}(x,t,\lambda), \qquad C_2(\tilde{V}^\dagger(x,t,(\lambda\omega)^*)) = \tilde{V}(x,t,\lambda).
\end{aligned} \tag{40}$$

Imposing both reductions (40) we obtain that $\tilde{L}$ is a subject also of the $\mathbb{Z}_h$-reduction

$$\tilde{C}(\tilde{U}(x,t,\lambda)) = \tilde{U}(x,t,\omega\lambda), \qquad \tilde{C}(\tilde{V}(x,t,\lambda)) = \tilde{V}(x,t,\omega\lambda). \tag{41}$$

Thus this realization of the reductions effectively gives rise to the additional conditions on q_α and q_β (38) so our reduction group is $\mathbb{D}_h$. In order to check that V_s also satisfy the reduction conditions one needs to use equation (38) above.

3.1. The Spectral Properties of the Lax Operator

In our paper [22] we extended the construction of Caudrey-Beals-Coifman to any simple Lie algebra and constructed the FAS of generalized Zakharov-Shabat systems whose U_1 has complex eigenvalues. Imposing the $\mathbb{Z}_h$-reduction we obtain the Lax operator (36). Particular cases of such operators have been considered in [23, 38].

The Jost solutions $\tilde{\psi}(x,t,\lambda)$ and $\tilde{\phi}(x,t,\lambda)$ and the scattering matrix $T(\lambda,t)$ of $\tilde{L}$ (36) exist for large class of potentials, in particular for potentials on compact support and are determined uniquely by the following conditions

$$\begin{aligned}
&\lim_{x\to\infty} \tilde{\psi}(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\mathcal{J}x} = \mathbb{1}, \qquad \lim_{x\to-\infty} \tilde{\phi}(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\mathcal{J}x} = \mathbb{1} \\
&T(\lambda,t) = \tilde{\psi}^{-1}(x,t,\lambda)\tilde{\phi}(x,t,\lambda).
\end{aligned} \tag{42}$$

When the potential is on compact support both Jost solutions, as well as the scattering matrix are rational functions of λ.

It can be shown that the continuous spectrum of $\tilde{L}$ lies on those lines in $\mathbb{C}$ on which one or more of the entries of $\mathrm{e}^{-i\lambda\mathcal{J}x}$ oscillate. In other words the continuous spectrum of $\tilde{L}$ is determined by the condition λ belongs to the continuous spectrum $\mathcal{S}_{\mathrm{cont}}$ if and only if $\operatorname{Im}\lambda\alpha(\mathcal{J}) = 0$ for some root α of the algebra $\mathfrak{g}$. Using the explicit form of $\mathcal{J}$ from equation (25) one derives that the continuous spectrum of L fills up the set of rays

$$\mathcal{S}_{\mathrm{cont}} \equiv \bigcup_{\nu=1}^{2h} l_\nu, \qquad l_\nu \equiv \{\lambda;\ \arg\lambda = -\frac{\pi}{2} + (\nu-1)\frac{\pi}{h}, \quad \nu = 1,\dots,2h\}. \tag{43}$$

These rays split $\mathbb{C}$ into $2h$ sectors Ω_ν situated between the rays l_ν and $l_{\nu+1}$. In addition, for each fixed value of ν one finds that the set of roots

$$\Delta_\nu \equiv \{\alpha;\ \operatorname{Im}\lambda\alpha(\mathcal{J}) = 0 \ \text{ for } \ \lambda \in l_\nu\} \tag{44}$$

forms a root system for a subalgebra $\mathfrak{g}_\nu \subset \mathfrak{g}$.

With the above choice of $\mathcal{J}$ one can check that the subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ associated with the rays l_1 and l_2 respectively, have as root systems Δ_1 and Δ_2 where

$$\Delta_1 \equiv \{\pm\alpha;\ \alpha \in A_1\}, \quad \Delta_2 \equiv \{\pm\beta;\ \beta \in A_2\}. \tag{45}$$

Since any pair of roots $\alpha_i, \alpha_j \in A_0$ and $\beta_i, \beta_j \in A_1$ are orthogonal, then each of the subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are direct sums of $\mathfrak{sl}(2)$ subalgebras. The $\mathbb{Z}_h$ reduction condition allows one to check that the subalgebras related to the other rays are obtained from $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by acting with the Coxeter automorphism

$$\Delta_{2\nu+1} \equiv \{\pm\tilde{C}^{\nu-1}\alpha;\ \alpha \in A_1\}, \qquad \Delta_{2\nu} \equiv \{\pm\tilde{C}^{\nu-1}\beta;\ \beta \in A_2\}. \tag{46}$$

Remark 5. *The scattering matrix $T(\lambda,t)$ of the scattering problem L for $\lambda \in l_\nu$ takes values in the subgroup $\mathcal{G}_\nu$ whose Lie algebra is the subalgebra $\mathfrak{g}_\nu$.*

The next step is to construct FAS $\chi_\nu(x,t,\lambda)$ which retain their analyticity properties inside the sector Ω_ν, closed between the rays l_ν and $l_{\nu+1}$, in the case when the potential is not on compact support. Skipping the details (see [22]) we note that these FAS are related to the Jost solutions by

$$\begin{aligned}
\tilde{\chi}_\nu(x,t,\lambda) &= \tilde{\phi}(x,t,\lambda)S_\nu^+(\lambda,t) \\
&= \tilde{\psi}(x,t,\lambda)T_\nu^-(\lambda,t)D_\nu^+(\lambda), \qquad \lambda \in l_\nu \mathrm{e}^{+\mathrm{i}0} \\
\tilde{\chi}_\nu(x,t,\lambda) &= \tilde{\phi}(x,t,\lambda)S_{\nu+1}^-(\lambda,t) \\
&= \tilde{\psi}(x,t,\lambda)T_{\nu+1}^+(\lambda,t)D_{\nu+1}^-(\lambda), \quad \lambda \in l_{\nu+1}\mathrm{e}^{-\mathrm{i}0}
\end{aligned} \tag{47}$$

where the factors $S_\nu^\pm(\lambda,t)$, $T_\nu^\pm(\lambda,t)$ and $D_\nu^\pm(\lambda)$ are related to $T(\lambda,t)$ by its Gauss decomposition

$$T(\lambda,t) = T_\nu^+(\lambda,t) D_\nu^+(\lambda) \hat{S}_\nu^+(\lambda,t) = T_\nu^+(\lambda,t) D_\nu^-(\lambda) \hat{S}_\nu^-(\lambda,t), \qquad \lambda \in l_\nu. \tag{48}$$

More specifically, using for each of the $\mathfrak{sl}(2)$-subalgebras the Cartan-Weyl basis $E_\alpha, E_{-\alpha}, H_\alpha$ or $E_\beta, E_{-\beta}, H_\beta$ respectively we have

$$S_\nu^\pm(\lambda,t) = \exp \boldsymbol{s}_\nu^\pm(\lambda,t), \quad T_\nu^\pm(\lambda,t) = \exp \boldsymbol{\tau}_\nu^\pm(\lambda,t), \quad D_\nu^\pm(\lambda,t) = \exp \boldsymbol{d}_\nu^\pm(\lambda)$$

where

$$\begin{aligned}
\boldsymbol{s}_{2\nu-1}^\pm(\lambda,t) &= \sum_{\alpha\in\Delta_{2\nu-1}^+} \sigma_{2\nu-1;\alpha}^\pm(\lambda,t) E_{\pm\alpha}, & \boldsymbol{s}_{2\nu}^\pm(\lambda,t) &= \sum_{\beta\in\Delta_{2\nu}^+} \sigma_{2\nu;\beta}^\pm(\lambda,t) E_{\pm\beta} \\
\boldsymbol{\tau}_{2\nu-1}^\pm(\lambda,t) &= \sum_{\alpha\in\Delta_{2\nu-1}^+} \tau_{2\nu-1;\alpha}^\pm(\lambda,t) E_{\pm\alpha}, & \boldsymbol{\tau}_{2\nu}^\pm(\lambda,t) &= \sum_{\beta\in\Delta_{2\nu}^+} \tau_{2\nu;\beta}^\pm(\lambda,t) E_{\pm\beta} \\
\boldsymbol{d}_{2\nu-1}^+(\lambda) &= \sum_{\alpha\in\Delta_{2\nu-1}^+} d_{2\nu-1;\alpha}^\pm H_\alpha, & \boldsymbol{d}_{2\nu}^+(\lambda) &= \sum_{\beta\in\Delta_{2\nu}^+} d_{2\nu;\beta}^\pm H_\beta.
\end{aligned} \tag{49}$$

While the triangular Gauss factors $S_\nu^\pm(\lambda,t)$ and $T_\nu^\pm(\lambda,t)$ exist only for $\lambda \in l_\nu$, the diagonal Gauss factors $D_\nu^+(\lambda)$ and $D_{\nu+1}^-(\lambda)$ allow analytic extension inside the whole sector Ω_ν.

As mentioned above, the reduction conditions (31) on the Lax pair impose constraints on the scattering data as follows

i) the $\mathbb{Z}_h$ reduction

$$\begin{aligned}
S_{2\nu+1}^\pm(\lambda,t) &= C^{\nu-1} S_1^\pm(\omega^{\nu-1}\lambda,t), & T_{2\nu+1}^\pm(\lambda,t) &= C^{\nu-1} T_1^\pm(\omega^{\nu-1}\lambda,t) \\
S_{2\nu}^\pm(\lambda,t) &= C^{\nu-1} S_2^\pm(\omega^{\nu-1}\lambda,t), & T_{2\nu}^\pm(\lambda,t) &= C^{\nu-1} T_2^\pm(\omega^{\nu-1}\lambda,t) \\
D_{2\nu+1}^\pm(\lambda) &= C^{\nu-1} D_1^\pm(\omega^{\nu-1}\lambda), & D_{2\nu}^\pm(\lambda) &= C^{\nu-1} D_2^\pm(\omega^{\nu-1}\lambda).
\end{aligned} \tag{50}$$

ii) the first $\mathbb{Z}_2$-reduction acts on the complex λ-plane by $\lambda \to \lambda^*$. This means that it acts on the sectors as $\Omega_\nu \to \Omega_{h-\nu+1}$ and on the rays as $l_\nu \to l_{h+2-\nu}$. On the Gauss factors of $T(\lambda,t)$ it acts in the following way

$$\begin{aligned}
&C_1(S_\nu^{+,\dagger}(\lambda,t)) = \hat{S}_{h-\nu+2}^-(\lambda^*,t), \qquad C_1(D_\nu^{+,\dagger}(\lambda)) = \hat{D}_{h-\nu+2}^-(\lambda^*) \\
&C_1(T_\nu^{-,\dagger}(\lambda,t)) = \hat{T}_{h-\nu+2}^+(\lambda^*,t).
\end{aligned} \tag{51}$$

Consequently the coefficients $\tau_{2\nu-1,\alpha}^\pm(\lambda,t)$, $\tau_{2\nu,\beta}^\pm(\lambda,t)$, $\sigma_{2\nu-1,\alpha}^\pm(\lambda,t)$ and $\sigma_{2\nu,\beta}^\pm(\lambda,t)$ are related as follows

$$\begin{aligned}
\sigma_{2\nu-1,\alpha}^-(\lambda,t) &= -\sigma_{h-2\nu+1,C_1(\alpha)}^{+,*}(\lambda^*,t), \qquad \lambda \in l_{2\nu-1}, \quad \alpha \in \Delta_{2\nu-1}^+ \\
\tau_{2\nu,\beta}^+(\lambda,t) &= -\tau_{h-2\nu+2,C_1(\beta)}^{-,*}(\lambda^*,t), \qquad \lambda \in l_{2\nu}, \qquad \beta \in \Delta_{2\nu}^+.
\end{aligned} \tag{52}$$

iii) the second $\mathbb{Z}_2$-reduction acts on λ as $\lambda \to \lambda^* \omega^{-1}$. This means that it acts on the sectors $\Omega_\nu \to \Omega_{h-\nu-1}$ and on the rays as $l_\nu \to l_{h-\nu}$. The action on the Gauss factors is then given by

$$\begin{aligned} &C_2(S_\nu^{+,\dagger}(\lambda,t)) = \hat{S}_{h-\nu}^-(\omega^{-1}\lambda^*,t), \quad C_2(D_\nu^{+,\dagger}(\lambda)) = \hat{D}_{h-\nu}^-(\omega^{-1}\lambda^*) \\ &C_2(T_\nu^{-,\dagger}(\lambda,t)) = \hat{T}_{h-\nu}^+(\omega^{-1}\lambda^*,t) \end{aligned} \tag{53}$$

and the coefficients $\tau^\pm_{2\nu-1,\alpha}(\lambda,t)$, $\tau^\pm_{2\nu,\beta}(\lambda,t)$, $\sigma^\pm_{2\nu-1,\alpha}(\lambda,t)$ and $\sigma^\pm_{2\nu,\beta}(\lambda,t)$ are related by

$$\begin{aligned} \sigma^-_{2\nu-1,\alpha}(\lambda,t) &= -\sigma^{+,*}_{h-2\nu-3,C_2(\alpha)}(\omega^{-1}\lambda^*,t), \qquad \lambda \in l_{2\nu-1}, \quad \alpha \in \Delta^+_{2\nu-1} \\ \tau^+_{2\nu,\beta}(\lambda,t) &= -\tau^{-,*}_{h-2\nu-1,C_2(\beta)}(\omega^{-1}\lambda^*,t), \qquad \lambda \in l_{2\nu}, \quad \beta \in \Delta^+_{2\nu}. \end{aligned} \tag{54}$$

3.2. The Time Evolution of the Scattering Data

The Lax representation with L and M as in (32), (35) allows one to solve a system of NLEEs for $q_j(x,t)$. We will give examples of such systems in the next sections. Here we just note that the Lax representation determines the t-dependence of the scattering matrix (and its Gauss factors) as follows

$$\begin{aligned} &\mathrm{i}\frac{\partial T_\nu}{\partial t} - \lambda^N[\mathcal{K}_N, T_\nu(\lambda,t)] = 0, \qquad \mathrm{i}\frac{\partial S_\nu^\pm}{\partial t} - \lambda^N[\mathcal{K}_N, S_\nu^\pm(\lambda,t)] = 0 \\ &\mathrm{i}\frac{\partial T_\nu^\pm}{\partial t} - \lambda^N[\mathcal{K}_N, T_\nu^\pm(\lambda,t)] = 0, \qquad \mathrm{i}\frac{\partial D_\nu^\pm}{\partial t} = 0. \end{aligned} \tag{55}$$

Since these equations can be immediately solved, thus one finds the evolution in time. In particular, the last equations in (55) show that the functions $D_\nu^\pm(\lambda)$ are time-independent, i.e., they can be viewed as generating functionals of the integrals of motion of the corresponding NLEEs.

4. The Inverse Scattering Problem and the Riemann-Hilbert Problem.

Of course, finding the time evolution of the Gauss factors is only a step towards finding the solutions for the corresponding NLEE. One should be able to construct from the Gauss factors the solutions, a process called Inverse Scattering Transform (IST). We shall outline here how one can do it reducing the IST for the GZS system to a local Riemann-Hilbert problem (RHP). Indeed, on the ray l_ν we have

$$\begin{aligned} \xi_\nu(x,t,\lambda) &= \xi_{\nu-1}(x,t,\lambda) G_\nu(x,t,\lambda), \qquad \lambda \in l_\nu \\ G_\nu(x,t,\lambda) &= \mathrm{e}^{-\mathrm{i}(\lambda \mathcal{J} x + \lambda^N \mathcal{K}_N t)} G_{0,\nu}(\lambda) \mathrm{e}^{\mathrm{i}(\lambda \mathcal{J} x + \lambda^N \mathcal{K}_N t)} \end{aligned} \tag{56}$$

where $G_{0,\nu}(\lambda) = \hat{S}_\nu^- S_\nu^+(\lambda,t)\Big|_{t=0}$ and $\xi_\nu, \xi_{\nu-1}$ are functions analytic in the sectors $\Omega_\nu, \Omega_{\nu-1}$. The collection of all relations (56) for $\nu = 1, 2, \dots, 2N$ together with the condition

$$\lim_{\lambda\to\infty} \xi_\nu(x,t,\lambda) = \mathbb{1} \tag{57}$$

can be viewed as a local RHP with canonical normalization posed on the collection of rays $\Sigma \equiv \{l_\nu\}_{\nu=1}^{2N}$. The canonical normalization implies that each solution of the RHP possesses asymptotic expansion of the form

$$\xi_\nu(x,t,\lambda) = \exp\left(\sum_{s=1}^{\infty} \lambda^{-s} Q_s(x,t)\right) \tag{58}$$

where $Q_s(x,t)$ are elements of the Lie algebra $\mathfrak{g}$. The $\mathbb{Z}_h$-reduction means that $Q_s(x,t) \in \mathfrak{g}^{(k)}$. Quite straightforwardly one can prove that if $\xi_\nu(x,\lambda)$ is a solution of the RHP (56), (57) then $\chi_\nu(x,\lambda) = \xi_\nu(x,\lambda)\mathrm{e}^{-\mathrm{i}\lambda\mathcal{J}x}$ is a FAS of $\tilde{L}$ with potential

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda\left(\mathcal{J} - \xi_\nu(x,t,\lambda)\mathcal{J}\hat{\xi}_\nu(x,t,\lambda)\right) = [\mathcal{J}, Q_1(x,t)]. \tag{59}$$

In what follows we will consider two classes of solutions to the RHP: i) the class of regular solutions which have no singularities in their sectors of analyticity, ii) the class of singular solutions, which allow both poles and zeros[1] in their regions of analyticity.

Each regular solution of the RHP (56), (57) is determined uniquely by the sewing functions $G_\nu(x,t,\lambda)$, which due to (50) also satisfy the $\mathbb{Z}_h$ reduction condition. Therefore it is enough to know the sewing functions on the rays l_1 and l_2 in order to calculate the whole set of sewing functions $G_\nu(x,t,\lambda)$. Thus the minimal set of scattering data that determines the regular solution $\xi^\nu(x,t,\lambda)$ are given by

$$\mathcal{T} \equiv \{\sigma^{\pm}_{1;\alpha}(\lambda,t);\ \alpha\in A_1,\ \lambda\in l_1\} \cup \{\sigma^{\pm}_{2;\beta}(\lambda,t);\ \beta\in A_2,\ \lambda\in l_2\}. \tag{60}$$

In other words the set $\mathcal{T}$ contains r functions of λ and t, each defined on a certain ray and from $\mathcal{T}$ one must recover r scalar functions defined on the real axis – the potential $Q(x,t)$, see equation (38). The singular solutions to the RHP and the related soliton solutions of the corresponding NLEE can be derived using the the dressing Zakharov-Shabat method [32,45]. Starting from the trivial solution of the RHP we obtain explicit rational solutions of the RHP.

5. The Recursion Operators and the NLEEs

One of the important steps in the theory is the explicit derivation of the relevant NLEEs. Below we assume that $N \mod (h)$ is an exponent of $\mathfrak{g}$. We will follow [1] and construct recurrent relations for calculating the coefficients $\tilde{V}_s(x,t)$ in

[1] by a zero of $\xi^\nu(x,t,\lambda)$ at $\lambda_{\nu,k}\in\Omega_\nu$ here we mean that $\det\xi^\nu(x,t,\lambda_{\nu,k}) = 0$.

the $\tilde{M}$-operator (35) in terms of $Q(x,t)$ and its derivatives. Doing this it is natural to use the condition that $\tilde{L}$ and $\tilde{M}$ commute *identically* with respect to λ. Equating to zero all the coefficients of the positive powers in λ in $[\tilde{L},\tilde{M}]$ we obtain a set of recurrent relations for the functions $\tilde{V}_s(x,t)$. Note, that due to the $\mathbb{Z}_h$ reduction, the potentials of both $\tilde{L}$ and $\tilde{M}$ take values in the graded algebra, which requires generalization of the AKNS method and a substantially new structure of the recursion operators. Thus we have

$$\begin{aligned}
\lambda^{N+1}: &\qquad [\mathcal{J},\mathcal{K}_N]=0\\
\lambda^{N}: &\qquad [\mathcal{J},\tilde{V}_1(x,t)]+[Q(x,t),\mathcal{K}_N]=0\\
\lambda^{N-s}: &\qquad \mathrm{i}\frac{\partial\tilde{V}_s}{\partial x}+[Q(x,t),\tilde{V}_s(x,t)]-[\mathcal{J},\tilde{V}_{s+1}(x,t)]=0\\
\lambda^{0}: &\qquad -\mathrm{i}\frac{\partial Q}{\partial t}+\mathrm{i}\frac{\partial\tilde{V}_N}{\partial x}+[Q(x,t),\tilde{V}_N(x,t)]=0.
\end{aligned}\tag{61}$$

The first of the above equations is satisfied identically. The second can be resolved as

$$\tilde{V}_1(x,t)=\mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_N,Q(x,t)\tag{62}$$

where $\mathrm{ad}_{\mathcal{J}}\tilde{X}\equiv[\mathcal{J},X]$. The operator $\mathrm{ad}_{\mathcal{J}}$ obviously has a kernel, and its inverse $\mathrm{ad}_{\mathcal{J}}^{-1}$ is defined only on its image. Thus there naturally appear the necessity to split each of the coefficients $\tilde{V}_s(x,t)$ into 'orthogonal' and 'parallel' parts

$$\tilde{V}_s(x,t)=\tilde{V}_s^{\perp}(x,t)+\tilde{V}_s^{\|}(x,t),\quad [\tilde{V}_s^{\|}(x,t),\mathcal{J}]=0,\quad \langle\tilde{V}_s^{\perp}(x,t),\mathcal{H}\rangle=0.\tag{63}$$

where $\mathcal{H}$ is any element of the Cartan subalgebra. Since $\tilde{V}_s(x,t)\in\mathfrak{g}^{(N-s)}$ in fact we need to split each of the subspaces $\mathfrak{g}^{(N-s)}$ into

$$\begin{aligned}
&\mathfrak{g}^{(N-s)}=\mathfrak{g}^{(N-s)\perp}\oplus\mathfrak{g}^{(N-s)\|}\\
&V_s=V_s^{\perp}+V_s^{\|},\quad V_s^{\|}=\begin{cases}0 & \text{if } s \text{ is not an exponent}\\ c_s^{-1}\mathcal{J}^s\langle V_s,\mathcal{J}^{h-N+s}\rangle & \text{if } s \text{ is an exponent}\end{cases}
\end{aligned}\tag{64}$$

where $c_s=\langle\mathcal{J}^{h-N+s}s,\mathcal{J}^{N-s}\rangle$.

The above formulae hold true for the classical series of algebras A_r, B_r and C_r while the D_r series requires more care and will be discussed elsewhere.

Note that we can always fix up the gauge so that $Q(x,t)\equiv Q^{\perp}(x,t)$. Then we have

$$\tilde{V}_1(x,t)=\sum_{\alpha\in A_1}\frac{\alpha(\mathcal{K}_N)}{\alpha(\mathcal{J})}\,q_\alpha(x,t)\mathcal{E}_\alpha^{(N-1)}+\sum_{\beta\in A_2}\frac{\alpha(\mathcal{K}_N)}{\beta(\mathcal{J})}\,q_\beta(x,t)\mathcal{E}_{-\beta}^{(N-1)}.\tag{65}$$

In doing this we used the commutation relations (28).

Equation (65) provides the initial condition for the recurrent relations. They are determined from the third line of (61) where we must insert the splitting of $V_s(x,t)$ and $V_{s+1}(x,t)$ according to (64).

5.1. The Case of A_r

Let us consider first the A_r series, for which all numbers $1, 2, \dots, r$ are exponents. Then we obtain the following two equations

$$\begin{aligned} \tilde{V}_{s+1}^{\perp}(x,t) &= \mathrm{ad}_{\mathcal{J}}^{-1}\left(\mathrm{i}\frac{\partial \tilde{V}_s^{\perp}}{\partial x} + [Q(x,t), \tilde{V}_s^{\perp}] + [Q(x,t), \tilde{V}_s^{\|}]\right) \\ \mathrm{i}\left\langle \frac{\partial \tilde{V}_s^{\|}}{\partial x}, \mathcal{J}^s \right\rangle &= \langle [Q(x,t), \tilde{V}_s^{\perp}]\mathcal{J}^s\rangle. \end{aligned} \tag{66}$$

Integrating formally the second one we obtain

$$\begin{aligned} \tilde{V}_s^{\|}(x,t) &= \mathrm{i}(\partial_x)_{\pm}^{-1}\left([Q(x,t), \tilde{V}_s^{\perp}]\right)^{\|} \\ &= \mathrm{i}c_s^{-1}\mathcal{J}^{N-s}\partial_x^{-1}\langle [Q(x,t), \tilde{V}_s^{\perp}], \mathcal{J}^{h-N+s}\rangle + \tilde{v}_{s0} \end{aligned} \tag{67}$$

where $(\partial_x)_{\pm}^{-1}\cdot = \int_{\pm\infty}^{x} \mathrm{d}y \cdot$ and $\tilde{v}_{s0} = \mathrm{const}$. Then we can write down the formal solution of the recurrent relations in the form

$$\begin{aligned} &\tilde{V}_{s+1}^{\perp}(x,t) = \Lambda_{N-s}^{\pm}\tilde{V}_s^{\perp}(x,t) + \tilde{v}_{s0}[Q(x,t), \mathcal{J}^{N-s}] \\ &\Lambda_{N-s}^{\pm}X_s^{\perp} = \mathrm{ad}_{\mathcal{J}}^{-1}\left(\mathrm{i}\frac{\partial \tilde{X}_s^{\perp}}{\partial x} + [Q(x,t), \tilde{X}_s^{\perp}]\right. \\ &\qquad\left. + \mathrm{i}c_s^{-1}[Q(x,t), \mathcal{J}^{N-s}](\partial_x)_{\pm}^{-1}\langle [Q(y,t), X_s^{\perp}(y)]\mathcal{J}^{h-N+s}\rangle\right). \end{aligned} \tag{68}$$

Further, taking for simplicity $\tilde{v}_{s0} = 0$ the solution to these recursion relation is

$$\tilde{V}_s(x,t) = \Lambda_{N-s+1}^{\pm}\Lambda_{N-s+2}^{\pm}\cdots\Lambda_{N-1}^{\pm}\mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_N, Q(x,t)] \tag{69}$$

for $s = 2, \dots, N$. The corresponding NLEEs can be written in compact form as

$$\mathrm{i}\ \mathrm{ad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_N\Lambda_0\Lambda_1^{\pm}\Lambda_2^{\pm}\cdots\Lambda_{N-1}^{\pm}\mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_N, Q(x,t)] = 0 \tag{70}$$

where

$$\Lambda_0\tilde{X}_{2p+1}^{\perp} = \mathrm{ad}_{\mathcal{J}}^{-1}\left(\mathrm{i}\frac{\partial \tilde{X}_{2p+1}^{\perp}}{\partial x} + [Q(x,t), \tilde{X}_{2p+1}^{\perp}]\right). \tag{71}$$

The dispersion law of the NLEE (70) is $f_N\lambda^N$. In the simplest cases $N=2$ and $N=3$ we get

$$\begin{aligned} \tilde{V}_N(x,t) &= \Lambda_1^{\pm}\mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_2, Q(x,t)], \quad & \tilde{V}_N(x,t) &= \Lambda_1^{\pm}\Lambda_2^{\pm}\mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_3, Q(x,t)] \\ \mathcal{K}_2 &= f_2\mathcal{J}^2, & \mathcal{K}_3 &= f_3\mathcal{J}^3. \end{aligned} \tag{72}$$

The recursion operators $\Lambda_{N-s}^{\pm}$ can be obtained as the restriction of the operators $\Lambda_{\pm}$ which are the recursion operators for potential without any restrictions

$$\begin{aligned} \Lambda^{\pm}\tilde{X}^{\perp} = \mathrm{ad}_{\mathcal{J}}^{-1}\Bigg(&\mathrm{i}\frac{\partial \tilde{X}^{\perp}}{\partial x} + [Q(x,t), \tilde{X}_s^{\perp}] \\ &+ \mathrm{i}\sum_{s=1}^{r} c_{N-s}^{-1}[Q(x,t), \mathcal{J}^s](\partial_x)_{\pm}^{-1}\langle [Q(y,t), \tilde{X}_s^{\perp}(y)], \mathcal{J}^{h-s}\rangle \Bigg) \end{aligned} \tag{73}$$

by restricting them onto the subspaces $\mathfrak{g}^{(N-s)}$. Indeed, as readily seen, $\Lambda_{\pm}$ maps $\mathfrak{g}^{(s)}$ into $\mathfrak{g}^{(s-1)}$. However, practically it is much better to have expressions where the grading can be observed explicitly.

We end this subsection by giving an example of integrable NLEE known as the $\mathbb{Z}_h$-reduced derivative NLS equation. The Lax operator $\tilde{L}$ is parametrized by

$$Q(x,t) = \sum_{j=1}^{N-1}\psi_j(x,t)J_j^{(0)}, \qquad \mathcal{J} = -a\omega^{-1/2}J_0^{(1)}. \tag{74}$$

The $\tilde{M}$-operator is quadratic in λ with

$$\begin{aligned} V_1(x,t) &= \sum_{k=1}^{N} v_{1,k}(x,t)J_j^{(1)}, \qquad & v_{1,p} &= -\frac{b}{a}\omega^{(p+1)/2}\cos\left(\frac{p\pi}{N}\right)\psi_p(x,t) \\ V_0(x,t) &= \sum_{k=1}^{N-1} v_{0,k}(x,t)J_j^{(0)}, & V_2 &= -bJ_0^{(2)} \end{aligned} \tag{75}$$

where

$$v_{0,p} = \gamma\left(\mathrm{i}\,\mathrm{cotan}\,\frac{p\pi}{N}\,\psi_{p,x} - \sum_{k+s=p}^{N}\psi_k\psi_s(x,t)\right), \qquad \gamma = \frac{b\omega}{a^2}\cdot \tag{76}$$

The λ-independent term in the Lax representation vanishes whenever the functions ψ_k satisfy the $\mathbb{Z}_h$-reduced derivative NLS equation [13, 17]

$$\frac{\partial q_k}{\partial t} + \gamma\,\mathrm{cotan}\left(\frac{\pi k}{N}\right)\frac{\partial^2 q_k}{\partial x^2} - \gamma\sum_{p=1}^{N-1}\frac{\partial}{\partial x}(q_p q_{k-p}) = 0, \qquad k = 1, 2, \ldots, r. \tag{77}$$

Its dispersion law is $\gamma\lambda^2$.

5.2. The Case of B_r and C_r

Consider now the series B_r and C_r. For them all the odd numbers $1, 3, \dots, 2r-1$ are exponents. Note also that now N must be odd: $N = 2k+1$. Again we obtain two equations for each s

$$\begin{aligned} \tilde{V}_{s+1}^{\perp}(x,t) &= \operatorname{ad}_{\mathcal{J}}^{-1}\left(\mathrm{i}\frac{\partial \tilde{V}_s^{\perp}}{\partial x} + [Q(x,t), \tilde{V}_s^{\perp}] + [Q(x,t), \tilde{V}_s^{\|}]\right) \\ \mathrm{i}\left\langle \frac{\partial \tilde{V}_s^{\|}}{\partial x}, \mathcal{J}^s \right\rangle &= \langle [Q(x,t), \tilde{V}_s^{\perp}]\mathcal{J}^s \rangle \end{aligned} \tag{78}$$

but this time they have to be considered separately for even $s = 2p$ and odd $s = 2p+1$ values of s. Indeed, $\tilde{V}_{2p} \in \mathfrak{g}^{(2k-2p+1)}$ and therefore $\tilde{V}_{2p}^{\|}$ are nontrivial, while $\tilde{V}_{2p+1} \in \mathfrak{g}^{(2k-2p)}$ and therefore $\tilde{V}_{2p+1}^{\|} = 0$. Applying similar technique as above we obtain

$$\begin{aligned} \tilde{V}_{2p+2}^{\perp}(x,t) &= \Lambda_0 \tilde{V}_{2p+1}^{\perp}(x,t) \\ \tilde{V}_{2p+1}^{\perp}(x,t) &= \Lambda_{N-2p}^{\pm} \tilde{V}_{2p}^{\perp}(x,t) + \tilde{v}_{2p,0}[Q(x,t), \tilde{\mathcal{J}}^{N-2p}] \end{aligned} \tag{79}$$

where

$$\Lambda_0 \tilde{X}_{2p+1}^{\perp} = \operatorname{ad}_{\mathcal{J}}^{-1}\left(\mathrm{i}\frac{\partial \tilde{X}_{2p+1}^{\perp}}{\partial x} + [Q(x,t), \tilde{X}_{2p+1}^{\perp}]\right) \tag{80}$$

$$\begin{aligned} \Lambda_{N-2p}^{\pm} \tilde{X}_{2p}^{\perp} = \operatorname{ad}_{\mathcal{J}}^{-1}\Bigg(&\mathrm{i}\frac{\partial \tilde{X}_{2p}^{\perp}}{\partial x} + [Q(x,t), \tilde{X}_{2p}^{\perp}] \\ &+ \frac{1\mathrm{i}}{c_{2s}}[Q(x,t), \mathcal{J}^{N-2p}](\partial_x)_{\pm}^{-1}\langle [Q(y,t), \tilde{X}_{2p}^{\perp}(y)], \mathcal{J}^{h-N+2p}\rangle\Bigg). \end{aligned} \tag{81}$$

The formal solutions for $\tilde{V}_s(x,t)$ in terms of the recursion operators, using again for simplicity $\tilde{v}_{s0} = 0$ is

$$\begin{aligned} \tilde{V}_{2p}(x,t) &= \Lambda_0 \Lambda_{N-2p+2}^{\pm} \Lambda_0 \Lambda_{N-2p+4}^{\pm} \cdots \Lambda_0 \Lambda_{N-2}^{\pm} \Lambda_0 \operatorname{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_N, Q(x,t)] \\ \tilde{V}_{2p+1}(x,t) &= \Lambda_{N-2p}^{\pm} \Lambda_0 \Lambda_{N-2p+2}^{\pm} \Lambda_0 \cdots \Lambda_{N-2}^{\pm} \Lambda_0 \operatorname{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_N, Q(x,t)] \end{aligned} \tag{82}$$

for $p = 2, \dots, k$, $N = 2k+1$. The corresponding NLEEs can be written implicitly as

$$\mathrm{i}\operatorname{ad}_{\mathcal{J}}^{-1} \frac{\partial Q}{\partial t} - f_N \Lambda_0 \tilde{V}_N(x,t) = 0 \tag{83}$$

or more explicitly

$$\mathrm{iad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_N \Lambda_0 \Lambda_1^{\pm} \Lambda_0 \Lambda_3^{\pm} \Lambda_0 \cdots \Lambda_{N-2}^{\pm} \Lambda_0 \mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_N, Q(x,t)] = 0. \tag{84}$$

In the simplest cases $N = 3$ and $N = 5$ we get

$$\begin{aligned} \tilde{V}_3(x,t) &= \Lambda_1^{\pm}\Lambda_0 \mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_3, Q(x,t)], & \mathcal{K}_3 &= f_3\mathcal{J}^3 \\ \tilde{V}_5(x,t) &= \Lambda_1^{\pm}\Lambda_0\Lambda_3^{\pm}\Lambda_0 \mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{K}_5, Q(x,t)], & \mathcal{K}_5 &= f_5\mathcal{J}^5. \end{aligned} \tag{85}$$

Then the corresponding NLEEs

$$\begin{aligned} &\mathrm{iad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_3\Lambda_0\Lambda_1^{\pm}\Lambda_0 \mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{J}^3, Q(x,t)] = 0 \\ &\mathrm{iad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_5\Lambda_0\Lambda_1^{\pm}\Lambda_0\Lambda_3^{\pm}\Lambda_0 \mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{J}^5, Q(x,t)] = 0 \end{aligned} \tag{86}$$

will be systems of differential equations of order 3 and 5 respectively for the r independent functions q_α and q_β.

We will call Λ_0 and Λ_{2k-1} elementary recursion operators. Along with them we will introduce r fundamental recursion operators

$$\begin{aligned} \mathbf{\Lambda}_{(1)} &= \Lambda_0, & \mathbf{\Lambda}_{(3)} &= \Lambda_0\Lambda_1\Lambda_0 \\ \mathbf{\Lambda}_{(m_k)} &= \Lambda_0\Lambda_1\Lambda_0\cdots\Lambda_{m_k-2}\Lambda_0, & k &= 1,\ldots r. \end{aligned} \tag{87}$$

Each of these recursion operators generates an MKdV-type of NLEE

$$\mathrm{iad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_{m_k}\mathbf{\Lambda}_{m_k}\mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{J}^{m_k}, Q(x,t)] = 0 \tag{88}$$

where $m_k = 2k-1$ is an exponent of $\mathfrak{g}$. The equation (88) is a system of r equations whose highest order derivative with respect to x equals m_k. Each of them is a simplest member of a hierarchy of NLEE generated by the master recursion operator

$$\mathbf{\Lambda}_{\pm} = \Lambda_0\Lambda_1^{\pm}\Lambda_0\Lambda_3^{\pm}\Lambda_0\cdots\Lambda_{h-1}^{\pm}\Lambda_0 \tag{89}$$

namely

$$\mathrm{iad}_{\mathcal{J}}^{-1}\frac{\partial Q}{\partial t} - f_{m_k+hp}\mathbf{\Lambda}^p\mathbf{\Lambda}_{m_k}\mathrm{ad}_{\mathcal{J}}^{-1}[\mathcal{J}^{m_k}, Q(x,t)] = 0, \qquad p = 1,2,\ldots. \tag{90}$$

The corresponding M-operators are polynomials in λ of degree $m_k + ph$.

Remark 6. *The equation* (88) *with* $m_k = 1$ *is in fact linear evolution equation. However its hierarchy* (90) *starting with* $p = 1$ *is nontrivial.*

Remark 7. *The algebra* $\mathfrak{so}(3)$ *is of rank* 1 *and its Coxeter number is* $h = 2$. *It has only one exponent equal to* 1. *Thus it has only one hierarchy. The* MKdV *equation is of the form* (90) *with* $p = 1$.

Again the recursion operators Λ_0 and $\Lambda^{\pm}_{N-2p}$ can be obtained from the operators $\Lambda_{\pm}$ for the general case

$$\begin{aligned} \Lambda^{\pm}\tilde{X}^{\perp} = \mathrm{ad}_{\mathcal{J}}^{-1} \Bigg(&\mathrm{i}\frac{\partial \tilde{X}^{\perp}}{\partial x} + [Q(x,t), \tilde{X}_s^{\perp}] \\ &+ \mathrm{i}\sum_{s=1}^{r} c_{N-s}^{-1} [Q(x,t), \mathcal{J}^s](\partial_x)_{\pm}^{-1} \langle [Q(y,t), \tilde{X}_s^{\perp}(y)], \mathcal{J}^{h-s}\rangle \Bigg) \end{aligned} \tag{91}$$

by restricting them onto the subspace $\mathfrak{g}^{(N-2p)}$. Note that $\Lambda_{\pm}$ maps $\mathfrak{g}^{(s)}$ into $\mathfrak{g}^{(s-1)}$. The simplest NLEE we obtain in the above way would be of MKdV-type, i.e., this would be systems of r equations which contains third derivative with respect to x. We end this Section recalling briefly the equivalence of the inverse scattering problem for $\tilde{L}$ to the RHP (56), (57). As we mentioned above the solution of the RHP allows the asymptotic expansion (58) which can be used to prove the relation

$$\begin{aligned} W_{\nu,N} &= \xi_\nu \mathcal{J}^N \hat{\xi}_\nu(x,t,\lambda) = \mathcal{J}^N + \sum_{s=1}^{\infty} \frac{1}{s!} \mathrm{ad}_{\mathcal{Q}}^{s} \mathcal{J} \\ &= \mathcal{J}^N + \lambda^{-1} \mathrm{ad}_{Q_1} \mathcal{J}^N + \lambda^{-2} \left(\mathrm{ad}_{Q_2} \mathcal{J}^N + \mathrm{ad}_{Q_1}^2 \mathcal{J}^N \right) + \cdots \end{aligned} \tag{92}$$

where $\mathcal{Q}(x,t,\lambda) = \sum_{s=1}^{\infty} \lambda^{-s} Q_s(x,t)$. It is easy to check that i) for $N = m_k + s_0 h$, where m_k is an exponent of $\mathfrak{g}$, we have that $W_{\nu,N}(x,t,\lambda) \in \mathfrak{g}^{(m_k)}$ is analytic function of λ in the sector Ω_ν; ii) the right hand sides of the expansions (92) in fact do not depend on ν so we will skip the index ν in what follows; iii) we can split $Y_N = \lambda^N W_N(x,t,\lambda)$ into

$$\begin{aligned} \lambda^N W_N(x,t,\lambda) &= (Y_N(x,t,\lambda))_+ + (Y_N(x,t,\lambda))_- \\ (Y_N(x,t,\lambda))_+ &= \lambda^N \mathcal{J} - \sum_{s=1}^{N} \lambda^{N-s} \tilde{V}_s(x,t) \\ (Y_N(x,t,\lambda))_- &= - \sum_{s=N+1}^{\infty} \lambda^{N-s} \tilde{V}_s(x,t). \end{aligned} \tag{93}$$

The importance of the splitting (93) is demonstrated by the next Lemma.

Lemma 8. *The generating functional of the* M*-operators of the* $\mathbb{Z}_h$ *and* $\mathbb{D}_h$*-reduced* NLEEs *are provided by* $Y_N(x,t,\lambda)$ *where* N *can take the values* $N = m_k + s_0 h$

and m_k is an exponent of the algebra $\mathfrak{g}$. The corresponding M-operator takes the form

$$M_N \equiv \mathrm{i}\frac{\partial}{\partial t} - (Y_N(x,t,\lambda))_+ \tag{94}$$

and the corresponding NLEEs *can be written in the form*

$$\mathrm{i}\frac{\partial Q}{\partial t} + [\mathcal{J}, V_{N+1}] = 0. \tag{95}$$

Proof: It is easy to check, using the relation between ξ_ν and χ_ν

$$\xi_\nu(x,t,\lambda) = \chi(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\mathcal{J}x} \tag{96}$$

that $Y_N(x,t,\lambda)$ is a solution to the equation

$$\mathrm{i}\frac{\partial Y_N}{\partial x} + [Q - \lambda\mathcal{J}, Y_N(x,t,\lambda)] = 0. \tag{97}$$

Therefore the compatibility condition $[\tilde{L}, \tilde{M}] = 0$ with $\tilde{M}$ chosen as in (94) gives

$$\begin{aligned} &\mathrm{i}\frac{\partial (Y_N)_+}{\partial t} - \mathrm{i}\frac{\partial Q}{\partial t} + [Q - \lambda\mathcal{J}, (Y_N)_+] \\ &\qquad = -\mathrm{i}\frac{\partial (Y_N)_-}{\partial t} - \mathrm{i}\frac{\partial Q}{\partial t} - [Q - \lambda\mathcal{J}, (Y_N)_+] \\ &\qquad = -\mathrm{i}\frac{\partial Q}{\partial t} + [\mathcal{J}, \tilde{V}_N] + \mathcal{O}(\lambda^{-1}). \end{aligned} \tag{98}$$

But by definition L and M are polynomial in λ. The lemma is proved. ■

6. The Wronskian Relations and the Effects of Reduction

6.1. The Mapping $\mathcal{F}$

We start with the Wronskian relations

$$\left(\hat{\tilde{\chi}}_\nu \mathcal{K} \tilde{\chi}_\nu(x,t,\lambda) - \mathcal{K}\right)\Big|_{x=-\infty}^{\infty} = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}y\, \left(\hat{\tilde{\chi}}_\nu [\mathcal{K}, Q(y,t)] \tilde{\chi}_\nu(y,t,\lambda)\right). \tag{99}$$

Let us outline the technicalities in deriving the basic relations describing the mapping $\mathcal{F}$. In doing this it will be enough to consider these relations for the two rays l_1 and l_2. The left hand sides of (99) take the form

$$\begin{aligned} &\mathrm{i}(\hat{D}_1^+ \hat{T}_1^- \mathcal{K}(T_1^- D_1^+) - \hat{S}_1^+ \mathcal{K} S_1^+), \qquad \text{for } \lambda \in l_1 \\ &\mathrm{i}(\hat{D}_2^- \hat{T}_2^+ \mathcal{K}(T_2^+ D_2^-) - \hat{S}_2^- \mathcal{K} S_2^-), \qquad \text{for } \lambda \in l_2. \end{aligned} \tag{100}$$

Next we will multiply both sides of (99) by E_γ and apply the Killing form. After this operation the right hand side of (99) acquires the form

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{d}y \ \left\langle \hat{\tilde{\chi}}_\nu [\mathcal{K}, Q(y,t)] \tilde{\chi}_\nu(x,t,\lambda), E_\gamma \right\rangle \\ &\qquad = \int_{-\infty}^{\infty} \mathrm{d}y \ \langle [\mathcal{K}, Q(y,t)], e_{\nu,\gamma}(x,t,\lambda)\rangle = - [\![[\mathcal{K}, \mathrm{ad}_{\mathcal{J}} Q(y,t)], e_{\nu,\gamma}(y,t,\lambda)]\!]\end{aligned} \tag{101}$$

where we have introduced the 'squared solutions' $e_{\nu,\gamma}(x,t,\lambda)$ and the skew-scalar product $[\![X,Y]\!]$ as follows

$$\begin{aligned} e_{\nu,\gamma}(x,t,\lambda) &= (\tilde{\chi}_\nu(x,t,\lambda) E_\gamma \hat{\tilde{\chi}}_\nu)^\perp \\ [\![X,Y]\!] &= \int_{-\infty}^{\infty} \mathrm{d}x \ \langle X(x), [\mathcal{J}, Y(y)]\rangle \,. \end{aligned} \tag{102}$$

Then using equations (49) after some calculations we obtain the following relations

$$\begin{aligned} \sigma^+_{1,\alpha}(\lambda,t) &= \frac{\mathrm{i}}{\alpha(\mathcal{K})} [\![[\mathcal{K}, \mathrm{ad}_{\mathcal{J}}^{-1} Q(x,t)], e_{1,-\alpha}(x,t,\lambda)]\!], \qquad \lambda \in l_1 \\ \sigma^-_{2,\beta}(\lambda,t) &= \frac{1}{\mathrm{i}\beta(\mathcal{K})} [\![[\mathcal{K}, \mathrm{ad}_{\mathcal{J}}^{-1} Q(x,t)], e_{1,\beta}(x,t,\lambda)]\!], \qquad \lambda \in l_2 \\ \tilde{\tau}^-_{1,\alpha}(\lambda,t) &= \frac{\mathrm{i}}{\alpha(\mathcal{K})} [\![[\mathcal{K}, \mathrm{ad}_{\mathcal{J}}^{-1} Q(x,t)], e_{1,\alpha}(x,t,\lambda)]\!], \qquad \lambda \in l_1 \\ \tilde{\tau}^+_{2,\beta}(\lambda,t) &= \frac{1}{\mathrm{i}\beta(\mathcal{K})} [\![[\mathcal{K}, \mathrm{ad}_{\mathcal{J}}^{-1} Q(x,t)], e_{1,-\beta}(x,t,\lambda)]\!], \qquad \lambda \in l_2 \end{aligned} \tag{103}$$

where $\alpha \in A_1$, $\beta \in A_2$ and

$$\tilde{\tau}^-_{1,\alpha}(\lambda,t) = \tau^-_{1,\alpha}(\lambda,t) \mathrm{e}^{(\alpha,\alpha) d^+_{1,\alpha}}, \qquad \tilde{\tau}^+_{2,\beta}(\lambda,t) = \tau^+_{2,\beta}(\lambda,t) \mathrm{e}^{-(\beta,\beta) d^-_{2,\beta}}.$$

These relations are fundamental for the analysis of the mapping $\mathcal{F}$ between the space of allowed potentials $Q(x,t)$ and the minimal set of scattering data. The main conclusion from them is that $\mathcal{F}$ has the meaning of generalized Fourier transform in which the 'squared solutions' $e_{\nu,\gamma}(x,t,\lambda)$ play the role of generalized exponents. Of course one must prove that $e_{\nu,\gamma}(x,t,\lambda)$ form complete set of functions in the space of allowed potentials. This can be done applying the contour integration method to a certain Green functions. This will be done elsewhere. Here we remark that the 'squared solutions' are eigenfunctions of the recursion operators. To state this more precisely we write down each 'squared solution' as sum of its projections according to the grading of $\mathfrak{g}$

$$e_{\nu,\gamma}(x,t,\lambda) = \sum_{s=0}^{h-1} e^{(s)}_{\nu,\gamma}(x,t,\lambda), \qquad e^{(s)}_{\nu,\gamma}(x,t,\lambda) \in \mathfrak{g}^{(s)} \tag{104}$$

and each of this projections should be split into orthogonal and parallel part as in equation (63)

$$\begin{aligned} e^{(s)}_{\nu,\gamma}(x,t,\lambda) &= e^{(s),\perp}_{\nu,\gamma}(x,t,\lambda) + e^{(s),\parallel}_{\nu,\gamma}(x,t,\lambda), \qquad e^{(2s),\parallel}_{\nu,\gamma}(x,t,\lambda) = 0 \\ e^{(2s+1),\parallel}_{\nu,\gamma}(x,t,\lambda) &= \frac{1}{c_s}\mathcal{J}^{2s+1}\left\langle \mathcal{J}^{h-2s-1}, e^{(2s+1)}_{\nu,\gamma}(x,t,\lambda)\right\rangle . \end{aligned} \tag{105}$$

It is also easy to check that the squared solutions satisfy the equation

$$\mathrm{i}\,\frac{\partial e_{\nu,\gamma}}{\partial x} + [Q(x), e_{\nu,\gamma}(x,t,\lambda)] - \lambda[\mathcal{J}, e_{\nu,\gamma}(x,t,\lambda)] = 0. \tag{106}$$

Inserting the splitting (104) into (106) we get

$$\mathrm{i}\,\frac{\partial e^{(s}_{\nu,\gamma})}{\partial x} + [Q(x), e^{(s}_{\nu,\gamma})(x,t,\lambda)] - \lambda[\mathcal{J}, e^{(s-1}_{\nu,\gamma})(x,t,\lambda)] = 0 \tag{107}$$

$s = 0, 1, \ldots, h-1$. Next we insert the splitting (105) and express the parallel parts of the squared solutions through the orthogonal ones. The calculations are similar to the ones in Subsection 5.1. Skipping the details we obtain

$$\begin{aligned} \Lambda_0 e^{(2k),\perp}_{\nu,\gamma}(x,t,\lambda) &= \lambda e^{(2k-1),\perp}_{\nu,\gamma}(x,t,\lambda) \\ \Lambda^{\pm}_{2k-1} e^{(2k-1),\perp}_{\nu,\gamma}(x,t,\lambda) &= \lambda e^{(2k),\perp}_{\nu,\gamma}(x,t,\lambda) - \frac{a^{\pm}_k}{c_k}\mathrm{ad}\,_{\mathcal{J}}^{-1}[Q(x), \mathcal{J}^{2k-1}] \end{aligned} \tag{108}$$

where

$$a^{\pm}_k = \lim_{x\to\pm\infty}\left\langle \mathcal{J}^{h-2k+1}, e^{(2k-1)}_{\nu,\gamma}(x,t,\lambda)\right\rangle . \tag{109}$$

If we choose ν and γ in such a way, that the constants $a^{\pm}_k = 0$ we find that $e^{(h-1),\perp}_{\nu,\gamma}(x,t,\lambda)$ are eigenfunctions of the master recursion operator (89)

$$\boldsymbol{\Lambda}_{\pm} e^{(h-1),\perp}_{\nu,\gamma}(x,t,\lambda) = \lambda^h e^{(h-1)}_{\nu,\gamma}(x,t,\lambda). \tag{110}$$

6.2. The Mapping $\delta\mathcal{F}$

The mapping between the variations of the potential and the variation of the scattering data is based on the following Wronskian relation

$$\left(\mathrm{i}\hat{\tilde{\chi}}_\nu \delta\tilde{\chi}_\nu(x,t,\lambda)\right)\Big|_{x=-\infty}^{\infty} = -\int_{-\infty}^{\infty} \mathrm{d}y\; \hat{\tilde{\chi}}_\nu \delta Q(y,t)\tilde{\chi}_\nu(y,t,\lambda). \tag{111}$$

Its left hand side on the rays l_1 and l_2 is given by

$$\begin{aligned} &\mathrm{i}(\hat{D}^+_1\hat{T}^-_1\delta(T^-_1 D^+_1) - \hat{S}^+_1\delta S^+_1), \qquad \text{for } \lambda\in l_1 \\ &\mathrm{i}(\hat{D}^-_2\hat{T}^+_2\delta(T^+_2 D^-_2) - \hat{S}^-_2\delta S^-_2), \qquad \text{for } \lambda\in l_2. \end{aligned} \tag{112}$$

Next we multiply both sides of (111) by E_γ and apply the Killing form. After this operation the right hand side of (111) acquires the form

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}y \left\langle \hat{\tilde{\chi}}_\nu \delta Q(y,t) \tilde{\chi}_\nu(y,t,\lambda), E_\gamma \right\rangle \\ &\qquad = \int_{-\infty}^{\infty} \mathrm{d}y \left\langle \delta Q(y,t), e_{\nu,\gamma}(y,t,\lambda) \right\rangle = -\left[\!\left[\mathrm{ad}_{\mathcal{J}}^{-1} Q(y,t), e_{\nu,\gamma}(y,t,\lambda) \right]\!\right] \end{aligned} \tag{113}$$

where we have used the 'squared solutions' $e_{\nu,\gamma}(x,t,\lambda)$ and the skew-scalar product $[\![X,Y]\!]$ introduced above. As a result we get

$$\begin{aligned} \delta\sigma_{1,\alpha}^{+}(\lambda,t) &= \mathrm{i}\left[\!\left[\mathrm{ad}_{\mathcal{J}}^{-1}\delta Q(x), e_{1,-\alpha}(x,t,\lambda) \right]\!\right], \qquad && \lambda\in l_1,\ \alpha\in A_1 \\ \delta\sigma_{2,\beta}^{-}(\lambda,t) &= \mathrm{i}\left[\!\left[\mathrm{ad}_{\mathcal{J}}^{-1}\delta Q(x), e_{1,\beta}(x,t,\lambda) \right]\!\right], && \lambda\in l_2,\ \beta\in A_2 \\ \delta'\tau_{1,\alpha}^{-}(\lambda,t) &= -\mathrm{i}\left[\!\left[\mathrm{ad}_{\mathcal{J}}^{-1}\delta Q(x), e_{1,\alpha}(x,t,\lambda) \right]\!\right], && \lambda\in l_1,\ \alpha\in A_1 \\ \delta\tau_{2,\beta}^{+}(\lambda,t) &= -\mathrm{i}\left[\!\left[\mathrm{ad}_{\mathcal{J}}^{-1}\delta Q(x), e_{1,-\beta}(x,t,\lambda) \right]\!\right], && \lambda\in l_2,\ \beta\in A_2. \end{aligned} \tag{114}$$

where

$$\delta'\tau_{1,\alpha}^{-}(\lambda,t) = \delta\tau_{1,\alpha}^{-}(\lambda,t) e^{(\alpha,\alpha)d_{1,\alpha}^{+}}, \qquad \delta'\tau_{2,\beta}^{+}(\lambda,t) = \delta\tau_{2,\beta}^{+}(\lambda,t) e^{-(\beta,\beta)d_{2,\beta}^{-}}.$$

Thus we conclude that the mapping $\delta\mathcal{F}$ also has the meaning of a generalized Fourier transform based on the same 'squared solutions' $e_{\nu,\gamma}(x,t,\lambda)$ as generalized exponents. This mapping and the formulae (114) are very important for analyzing the Hamiltonian properties of the relevant NLEEs.

We can derive useful relations also by multiplying both sides of (111) by $H_\alpha^\vee$ and applying the Killing form. The result is

$$\begin{aligned} \langle \hat{D}_{2\nu-1}^{+}\delta D_{2\nu-1}^{+}, H_\alpha^\vee\rangle &= \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}y\ \langle \delta Q(y,t) h_{2\nu-1,\alpha}^\vee(y,t,\lambda)\rangle, \qquad \lambda\in l_{2\nu-1} \\ \langle \hat{D}_{2\nu}^{-}\delta D_{2\nu}^{-}, H_\beta^\vee\rangle &= \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}y\ \langle \delta Q(y,t) h_{2\nu-1,\beta}^\vee(y,t,\lambda)\rangle, \qquad \lambda\in l_{2\nu} \end{aligned} \tag{115}$$

where $h_{\nu,\alpha}^\vee(x,t,\lambda) = \tilde{\chi}_\nu H_\alpha^\vee \hat{\tilde{\chi}}_\nu(x,t,\lambda)$ and $\alpha\in C^{\nu-1}A_1$ and $\beta\in C^{\nu-1}A_2$. Putting $\nu=1$ we have

$$\begin{aligned} \delta d_{1,\alpha}^{+}(\lambda) &= -\mathrm{i}\left[\!\left[\mathrm{ad}_{\mathcal{J}}\delta Q(x), h_{1,\alpha}^\vee(x,t,\lambda) \right]\!\right], \quad && \lambda\in l_1, \quad \alpha\in A_1 \\ \delta d_{2,\beta}^{-}(\lambda) &= -\mathrm{i}\left[\!\left[\mathrm{ad}_{\mathcal{J}}\delta Q(x), h_{1,\beta}^\vee(x,t,\lambda) \right]\!\right], && \lambda\in l_2, \quad \beta\in A_2. \end{aligned} \tag{116}$$

7. The Conservation Laws and Hamiltonian Structures

In order to treat the question of the conservation laws we need to introduce yet another type of Wronskian relations. They have the form

$$\left(\mathrm{i}\hat{\dot{\chi}}_\nu \dot{\chi}_\nu(x,t,\lambda) - x\mathcal{J}\right)\Big|_{x=-\infty}^{\infty} = \int_{x=-\infty}^{\infty} \mathrm{d}x \; \left(\hat{\chi}_\nu \mathcal{J} \chi_\nu(x,t,\lambda) - \mathcal{J}\right) \qquad (117)$$

where by 'dot' we denote the derivative with respect to λ.

The reason for considering these Wronskian relations is that they are related with the factors $D_\nu^\pm$ and from here to the conservation laws. Indeed, the left hand side of equation (117) is expressed through the scattering data of L as follows

$$\begin{aligned} &\left(\mathrm{i}\hat{\dot{\chi}}_1 \dot{\chi}_1(x,t,\lambda) - x\mathcal{J}\right)\Big|_{x=-\infty}^{\infty} = \hat{D}_1^\pm \dot{D}_1^\pm, \qquad \lambda \in l_1 \mathrm{e}^{\pm\mathrm{i}0} \\ &\left(\mathrm{i}\hat{\dot{\chi}}_2 \dot{\chi}_2(x,t,\lambda) - x\mathcal{J}\right)\Big|_{x=-\infty}^{\infty} = \hat{D}_2^\pm \dot{D}_2^\pm, \qquad \lambda \in l_2 \mathrm{e}^{\mp\mathrm{i}0}. \end{aligned} \qquad (118)$$

In order to evaluate the functions $d_{1,\alpha}^\pm(\lambda)$ and $d_{2,\beta}^\pm(\lambda)$ we can use the Killing form

$$\begin{aligned} \dot{d}_{1,\alpha_j}^\pm(\lambda) &= \langle \hat{D}_1^\pm \dot{D}_1^\pm, H_{\alpha_j}^\vee \rangle, \qquad \lambda \in l_1 \mathrm{e}^{\pm\mathrm{i}0}, \qquad \alpha_j \in A_1 \\ \dot{d}_{2,\beta_j}^\pm(\lambda) &= \langle \hat{D}_2^\pm \dot{D}_2^\pm, H_{\beta_j}^\vee \rangle, \qquad \lambda \in l_2 \mathrm{e}^{\pm\mathrm{i}0}, \qquad \alpha_j \in A_2 \end{aligned} \qquad (119)$$

where $H_{\alpha_j}^\vee$ and $H_{\beta_j}^\vee$ are dual to H_{α_j} and H_{β_j}

$$\langle H_{\alpha_j}^\vee, H_{\alpha_k} \rangle = \delta_{jk}, \qquad \langle H_{\beta_j}^\vee, H_{\beta_k} \rangle = \delta_{jk}. \qquad (120)$$

Thus we obtain

$$\begin{aligned} \dot{d}_{1,\alpha_j}^\pm(\lambda) &= \int_{-\infty}^{\infty} \mathrm{d}x \; \left(\langle \hat{\chi}_1 \mathcal{J} \chi_1(x,t,\lambda), H_{\alpha_j}^\vee \rangle - \langle \mathcal{J}, H_{\alpha_j}^\vee \rangle\right) \\ \dot{d}_{2,\beta_j}^\pm(\lambda) &= \int_{-\infty}^{\infty} \mathrm{d}x \; \left(\langle \hat{\chi}_2 \mathcal{J} \chi_2(x,t,\lambda), H_{\beta_j}^\vee \rangle - \langle \mathcal{J}, H_{\beta_j}^\vee \rangle\right). \end{aligned} \qquad (121)$$

The analyticity properties of $D_k^\pm(\lambda)$ allow one to reconstruct them from the sewing function $G(\lambda)$ (56) and from the locations of their simple zeros and poles but we are not going to treat these questions here.

It is well known, see for example [9], that the evolution equations related to the L operators we consider, (32), (36), possess $r = \operatorname{rank} \mathfrak{g}$ series of conservation laws. We will present below the formulae for the conservation laws obtained through the theory of the recursion operators. Their advantage, comparing with the formulae obtained via another approaches, is that they are compact in and give us the possibility to understand which of the conservation laws trivialize if we have reductions.

We are speaking below about the linear problem (36). So the constant element in it is $\mathcal{J}$ in (36)). The Cartan subalgebra that is relevant to the corresponding L will be denoted by $\mathfrak{h}$, it equals $\ker(\mathrm{ad}_{\mathcal{J}})$. Its orthogonal complement $((\ker(\mathrm{ad}_{\mathcal{J}}))^{\perp}$ will be denoted by $\overline{\mathfrak{g}}$ and the orthogonal projection on it by π_0. The potential function in (36) Q, it takes values in $\overline{\mathfrak{g}}$. The conservation laws are closely related to the adjoint solutions $h^{\vee}_{\nu,\alpha}(x,t,\lambda) = \tilde{\chi}_\nu H^{\vee}_\alpha \hat{\tilde{\chi}}_\nu(x,t,\lambda)$, which are defined above after the formula (115), or more generally, to the solutions $h_{\nu,H}(x,t,\lambda) = \tilde{\chi}_\nu H \hat{\tilde{\chi}}_\nu(x,\lambda)$, $H \in \mathfrak{h}$, $x,t \in \mathbb{R}$, $\lambda \in \Omega_\nu$. Even more precisely, relevant to us are the projections of these functions, namely $h^a_{\nu,H}(x,\lambda) = \pi_0 h_{\nu,H}(x,\lambda)$. Here of course $\tilde{\chi}_\nu(x,\lambda)$ is a FAS to the CBC system analytic in the sector Ω_ν. We have obvious analogs of the functions entering (121) which are defined for arbitrary $H \in \mathfrak{h}$, let us denote them by $d^{\pm}_{\nu,H}(\lambda)$. They are analytic in the sectors Ω_ν and for their λ-derivatives we have analogs of the relations (121)

$$\dot{d}^{\pm}_{\nu,H}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x \, \left(\langle \hat{\tilde{\chi}}_\nu \mathcal{J} \tilde{\chi}_\nu(x,t,\lambda), H\rangle - \langle \mathcal{J}, H\rangle \right). \tag{122}$$

In the sectors Ω_ν the functions $d^{\pm}_{\nu,H}(\lambda)$ have the following asymptotic behavior

$$d^{\pm}_{\nu,H}(\lambda) = \sum_{k=1}^{\infty} d^{\pm}_{H,k}\lambda^{-k}, \qquad |\lambda| \gg 1. \tag{123}$$

One can prove that actually the coefficients in the asymptotic expansion do not depend on the sector so we denote them by $d_{H,s}$.

The analytic and asymptotic properties of the functions $h^a_{\nu,H}(x,\lambda)$ for large $|\lambda|$ are crucial for the derivation of the conservation laws and they can be found in analogy with the case when the constant element in the operator L is real. However, though the final formulae we obtain are the same as in the real case and the main steps in the calculations are the same, there are some difficulties to overcome. We cannot go into more detais so we shall just sketch the main steps in this calculation and present the final results.

The first way to obtain the coefficients $d_{H,s}$ is to use the Wronskian-type relations (117) and the analytic properties of the functions $h^a_{\nu,H} = \pi_0 \tilde{\chi}_\nu H \hat{\tilde{\chi}}_\nu$, $H \in \mathfrak{h}$. In this way one is able to link the expansions of q over the adjoint solutions (obtained using the map $\mathcal{F}$, see the Wronskian relations (99)) with the functions $h^a_{\nu,H}$. Next one needs to use asymptotic formulae for $h^a_{\nu,H}$ and from there to calculate the quantities $d_{H,s}$. Thus one can obtain the formula

$$d_{H,s} = \frac{1}{s} \int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{-\infty}^{x} \langle [\mathcal{J}, Q], \Lambda^s_{\pm} \mathrm{ad}^{-1}_{\mathcal{J}} [H, Q] \rangle \mathrm{d}y, \quad s = 1,2,\ldots \tag{124}$$

where $\Lambda_\pm$ are the recursion operators for the case when on q are not imposed any conditions. One can prove that these conservation laws have local densities and are in involution with respect to a hierarchy of symplectic forms, see below.

The second way of obtaining the conservation laws is to use the Wronskian type relations involving the variation of the potential δQ, see (111). In this way we get the formula

$$\delta d_{H,s} = -\mathrm{i} \int\limits_{-\infty}^{+\infty} \langle \delta Q, \Lambda_\pm^{s-1} \mathrm{ad}_{\mathcal{J}}^{-1} [H, Q] \rangle \mathrm{d}x, \quad s = 1, 2, \dots \tag{125}$$

which is more popular in another form. In order to obtain it, let us identify the space $\mathcal{M}_{\mathcal{J}}$ consisting of Schwartz-type functions on the line with values in $\mathfrak{h}^\perp = \overline{\mathfrak{g}}$ and its dual $\mathcal{M}_{\mathcal{J}}^*$ through the bilinear form

$$\langle\langle X, Y \rangle\rangle = \int\limits_{-\infty}^{+\infty} \langle X(x), Y(x) \rangle \mathrm{d}x.$$

In other words, we shall consider the elements from $\mathcal{M}_{\mathcal{J}}^*$ as generalized functions (distributions) and a generalized function, say ξ, will be written as

$$\langle\langle \xi, Y \rangle\rangle = \int\limits_{-\infty}^{+\infty} \langle \xi(x), Y(x) \rangle \mathrm{d}x, \qquad Y(x) \in \mathcal{M}_{\mathcal{J}}.$$

As a matter of fact the generalized functions we have are regular, that is represented by locally Lebesgue integrable functions over $\mathbb{R}$, and even most of them belong to $\mathcal{M}_{\mathcal{J}}$. Taking into account the identification for the differentials of the conservation laws we get

$$\mathrm{d}d_{H,s} = -\mathrm{i}\Lambda_\pm^{s-1} \mathrm{ad}_{\mathcal{J}}^{-1} [H, Q] \tag{126}$$

and hence

$$\mathrm{d}d_{H,s} = \Lambda_\pm \mathrm{d}d_{H,s-1}, \qquad s = 2, 3, \dots. \tag{127}$$

The above relations in the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ are called *Lenart relations*, see [2], so we shall call them Lenart-type relations or Lenart chains.

In fact one can prove that with the above identification $\mathrm{d}d_{\nu,H} = \mathrm{i}h_{\nu,H}^a$ which explains why the functions $h_{\nu,H}^a$ are so important in the study of the conservation laws. Using (127) and the Poincaré lemma for closed one-forms one gets another formula

$$d_{H,s} = -\mathrm{i} \int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_0^1 \langle Q, \Lambda_\pm^{s-1}|_{(\zeta Q)} \, \mathrm{ad}_{\mathcal{J}}^{-1} [H, \zeta Q] \rangle \mathrm{d}\zeta \tag{128}$$

where by $\Lambda_{\pm}|_{(\zeta Q)}$ is denoted the recursion operator in which Q is substituted by $Q' = \zeta Q$. This formula can be obtained also directly. Indeed, let us consider instead of the potential q the potential $Q' = \zeta Q$ where $0 \leq \zeta \leq 1$ is a real parameter. Let us consider variation of the potential Q' of the form $\delta Q' = Q\delta\zeta$. Then (125) implies

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} d_{H,s} = -\mathrm{i} \int\limits_{-\infty}^{+\infty} \langle Q, \Lambda_{\pm}^{s-1}|_{Q\to Q'} \mathrm{ad}_{\mathcal{J}}^{-1}[H, Q'] \rangle \mathrm{d}x, \quad s = 1, 2, \dots. \tag{129}$$

Integrating over ζ between 0 and 1 and taking into account that for $\zeta = 0$ we have $Q' = 0$, $d_{H,s} = 0$ and for $\zeta = 1$ we have $Q' = Q$ we obtain the formula (128).

When one calculates the hierarchy of conservation laws the last form of the conservation laws can be a real advantage as the expressions become more and more complicated when s increases and in it $\Lambda_{\pm}$ enters with power $s-1$ while in (124) $\Lambda_{\pm}$ enters with power s.

When one has $\mathbb{Z}_h$ reductions the above formulae for the conservation laws remain true and naturally the conservation laws continue to have local densities. However, one can observe that some of them trivialize, that is they become identically zero. Indeed, since $\mathrm{ad}_{\mathcal{J}}^{-1}$ takes $\overline{\mathfrak{g}}^{(k)}$ into $\overline{\mathfrak{g}}^{(k-1)}$, $\Lambda_{\pm}^{m}$ takes functions with values in $\overline{\mathfrak{g}}^{(k)}$ to functions with values in $\overline{\mathfrak{g}}^{(k-m)}$ so if $H \in \mathfrak{h}^{(k)} = \mathfrak{g}^{(k)} \cap \mathfrak{h}$ (k must be exponent of course) then unless $k - s = 0 \ (\mathrm{mod}(\mathrm{h}))$ the expression (124) is identically zero. Of course, the same conclusion is obtained if one uses the expression (128). Then assumming that $0 \leq k \leq h-1$ for example in the hierarchies (124) 'survive' only the following integrals of motion

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{-\infty}^{x} \langle [\mathcal{J}, Q], (\Lambda_{\pm}^{p})^{n} \Lambda_{\pm}^{h-k} \mathrm{ad}_{\mathcal{J}}^{-1}[H, Q] \rangle \mathrm{d}y, \quad n = 1, 2, \dots \tag{130}$$

or, if one prefers the notation through $\mathbf{\Lambda}_{\pm}$, as for example in (110), from the hierarchy of integrals of motion remain only

$$\begin{aligned} &\int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{-\infty}^{x} \langle [\mathcal{J}, Q], \mathbf{\Lambda}_{\pm}^{n}(Q[H, k]) \rangle \mathrm{d}y, \qquad n = 1, 2, \dots \\ &Q[H, k] = \Lambda_{\pm}^{h-k} \mathrm{ad}_{\mathcal{J}}^{-1}[H, Q]. \end{aligned} \tag{131}$$

One sees that $Q[H, k]$ takes values in $\mathfrak{g}^{(h-1)}$ and the role of the recursion operator is played now by $\mathbf{\Lambda}_{\pm}$. The observation that in case of reductions $\mathbb{Z}_h$ reductions have gaps in the conservation laws sequencie has been made in [32].

Maybe we must say here that we went a bit too quickly into the discussion of the conservation laws. In fact we ought to discuss first equation (126) and the

Lenart chains. One can see that when we have reductions (126) should be modified because when one identifies a linear functional over $\mathfrak{g}^{(0)}$ with an element from $\mathfrak{g}$ this element should belong to $\mathfrak{g}^{(0)}$, because $\mathfrak{g}^{(0)}$ is orthogonal to $\mathfrak{g}^{(k)}$ for $k \neq 0$. So in case we have reductions (126) should be replaced by

$$\mathrm{d}d_{H,s} = -\mathrm{i}p_0\Lambda_{\pm}^{s-1}\mathrm{ad}\,_{\mathcal{J}}^{-1}[H,Q] \tag{132}$$

where p_0 is the orthogonal projection on $\mathfrak{g}^{(0)}$. Thus one sees again that if $H \in \mathfrak{h}^{(k)}$ then unless $k - s = 0\ (\mathrm{mod(h)})$ the above expression is identically zero. It is also readily seen that the Lenart chains generate from a given nontrivial conservation law a nontrivial one if one acts not with $\Lambda_{\pm}$ but with $\Lambda_{\pm}^{h}$, or if one prefers by $\boldsymbol{\Lambda}_{\pm}$ as for example in (110). As an illustration we give the two first integrals of motion of the $\mathbb{Z}_h$-DNLS equation

$$I_{1,1}^{(1)} = \frac{1}{2\omega}\int_{-\infty}^{\infty}\mathrm{d}x\,\sum_{p=1}^{n}\psi_p\psi_{n-p}(x,t) \tag{133}$$

$$\begin{aligned} I_{1,1}^{(2)} = \frac{1}{2\omega^2}\int_{-\infty}^{\infty}\mathrm{d}x\,\Bigg\{ & \sum_{p=1}^{n}\mathrm{i}\,\mathrm{cotan}\,\left(\frac{\pi p}{n}\right)\left(\frac{\mathrm{d}\psi_p}{\mathrm{d}x}\psi_{n-p} - \psi_p\frac{\mathrm{d}\psi_{n-p}}{\mathrm{d}x}\right) \\ & -\frac{2}{3}\sum_{p+k+l=n}\psi_p\psi_k\psi_l(x,t)\Bigg\}. \end{aligned} \tag{134}$$

Let us mention very briefly the hierarchies of the Hamiltonian structures for the soliton equations we study. Two approaches are possible here. One is based on the hierarchy of Poisson structures, the other on the hierarchy of symplectic structures. The Poisson structures are easier to construct in the general case of non-restricted systems, see for example [29]. From the other side, if one has reductions then one must calculate the corresponding Dirac brackets. The symplectic structures seem more complicated to construct but offer the advantage that the restrictions are immediate, provided they do not degenerate. It is well-known that the equations without any reductions possess a hierarchy of Hamiltonian structures with symplectic forms Ω_m that can be written as follows

$$\Omega_m(X,Y) = \int\limits_{-\infty}^{+\infty}\langle X, \Lambda_{\pm}^{m}\mathrm{ad}\,_{\mathcal{J}}^{-1}(Y)\rangle\mathrm{d}x \tag{135}$$

where $X(x), Y(x)$ are smooth functions with values in $\overline{\mathfrak{g}}$ – the orthogonal complement of $\mathfrak{h}$, see [20] and the numerous citations therein. However, due to the fact that q takes values in $\overline{\mathfrak{g}}^{(0)}$ some of these structures degenerate. Indeed, if

X, Y take values in $\overline{\mathfrak{g}}^{(0)}$ then $\mathrm{ad}_{\mathcal{J}}^{-1}Y$ takes values in $\overline{\mathfrak{g}}^{(-1)}$, $\Lambda_{\pm}^{m}\mathrm{ad}_{\mathcal{J}}^{-1}Y$ takes values in $\overline{\mathfrak{g}}^{(-m-1)}$ and unless $m+1 = 0\ (\mathrm{mod(h)})$ the form is identically zero. Let $m = kh - 1$. Then

$$\Omega_m(X,Y) = \int\limits_{-\infty}^{+\infty} \langle X, (\Lambda_{\pm}^{h})^k \Lambda_{\pm}^{-1}\mathrm{ad}_{\mathcal{J}}^{-1}(Y)\rangle \mathrm{d}x = \Omega_{-1}(X, (\Lambda_{\pm}^{h})^k(Y)) \qquad (136)$$

again demonstrating that the operator that generates the symplectic structures is now $\mathbf{\Lambda}_{\pm} = \Lambda_{\pm}^{h}$. This fact has been also proved with geometric methods in [40] using the theory of the so-called Poisson-Nijenhuis manifolds.

8. Conclusions

As it is well-known, the generic integrable NLEEs related with the generalized Zakharov-Shabat system and its generalization – the CBC system defined on a semisimple Lie algebra of rank r possesses a number of interesting properties. Among them are: they possess a hierarchy of Hamiltonian structures, r series of conservation laws etc. Both the Hamiltonian hierarchy and the series of conservation laws are generated by a certain recursion operator (called also Λ-operator) [1,11,14,15,19,20]. Its spectral properties and the expansions over the eigenvectors of Λ have deep applications to the theory of the corresponding NLEEs, see e.g. [20]. The case when we have reductions is of big importance and in this case we have some specifics, see [11], which must be taken care of when studying the properties of the corresponding Λ-operator. In this article we analysed $\mathbb{Z}_h$ and $\mathbb{D}_h$ reductions. In case we have these type of reductions and we have NLEE represented in a Lax form $[L, M] = 0$ with $L\psi = 0$ being a CBC type system with reductions, the coefficient in front of the leading power of λ in the operator M see (35) should be an exponent of the corresponding algebra $\mathfrak{g}$. Since 2 is an exponent only for the series A_r, there would be no new examples of $\mathbb{Z}_h$-reduced Nonlinear Schrödinger type equations. However, 3 is an exponent for all the algebras from the classical series. Therefore choosing M in the Lax representation to be cubic in λ one gets series of KdV type equations and the simplest of them are already known, see [8,9,38]. The theory we develop applies to these type of equations and of course to those corresponding to exponents larger than 2 and 3.

Our results show the specific way the generating operator factorizes in case of reductions. Along with the elementary recursion operators – the restriction of the standard Λ on $\mathfrak{g}^{(k)}$, we have introduced fundamental recursion operators $\mathbf{\Lambda}_{(m_k)}$ and the master recursion operator $\mathbf{\Lambda}$. The operators $\mathbf{\Lambda}_{(m_k)}$ generate the MKdV-type NLEEs, and combined with $\mathbf{\Lambda}$ – the hierarchy of their Hamiltonian structures. Thus we have outlined the effects of the reduction group on the recursion operators. Important questions, that will be answered in next publications concern the spectral theory of the recursion operators, see [14–16, 19, 22] and their geometrical properties, see [31, 40, 41, 46] and the monograph [20].

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